HEC MONTRÉAL
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Evaluation of Counterparty Credit Risk under Netting Agreements

par
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Résumé

La crise de 2008 a révélé les implications systémiques du risque de contrepartie lorsque la faillite de Lehman Brothers, un acteur majeur sur le marché de gré à gré, a déclenché une instabilité généralisée. En réponse, les institutions financières et régulateurs ont adopté des mesures permettant de se protéger du risque de contrepartie, dont l'ajustement de la valeur de crédit (AVC, mieux connu sous le vocable de CVA en anglais).

Le CVA représente la valeur de marché du risque de contrepartie. Il permet d'ajuster la valeur d'un contrat pour tenir compte d'un défaut potentiel. Le Comité de Bâle préconise l'utilisation du CVA pour gérer le risque de contrepartie, et recommande aux institutions financières de maintenir des réserves de capital adéquates sur la base de la valeur du CVA. Estimer avec précision le CVA est par conséquent important pour assurer la compétitivité et la résilience des institutions.

Cette thèse traite du calcul du CVA dans le cas de portefeuilles sujets à un accord de compensation comportant des titres avec possibilité d'exercice anticipé. La compensation, en consolidant la valeur de contrats bilatéraux en cas de défaut, est une pratique d'atténuation du risque de contrepartie. Elle se traduit généralement par un CVA inférieur à la somme des CVA individuels, du fait de la diminution globale du risque de contrepartie. Cependant, un accord de compensation peut modifier la stratégie d'exercice des instruments financiers avec possibilité d'exercice anticipé.

Cette thèse présente une approche pour l'évaluation du CVA (ou BVA dans le cas de risque bilatéral) pour des portefeuilles d'instruments dérivés comportant des caractéristiques d'exercice anticipé, lorsque ces portefeuilles sont soumis à un accord de compensation. À notre connaissance, il s'agit de la première analyse des effets de la compensation sur l'évaluation du risque de contrepartie lorsque les deux parties ont des droits d'exercice anticipé.

Dans le premier essai, nous développons un modèle récursif pour déterminer la valeur d'un jeu à somme nulle représentant l'interaction entre les deux parties. Cette valeur permet de calculer le CVA/BVA en la comparant à celle d'un portefeuille équivalent, mais sans risque. La solution par programmation dynamique caractérise l'ajustement comme une fonction des divers facteurs risque, de la composition du portefeuille, en tout moment jusqu'à l'échéance.

Dans le deuxième essai, nous montrons comment les accords de compensation peuvent modifier les stratégies d'exercice, même en cas de risque unilatéral. Nous illustrons également l'impact de la variation de divers paramètres sur les stratégies d'exercice. Nos résultats remettent en question les méthodes conventionnelles d'évaluation du CVA qui négligent l'effet d'un accord de compensation sur le risque de contrepartie et sur les stratégies d'exercice.

Dans le troisième essai, nous estimons le quantile (VaR) du CVA pour divers horizons dans le cas de portefeuilles comportant un accord de compensation. Contrairement à ce qui pourrait être attendu, nous constatons que la présence d'un accord de compensation n'atténue pas systématiquement les valeurs extrêmes du CVA. Il est même possible que la volatilité du CVA augmente en cas de compensation, en raison de la modification des décisions d'exercice anticipé, qui affectant l'exposition au risque.

Mots-clés

Risque de contrepartie, CVA, BVA, exercice anticipé, accords de compensation, valeurs extrêmes.

Méthodes de recherche

Programmation dynamique, théorie des jeux, modélisation mathématique.

Abstract

The 2008 financial crisis revealed the systemic implications of counterparty credit risk (CCR) when the bankruptcy of Lehman Brothers, a major over-the-counter (OTC) market player, triggered widespread instability. In response, financial institutions and regulators adopted measures to protect against CCR, including credit valuation adjustment (CVA).

The CVA represents the market value of counterparty credit risk. It adjusts the value of a contract to account for the counterparty's potential default. The Basel Committee advocates for using CVA to manage CCR and mandates financial institutions to maintain adequate capital reserves based on the CVA calculation. Thus, accurately estimating CVA is important to ensure the competitiveness and resilience of institutions.

This thesis addresses the calculation of CVA for portfolios subject to a netting agreement involving securities with early exercise features. Netting, by consolidating the value of bilateral contracts in the event of default, is a CCR mitigation practice. It generally results in a lower CVA than the sum of individual CVAs, due to offsetting positions reducing overall CCR exposure. However, a netting agreement can change the exercise strategy of financial instruments with early exercise features.

This thesis presents an approach for the valuation of CVA (or BVA in the case of bilateral risk) for portfolios of derivative instruments with early exercise features, when these portfolios are subject to a netting agreement. To our knowledge, this is the first analysis of the effects of netting on CCR valuation when both parties have early exercise rights.

In the first essay, we develop a recursive model to determine the value of a zero-sum game representing the interaction between the two parties. This value allows the calculation of CVA/BVA by comparing it to that of an equivalent portfolio, but without risk. The dynamic programming solution characterizes the adjustment as a function of the various risk factors, the portfolio composition, at any time until maturity.

In the second essay, we show how netting agreements can change exercise strategies, even for the party that is not exposed to CCR. We also illustrate the impact of changing various parameters on exercise strategies. Our results call into question conventional CVA valuation methods that neglect the effect of a netting agreement on CCR and exercise strategies.

In the third essay, we estimate the Value-at-Risk (VaR) of the CVA for various horizons in the case of portfolios with a netting agreement. Contrary to expectations, we find that the presence of a netting agreement does not systematically mitigate CVA extreme values. It is even possible for CVA volatility to increase with netting, due to changes in early exercise decisions affecting risk exposure.

Keywords

Counterparty risk, CVA, BVA, Early exercise, Netting, CVA VaR.

Research methods

Dynamic programming, Game theory, Mathematical modeling.

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List of Acronyms

BVA Bilateral Valuation Adjustment

CCR Counterparty Credit Risk

CVA Credit Valuation Adjustment

DP Dynamic Programming

DVA Debt Valuation Adjustment

GBM Geometric Brownian Motion

LSM Least-Squares Monte Carlo

VaR Value at Risk

To my cherished mother, Fereshteh,
my origin of affection and inspiration.
To my beloved wife, Maedeh,
my fountain of boundless love and support.
To my wonderful brother, Amirsalar,
my spring of generosity and wisdom.
To my incredible family, my universe.

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Introduction

Risk management has evolved from a protection mechanism into a central component in financial decision-making. This evolution is due to the rising complexity of financial markets, the emergence of innovative instruments, and the increased interconnectedness of global economies. Today, risk management enables informed decisions that shape institutions' trajectories, and influence capital allocation, product development, and market entry.

One critical aspect of risk management that has gained prominence is Counterparty Credit Risk (CCR). CCR is the risk that a counterparty in a financial contract may default or fail to meet its obligations, leading to potential financial loss for the other party. This element of risk management has gained significant prominence, particularly in the context of the Over-the-Counter (OTC) market. The OTC market is a domain of customized financial contracts, traded away from public view, yet holding significant impact on global financial stability.

The over-the-counter (OTC) market facilitates the direct exchange of various financial instruments such as derivatives, swaps, and forwards. These instruments address specific risk management, hedging, and speculation needs, making them key tools for financial institutions and companies. The scale of the OTC derivatives market is immense - the Bank for International Settlements estimates the notional value of outstanding OTC derivatives exceeded \$600 trillion as of June 2022, see *OTC derivatives statistics at end-June 2022* (2022). This vast market not only offers companies avenues for raising capital but also provides investors with a broader investment spectrum, enhancing liquidity. However, the decentralized nature and lack of a centralized clearinghouse amplify counterparty credit risk in the OTC market. Consequently, implementing rigorous risk management strategies and regulatory measures is necessary to mitigate adverse effects and ensure financial stability.

The 2008 financial crisis highlighted the far-reaching impacts of CCR on the financial system. The 2008 global financial crisis highlighted the systemic risks posed by counterparty credit risk (CCR) in the OTC market. The failure of Lehman Brothers, a major OTC derivatives participant, triggered a chain reaction of defaults that led to widespread instability. This demonstrated the potential for a single institution's failure to produce a systemic crisis in an interconnected system. The crisis underscored inherent OTC market vulnerabilities and the critical need for robust risk management tools and strict regulations to protect against such events. In response, financial institutions and regulators worldwide focused on risk management tools, one being the credit valuation adjustment (CVA). While CVA was not a new concept (refer to Duffie and Singleton (2003), Bielecki and Rutkowski (2004), and Brigo and Masetti (2005) for early review), the crisis emphasized its critical role and need for its careful implementation.

CVA serves as a modification to the fair value of derivative contracts, accounting for CCR. It quantifies the market value of credit risk embedded in a transaction. CVA can also be viewed as an expected discounted loss - a measure of the potential loss a financial institution could incur due to a counterparty's default, discounted to its present value. This perspective underscores CVA's risk management role.

The Basel Committee on Banking Supervision has introduced the Basel Accords, regulatory frameworks aimed at strengthening banking regulation, supervision, and risk management. The Basel II Accord marked a significant shift in the regulatory landscape for financial institutions in CCR management. It mandated that banks maintain adequate capital to cover potential counterparty credit losses, a requirement directly influenced by CVA. By quantifying risk for each counterparty, CVA plays a crucial role in determining the capital banks must set aside.

The Basel III Accord, see *Basel III: A global regulatory framework for more resilient banks and banking systems* (2011), has placed a renewed emphasis on CVA. It requires financial institutions to calculate a CVA risk capital charge, which is an amount of capital that banks must set aside to absorb potential losses from CVA fluctuations due to changes in counterparty credit spreads and market factors. Introducing the CVA risk capital charge has not only increased capital requirements but also necessitated a more integrated approach to accurately measure and manage this risk.

The accuracy of CVA calculations holds significant implications for an institution's financial health. Overestimating CVA can lead to excessive risk provisions, over-allocation of capital, and diminished returns and competitiveness. Conversely, underestimating CVA results in an inadequate buffer against potential losses, exposing the institution to heightened risk and instability. Therefore, a precise CVA level protects the institution's interests and ensures sound risk management.

Several factors shape CVA computation, including the counterparty's default likelihood, exposure at default, and underlying asset market risk. Exposure at default (EAD) assesses the potential loss if a counterparty defaults, representing the positive mark-to-market value of a contract at the default date. This introduces non-linearities in counterparty risk pricing, making CVA akin to a zero-strike call option on a default-free asset, but with a twist. The "option's" maturity date is not fixed but tied to the counterparty's potential default date. This means pricing a CVA can be as, if not more, intricate as pricing the derivative instrument itself.

Calculating CVA is relatively straightforward for simple financial instruments like bonds or European options, thanks to analytical formulas. However, complexity increases for products with early exercise features. Here, counterparty risk can alter the exercise strategy, making CVA more than a simple expected loss. The early exercise decision depends not just on market conditions but also on counterparty credit risk. If that risk is high, the option holder may exercise earlier to reduce exposure, significantly affecting the CVA value.

The CVA computation becomes even more complex when considering a portfolio of contracts under a netting agreement. Netting, commonly used to mitigate counterparty credit risk, refers to the agreement that should a default occur, all transactions between the two parties will be consolidated and treated as one. This is particularly beneficial when a financial institution has multiple derivatives with the same counterparty, as it allows balancing positive and negative exposures, thereby reducing potential loss upon default.

CVA assessment fulfills two primary roles: modifying pricing to reflect counterparty risk and determining regulatory capital needs for effective risk management. In the context of netted portfolios, the emphasis is predominantly on calculating the capital charge, owing to the risk mitigation advantages provided by netting agreements. Effective CVA assessment enables firms to balance risk management, capital planning, and regulatory compliance for consolidated exposures across netted portfolios.

When the netted portfolio contains derivatives with early exercise opportunities, the exercise decision for each claim cannot be made individually. It can depend on the condition of all contracts and the default probabilities of parties. This interdependence within a portfolio adds another layer of complexity to CVA computation. To benefit from the mitigating effect of netting on counterparty credit risk, it is imperative to compute CVA values at the portfolio level rather than individual contracts, accounting for the complex interplay between various risk factors, default probabilities, and exercise decisions.

Moreover, achieving this level of accuracy in CVA evaluation can be computationally challenging. The portfolio-level calculation may necessitate advanced computational techniques to handle the high dimensionality. High dimensionality arises from the multitude of risk factors influencing the contracts within the netted portfolio. As the number of factors increases, computational complexity grows exponentially, making it difficult for traditional methods to cope. Despite these challenges, the benefits of accurate portfolio-level CVA computation far outweigh the costs, justifying the investment for financial institutions.

In situations where both contract parties have a default probability, CVA alone does not fully capture the intricacies of counterparty credit risk. It neglects the potential impact of the institution's own default risk on the risk adjustment value. To address this gap, the Bilateral Valuation Adjustment (BVA) was introduced, incorporating both CVA and the institution's credit risk, known as Debt Valuation Adjustment (DVA). This dual risk perspective embodied in BVA provides a more comprehensive CCR measure, acknowledging risk as a two-way street where both parties' creditworthiness plays a crucial role in determining the overall contract risk.

This thesis comprehensively examines CCR within netting agreements. It offers an in-depth understanding of the dynamics of CCR and netting agreements on managing portfolios with derivatives having early exercise features, and evaluating risk adjustments for these portfolios.

The first essay develops an approach for calculating risk adjustment values (CVA or BVA) for a portfolio of options subject to CCR. It focuses on general situations where both parties have early exercise opportunities and are exposed to default risk. We show netted sets are dynamic games when both parties have early exercise chances. We introduce a model embodying strategic interactions among parties and a dynamic programming algorithm to determine the netted portfolio value and optimal exercise strategies. It also contains numerous illustrations offering a deeper understanding of risk adjustment values within the context of netting agreements.

Our goal in the second essay is to fill an existing literature gap by providing valuable insights into managing options with early exercise opportunities within a netted portfolio. We offer a thorough analysis of the impact of default risk under netting agreements on the exercise strategy of Bermudan options. The essay also critically examines conventional methodologies used for assessing the risk adjustment value of netted portfolios, arguing these methods often neglect the role of netting on the exercise mechanism, potentially leading to misestimation of CVA and BVA.

In the third essay, we leverage the CVA pricing model from the first essay to assess CVA variability across risk horizons in netted portfolios. Our objective is exploring the impact of netting agreements on CVA tail risk. Through numerical experiments, we show the influence of netting on CVA tail risk is not as straightforward as its general CVA reduction ability. Complex dynamics from the interplay of netting agreements, early exercise frequencies, and default probabilities result in distinct outcomes for each case. Therefore, conducting thorough, portfolio-specific analyses is crucial to accurately evaluate these effects.

The remainder of this thesis is organized into five chapters. Chapter One provides a literature review. Chapter Two presents the first essay, introducing the approach for calculating risk adjustment values for netted portfolios subject to CCR. Chapter Three, including the second essay, examines how default risk influences exercise strategies for Bermudan options under netting. Chapter Four, presenting the third essay, explores the impact of netting on CVA tail risk. The thesis concludes with Chapter Five, summarizing key findings and implications.

Chapter 1

Literature Review

1.1 The Evolution of CVA and Netting Research

The computation of Credit Valuation Adjustment (CVA) has been a key focus of extensive research, as various models have been proposed to address the impact of Counterparty Credit Risk (CCR) on financial contract values. These models can generally be divided into two main categories: structural and intensity models.

Structural models, as the name suggests, are based on the structure of the firm's balance sheet. First introduced by Merton (1974), these models utilize variables such as a firm's debt-to-equity ratio or asset value to predict default likelihood. The fundamental assumption is that default occurs when a firm's liabilities exceed its assets at debt maturity. This approach offers a comprehensive perspective on credit risk by considering the complex interplay between a firm's assets, liabilities, and equity. Notable works on the structural credit risk framework include Hull and White (1995), Klein (1996), Leland (1998), and Zhou (2001).

Intensity models, on the other hand, are based on the hazard rate, which represents the instantaneous probability of default. First proposed by Jarrow and Turnbull (1995), these models assume default events result from exogenous jump processes. The key advantage is their ability to capture the randomness of defaults and ease of calibration to market data. Significant contributions to the literature on credit risk intensity models include Lando (1998), Duffie and Singleton (1999), and Brigo and Alfonsi (2005).

Regardless of the chosen model—structural or intensity—the CVA for European-style options can be achieved through an analytical formula. Moreover, for advanced cases with many stochastic factors or complex correlation structures, semi-analytical approaches have recently been proposed to compute the CVA for European-style options (see Kim and Leung (2016), Brigo and Vrins (2018), and Antonelli, Ramponi, and Scarlatti (2022)). However, American-style options, exercisable before expiration, lack a closed-form solution. The potential for early exercise requires incorporating an optimal stopping problem into the CVA computation, adding complexity. The holder must determine the best exercise time, and the interplay between defaults and early exercise must be considered.

In the context of contracts with optional exercise features, financial institutions often employ simulation-regression techniques, primarily based on Least Square Monte Carlo (LSMC), for CVA computation (see Longstaff and Schwartz (2001), Tsitsiklis and Van Roy (2001), Glasserman (2004), and Broadie, Glasserman, et al. (2004)). As explained in Cesari et al. (2009) and Brigo, Morini, and Pallavicini (2013), these methods consider exercise policy and counterparty risk as independent phenomena. However, Klein and Yang (2013) and Breton and Marzouk (2018) have shown that counterparty risk can influence exercise behavior, suggesting they should not be treated in isolation.

Building on the topic of netted portfolio valuation, this review turns to exploring various issues in calculating risk adjustments under netting agreements. As noted by Brigo and Masetti (2005), netting agreements offer a means to decrease counterparty credit risk, as the CVA of a netted portfolio is often less than the sum of individual CVAs. This is predominantly driven by offsetting positions within the portfolio, which reduce overall exposure. Notably for derivatives products, this benefit is evident in portfolios with a combination of long and short positions towards a counterparty, underscoring the importance of calculating CVA at the portfolio level rather than per claim.

To address this need, Brigo and Masetti (2005) introduced an approximate formula to assess the CVA of a netted portfolio of interest rate swaps, assuming counterparty default follows an intensity model. Building on this, Brigo and Pallavicini (2007) employed Monte Carlo techniques to evaluate the CVA for a netted portfolio of swaps, accounting for the correlation between defaults and interest rates.

The exploration of bilateral risk for netted portfolios has also been the focus of numerous studies. Key contributions to discussions around assessing bilateral counterparty risk for individual claims have been made by Brigo and Capponi (2008), Brigo, Buescu, and Morini (2011), and Gregory (2017). For netted portfolios, Brigo, Pallavicini, and Papatheodorou (2009) extended the work of Brigo and Pallavicini (2007), showing that risk adjustment with both parties at default risk involves a long position in a put option and a short position in a call option, both with zero strikes. These options are written on the residual net portfolio value at relevant default times. Furthering this, Durand (2010) proposed an iterative evaluation procedure for the bilateral CVA (BVA) of a netted portfolio.

Recent studies have delved deeper into the challenges of evaluating netted portfolios, focusing on complications from netting agreements. Burgard and Kjaer (2017) and Brigo, Francischello, and Pallavicini (2019) studied portfolios across various netting sets with different defaultable counterparties. As emphasized by Brigo, Francischello, and Pallavicini (2019), counterparty credit risk introduces inherent nonlinearity, such that overall portfolio value differs from the sum of netting set values. Additionally, Ballotta, Fusai, and Marazzina (2019) introduced a structural method to compute CVA for a netted portfolio, analyzing the impacts of collateralization and wrong-way risk on CVA.

While the existing literature on risk adjustment evaluation with netting agreements provides a substantial foundation, it predominantly focuses on European-style contracts. This leaves a significant gap concerning portfolios incorporating early exercisable contracts. To the best of our knowledge, the only study venturing into this domain is Andersson and Oosterlee (2020), which considers the early exercise feature for only one party in the netting agreement. It proposes a deep learning technique to estimate the exercise policy of portfolio options and the CVA. Therefore, there is a clear opportunity for further research to delve deeper into this under-explored area. By doing so, we can gain a more comprehensive understanding of the intricate interplay between CVA, netting, and early exercise features, significantly enhancing our knowledge in managing counterparty credit risk.

1.2 The Methodological Evolution in this research

The CVA can be determined by differentiating between the value of a position exposed to counterparty credit risk (CCR) and a similar position not subject to this risk. This perspective allows us to view CVA calculation as valuing a contract or portfolio of contracts in the presence of CCR.

Dynamic programming (DP) is a well-established technique for pricing American-style financial contracts and determining optimal exercise policies. As an optimization technique, DP excels at solving recursive problems in *Markov decision processes* (MDPs). MDPs are mathematical structures used for modeling decision-making in stochastic dynamic environments. For a detailed understanding of DP and MDPs, refer to Bertsekas (2012). Formulating American financial derivatives as MDPs enables applying DP for pricing such contracts and determining optimal exercise policies. For further reading on evaluating derivatives with early exercise features in the absence of CCR, consider Ben-Ameur, Breton, and L'Ecuyer (2002), chapter 8 of Glasserman (2004), Ben-Ameur, Breton, Karoui, et al. (2007), and Breton and Frutos (2012).

1.2.1 Dynamic Games: A Multi-Agent Framework for Portfolio Evaluation

In the presence of counterparty credit risk (CCR), Breton and Marzouk (2018) demonstrates that dynamic programming (DP) is an effective method for valuing a single contract. This method accounts for the interdependence of exercise behavior and counterparty risk, enabling the accurate computation of credit valuation adjustment (CVA) and optimal exercise strategies throughout the contract's lifespan. Thus, it overcomes the limitations of prevalent simulation-regression techniques, which often incorrectly assume exercise policy and default events are independent occurrences. This can potentially result in suboptimal exercise behavior and imprecise CVA calculations.

The evaluation scenario evolves when transitioning from a single-agent dynamic problem to a multi-agent environment with multiple decision makers, similar to portfolio evaluation under a netting agreement. Here, both parties can make decisions regarding exercising the contracts within the portfolio. This shift necessitates introducing a *Dynamic Game* to accommodate the interdependence of decision-makers.

Dynamic games model sequential strategic interactions between agents selecting actions from a set of possible choices. These strategic decisions are made simultaneously and independently, but their outcomes are interdependent, creating a dynamic interaction. In a dynamic game, the prevailing conditions of the game at a particular moment are represented by states. The actions undertaken by the agents cause these states to evolve, leading to different possible future states. The model's future state then depends not only on the current state and one party's decisions but also on other parties' decisions.

Game theory, a branch of mathematics examining strategic interactions, provides the theoretical foundation for dynamic games. It offers a framework to analyze how rational decision-makers select strategies to maximize their benefits given others' choices. In this context, equilibrium strategies represent joint decisions where no agent has an incentive to deviate. For a comprehensive review of games and dynamic games, refer to Fudenberg and Tirole (1991) and Haurie, Krawczyk, and Zaccour (2012).

Dynamic games offer a relevant framework for evaluating netted portfolios, where one party's actions can significantly impact portfolio value and the other party's decisions. This section traced the progression from dynamic programming to dynamic games, having the potential to evaluate netted portfolios with early exercisable contracts under counterparty credit risk (CCR). This study aims to apply these methodologies to unravel CVA evaluation complexities under netting agreements, offering a unique perspective not previously explored. Further sections will demonstrate applying dynamic games and refinements to address real-world complexities in netted portfolio valuation.

Chapter 2

Counterparty Risk under Netting Agreements: A Dynamic Game Interpretation

2.1 Introduction

In the complex world of over-the-counter (OTC) contracts, the Credit Valuation Adjustment (CVA) has become an essential tool for accounting for counterparty credit risk (CCR). The CVA modifies the default-free valuation of an OTC contract to reflect the potential losses if the counterparty fails to meet its payment obligations. Calculating the CVA requires differentiating between the value of a position exposed to CCR (*defaultable*) and an equivalent position without counterparty risk (*default-free*). Brigo, Morini, and Pallavicini (2013) and Gregory (2012) provide an extensive review of credit risk evaluation.

As explained in Breton and Marzouk (2018), evaluating claims with early-exercise features under CCR poses challenges, mainly due to the influence of counterparty risk on the optimal exercise strategy. A dynamic programming (DP) approach offers a viable solution, allowing concurrent computation of a defaultable claim's value and the exercise strategy adjusted for CCR. Breton and Marzouk (2018) demonstrates this methodology's efficiency for low-dimensional state spaces.

A pivotal mechanism for alleviating CCR is *netting*, a common agreement among counterparties managing large portfolios of OTC derivative products. Netting involves the aggregation of contractual obligations, translating them into net cash flows from one party to the other over the contract's lifespan. Under a netting agreement, all financial obligations are offset upon one counterparty's default. The inclusion of derivatives with early exercise features in the netted portfolio adds another layer of complexity to credit risk evaluation, as the exercise strategies of both parties could be influenced by netting.

As we broaden our analytical lens, we begin to see the emergence of a new paradigm where both contractual parties are subject to CCR. This bilateral perspective, as Brigo, Buescu, and Morini (2011) points out, has gained relevance in the post-2008 financial crisis era, an era marked by multiple default events involving financial institutions. The traditional assumption of unilateral default risk seems increasingly untenable in the contemporary financial environment. In such bilateral risk contexts, the adjustment, referred to as the Bilateral Valuation Adjustment (BVA), is contingent upon the first-to-default risk. (see Brigo, Buescu, and Morini (2011)).

In light of these complexities, this chapter aims to propose a novel approach for computing the risk adjustment value (CVA or BVA) for a portfolio of options subject to CCR when the parties have early exercise opportunities and are involved in a netting agreement. We introduce a model that captures the strategic interactions among the parties in the evaluation of a vulnerable options portfolio within a netting agreement. Complementing this, we introduce a recursive algorithm for determining the netted portfolio value and optimal exercise strategies of both parties, thereby offering an approach to tackling CCR challenges within netting agreements.

2.1.1 Content and Organization

In this chapter, we initiate our discussion with a simplified example to illuminate the strategic aspect of netting when parties hold options with early exercise features, giving rise to a dynamic zero-sum game between the counterparties. Subsequently, we introduce a model, underpinned by a dynamic programming technique, that facilitates the evaluation of a netted derivatives portfolio under Counterparty Credit Risk (CCR) and, correspondingly, the risk adjustment for both parties. This approach enables the efficient computation of the CVA and the BVA for a low-dimensional netted portfolio.

We employ a robust array of numerical illustrations to analyze risk adjustments within the framework of netting agreements. We delve into the implications of neglecting netting in risk adjustment evaluations for a netted portfolio, supported by relevant examples. Moreover, we elucidate the influence of variation in various elements, notably the default risk of the involved parties, on the magnitude of the final adjustment applicable to them.

The chapter is structured as follows: Section 2.2 serves as the motivation, illustrating the effect of netting and CCR on optimal exercise behavior. Section 2.3 introduces the dynamic game model, a tool designed to evaluate the risk adjustment of the portfolio within the context of the netting agreement. Section 2.4 provides illustrative examples and numerical results, while Section 2.5 concludes the chapter, summarizing the key insights and implications of our research.

2.2 Motivating Example

Consider two parties, named C_1 and C_2, involved as counterparties in a portfolio of two Bermudan put options, identified by O_1 and O_2, where C_i has a long position on O_i and a short position on O_{3-i}, $i = 1,2$. Since all the cashflows generated by this portfolio are from C_1 to C_2 or conversely, the value of the portfolio from the point of view of C_1 is the negative of its value from that of C_2. Assume that both parties can exercise their option at a single given date before maturity (the *decision date*).

Table 2.1 provides the exercise and holding values of the two Bermudan put options at the decision date, in the absence of CCR. From these values, it is straightforward to conclude that both parties should hold. The value of the portfolio is then $v_1 = h_1 - h_2 = 1$ for C_1 (and $v_2 = -1$ for C_2).

	O_1	O_2
Exercise value	$e_1 = 8$	$e_2 = 6$
Holding value	$h_1 = 10$	$h_2 = 9$

Table 2.1: Exercise and holding value of options O_1 and O_2 at the decision date from the respective viewpoints of their holder, when both parties are risk-free.

Now suppose that there is a $p_2 = 0.3$ probability that C_2 defaults, so that C_1 does not recover anything upon maturity of O_1. Table 2.2 shows the updated exercise and holding values of both options, where the expected holding value $\hat{h}_1$ accounts for counterparty default risk. In that case, it becomes optimal for C_1 to exercise O_1, and the portfolio value for C_1 is now $v_1 = e_1 - h_2 = -1$.

	O_1	O_2
Exercise value	$e_1 = 8$	$e_2 = 6$
Holding value	$\hat{h}_1 = (1 - p_2)h_1 = 7$	$h_2 = 9$

Table 2.2: Exercise and holding value of options O_1 and O_2 at the decision date from the respective viewpoints of their holder when the default probability of C_2 is $p_2 = 0.3$.

Note that the exercise strategy is modified by the presence of CCR. Using the risk-free exercise strategy to value this portfolio would result in the erroneous value of $\hat{h}_1 - h_2 = -2$.

Now suppose that the portfolio is subject to a netting agreement, so that the contractual cash-flows are no longer independent. Assuming that the probability of default by C_2 is $p_2 = 0.3$, Table 2.3 contains the cash-flows from C_2 to C_1 under a netting agreement in the form of a matrix game, where C_1 *(resp.* C_2*)* is the row *(resp. column)* player and where E *(resp. H)* stands for "exercise" *(resp. "hold")*.

		C_2: E	C_2: H
C_1	E	$e_1 - e_2 = 2$	$e_1 - h_2 = -1$
	H	$\hat{h}_1 - e_2 = 1$	$(1 - p_2)\big(h_1 - h_2\big) = 0.7$

Table 2.3: Matrix-game representation of the netted portfolio's value from the viewpoint of C_1 at the decision date. Risk-free holding and exercise values are provided in Table 2.1. Default probability of C_2 is $p_2 = 0.3$.

Table 2.3 is a representation of a *zero-sum* matrix game where C_1 is the maximizer and C_2 is the minimizer. The security strategy for C_1, maximizing the worst (smallest) outcome, is to hold, which guarantees an outcome of at least 0.7. Conversely, the security strategy for C_2, minimizing the worst (largest) outcome, is also to hold, which guarantees an outcome of at most 0.7. In that specific example, the security strategies yield the same expected outcome, so that each party's decision is the optimal response to the other's, and neither party has an incentive to depart from it, yielding a *Nash equilibrium*. In that case, the equilibrium strategy consists of holding both options, and the equilibrium value of the netted portfolio is 0.7.

Since C_1 is the only party vulnerable to CCR, the CVA for C_1 is computed by deducting the value of the vulnerable portfolio from that of the corresponding risk-free portfolio. Results are reported in Table 2.4. Examination of Table 2.4 shows that both CCR and

Case	Decisions		portfolio value (C_1)	CVA
	O_1	O_2		
Risk-free	H	H	$h_1 - h_2 = 1$	–
CCR without netting	E	H	$e_1 - h_2 = -1$	2
CCR with netting	H	H	$(1-p_2)\big(h_1 - h_2\big) = 0.7$	0.3

Table 2.4: Impact of CCR and of netting on exercise decisions and CVA. Risk-free holding and exercise values are provided in Table 2.1 and default probability of C_2 is $p_2 = 0.3$.

netting affect the exercise strategy and, therefore, the CVA. One also observes that netting does reduce the portfolio's CVA in this case.

We now consider the case of bilateral counterparty risk by assuming a default probability of $p_1 = 0.25$ for C_1 and $p_2 = 0.15$ for C_2. Table 2.5 reports the exercise and holding values of the vulnerable options, without netting, at the decision date. In that case, the optimal decision for both parties is to hold its option, and the value of the portfolio for C_1 is $v_1 = \hat{h}_1 - \hat{h}_2 = 1.75$.

	O_1	O_2
Exercise value	$e_1 = 8$	$e_2 = 6$
Holding value	$\hat{h}_1 = (1-p_2)h_1 = 8.5$	$\hat{h}_2 = (1-p_1)h_2 = 6.75$

Table 2.5: Exercise and expected holding values of O_1 and O_2 at the decision date from the respective viewpoints or their holder when the default probabilities of C_1 and C_2 are respectively $p_1 = 0.25$ and $p_2 = 0.15$.

Table 2.6 reports the cash-flows from C_2 to C_1 in the presence of a netting agreement and bilateral counterparty risk. According to these values, the security strategy of C_1 is to exercise O_1 and that of C_2 is to hold O_2, guaranteeing in both cases a portfolio value of $v_1 = 1.25$. The strategy pair (E,H) is then a Nash equilibrium for the matrix game, and differs from the optimal strategies obtained when there is no netting agreement.

Similar to the CVA, the BVA is computed by subtracting the vulnerable portfolio's value from its non-vulnerable counterpart. The BVA can be negative or positive; its sign

		C_2	
		E	H
C_1	E	$e_1 - e_2 = 2$	$e_1 - \hat{h}_2 = 1.25$
	H	$\hat{h}_1 - e_2 = 2.5$	$(1 - p_2)\big(h_1 - h_2\big) = 0.85$

Table 2.6: Matrix-game representation of the netted portfolio's value from the viewpoint of C_1 at the decision date. Risk-free holding and exercise values are provided in Table 2.1. Default probabilities of C_1 and C_2 are respectively $p_1 = 0.25$ and $p_2 = 0.15$.

depends on the two parties' relative vulnerability to CCR. The impact of a netting agreement on the exercise decisions of defaultable parties and on the BVA for this example is summarized in Table 2.7. Again, one observes that netting affects the portfolio's risk adjustment value and decreases the absolute value of the BVA.

Case	Decision		portfolio value (C_1)	BVA
	O_1	O_2		
Risk-free	H	H	$h_1 - h_2 = 1$	–
CCR without netting	H	H	$\hat{h}_1 - \hat{h}_2 = 1.75$	-0.75
CCR with netting	E	H	$e_1 - \hat{h}_2 = 1.25$	$-.25$

Table 2.7: Impact of CCR and netting agreement on exercise decisions and BVA. Risk-free holding and exercise values are provided in Table 2.1. Default probabilities of C_1 and C_2 are respectively $p_1 = 0.25$ and $p_2 = 0.15$.

Finally, Table 2.8 reports an instance where the default probabilities for C_1 and C_2 are respectively $p_1 = 0.1$ and $p_2 = 0.35$. In that case, the security maxmin strategy of C_1 is to hold, guaranteeing a payoff of at least 0.5, while the security minmax strategy of C_2 is to hold, guaranteeing a payoff of at most 0.65. The matrix game does not admit a Nash equilibrium in pure strategies since the minmax and maxmin values do not coincide.[1]

In such a situation, we can propose various conjectures about the way the parties will act, which will affect the value of the portfolio. One plausible assumption is that each

[1] The best response of C_2 when C_1 holds is to exercise.

party will adhere to its own security strategy. In the case of the matrix game in Table 2.8, both parties would then choose to hold their option, resulting in a portfolio value of $v_1 = 0.65$ for C_1 (and $v_2 = -0.65$ for C_2).

A second assumption is that parties adopt a mixed strategy, that is, they randomize their decision by choosing a probability to exercise their option at the decision date. This assumption is founded on the game-theoretical interpretation of managing the netted portfolio, as zero-sum matrix games always admit a Nash equilibrium in mixed strategies. For the game presented in Table 2.8, the equilibrium mixed strategy is for C_1 to exercise with a probability of $2/30$ and for C_2 to exercise with a probability of $1/3$. It is straightforward to check that, if $C1$ exercises with a probability of $2/30$, $C2$ cannot reduce the expected value of the portfolio below 0.6 (actually, the value of the portfolio is 0.6 whether $C2$ exercises or holds). In the same way, if $C2$ exercises with a probability of $1/3$, $C1$ can not do better than an expected value of 0.6. Under this equilibrium mixed strategy, the portfolio value is then $v_1 = 0.6$.

		C_2: E	C_2: H
C_1	E	$e_1 - e_2 = 2$	$e_1 - \hat{h}_2 = -0.1$
	H	$\hat{h}_1 - e_2 = 0.5$	$(1-p_2)\big(h_1 - h_2\big) = 0.65$

Table 2.8: Matrix-game representation of the netted portfolio's value from the viewpoint of C_1 at the decision date. Risk-free holding and exercise values are provided in Table 2.1. Default probabilities of C_1 and C_2 are respectively $p_1 = 0.1$ and $p_2 = 0.35$.

It is interesting to note that in a two-party zero-sum game, if one party has a dominant strategy, a pure strategy Nash equilibrium exists. A dominant strategy represents the uniformly best choice for a player, regardless of the other's decision. If counterparty $C_i, i = 1,2$ has a dominant strategy, counterparty C_{3-i}'s best response is the pure strategy that minimizes C_i's maximum payoff against that dominant strategy. When C_i chooses their dominant strategy and C_{3-i} selects the minimizing response, this forms an equilibrium in pure strategies, since neither counterparty can unilaterally improve their payoff. Thereby, the zero-sum property aligns incentives such that one counterparty's dominance introduces sufficient structure to yield a mutually optimal equilibrium.

To demonstrate, consider the case where the exercise value of option O_1 in our example model is $e_1 = 3$ instead of 2. The new matrix game payoffs are shown in Table 2.9, replacing the prior values in Table 2.8. Here, C_1 has a dominant strategy to exercise O_1. The best response for C_2 is then to hold O_2. The strategy pair (E,H) forms a Nash equilibrium for this matrix game, and the portfolio value is $v_1 = 0.9$. This illustrates how a dominant strategy for one counterparty provides a structure for both parties to play mutual best responses, creating a pure strategy Nash equilibrium.

		C_2: E	C_2: H
C_1	E	$e_1 - e_2 = 3$	$e_1 - \hat{h}_2 = 0.9$
	H	$\hat{h}_1 - e_2 = 0.5$	$(1-p_2)\big(h_1 - h_2\big) = 0.65$

Table 2.9: Matrix-game representation of the netted portfolio's value from the viewpoint of C_1 at the decision date. Risk-free holding and exercise values are provided in Table 2.1, but with modified exercise value $e_1 = 3$ for $O-1$. Default probabilities of C_1 and C_2 are respectively $p_1 = 0.1$ and $p_2 = 0.35$.

These simple examples reported in this section illustrate the impact of CCR and netting on the exercise decisions of the parties and, therefore, on the value of the portfolio. In the absence of a netting agreement, each claim is examined individually to determine the optimal exercise strategy, where the holding value of each individual claim is adjusted to account for the possibility of loss upon default. Under a netting agreement, however, losses upon default are applied to the net value of the portfolio; specifically, upon default of C_2, losses are only incurred if the net value of the portfolio claims is positive for C_1. This makes the expected payoff for each counterparty dependent on the exercise decisions of the other. Clearly, a portfolio of claims should include both positive and negative cashflows for netting to have an impact on the expected payoffs to the counterparties. When, in addition, the portfolio includes claims with early exercise features, netting can also impact the exercise strategy and, therefore, further modify the risk adjustment.

The following section proposes a general model to evaluate a portfolio of claims having early exercise features under CCR and netting.

2.3 Risk valuation adjustment under CCR and netting

We consider two parties (C_1 and C_2) involved in a portfolio of claims under a netting agreement, where the portfolio includes contractual payments in both directions (from C_2 to C_1 and from C_1 to C_2), possibly with early exercise features. We assume that both parties have at least one early exercise opportunity.

Since all cash flows from this portfolio are from one party to the other, its value from the perspective of one party is the negative of that of the other. In the sequel, the netted portfolio value is expressed from the perspective of C_1.

The essential feature of a netting agreement is the consolidation of contractual obligations upon default of one of the parties. Accordingly, the agreement and the portfolio ceases to exist on the date of the first default event; at that date, the values of the claims are netted and the result is recovered by C_1 (if positive) or C_2 (if negative). Since, on the date of the first default event, some claims can include optional rights, we will assume in this paper that, upon default, the value of the netted portfolio corresponds to the expected value of its future cash flows under a risk- and netting-adjusted exercise strategy.

2.3.1 Notation

To simplify the exposition, we assume that the portfolio is composed of $n = n_1 + n_2$ Bermudan options with different features (maturity, exercise payoffs and dates, underlying asset), where C_1 holds the optional rights of the first n_1 options and C_2 holds the optional rights of the remaining n_2 options.[2] Let $t = 0$ denote the inception of the netting contract and $t = T$ the longest maturity among the n options included in the portfolio. Denote by $(X_t)_{0 \leq t \leq T}$ the (possibly multidimensional) process of the underlying risk factors, including the price process of the options' underlying assets. We assume that $(X_t)_{0 \leq t \leq T}$ is a finite Markov process, where $(\mathcal{F}_t)_{0 \leq t \leq T}$ is the filtration generated by $(X_t)_{0 \leq t \leq T}$.

Let $\mathcal{T} = \{t_m, m = 0, 1, ..., M\}$ be a set of discrete *evaluation dates* that includes all possible exercise dates for all options in the portfolio, where $t_M \equiv T$. The notation $\mathbb{E}_m[\cdot]$ represents the expectation at date t_m, under the risk-neutral measure, conditional on no prior default and on the filtration $(\mathcal{F}_{t_m})$. For $j = 1, ..., n$, $F_{mj}(x)$ then denotes the exercise payoff of option j at $(t_m, X_{t_m} = x)$ from the perspective of C_1, where $F_{mj}(x) = 0$ when exercise of option j is not allowed at t_m.

Let r denote the risk-free interest rate, assumed constant. To simplify the notation, we assume that evaluation dates are evenly distributed in $[0, T]$, so that the discount factor corresponding to a single time step $\Delta \equiv t_{m+1} - t_m, m = 0, ..., M-1$, is given by $\beta \equiv e^{-r\Delta}$.

We denote by τ_i the stochastic default date (possibly infinite) of C_i and by $\rho_i \in [0,1]$ the deterministic recovery rate upon default by C_i, $i \in \{1,2\}$. The recovery rate is applied to the netted portfolio value eventually recovered by C_{3-i}.

To compute the CCR valuation adjustment, one needs to compare the value of the vulnerable portfolio with that of a risk-free portfolio with the same characteristics. It is important to emphasize that by "risk-free portfolio" in this thesis, we refer to a portfolio

[2] It is straightforward to adapt the model to the general case of derivatives with multiple contractual cash flows.

free of counterparty risk. To this end, we introduce a state vector $b = (b_1, b_2)$ of binary variables indicating which options of the portfolio are still alive, that is, for $j = 1,...,n$, $b_j = 1$ if option j has not yet been exercised or expired, whereas $b_j = 0$ indicates that option j no longer exists in the portfolio. At a given evaluation date t_m where $X_{t_m} = x$ and given no prior default, let $\hat{V}_m(x,b)$ and $V_m(x,b)$ denote respectively the value of the vulnerable portfolio and that of the corresponding risk-free portfolio, under the risk-neutral measure.

Finally, the indicator function 1_A is defined by

$$1_A \equiv \begin{cases} 1 & \text{if } A \text{ is true} \\ 0 & \text{otherwise,} \end{cases}$$

and, for a given $y \in \mathbb{R}$,

$$y^+ \equiv \max\{0, y\}$$

$$y^- \equiv \min\{0, y\}.$$

2.3.2 The risk-free portfolio

It is easy to show that netting has no impact on the optimal exercise of the individual options in a risk-free portfolio, so that

$$V_m(x,b) = \sum_{j=1}^{n} b_j V_{mj}(x) \tag{2.1}$$

where, for $j = 1,...,n$, $V_{mj}(x)$ is the value (from the perspective of C_1) of Option j at $(t_m, X_{t_m} = x)$, under its holder's optimal exercise strategy, assuming Option j has not been exercised yet. The risk-free value of the n options satisfy the following recursive equations

$$V_{mj}(x) = \max\{F_{mj}(x); \beta \mathbb{E}_m[V_{m+1,j}(X_{t_{m+1}})]\} \text{ for } j = 1,...,n_1 \text{ and } m < M \quad (2.2)$$

$$V_{mj}(x) = \min\{F_{mj}(x); \beta \mathbb{E}_m[V_{m+1,j}(X_{t_{m+1}})]\} \text{ for } j = n_{1+1},...,n \text{ and } m < M \quad (2.3)$$

$$V_{Mj}(x) = F_{Mj}(x) \text{ for } j = 1,...,n. \quad (2.4)$$

2.3.3 The vulnerable netted portfolio

However, as shown in Section 2.2, netting can impact the exercise strategies of the vulnerable portfolio's claims, giving rise to a dynamic game interpretation for the value of the netted portfolio. We therefore proceed to characterize the payoffs and exercise strategies of the counterparties involved in a netting agreement in order to obtain the value of a netted portfolio of vulnerable options.

Exercise payoff

At a given evaluation date, let $a = (a_1, a_2)$ represent a vector of binary decisions with respect to each of the n options, where, for $j = 1,...,n$, option j is exercised by its holder if $a_j = 1$. Note that feasible decision vectors satisfy $a \leq b$, and recall that $F_{mj}(x) = 0$ if exercise of option j is not allowed at t_m. The *exercise payoff* $R_m(x,a)$ corresponding to a feasible action vector a at $(t_m, X_{t_m} = x)$ is defined by

$$R_m(x,a) \equiv \sum_{j=1}^{n} a_j F_{mj}(x). \quad (2.5)$$

Holding value

The *holding value* $W_m(x,b)$ of the portfolio at $(t_m, X_{t_m} = x)$, given no prior default, is the expected value of all the remaining options in the netted portfolio, described by the vector b. Accordingly, using a recursive interpretation and assuming that the value of the vulnerable portfolio is known at the next evaluation date as a function of the state vector, the holding value is computed by considering the expected discounted value of the portfolio upon three mutually exclusive and collectively exhaustive events during the time interval until the next evaluation date, namely, survival of both parties, first default of C_1, or first default of C_2, given no prior default. We can then write

$$W_m(x,b) = W_m^0(x,b) + W_m^1(x,b) + W_m^2(x,b), \tag{2.6}$$

where $W_m^0(x,b)$, $W_m^1(x,b)$ and $W_m^2(x,b)$ correspond respectively to the holding value upon each of these three mutually exclusive events, defined as follows:

Case 0: Let $D_m^0 = 1_{t_{m+1}<\tau_2} 1_{t_{m+1}<\tau_1}$ indicate the event that both parties will survive until t_{m+1}. In this case, the holding value at t_m is the discounted value of the portfolio value at the next evaluation date, yielding

$$W_m^0(x,b) = \beta \mathbb{E}_m \left[D_m^0 \hat{V}_{m+1}(X_{t_{m+1}}, b) \right]. \tag{2.7}$$

Case 1: Let $D_m^1 = 1_{t_m<\tau_1\leq t_{m+1}} 1_{\tau_1<\tau_2}$ indicate the event that C_1 is the first to default during the time interval $(t_m, t_{m+1}]$. In this case, if the expected value of the portfolio at $(t_m+1, X_{t_{m+1}})$ is negative, C_2 will recover a portion ρ_1 of this (discounted) value at τ_1; otherwise, C_2 will deliver the total of the portfolio's expected discounted value to C_1 at τ_1. We then have

$$W_m^1(x,b) = \beta \mathbb{E}_m \left[D_m^1 \left(\hat{V}_{m+1}(X_{t_{m+1}}, b)^+ + \rho_1 \hat{V}_{m+1}(X_{t_{m+1}}, b)^- \right) \right]. \tag{2.8}$$

Case 2: Let $D_m^2 = 1_{t_m<\tau_2\leq t_{m+1}} 1_{\tau_2<\tau_1}$ indicate the event that C_2 is the first to default during the time interval $(t_m, t_{m+1}]$. Similarly to Case 1, if the expected value of the portfolio at the next evaluation date is positive, C_1 will recover a portion ρ_1 of it, otherwise C_2 will recover the total value, yielding

$$W_m^2(x,b) = \beta \mathbb{E}_m \left[D_m^2 \left(\rho_2 \hat{V}_{m+1}(X_{t_{m+1}}, b)^+ + \hat{V}_{m+1}(X_{t_{m+1}}, b)^- \right) \right]. \tag{2.9}$$

Using (2.7)-(2.9), Equation (2.6) reduces to

$$\begin{aligned} W_m(x,b) &= \beta \mathbb{E}_m \left[\left(1-(1-\rho_2)D_m^2\right) \hat{V}_{m+1}(X_{t_{m+1}},b)^+ \right. \\ &\quad \left. + \left(1-(1-\rho_1)D_m^1\right) \hat{V}_{m+1}(X_{t_{m+1}},b)^- \right]. \end{aligned} \tag{2.10}$$

It is important to note that the above characterization of the holding value implicitly assumes that, upon default, both parties agree on the value of the portfolio, that is, on the expected discounted value of its future cash flows. Clearly, the future cash flows of an option with early exercise opportunities depend on the exercise strategy of its holder, and the value of an option is obtained by assuming an optimal exercise strategy. As shown in Section 2.2, the holder's exercise strategy should account for counterparty risk and for the impact of netting on its exposure.

Security strategies

A *security strategy* for C_1 at $(t_m, X_{t_m} = x, b)$ prescribes a decision vector maximizing the outcome against all the possible decisions of the other party. The *lower value* of the portfolio at (m,x,b) is defined by

$$V_m^{S1}(x,b) \equiv \max_{a_1 \le b_1} \left\{ \min_{a_2 \le b_2} \left\{ R_m(x,a) + W_m(x,b-a) \right\} \right\}, \tag{2.11}$$

where $b-a$ indicates the contracts remaining in the portfolio after the exercise decisions designated by the vector $a = (a_1, a_2)$. A security strategy for C_1 then satisfies

$$a_m^{S1}(x,b) \in \arg\max_{a_1 \le b_1} \left\{ \min_{a_2 \le b_2} \left\{ R_m(x,a) + W_m(x,b-a) \right\} \right\}. \tag{2.12}$$

In the same way, a security strategy for C_2 at $(t_m, X_{t_m} = x, b)$ is a decision vector $a_m^{S2}(x,b)$ minimizing the outcome against all the possible decisions of C_1. The *upper value* of the portfolio at (m,x,b) is defined by

$$V_m^{S2}(x,b) \equiv \min_{a_2 \le b_2} \left\{ \max_{a_1 \le b_1} \left\{ R_m(x,a) + W_m(x,b-a) \right\} \right\}, \tag{2.13}$$

and a security strategy for C_2 satisfies

$$a_m^{S2}(x,b) \in \arg \min_{a_2 \leq b_2} \left\{ \max_{a_1 \leq b_1} \{R_m(x,a) + W_m(x,b-a)\} \right\}. \tag{2.14}$$

Security strategies are *pure* strategies, of dimension n_1 for C_1 and n_2 for C_2. They indicate the vector of decisions (exercise or hold) corresponding to all the options in the portfolio held by each counterparty, as a function of t_m, $x = X_{t_m}$ and b. Note that the feasibility condition $a \leq b$ ensures that options that are no longer alive cannot be exercised.

2.3.4 Equilibrium

At a given evaluation date t_m where $X_{t_m} = x$ and the options still included in the portfolio are described by the vector b, if the lower value and the upper value of the portfolio coincide, the security strategies of the counterparties define a Nash equilibrium at (m,x,b). In that case, it is reasonable to assume that the counterparties will use the strategy pair $(a_m^{S1}(x,b), a_m^{S2}(x,b))$ since neither party can improve its outcome by changing its strategy.[3] The value of the netted portfolio is then defined by

$$\hat{V}_m(x,b) \equiv V_m^{S1}(x,b) = V_m^{S2}(x,b). \tag{2.15}$$

If however, the upper and lower values do not coincide at (m,x,b), there exists no equilibrium in pure strategies at (m,x,b), and the value of the portfolio is open to interpretation. As illustrated in Section 2.2, we propose three ways to determine the value of the netted portfolio in that case, based on plausible conjectures about the exercise strategies used by the counterparties.

[3]To simplify the exposition, we assume in the sequel that the solutions to the optimization problems (2.11) and (2.13) are unique. Note that the portfolio value is well-defined even when this is not the case. The issue of multiple solutions is addressed in Section 2.4.

Robust intepretation

In the first case, we assume that each counterparty uses its security strategy, a robust behavior avoiding the worst possible outcomes and ensuring that the value of the portfolio lies between its lower and upper values. The strategy pair used by the counterparties is then $a_m^S(x,b) \equiv (a_m^{S1}(x,b), a_m^{S2}(x,b))$ and the value of the portfolio is given by

$$\begin{aligned} \hat{V}_m(x,b) &\equiv R_m(x,(a_m^S(x,b)) + W_m(x, b - a_m^S(x,b)) \\ &\in \left[V_m^{S1}(x,b), V_m^{S2}(m,b)\right]. \end{aligned} \tag{2.16}$$

Mixed strategies

In the second case, we consider the possibility that counterparties randomize their decisions by choosing a probability distribution over the set of actions available to them. A *mixed* strategy for C_i, $i \in \{1,2\}$ is a vector z_i of dimension 2^{n_i} such that each element is in $[0,1]$ and the elements sum to 1. The exercise payoff and holding value corresponding to a mixed strategy is the weighted average of the values corresponding to each of the 2^{n_i} pure strategy vectors available to counterparty C_i, denoted by a_{i_k}, $k = 1, \ldots 2^{n_i}$. Accordingly, under a mixed strategy z_1, the exercise payoff of the netted portfolio at $(t_m, X_{t_m} = x)$ when C_2 uses the action vector a_2 is defined by

$$\tilde{R}_m(x, z_1, a_2) = \sum_{k=1}^{2^{n_1}} z_{1_k} R_m(x, a_{1_k}, a_2). \tag{2.17}$$

In the same way, under a mixed strategy z_1, the holding value of the netted portfolio at $(t_m, X_{t_m} = x)$ when C_2 uses the action vector a_2 is defined by

$$\tilde{W}_m(x, z_1, a_2) = \sum_{k=1}^{2^{n_1}} z_{1_k} W_m(x, b - (a_{1_k}, a_2)). \tag{2.18}$$

The exercise payoff and holding value of the netted portfolio corresponding to the use of a mixed strategy by C_2 are defined similarly.

Note that a Nash equilibrium in mixed strategies always exists.[4] The value of the portfolio

$$\hat{V}_m(x,b) \equiv v \tag{2.19}$$

can be obtained by solving the following linear program at (m,x,b):

$$\max_{z,v} v \tag{2.20}$$

s.t.

$$v \leq \tilde{R}_m(x,z,a_{2_l}) + \tilde{W}_m(x,z,a_{2_l}) \text{ for } l = 1,...2^{n_2} \tag{2.21}$$

$$\sum_{k=1}^{2^{n_1}} z_k = 1 \tag{2.22}$$

$$z_k \geq 0, \;\; k = 1,...,2^{n_1}. \tag{2.23}$$

The equilibrium mixed strategy for C_1 is the vector $z \in \mathbb{R}^{2^{n_1}}$ solving (2.20)-(2.23). The equilibrium mixed strategy for C_2 is the vector of dual variables corresponding to the 2^{n_2} constraints (2.21).

Conservative values

Finally, we consider the possibility that parties do not agree on the value of the portfolio, so that each party computes its own estimation of the value of the vulnerable portfolio, a conservative value corresponding to either the lower (for C_1) or the upper (for C_2) value, obtained using Equations (2.11) or (2.13), respectively.

To summarize, we propose three distinct assumptions about the behavior of the parties in a netted agreement, leading to four different ways to compute the value of a vulnerable portfolio, namely:

[4] Again, the equilibrium value is unique even though multiple equilibrium strategies may exist.

A1 Parties agree on the value of the portfolio, which corresponds to a Nash equilibrium. In that scenario, parties use mixed strategies when a Nash equilibrium in pure strategies does not exist. The value of the portfolio is obtained using Equation (2.19).

A2 Parties agree on a robust interpretation of the value of the portfolio. In that scenario, each party uses its security strategy, which is not necessarily in equilibrium, but guarantees that the value of the portfolio, obtained using Equation (2.16) lies between the upper and the lower value.

A3 Parties do not agree on the value of the portfolio and use a conservative value obtained using Equations (2.11) for C_1 and Equation (2.13) for C_2.

Clearly, both counterparties should agree that the value of the portfolio lies between its lower and upper values. Note that Equations (2.16) and (2.19) satisfy this condition and yield the same result, corresponding to Equation (2.15), when the lower and the upper values coincide.

2.3.5 Computation of valuation adjustments

Given that the value of the vulnerable portfolio is a known function of (x,b) at maturity,

$$\hat{V}_M(x,b) = \sum_{j=1}^{n} b_j F_{Mj}(x), \tag{2.24}$$

Equations (2.6)-(2.9) provide a backward recursive formulation to compute the holding value $W_m(x,b)$ at t_m when the value of the vulnerable netted portfolio is known at t_{m+1} as a function of the state vector (x,b). Under Assumptions A1 or A2, the vulnerable portfolio value can then be obtained at t_m using Equations (2.16) or (2.19), respectively. Note that the two equations yield the same value when the upper and lower values of the vulnerable portfolio coincide.

The BVA at $(t_m, X_m = x, b)$ is then given by the difference

$$\mathrm{BVA}_m(x,b) = V_m(x,b) - \hat{V}_m(x,b). \tag{2.25}$$

When only one party is exposed to default risk, say C_i, the stochastic default time τ_i is set to $= \infty$ in Equations (2.7)-(2.9). The CVA at (m, x, b) is then given by

$$\mathrm{CVA}_m(x,b) = V_m(x,b) - \hat{V}_m(x,b). \tag{2.26}$$

When parties do not agree on the value of the vulnerable portfolio (Assumption A3), and, therefore, on the price of counterparty risk (BVA or CVA), each party will compute its own *conservative* value of the risk adjustment, yielding the conservative BVAs

$$\begin{aligned}\mathrm{BVA}^1_m(x,b) &= V_m(x,b) - V^{S1}_m(x,b)\\ \mathrm{BVA}^2_m(x,b) &= V_m(x,b) - V^{S2}_m(x,b).\end{aligned}$$

These conservative BVA values are likely to differ and to be higher (in absolute value) than the BVA computed using either the mixed strategy or the robust assumptions.

The general model proposed in this section provides an analytical characterization of the price of counterparty risk under various default risk models and various assumptions about the state process $(X_t)_{0 \le t \le T}$, provided the expectations in (2.7)-(2.9) can be computed or approximated efficiently. In particular, it can accommodate both intensity-based and structural default models by including the risk factors (e.g. structural or exogenous variables) in the state vector.

However, while analytic, Equations (2.2)-(2.4) and (2.11), (2.13), (2.16) or (2.19) do not admit closed-form solutions in general and require some form of numerical approximation. In the numerical illustrations presented in the next section, we solve Equations (2.2)-(2.4) and (2.11), (2.13), (2.16) or (2.19) on a set of grid points for the state vector X and approximate the value of the portfolio using linear spline interpolation (see Breton and Frutos 2012).

2.3.6 Implementation

In our implementation of the algorithm, we utilize the linear spline interpolation approach outlined in Ben-Ameur, Breton, and François (2006). This approximation technique has been shown in Breton and Frutos (2012) to be both efficient and numerically robust for pricing Bermudan options. The resulting approximation provides a good balance between accuracy and computational efficiency, making it well-suited for our purposes.

In this regard, we can write Equations (2.7)-(2.10) as the general form of

$$W_m(x,b) = \beta \mathbb{E}_m\left[G_m f_{m+1}(X_{t_{m+1}}, b)\right], \tag{2.27}$$

where G_m is an event, f_{m+1} is a known function, and the joint density of $(G_m, X_{t_{m+1}})$, conditional on $X_{t_m} = x$, is known under the risk-neutral measure.

For a clearer understanding, we consider the case where the state space related to the risk factor is unidimensional, with $x \in [0,\infty)$. We define a set $\mathcal{G} = \{x_k, k = 1,...,p\}$ of p grid points such that

$$0 < x_1 < x_2 < ... < x_p < \infty$$

and a family of p basis functions, denoted by $(\psi_k)_{\{k=1,..,p\}}$. An interpolation function is then defined by

$$\hat{W}_m(x,b) = \begin{cases} \sum_{k=1}^{p} c_k^m(b)\psi_k(x) & \text{if } x \in [x_1, x_p] \\ o(x) & \text{if } x \notin [x_1, x_p], \end{cases} \tag{2.28}$$

where $o(x)$ is an extrapolation function characterizing the behavior of W outside the localization interval, and where the coefficients c_j^m satisfy the linear system.

$$W_m(x_i,b) = \sum_{k=1}^{p} c_k^m(b)\psi_k(x_i), i = 1,...,p. \tag{2.29}$$

Knowing the function of $\hat{V}_{m+1}(X_{t_{m+1}}, b)$, we proceed to compute the expected values $\mathbb{E}_m[G_{1m}\hat{V}_{m+1}(X_{t_{m+1}}, b)|X_{t_m} = x]^+$ and $\mathbb{E}_m[G_{2m}\hat{V}_{m+1}(X_{t_{m+1}}, b)|X_{t_m} = x]^-$ over the grid points set $\mathcal{G}$. Subsequently, the holding value function $\hat{W}_m$ for the netted portfolio, indicated by b, is determined by applying the spectral interpolation scheme outlined in Equation (2.28).

To determine the value function $\hat{V}_m$ using either strategy, the conservative strategy as described in Equations (2.11) and (2.13), the security strategy in Equation (2.16), or the mixed strategy delineated in the linear program (2.19), it is essential to compute the holding value function for every subset of the portfolio. This necessitates that the function $\hat{V}_{m+1}$ be known for all sub-portfolios corresponding to $b' \leq b$. Therefore, the valuation process is initiated by recursively evaluating the smallest sub-portfolios, then progressively assessing larger subsets under the netting agreement, and culminating with the valuation of the entire netted portfolio. Consequently, with $\hat{W}_m$ known for all $b' \leq b$, we are equipped to derive the function of $\hat{V}_m(X_{t_m}, b)$ using any of these outlined methods.

The complete backward recursive algorithm to compute the value of a netted portfolio is shown in Algorithm 1. This algorithm provides the BVA as a function of the state vector (x, b), obtained by mixed strategy, for a set of discrete evaluation dates in $\mathcal{T}$ using dynamic programming.

In the pseudocode, we use decimal numbers $\phi_1 = 1, 2, ..., 2^{n_1}$ and $\phi_2 = 1, 2, ..., 2^{n_2}$ to describe the binary vectors b_1 and b_2, respectively, in ascending order. For a given binary vector u, we denote by $\Phi(u)$ as the set of positive indexes in the vector u. The notation $h(.)$ denotes a decimal number to binary vector converter and $\odot$ represents the element-wise multiplication.

2.4 Numerical illustration

This section reports on numerical experiments addressing the sensitivity of counterparty risk to various parameters and the impact of netting and CCR on adjustment values using the dynamic program proposed in Section 2.3.

Algorithm 1 Backward Recursive Algorithm to Evaluate BVA of Netted Portfolio Using Mixed Strategy

Input: $\{F_{mj}(x), m = 1, \ldots, M, j = 1, \ldots, N\}$, β, n_1, n_2, ρ_1, ρ_2, M, grid points in set of $\mathcal{G}$, $(\psi_k)_{\{k=1,..,p\}}$, and defaults models.
for $\phi_1 = 1$ to 2^{n_1} **do**
  for $\phi_2 = 1$ to 2^{n_2} **do**
    $b = (b_1, b_2) \leftarrow h(\phi_1 - 1, \phi_2 - 1)$
    $\hat{V}_M(x_k, b) = \sum_{j \in \Phi(b)} F_{Mj}(x_k)$, for $k = 1, \ldots, p$
    $\hat{W}_M(x_k, b) = 0$, for $k = 1, \ldots, p$
    for $m = M - 1$ down to 0 **do**
      for $\kappa_1 = 1$ to ϕ_1 **do**
        for $\kappa_2 = 1$ to ϕ_2 **do**
          $a^\kappa = (a_1^{\kappa_1}, a_2^{\kappa_2}) \leftarrow h(\kappa_1 - 1, \kappa_2 - 1)$
          $b' = b - b \odot a^\kappa$
          Compute $c_k^m(b')$ via Eq. (2.29), for $k = 1, \ldots, p$.
          Compute $\hat{W}_m(x_k, b')$ using Eq. (2.28) , for $k = 1, \ldots, p$.
          $R_m(x_k, a^\kappa) = \sum_{j \in \Phi(a^\kappa)} F_{Mj}(x_k)$, for $k = 1, ..., p$
          $\hat{V}_{\kappa_1 \kappa_2, m}(x_k, b) = R_m(x_k, a^\kappa) + \hat{W}_m(x_k, b')$, for $k = 1, \ldots, p$
      Compute $\hat{V}_m(x_k, b)$ using Eq. (2.19), for $k = 1, ..., p$
      Compute $V_m(x_k, b) = \sum_{j \in \Phi(b)} V_{mj}(x_k)$ using Eq. (2.1), for $k = 1, ..., p$
      $BVA_m(x_k, b) = V_m(x_k, b) - \hat{V}_m(x_k, b)$

2.4.1 Base case specification

We consider a portfolio consisting of $n = n_1 + n_2$ Bermudan put options written on the same underlying asset, with possibly distinct strike prices denoted by K_{ij}, $j = 1, ..., n_i$ for Option j, that is held by C_i. All options have the same maturity $T = 1$ and $Ne = 50$ equally spaced exercise opportunities, which, along with the inception date, form the set $\mathcal{T}$. Counterparty C_1 and C_2 are in a netting agreement, where C_1 holds a long position on n_1 options with the strike prices of K_{1j}, $j = 1, ..., n_1$ and a short position on n_2 options with the strike prices of K_{2j}, $j = 1, ..., n_2$ and C_2 holds the opposite position.

We assume that the underlying asset price process is described by a geometric Brownian motion, so that the price process under the risk-neutral measure is given by

$$X_t = X_0 \exp\left((r - \frac{\sigma^2}{2})t + \sigma B_t\right), \tag{2.30}$$

where X_0 is the asset price at inception, σ is the volatility ¸of the price process, and B denotes a standard Brownian motion. The benchmark values characterizing the underlying asset process are reported in Table 2.10.

Parameters	Underlying asset			T	Ne
	r	X_0	σ		
Base value	0.05	100	0.35	1	50

Table 2.10: Benchmark values for the numerical experiments.

We use an intensity-based model of default and assume that the parties' defaults are exogenous events governed by the first jump of independent Poisson processes with constant hazard intensities, denoted respectively by $\lambda_i, i = 1,2$. Accordingly, the probability that counterparty C_i defaults first during a time interval Δ, given that it has not defaulted yet, is a constant given by

$$\begin{aligned} p_i &\equiv \mathbb{E}_m\left[1_{t_m<\tau_i\le t_{m+1}} 1_{\tau_i<\tau_{3-i}}\right] \\ &= \frac{\lambda_i}{\lambda_1+\lambda_2}(1-\exp(-\Delta(\lambda_1+\lambda_2))), \quad m = 1,\ldots,M-1, \quad i \in \{1,2\}. \end{aligned} \tag{2.31}$$

In that case, Equation (2.10) simplifies to

$$\begin{aligned} W_m(x,b) = \beta\big((1-p_1(1-\rho_1))\mathbb{E}_m\left[\hat{V}_{m+1}(X_{t_{m+1}},b)^+\right] \\ +(1-p_2(1-\rho_2))\mathbb{E}_m\left[\hat{V}_{m+1}(X_{t_{m+1}},b)^-\right]\big). \end{aligned} \tag{2.32}$$

Finally, note that if the following condition is satisfied

$$p_1(1-\rho_1) = p_2(1-\rho_2) \equiv s, \tag{2.33}$$

the holding value further simplifies to

$$W_m(x,b) = \beta(1-s)\mathbb{E}_m\left[\hat{V}_{m+1}(X_{t_{m+1}},b)\right]. \tag{2.34}$$

In this specific instance, where both parties are subject to the exact same counterparty default risk, it is easy to see that netting has no impact on the valuation of the portfolio, risk adjustment value, and the early exercise strategy.

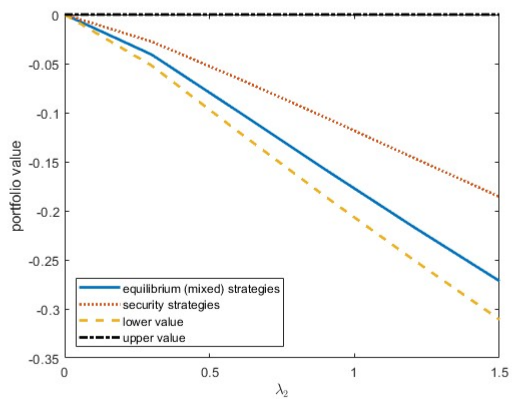
2.4.2 Behavioral assumptions

As delineated in Section 2.3.4, the dynamic-game model employed for evaluating the price of CCR within a netted portfolio—where both parties have optional rights - can be used under distinct assumptions about the way to compute the value of a vulnerable portfolio. The three assumptions proposed in Section 2.3.4 can yield four distinct values for the CVA/BVA. This holds true even in scenarios where the portfolio's upper and lower value functions coincide, provided that there exists, at minimum, one region within the state space over the portfolio's remaining horizon where such coincidence is absent. This is due to the fact that the portfolio value is an expectation of future cash flows, contingent to the exercise strategies of both parties.

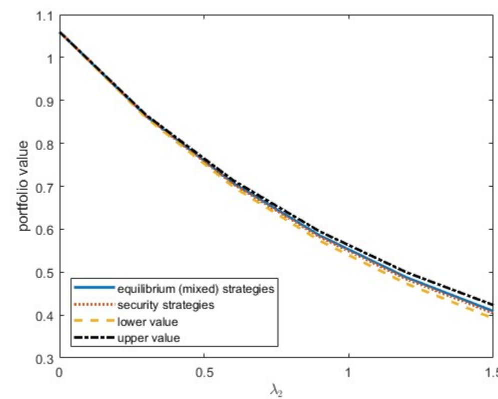
This section explores scenarios in which the upper and lower boundaries of the portfolio value are not equal. Specifically, the discussion focuses on how the default risk of counterparties and the portfolio value interact in these particular instances.

Unilateral risk

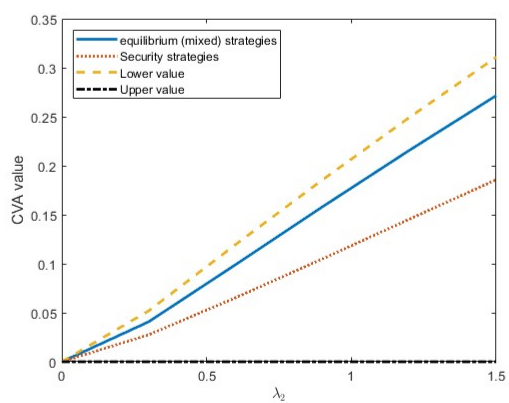
To analyze unilateral risk, we posit that C_1 is devoid of default risk ($\lambda_1 = 0$), while C_2 is susceptible to default, characterized by a recovery rate of $\rho_2 = 0$. For our initial set of experiments, we examine a portfolio consisting of two Bermudan put options with strike prices of K_{11}, as specified in each plot, and $K_{21} = 100$. These options are written on the same underlying asset, with parameter values specified in Table 2.10. Figure 2.1 illustrates the four portfolio values and their corresponding CVA calculations, undertaken under distinct assumptions proposed in Section 2.3.4 for the netted portfolio of two Bermudan put options. These are evaluated as functions of C_2's hazard rate under two distinct conditions: when $K_{11} = K_{21}$ (Panels a and c) and when $K_{11} = 1.02K_{21}$ (Panels b and d). As



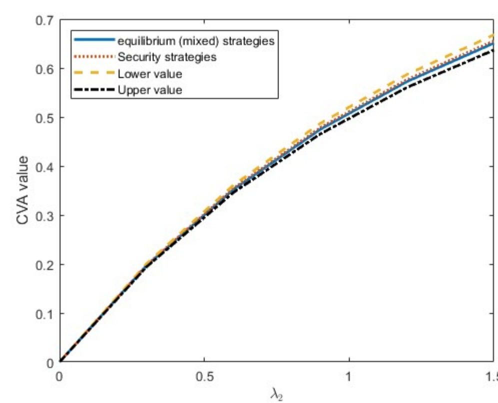
(a) $K_{11} = K_{21} = 100$



(b) $K_{11} = 1.02K_{21} = 102$



(c) $K_{11} = K_{21} = 100$



(d) $K_{11} = 1.02K_{21} = 102$

Figure 2.1: Comparison of the netted portfolio value (Panels a and b) and CVA value (Panels c and d) at inception, as a function of λ_2 when $\lambda_1 = 0$ and $\rho_2 = 0$, according to the assumption used to compute the portfolio value. Other parameter values are reported in Table 2.10

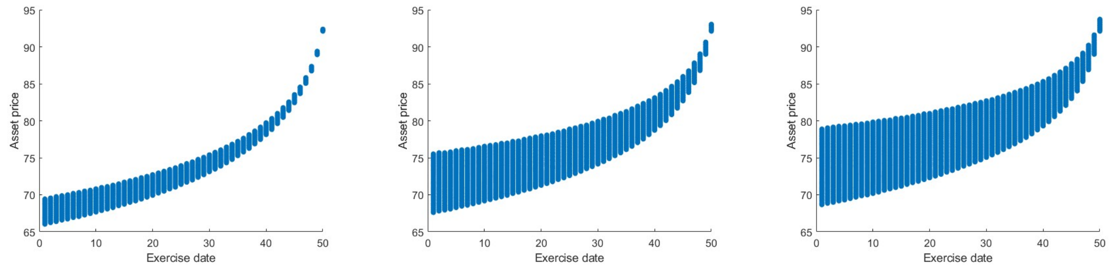
illustrated, when $K_{11} = K_{21}$, the CVA value is precisely the negative of the portfolio value calculated using the same exercise strategy. This inverse relationship arises due to the risk-free portfolio value becoming zero when the strike prices are identical.

One discerns that under Assumption A3, the CVA values manifest discrepancies between the counterparties. Both parties tend to overestimate their expected losses by adopting worst-case outcomes, particularly when a Nash equilibrium in pure strategies is not attainable.

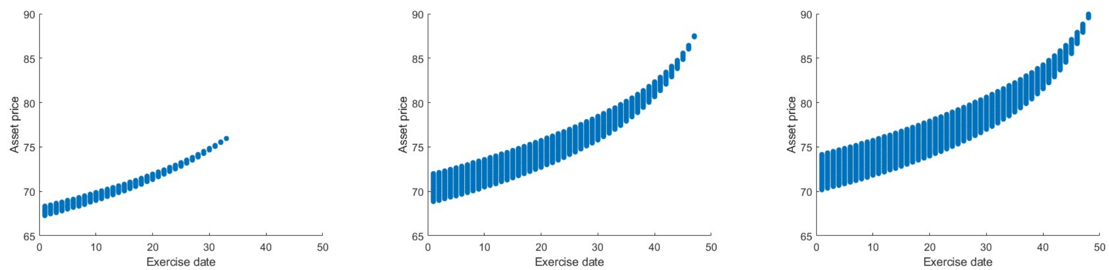
A noticeable consequence of increasing the parameter λ_2 is a more pronounced difference between the portfolio's upper and lower value functions. This difference is due to the increased number of states that lack a Nash equilibrium in pure strategies. Such states, within a portfolio of two Bermudan put options, are graphically represented in Figure 2.2. Panels a to c pertain to scenarios where $K_{11} = K_{21} = 100$, while Panels e to g correspond to those where $K_{11} = 1.02K_{21} = 102$. The depicted states represent the price of the underlying asset at each potential exercise date

Upon closely examining Figure 2.2, it is apparent that increasing the default risk of C_2 within the subject portfolio results in a greater number of states that lack a Nash equilibrium in pure strategies. This phenomenon can be traced back to the deterioration of the most unfavorable outcome for C_1 as C_2's default probability escalates, while the least favorable outcome for C_2 remains static. Consequently, in a heightened number of states, the portfolio's lower value tends to descend below its upper value, thus precluding the existence of a Nash equilibrium in pure strategies.

In a follow-up experiment, we delve into the interplay between the value of a risk-free portfolio and the divergence between the upper and lower values. Extending beyond the earlier portfolio comprising two Bermudan put options, a portfolio of four such options with strike prices of K_{11}, $K_{12} = 100$, $K_{21} = 105$, $K_{22} = 103$ is considered. The results are depicted in Figure 2.3, where Panels a and b showcase the vulnerable and risk-free



(a) $\lambda_2 = 0.1$ (b) $\lambda_2 = 0.5$ (c) $\lambda_2 = 1$

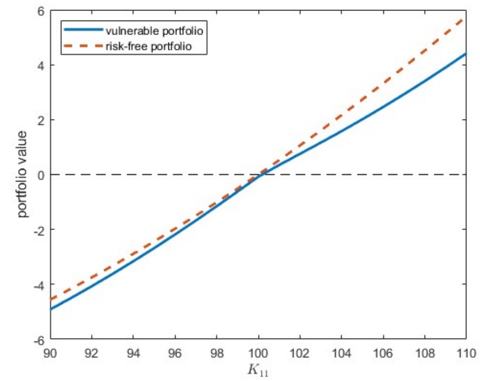


(d) $\lambda_2 = 0.1$ (e) $\lambda_2 = 0.5$ (f) $\lambda_2 = 1$

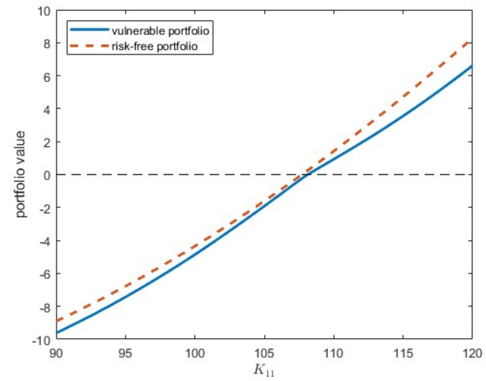
Figure 2.2: Representation of states without Nash equilibrium in pure strategy for a portfolio of two Bermudan put options when $K_{11} = K_{21} = 100$ (Panels a-c) and $K_{11} = 1.02K_{21} = 102$ (Panels e - f) for $\lambda_2 = 0.1$, $\lambda_2 = 0.5$, and $\lambda_2 = 1$ when $\lambda_1 = 0$ and $\rho_2 = 0$. Other parameter values are reported in Table 2.10.

portfolio values for portfolios of two and four Bermudan put options, respectively, as functions of the strike price K_{11} under the conditions $\lambda_2 = 0.5$ and $\rho_2 = 0$. Panels c and d offer a comparison of the lower, upper, security, and equilibrium values of the CVA for these portfolios. Furthermore, Figure 2.4 presents the states that lack a Nash equilibrium in pure strategies for portfolios as a function of varying K_{11}.

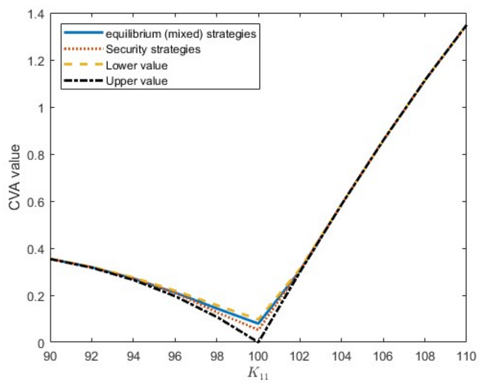
Our findings suggest that with a constant default probability, the gap between the upper and lower bounds tends to narrow as the absolute value of the portfolio increases. This phenomenon is attributed to a higher likelihood of dominant strategies emerging when the portfolio's value deviates substantially from zero. Specifically, a larger positive (negative) portfolio value from the viewpoint of C_1 implies that the relative moneyness of options within C_1's subportfolio exceeds (is below) those held by C_2. In such cases, there is a



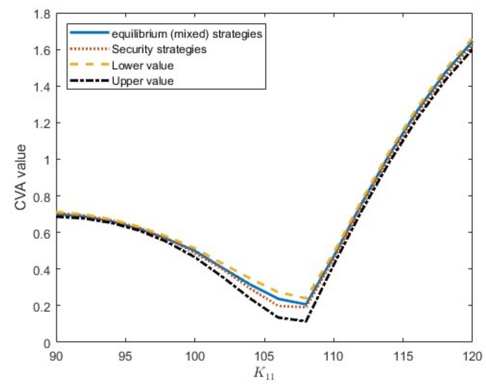
(a) $K_{21} = 100$



(b) $K_{12} = 100, K_{21} = 105, K_{22} = 103$



(c) $K_{21} = 100$

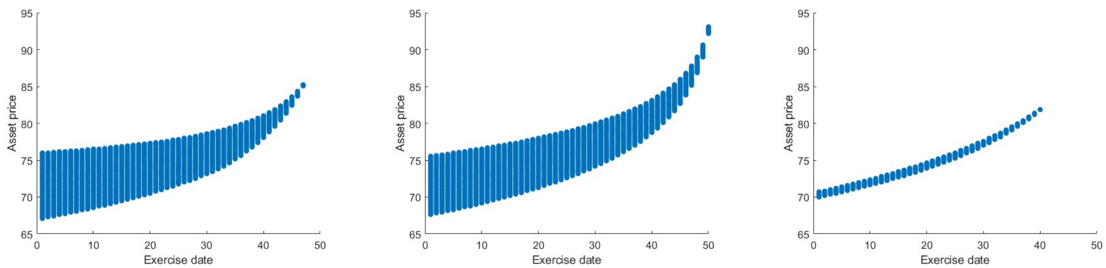


(d) $K_{12} = 100, K_{21} = 105, K_{22} = 103$

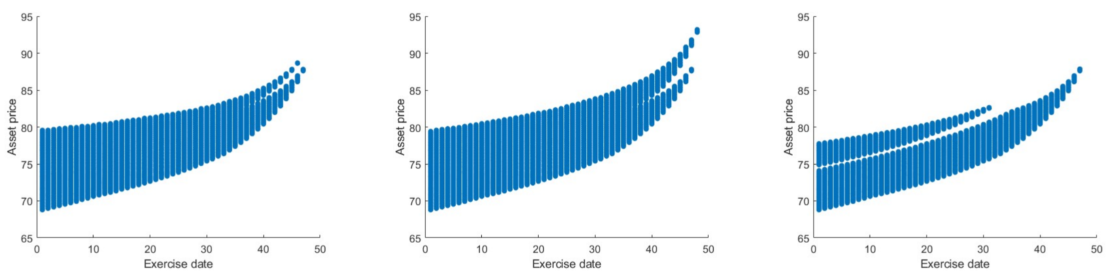
Figure 2.3: Illustration of the conservative value (from C_1's viewpoint) of the vulnerable and risk-free portfolio (Panels a and b) and CVA (Panels c and d) at inception for the portfolio of two Bermudan put options with strike prices of K_{11} and $K_{21} = 100$ and portfolio of four Bermudan put options with strike prices of K_{11}, $K_{12} = 100$, $K_{21} = 105$, $K_{22} = 103$ when $\lambda_2 = 0.5$ and $\rho_2 = 0$. All options are written on the same underlying asset with parameter values in Table 2.10.

greater probability that a dominant strategy exists for C_1, including early exercising its deeper ITM options (holding its deeper OTM options). As discussed in Section 2.2, the emergence of dominant pure strategies in this manner increases the likelihood of Nash equilibrium across more underlying risk factor states over the portfolio's lifespan, thereby reducing the spread between bounds.

This pattern is evident in Panel c of Figure 2.3 for the two-option portfolio, where a sufficient disparity in the degree of moneyness between options held by C_1 and C_2 leads to the convergence of the bounds. Similarly, in the four-option portfolio (Panel d of Figure 2.4), as the relative moneyness of C_1's options diverges further from those of C_2, influencing the portfolio's value, we observe a reduced distance between the lower and upper bounds.



(a) $K_{11} = 97$ (b) $K_{11} = 100$ (c) $K_{11} = 103$



(d) $K_{11} = 104$ (e) $K_{11} = 107$ (f) $K_{11} = 110$

Figure 2.4: Representation of states without Nash equilibrium in pure strategy for the portfolio of two Bermudan put options with strike prices of K_{11} and $K_{21} = 100$ (Panels a-c) and portfolio of four Bermudan put options with strike prices of K_{11}, $K_{12} = 100$, $K_{21} = 105$, $K_{22} = 103$ (Panels e - f) when $\lambda_2 = 0.5$ and $\rho_2 = 0$. All options are written on the same underlying asset with parameter values in Table 2.10.

Bilateral risk

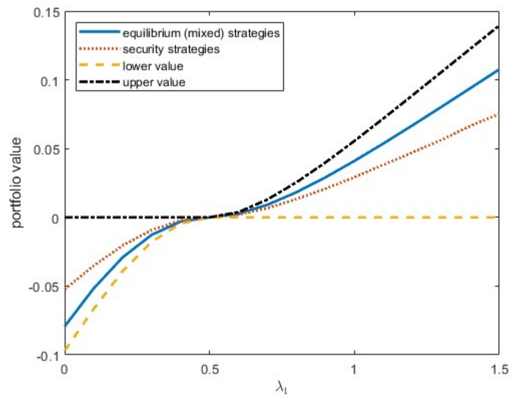
To explore the case of bilateral risk, we consider a scenario in which both counterparties are subject to default risk. Throughout the experiments detailed in this section, we fix $\lambda_2 = 0.5$ and set $\rho_1 = \rho_2 = 0$. For the predetermined portfolio of two options, we draw a comparative analysis between the four distinct portfolio values calculated based on assumptions articulated in Section 2.3.4 and their corresponding BVA, as a function of the parameter λ_1, which is illustrated in Figure 2.5.

It is worth recalling that when Condition (2.33) is met, the portfolio's valuation is equivalent to the aggregated values of the individual options, making netting effects irrelevant. This matching among the four interpretations becomes evident in Figure 2.5, particularly when $\lambda_2 = \lambda_1 = 0.5$.

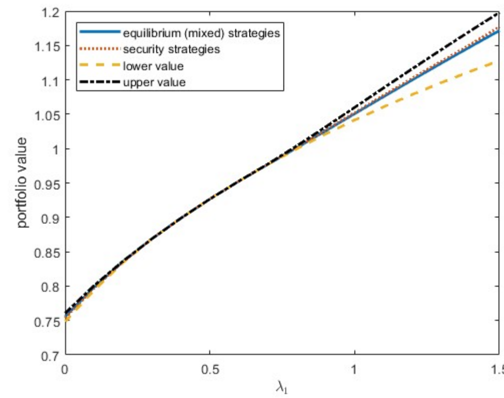
Figure 2.5 demonstrates that the BVA can be either negative or positive. This is contingent upon the interplay between the counterparties' relative default risks and the valuation of the sub-portfolios each party holds. A negative BVA signifies that Counterparty C_2 is more susceptible to the default risk of the other party and thus necessitates the allocation of additional capital as a contingency measure for possible losses. This effect is especially salient in Figure 2.5c, where the portfolio comprises options with identical riskless values, yet a negative BVA arises when the default risk of C_1 surpasses that of C_2 (i.e., $\lambda_1 > \lambda_2$).

Extending the observations from the unilateral risk scenario, it is evident that an augmented disparity between the default risks of the two counterparties results in a broader divergence between the portfolio's upper and lower valuations. This observation is coherent with earlier analyses; as the default risk of one counterparty (C_i) exceeds that of its opposite number (C_{2-i}), the least favorable financial outcomes against C_{2-i} deteriorate accordingly. Consequently, in an increased number of states, the portfolio's lower valuation is more likely to fall below its upper valuation. Figure 2.6 depicts these particular states, characterized by an absence of Nash equilibrium in pure strategy when both parties are exposed to bilateral risk.

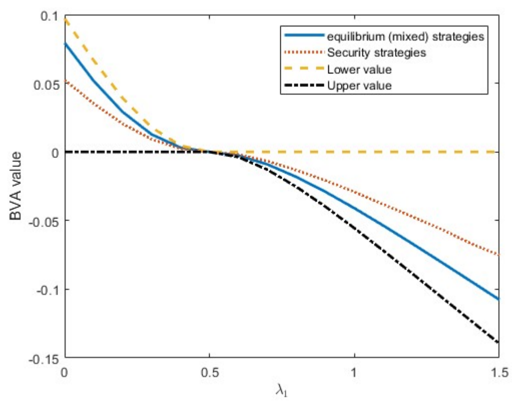
In Figure 2.7, we further illustrate the portfolio value and BVA as functions of the strike price K_{11}, for portfolios containing either two or four Bermudan put options when $\lambda_1 = 0.2$ and $\lambda_1 = 0.5$. In line with previous discussions of unilateral risk, the gap between



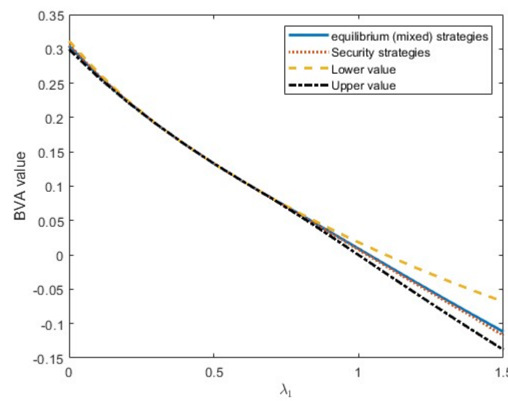
(a) $K_{11} = K_{21} = 100$



(b) $K_{11} = 1.02K_{21} = 102$



(c) $K_{11} = K_{21} = 100$

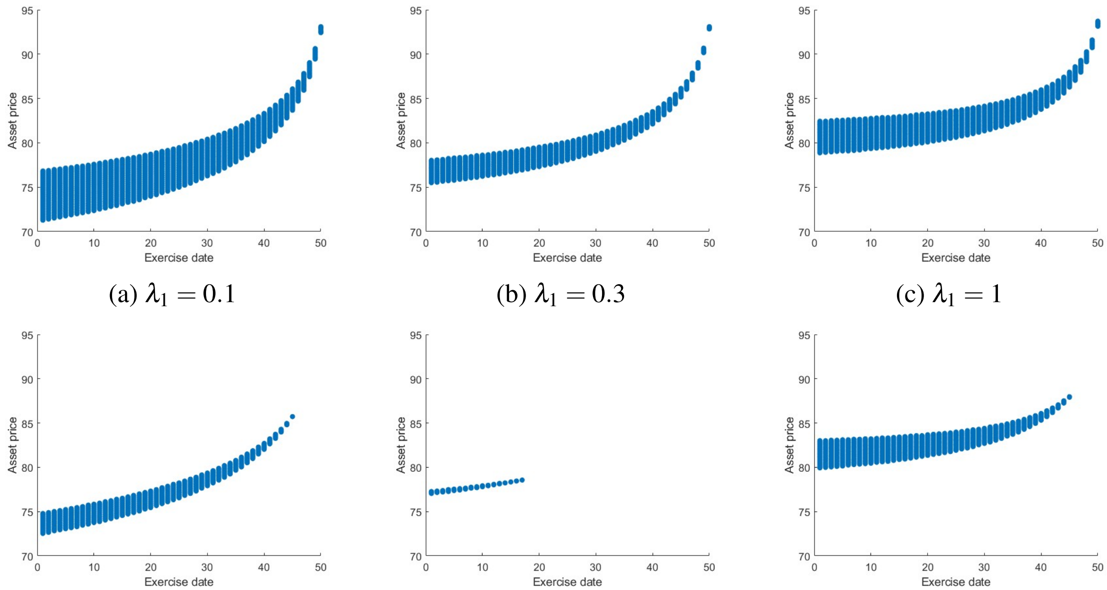


(d) $K_{11} = 1.02K_{21} = 102$

Figure 2.5: Comparison of the netted portfolio value (Panels a and b) and BVA value (Panels c and d) at inception, as a function of λ_1 when $\lambda_2 = 0.5$ and $\rho_1 = \rho_2 = 0$, according to the assumption used to compute the portfolio value. Other parameter values are reported in Table 2.10

the portfolio's upper and lower limits narrows when the portfolio's value substantially diverges sufficiently from zero.

Synthesizing the outcomes from both unilateral and bilateral scenarios, the divergence in the various interpretations of CCR pricing can be notably diminished under certain conditions. Specifically, our numerical experiments indicate that the gap between the lower and upper bounds of a vulnerable portfolio—and by extension, their associated risk ad-



(a) $\lambda_1 = 0.1$ (b) $\lambda_1 = 0.3$ (c) $\lambda_1 = 1$

(d) $\lambda_1 = 0.1$ (e) $\lambda_1 = 0.3$ (f) $\lambda_1 = 1$

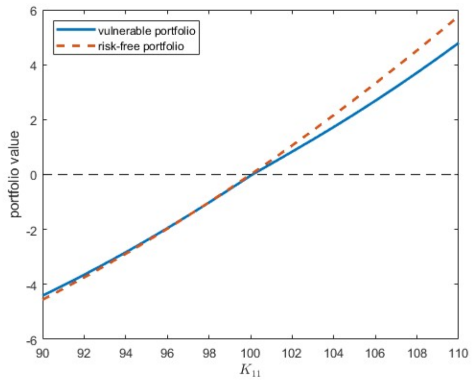
Figure 2.6: Representation of states without Nash equilibrium in pure strategy for a portfolio of two Bermudan put options when $K_{11} = K_{21} = 100$ (Panels a-c) and $K_{11} = 1.02K_{21} = 102$ (Panels e - f) for $\lambda_1 = 0.1$, $\lambda_1 = 0.3$, and $\lambda_1 = 1$ when $\lambda_2 = 0.5$ and $\rho_1 = \rho_2 = 0$. Other parameter values are reported in Table 2.10.

justment values— tends to narrow notably when:

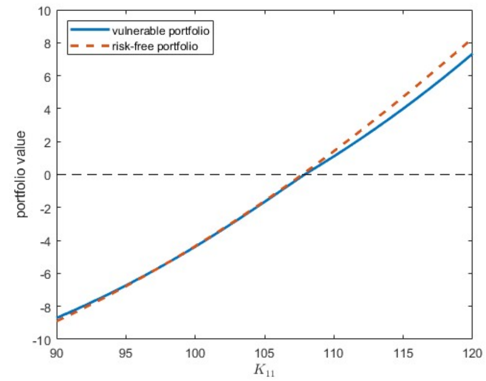
- The portfolio's value significantly deviates from zero;

- The default risks of both counterparties are more evenly balanced.

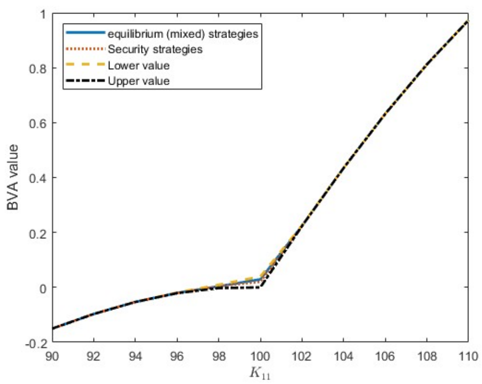
In the context of bilateral risk, a particularly intriguing phenomenon emerges when both counterparties adopt a conservative approach in portfolio evaluation, utilizing the BVA as the foundational metric for capital adjustment calculations. Figure 2.8 serves as an empirical illustration of this occurrence; specifically, the conservative BVA value for Counterparty C_1 and the negation of the conservative BVA value for Counterparty C_2, presented in Figure 2.8a), provide insights into the resultant capital adjustments, which are delineated in Figure 2.8b. Within the parameter interval where λ_2 varies from 0.21



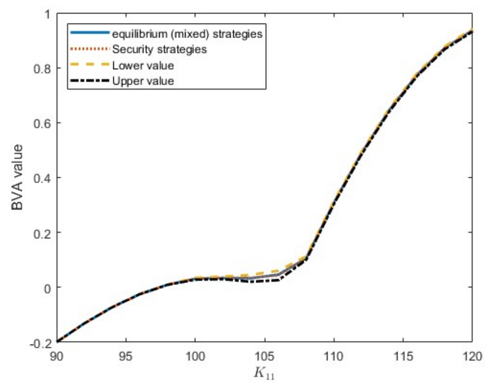
(a) $K_{21} = 100$



(b) $K_{12} = 100, K_{21} = 105, K_{22} = 103$



(c) $K_{21} = 100$

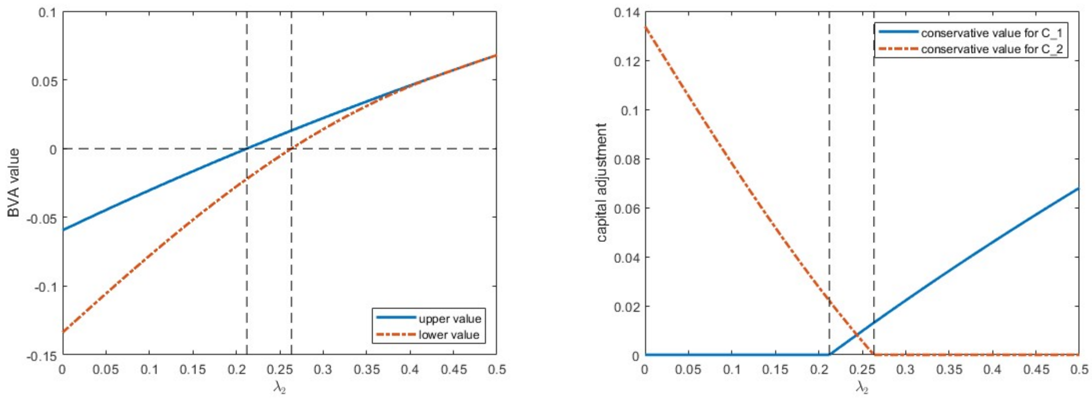


(d) $K_{12} = 100, K_{21} = 105, K_{22} = 103$

Figure 2.7: Illustration of the conservative value (from C_1's viewpoint) of the vulnerable and risk-free portfolio (Panels a and b) and BVA (Panels c and d) at inception for the portfolio of two Bermudan put options with strike prices of K_{11} and $K_{21} = 100$ and portfolio of four Bermudan put options with K_{11}, $K_{12} = 100$, $K_{21} = 105$, $K_{22} = 103$ when $\lambda_1 = 0.2$, $\lambda_2 = 0.5$ and $\rho_1 = \rho_2 = 0$. All options are written on the same underlying asset with parameter values in Table 2.10.

to 0.26, the capital adjustment for both parties attains a positive value. However, such an outcome is impossible when the portfolio is evaluated using a mixed strategy.

It is important to note that if counterparties employ different strategies in exercising their options, they may obtain disparate pricing for risk adjustment value. Nonetheless, this discrepancy can be relatively small when compared to the magnitude of error arising from disregarding the netting impact in the evaluations. The analyses presented in subse-



(a) BVA as a function of λ_2 (b) Capital adjustment as a function of λ_2

Figure 2.8: BVA value and capital adjustment at inception for both parties for a portfolio of two Bermudan put options with $K_{11} = 101$ and $K_{21} = 100$, $\lambda_1 = 0.5$ and $\rho_1 = \rho_2 = 0$. Other parameter values are reported in Table 2.10

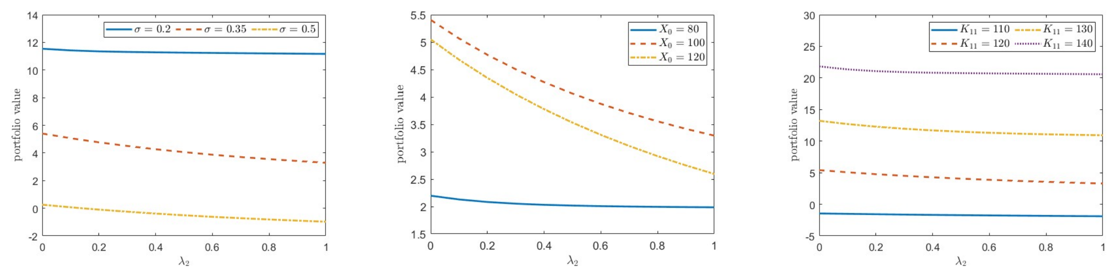
quent sections in this chapter are all obtained under the assumption that parties use mixed strategies.

2.4.3 Analyzing the Sensitivity of portfolio value and CVA/BVA in parameter values

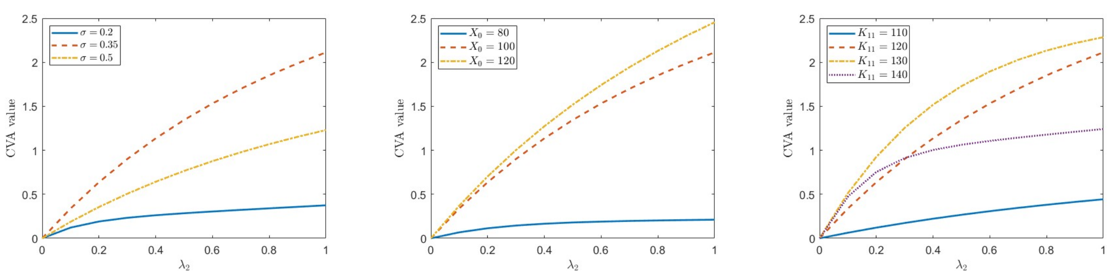
This section conducts an analytical examination to understand how variations in default probability and recovery rates for both counterparties influence the risk adjustment values. To enhance the robustness of our findings, sensitivity analyses related to variations in σ, X_0, and K_{11} are performed. To further elucidate the intricate behaviors of the CVA/BVA with respect to these parameter shifts, an investigation is executed on a portfolio that includes three Bermudan options. These options have benchmark strike prices $K_{11} = 120$, $K_{21} = 100$, and $K_{22} = 90$, all of which are written on the same underlying asset. The parameter values for this analysis are provided in Table 2.10.

Unilateral risk

In Figure 2.9, the netted portfolio value is depicted across Panels a-c, while CVA is presented in Panels d - f, as functions of λ_2 under the assumption that $\rho_2 = 0$. Figures 2.9a and 2.9d, serve to contrast the benchmark results, where the underlying asset volatility σ is 0.35, against cases when σ is altered to 0.2 and 0.5. Similarly, in Figures 2.9b, 2.9d, 2.9c, and 2.9f, comparative evaluations are performed for benchmark parameter values $X_0 = 100$ and $K_{11} = 120$ via scenarios where X_0 is modified to 80 and 120, and K_{11} is set to 110, 130, and 140 respectively.



(a) Changes in regard to σ (b) Changes in regard to X_0 (c) Changes in regard to K_{11}



(d) Changes in regard to σ (e) Changes in regard to X_0 (f) Changes in regard to K_{11}

Figure 2.9: Sensitivity analysis of the portfolio value (Panels a-c) and the CVA (Panels e - f) at inception as a function of λ_2, with $\rho_2 = 0$, for various values of σ, X_0, and K_{11} in the portfolio of three Bermudan options with benchmark strike prices of $K_{11} = 120$, $K_{21} = 100$, and $K_{22} = 90$. All options are written on the same underlying asset with parameter values reported in Table 2.10.

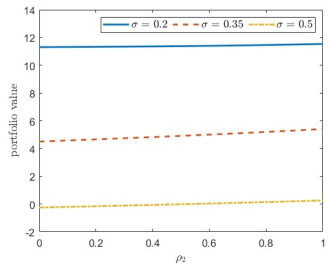
In line with expectations, our analysis substantiates that the CVA of a netted portfolio consistently escalates as counterparty risk increases. While it is possible to discern a pattern for the netted portfolio value in response to parameter modifications (σ, X_0, and K_{11}), such predictable behavior is absent for CVA. Changing these parameters affects both risk-free and risky portfolios, but the degree of impact is inconsistent across them. In certain portfolios, the parameter changes have a greater influence on the risk-free portfolio compared to the one exposed to counterparty risk, and vice versa. Consequently, it's difficult to generalize the CVA's sensitivity to these parameter adjustments. Our numerical investigations confirm this unpredictability, indicating that there are no broad trends in how such adjustments influence CVA.

In Figure 2.10, we illustrate the portfolio value and the CVA as functions of ρ_2 with a constant hazard rate $\lambda_2 = 0.3$. Similarly, the sensitivity of these results to variations in σ, X_0, and K_{11} is explored. The trends we see here are consistent with what we found in Figure 2.9.

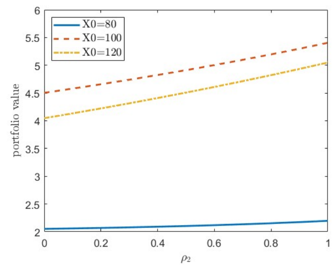
As we expected, the CVA value goes down when the counterparty's recovery rate goes up. This happens because a higher ρ_2 means that the counterparty (C_2) can take care of more debt if they default. This finding lines up with the basic theory behind calculating CVA. However, when we look at Figure 2.10, we don't see a clear pattern for how CVA changes with other parameters' values. This lack of a consistent pattern highlights that calculating CVA for portfolios under a netting agreement can be complicated.

Bilateral risk

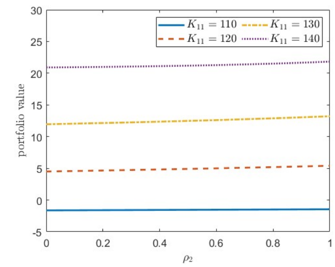
Figure 2.11 presents the value and the corresponding BVA level for the netted portfolio as a function of hazard rate λ_1, given that $\lambda_2 = 0.3$ and $\rho_1 = \rho_2 = 0$. Similar to the previous analysis, we compare the benchmark outcomes with different values of σ, X_0, and K_{11}. When we look at the BVA from C_1's point of view, we see a consistent and decreasing trend in all the graphs as λ_1 goes up. This shows that when C_1 considers its own risk of



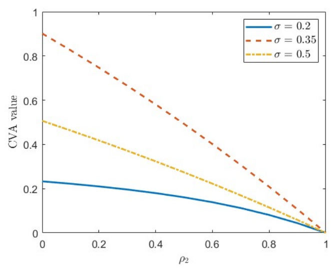
(a) Changes in regard to σ



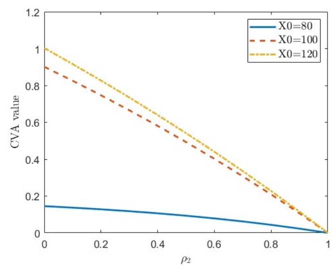
(b) Changes in regard to X_0



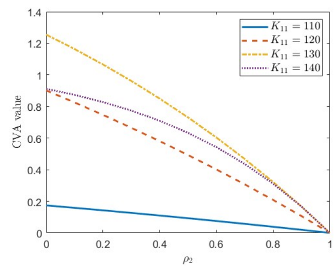
(c) Changes in regard to K_{11}



(d) Changes in regard to σ



(e) Changes in regard to X_0



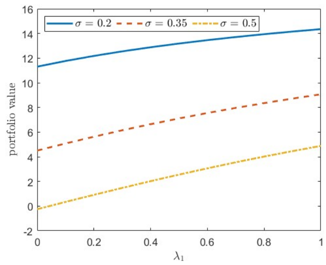
(f) Changes in regard to K_{11}

Figure 2.10: Sensitivity analysis of the portfolio value (Panels a-c) and the CVA (Panels e - f) at inception as a function of ρ_2 with fixed $\lambda_2 = 0.3$, for various values of σ, X_0, and K_{11} in the portfolio of three Bermudan options with benchmark strike prices of $K_{11} = 120$, $K_{21} = 100$, and $K_{22} = 90$. All options are written on the same underlying asset with parameter values reported in Table 2.10.

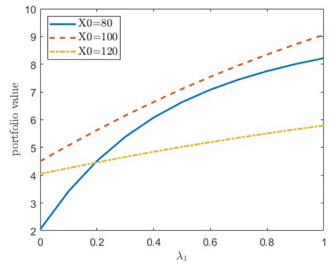
default, the need for capital reallocation to mitigate potential default risks is reduced.

Figure 2.12 shows how changing the recovery rate ρ_1 affects the benchmark portfolio's value and its BVA when $\lambda_1 = 0.2$, $\lambda_2 = 0.3$, and $\rho_2 = 0$. All the graphs show that BVA goes up as C_1's recovery rate increases, which makes sense because the risk of losing money if C_1 defaults goes down. In Figures 2.11 and 2.12, it's clear that, as we said earlier, there's no one rule that links how changes in σ, X_0, or K_{11} affect BVA.

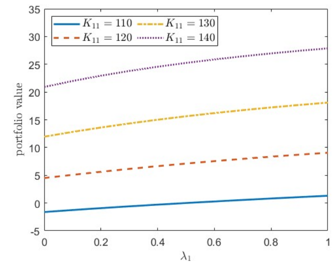
Numerical studies in both unilateral and bilateral cases reveal that there's no general trend in how risk valuation adjustments react to changes in model parameters in a netted portfolio. The only exceptions are the trends we see related to default risk and recovery rates of the counterparties, as mentioned earlier.



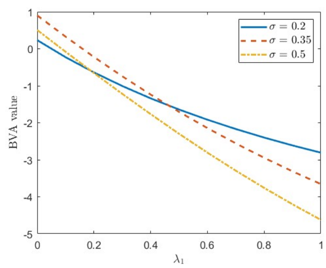
(a) Changes in regard to σ



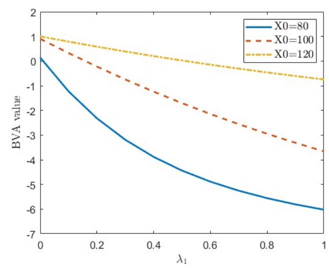
(b) Changes in regard to X_0



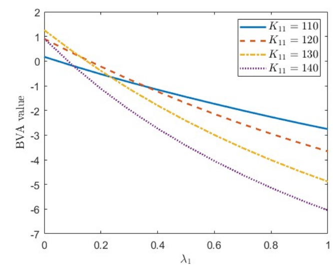
(c) Changes in regard to K_{11}



(d) Changes in regard to σ



(e) Changes in regard to X_0

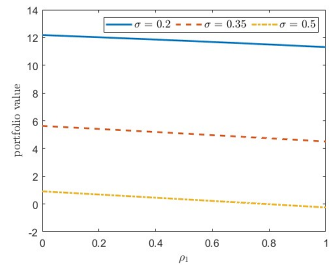


(f) Changes in regard to K_{11}

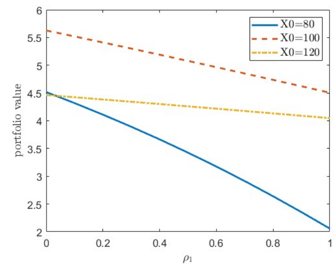
Figure 2.11: Sensitivity analysis of the portfolio value (Panels a-c) and the BVA (Panels e - f) at inception as a function of λ_1 with fixed $\lambda_2 = 0.3$ and $\rho_1 = \rho_2 = 0$, for various values of σ, X_0, and K_{11} in the portfolio of three Bermudan options with benchmark strike prices of $K_{11} = 120$, $K_{21} = 100$, and $K_{22} = 90$. All options are written on the same underlying asset with parameter values reported in Table 2.10.

2.4.4 Netting impact

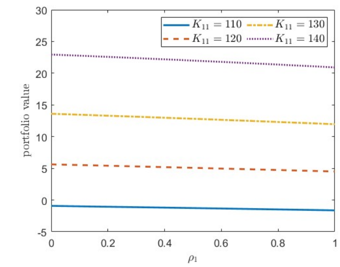
We conclude this chapter by discussing and illustrating the netting impact, which we define as the difference in CVA or BVA values for a netted portfolio compared to a non-netted one. In the absence of a netting agreement, the portfolio's value is computed by summing up the values of each individual vulnerable option, employing a risk-adjusted strategy. Our analysis specifically focuses on a portfolio containing two Bermudan options with strike prices $K_{11} = 120$ and $K_{21} = 100$. While our study is specific to this particular portfolio, the insights gained can be applied to other portfolios.



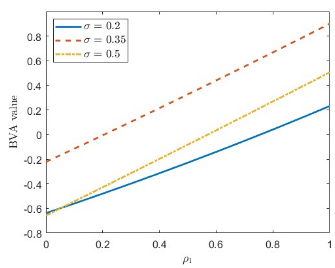
(a) Changes in regard to σ



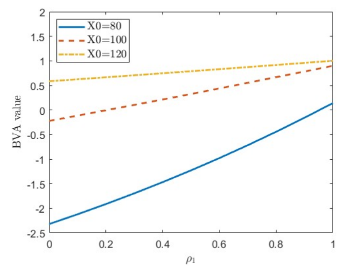
(b) Changes in regard to X_0



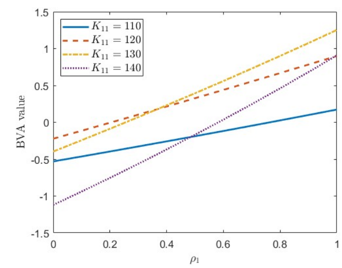
(c) Changes in regard to K_{11}



(d) Changes in regard to σ



(e) Changes in regard to X_0



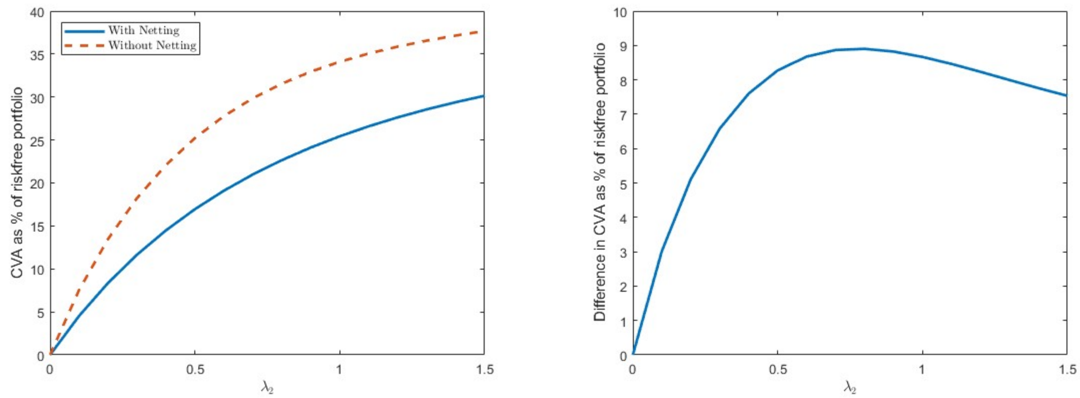
(f) Changes in regard to K_{11}

Figure 2.12: Sensitivity analysis of the portfolio value (Panels a-c) and the BVA (Panels e - f) at inception as a function of ρ_1 with fixed $\lambda_1 = 0.2$, $\lambda_2 = 0.3$ and $\rho_2 = 0$, for various values of σ, X_0, and K_{11} in the portfolio of three Bermudan options with benchmark strike prices of $K_{11} = 120$, $K_{21} = 100$, and $K_{22} = 90$. All options are written on the same underlying asset with parameter values reported in Table 2.10.

Unilateral risk

We contrast the CVA of the portfolio with and without netting, expressed as a percentage of the risk-free portfolio value, as a function of λ_2. Figure 2.13a displays this comparison, while Figure 2.13b demonstrates the netting impact, which is the difference between the two CVAs. Our analysis verifies that incorporating a netting agreement can substantially decrease the CVA, as the netted portfolio value is always greater than or equal to the sum of individual option values. Consequently, excluding netting from calculations can lead to a significant overestimation of the CVA for the netted portfolio.

To delve deeper, Figure 2.14 examines how the netting impact varies with changes



(a) CVA value (b) Difference

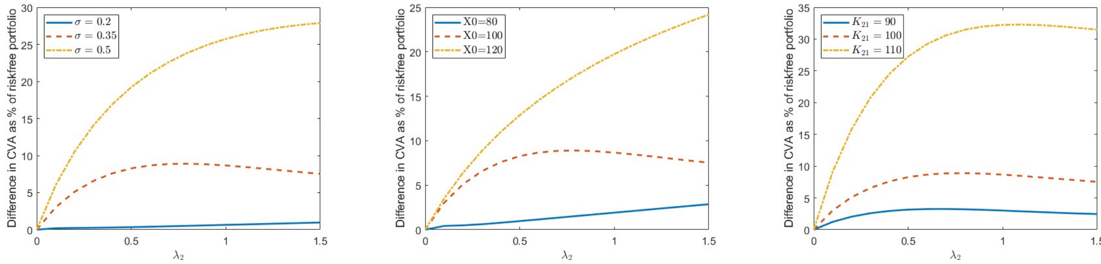
Figure 2.13: Netting impact on the CVA at inception as a function of λ_2: presented for a portfolio of two Bermudan put options with $K_{11} = 120$ and $K_{21} = 100$ when $\rho_2 = 0$. Other parameter values are reported in Table 2.10

in λ_2 for different levels of σ, X_0, and (K_{21}). The analysis reveals that the CVA gap between the netted and non-netted portfolios initially widens with increasing counterparty risk but later grows at a diminished rate. This slowing down occurs because, as C_2's default risk increases, C_1 is more likely to exercise the option with the strike price K_{11}, thereby reducing the netting impact.

Figure 2.15 extends the sensitivity analysis to explore how the netting impact varies with correlation parameter ρ_2. These findings are consistent with those from Figure 2.14. Generally, ignoring netting agreements amplifies the CVA overestimation, especially when there is an increase in the risk of loss from the counterparty's default, either due to a rise in default risk or a decrease in recovery rates.

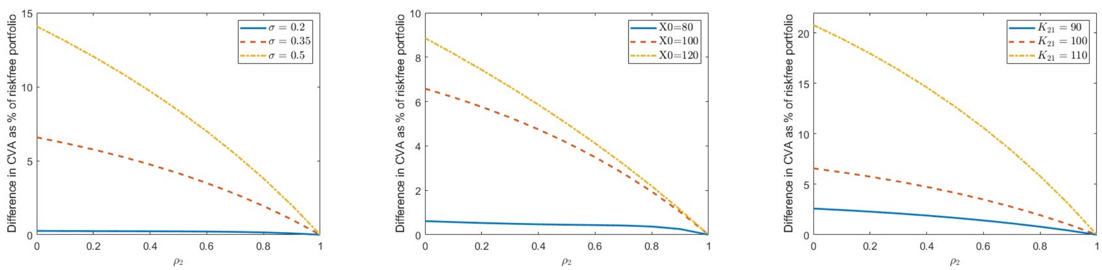
Bilateral risk

In Figure 2.16, the netting impact on the BVA is evaluated as a function of λ_1, assuming fixed values of $\lambda_2 = 0.3$ and $\rho_1 = \rho_2 = 0$. The figure indicates that integrating a netting



(a) Changes in regard to σ (b) Changes in regard to X_0 (c) Changes in regard to K_{21}

Figure 2.14: Sensitivity of the netting impact on the CVA at inception as a function of λ_2, when $\rho_2 = 0$, for various values of σ, X_0, and K_{21} in the portfolio of two Bermudan options with benchmark parameter values of $K_{11} = 120$, $K_{21} = 100$. Other benchmark parameter values are reported in Table 2.10.

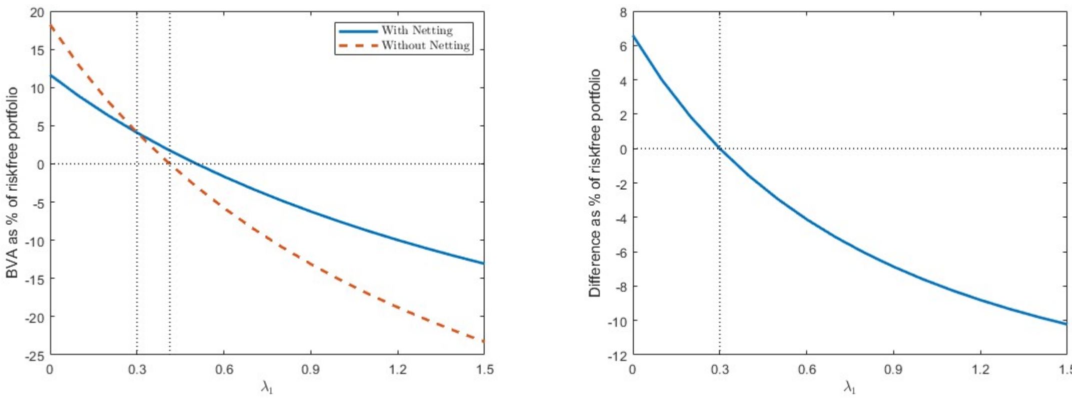


(a) Changes in regard to σ (b) Changes in regard to X_0 (c) Changes in regard to K_{21}

Figure 2.15: Sensitivity of the netting impact on the CVA at inception as a function of ρ_2, when $\lambda_2 = 0.3$, for various values of σ, X_0, and K_{21} in the portfolio of two Bermudan options with benchmark parameter values of $K_{11} = 120$, $K_{21} = 100$. Other benchmark parameter values are reported in Table 2.10.

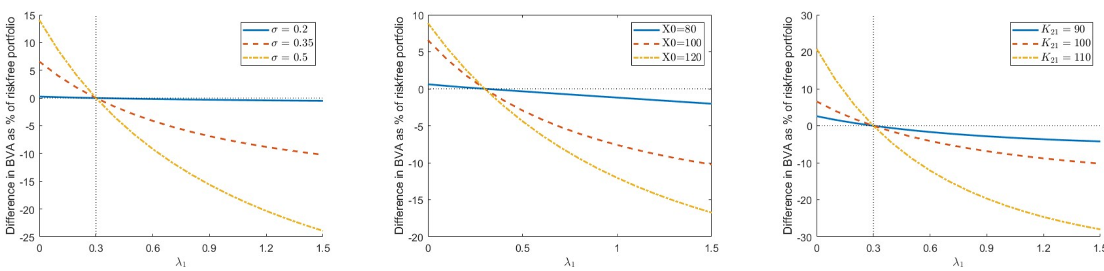
agreement notably reduces the risk adjustment for the counterparty with lower default risk. This observation is further supported by Figure 2.17, which performs a sensitivity analysis using different values of σ, X_0, and K_{21}.

Neglecting the netting effect can result in an overestimation of the BVA. Specifically, an overestimation occurs for C_1 when $\lambda_1 < \lambda_2$ and for C_2 when $\lambda_2 < \lambda_1$. Furthermore, the magnitude of this overestimation intensifies as the disparity in default probabilities between the two counterparties enlarges.



(a) BVA value (b) Difference

Figure 2.16: Netting impact on the BVA as a function of λ_1: presented for a portfolio of two Bermudan put options with $K_{11} = 120$ and $K_{21} = 100$ when $\lambda_2 = 0.3$ and $\rho_1 = \rho_2 = 0$. Other parameter values are reported in Table 2.10

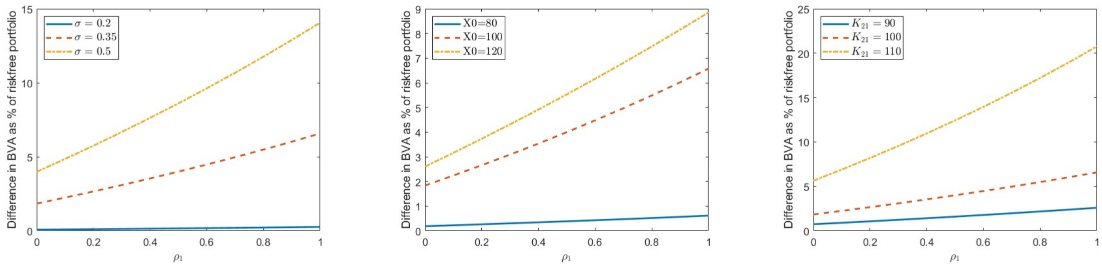


(a) Changes in regard to σ (b) Changes in regard to X_0 (c) Changes in regard to K_{21}

Figure 2.17: Sensitivity of the netting impact on the BVA at inception as a function of λ_1, when $\lambda_2 = 0.3$ and $\rho_1 = \rho_2 = 0$, for various values of σ, X_0, and K_{21} in the portfolio of two Bermudan options with benchmark parameter values of $K_{11} = 120$, $K_{21} = 100$. Other benchmark parameter values are reported in Table 2.10.

Note that the netting impact becomes null when Condition (2.33) is met. Specifically, this occurs when both parties exhibit identical default intensities, regardless of the differences in their relative exposures.

Figure 2.18 exhibits the sensitivity of the netting impact on the BVA at inception with respect to ρ_1. In line with Figure 2.17, increasing the recovery rate of C_1, which signifies



(a) Changes in regard to σ (b) Changes in regard to X_0 (c) Changes in regard to K_{21}

Figure 2.18: Sensitivity of the netting impact on the BVA at inception as a function of ρ_1, when $\lambda_1 = 0.2$, $\lambda_2 = 0.3$ and $\rho_2 = 0$, for various values of σ, X_0, and K_{21} in the portfolio of two Bermudan options with benchmark parameter values of $K_{11} = 120$, $K_{21} = 100$. Other benchmark parameter values are reported in Table 2.10.

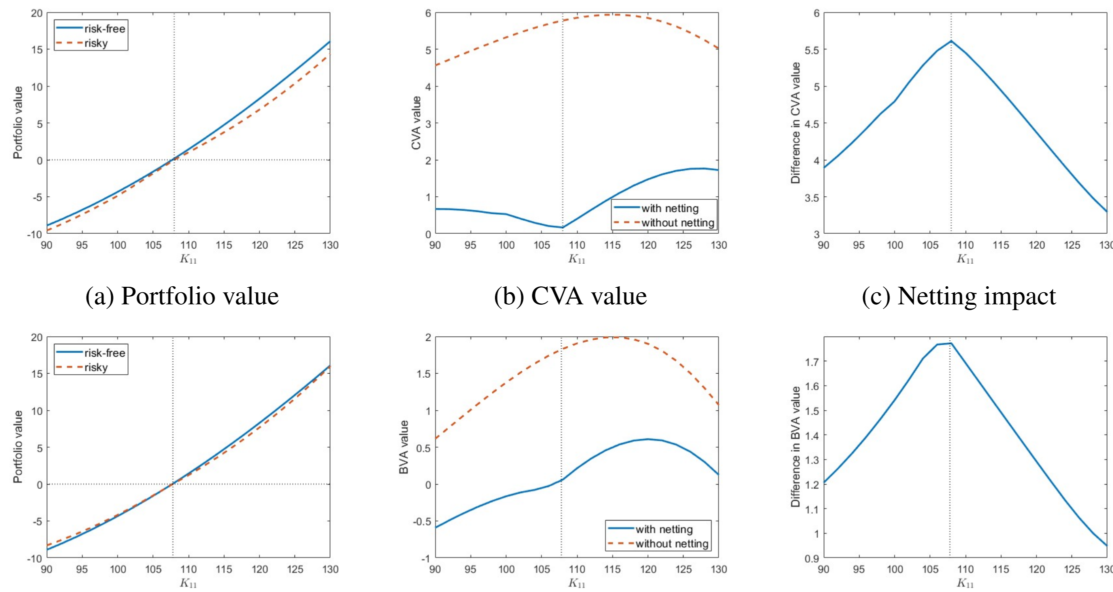
a decrease in the risk of loss from C_1 default, results in an increased netting impact.

Portfolio value

Figure 2.19 provides an analytical depiction of the netting impact on risk adjustment valuation in relation to K_{11}, considering a portfolio of four Bermudan put options on the underlying asset as outlined in Table 2.10. The strike prices for these options are $K_{12} = 100$, $K_{21} = 103$, and $K_{22} = 105$. The evaluation incorporates both unilateral and bilateral cases for a comprehensive understanding.

The figure highlights that the efficacy of netting is maximized when the portfolio's value nears zero. This phenomenon occurs due to the offsetting effect between options with positive and negative market values within the portfolio, thereby minimizing the netted expected loss. In contrast, if netting is disregarded, risk adjustment calculations solely focus on options with positive market value, leading to substantial inflation in the risk adjustment values.

Moreover, the analysis conveyed by the figure suggests a noteworthy trend. As the option with strike price K_{11} becomes increasingly in-the-money, a decline is observed in both CVA and BVA values, irrespective of whether netting is applied. This reduction in value is attributed to the increasing optimality of early exercise in numerous states, resulting in lower expected loss. This early exercise also eliminates the netting impact in those states, contributing to a narrowing gap in CVA/BVA between portfolios with and without netting.



(a) Portfolio value (b) CVA value (c) Netting impact

(d) Portfolio value (e) BVA value (f) Netting impact

Figure 2.19: Netting impact as a function of K_{11} for a portfolio of four Bermudan put options when $K_{12} = 100$, $K_{21} = 103$, and $K_{22} = 105$ for when $\lambda_1 = 0$ (Panels a-c) and $\lambda_1 = 0.3$ (Panels e - f), and $\lambda_2 = 0.5$, $\rho_1 = \rho_2 = 0$. All options are written on the same underlying asset with parameters values reported in Table 2.10.

Based on the empirical findings presented in this section, it can be concluded that the netting impact on risk adjustment valuation is likely to increase under specific conditions. These conditions include:

- When the portfolio value approaches zero, which indicates a balanced relationship between the positive and negative market values of the claims within the portfolio.

- When there is a widening disparity in potential loss exposure between the two parties involved. This amplification can be attributed to scenarios such as an increase in the default probability of one party in comparison to the other or a decrease (*increase*) in the recovery rate of party C_2 (C_1).

2.5 Conclusion

In the financial sector, netting is extensively used to reduce counterparty credit risk, especially among institutions handling multiple over-the-counter transactions. Yet, to fully benefit from the risk-mitigation advantages of netting, institutions must accurately determine risk adjustment values at the portfolio level. This becomes particularly complex for portfolios containing claims with early exercise features. In this chapter, we address the estimation of the CVA of portfolios of contracts under a netting agreement, when these contracts can include early-exercise features.

We introduce a dynamic programming methodology that comprehensively examines scenarios in which both counterparties have exercise rights and are susceptible to default. This methodology facilitates the accurate calculation of risk adjustment values at the portfolio level. Our analysis reveals that netted portfolios represent dynamic zero-sum games when parties have early exercise options. The proposed algorithm is enabled to compute exercise strategies for options within vulnerable portfolios while incorporating both netting and default risks. Furthermore, Our method allows us to characterize the CVA or BVA of a portfolio of contracts, for all possible values of the underlying market factors and all possible compositions of the portfolio, at all evaluation dates until maturity.

Our research identifies the possibility for divergence in the valuation of a risk adjustment for a netted portfolio, especially when counterparties adopt conservative strategies. Such divergence can occasionally result in a positive risk adjustment for both parties and is directly related to the absence of a Nash equilibrium in pure strategies within certain states. To address this issue, we propose the adoption of mixed strategies, which inherently ensure the existence of a Nash equilibrium, thereby aligning the valuation of the netted portfolio and risk adjustments from both parties' perspectives.

Further, our empirical analysis demonstrates that neglecting the role of netting can result in an overestimation of CVA/BVA values. This is notably true when there are significant disparities in default probabilities among counterparties. The findings also validate that netting efficacy increases when there is a greater balance between the positive and negative market values of the claims constituting the portfolio.

Finally, this chapter emphasizes the complex interplay involved in risk adjustments within the framework of netting agreements. Our findings suggest that while risk adjustment values generally exhibit an upward trend in relation to increasing counterparty risk, their response to changes in other parameters is less straightforward. This complexity challenges the feasibility of a universal approach to parameter changes at the portfolio level, underscoring the need for context-specific assessments.

Chapter 3

Beyond Counterparty Risk: The Influence of Netting Agreements on Exercise Strategies

3.1 Introduction

The early exercise feature in a derivative provides a distinctive combination of adaptability and potential profitability. However, this feature also introduces complexity to the derivative's evaluation, as the optimal exercise strategy, essential for maximizing the option's value, requires careful determination. In a scenario devoid of counterparty risk, the decision to exercise or retain these options is shaped by several factors, including the price of the underlying asset, time to maturity, volatility, and interest rates.

Computational methods have been extensively employed to solve the intricate optimization problems associated with determining the optimal exercise strategy. Longstaff and Schwartz (2001) develop a least squares Monte Carlo method for the determination of the exercise strategy and the evaluation of American-style options. Concurrently, Tsitsiklis and Van Roy (2001) devise a simulation-based approximate dynamic programming method for computing the optimal exercise strategy. These methods, which have been widely adopted in the industry, do not account for counterparty risk. Breton and Marzouk (2018) address this limitation by introducing a dynamic programming-based method that incorporates counterparty risk in the calculation of the exercise strategy.

3.1.1 Context and objectives

This chapter examines the often-overlooked impact of netting agreements between defaultable counterparties on the exercise strategy of Bermudan options, using various numerical experiments. The insights gained are crucial for financial institutions, investors, and regulators, shedding light on managing options with early exercise opportunities in a netted portfolio. The previous chapter discussed the effect of netting on the risk adjustment value of derivatives with early exercise options and introduced a dynamic game representation method. This chapter continues by exploring the influence of netting on portfolio management, particularly the exercise strategies of options.

3.1.2 Content and organization

This chapter begins with an exploration of the conservative and mixed strategies' implications on the optimal exercise decisions for Bermudan options in the netted portfolio, explained through a representative example. We then explore the complex dynamics of how changes in default hazard rate and the recovery rate of parties can significantly influence the exercise strategy of options within a netted portfolio. Our investigation continues to look at how a netting agreement can alter the exercise strategy, even for a party not exposed to counterparty default risk. Additionally, we examine the position of the exercise boundary within the netted portfolio, comparing it to the risk-free and risk-adjusted boundaries in the absence of netting. This comparison is conducted for both parties in both unilateral and bilateral default risk scenarios.

We also question traditional methodologies used to evaluate the risk adjustment value of netted portfolios. We argue that these methods, which are based on a risk-free exercise strategy or even the risky one in the absence of netting, often miss the impact of netting on the exercise process. This lack of attention can potentially lead to overestimations of the CVA and BVA of the netted portfolio.

The structure of this chapter is as follows: Section 3.2 compares the exercise behavior of options in the netted portfolio using the conservative strategy and mixed strategy in the presence of counterparty risk. Section 3.3 provides a detailed analysis of how changes in the default probability of parties and their recovery rate affect the exercise strategy of an option within a netted portfolio. Section 3.4 offers a foundational understanding of the exercise boundaries in scenarios involving unilateral and bilateral default risk, both with and without a netting agreement. Section 3.5 shows how using the risk-free exercise strategy or the isolated risky one, instead of the one that is adjusted for both netting and party's default risk, can lead to incorrect CVA or BVA of a netted portfolio. Finally, Section 3.6 concludes the chapter, highlighting the key findings.

3.1.3 Notation and settings

This chapter adheres to the notational conventions and symbols established in the preceding chapter, ensuring consistency across the discourse. The notations introduced in Chapter 2 persist throughout this chapter. The numerical experiments in subsequent sections are based on a portfolio comprising Bermudan put options, each written on the same underlying asset but potentially with distinct strike prices. The benchmark parameter values employed in the numerical experiments of the next sections are presented in Table 3.1.

Parameters	Underlying asset			T	Ne
	r	X_0	σ		
Base value	0.05	100	0.35	1	50

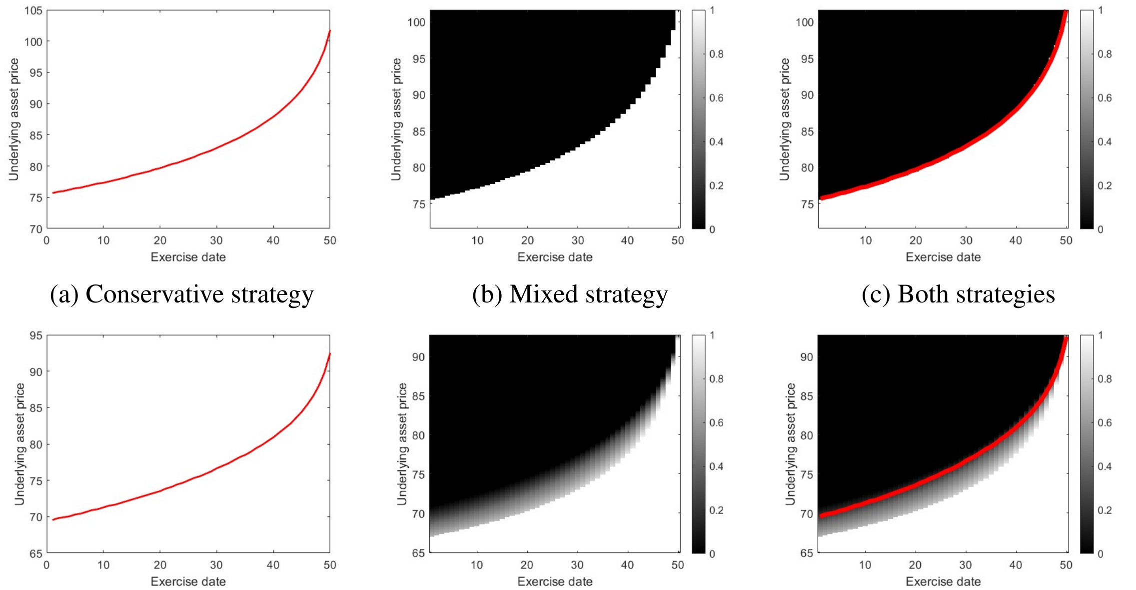
Table 3.1: Benchmark values for the numerical experiments.

3.2 Comparison of conservative and mixed strategies

In this section, we assume that C_1 is immune to default ($\lambda_1 = 0$), while C_2 is vulnerable to default with a constant hazard rate of $\lambda_2 = 0.5$ and a recovery rate of $\rho_2 = 0$. Our analysis centers around a portfolio comprising four Bermudan put options, each with distinct strike prices: $K_{11} = 110$, $K_{12} = 100$, $K_{21} = 105$, and $K_{22} = 100$. These options are written on the same underlying asset, with parameter values specified in Table 3.1. Figure 3.1 represents the exercise strategies of the two options held by C_1 over time, assuming that no option has been exercised yet.

The red curves in Figure 3.1 depict the security strategy of the C_1, that is $a_m^{S1}(.)$, representing the exercise barrier as a function of the date and the price of the underlying asset. These boundaries define the underlying asset price threshold, dependent on the exercise date, beneath which the immediate exercise of the put options held by C_1 becomes the optimal decision under the most adverse outcome.

Panels b and e of Figure 3.1 illustrate the exercise behavior using the mixed strategy for the options with strike prices K_{11} and K_{12}. It delineates the probability of exercising that option given a specific underlying asset price and exercise date, especially in the regions where the upper and lower values of the portfolio differ (in grey).



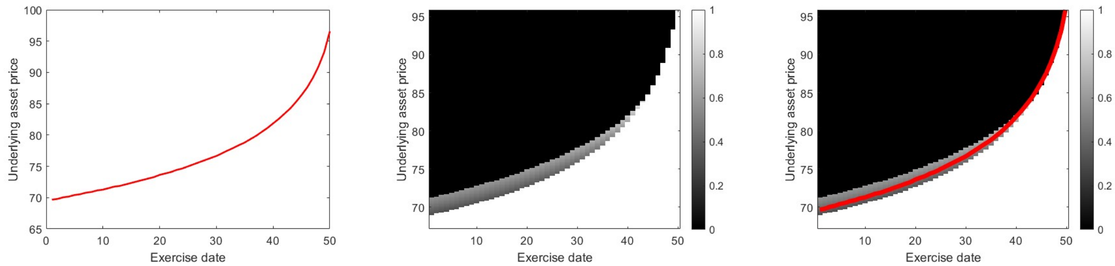
(a) Conservative strategy (b) Mixed strategy (c) Both strategies

(d) Conservative strategy (e) Mixed strategy (f) Both strategies

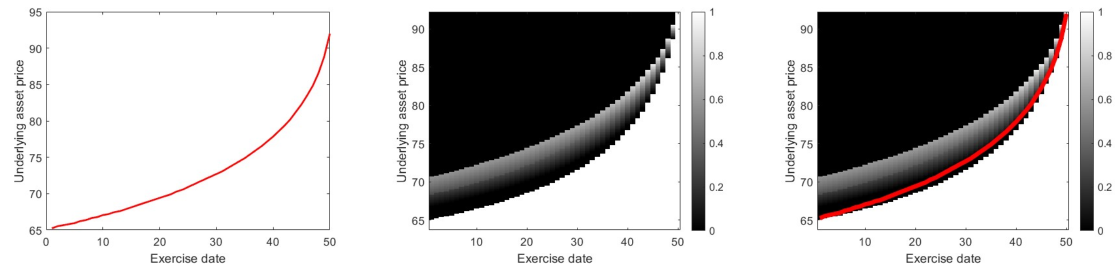
Figure 3.1: Equilibrium exercise strategies (security and mixed) under netting for the options with strike prices of K_{11} (panels a-c) and K_{12} (panels d-f), given $\lambda_2 = 0.5$, $\lambda_1 = 0$ and $\rho_2 = \rho_1 = 0$. The netted portfolio comprises four Bermudan put options with strike prices $K_{11} = 110$, $K_{12} = 100$, $K_{21} = 105$, and $K_{22} = 100$, written on the same underlying asset, with parameter values detailed in Table 3.1

From Panel c of Figure 3.1, it is clear that both the mixed and conservative strategies lead to the same decision-making for the option with a strike price of K_{11}. Yet, Panel f of Figure 3.1 shows that for the option with a strike price of K_{12}, the mixed strategy results in a different exercise approach. This shows that there can exist significant regions for the underlying asset price, along the life of the portfolio, where there exists no equilibrium in pure strategies.

It is noteworthy that the existence of a netting agreement can alter the exercise strategy of options held by a party, even when that party is not exposed to counterparty default risk. This is exemplified in Figure 3.2, where we illustrate the optimal exercise decisions for options held by C_2 using the conservative strategy for C_2, that is $a_m^{S2}(.)$, and mixed



(a) Conservative strategy (b) Mixed strategy (c) Both strategies



(d) Conservative strategy (e) Mixed strategy (f) Both strategies

Figure 3.2: Equilibrium exercise strategies (security and mixed) under netting for the options with strike prices of K_{21} (panels a-c) and K_{22} (panels d-f), given $\lambda_2 = 0.5$, $\lambda_1 = 0$ and $\rho_2 = \rho_1 = 0$. The netted portfolio comprises four Bermudan put options with strike prices $K_{11} = 110$, $K_{12} = 100$, $K_{21} = 105$, and $K_{22} = 100$, written on the same underlying asset, with parameter values detailed in Table 3.1

strategy, assuming that no option has been exercised yet. It is evident that the exercise strategy, derived from the mixed strategy for the option held by C_2, is influenced by the netting agreement. This can be attributed to the fact that C_2 no longer makes decisions for the options independently. Rather, the decision-making process is influenced by the other party's decisions and consequently its own default probability. We discuss this effect on decision-making using the conservative strategy in the following sections.

In the subsequent sections, we present our analysis on the exercise boundary based on the conservative strategy, primarily due to its straightforward representation. However, these results can be extrapolated when the mixed strategy is employed.

3.3 Impact of parameter values on exercise strategies

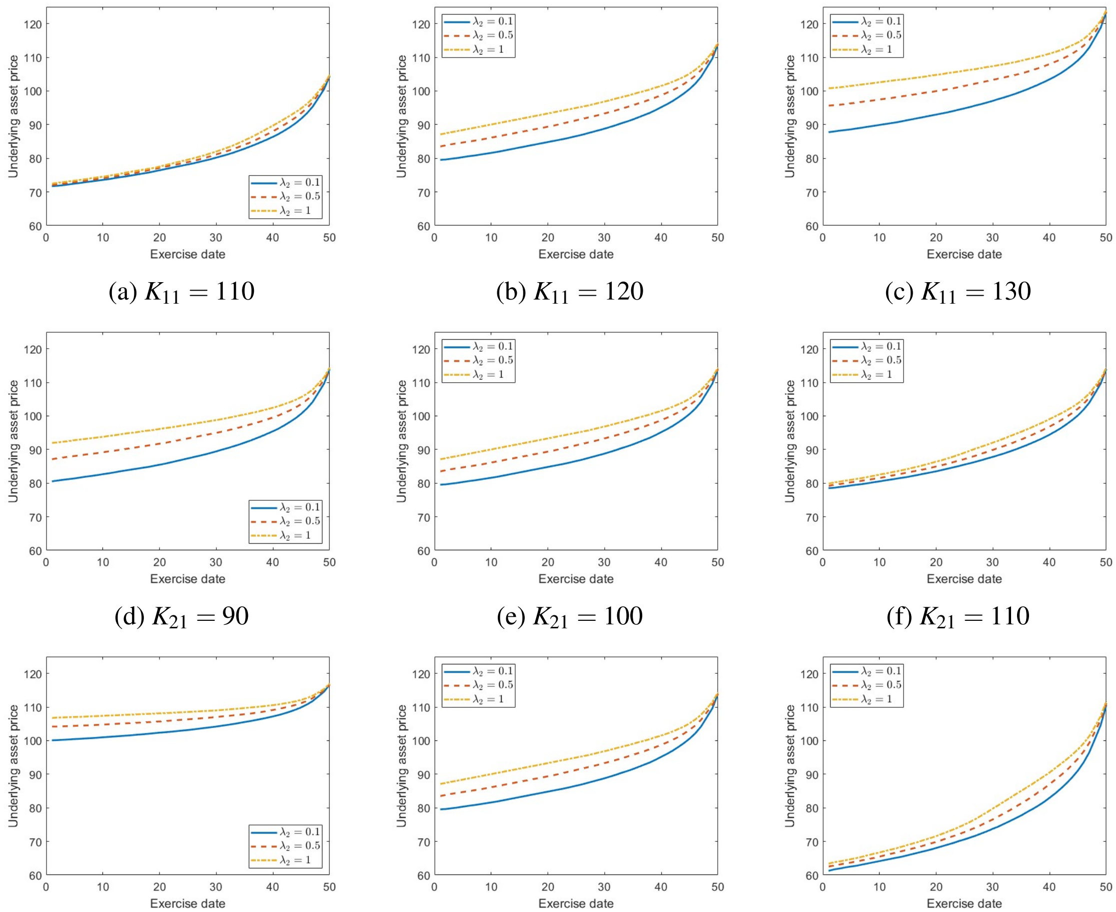
The next set of experiments illustrates the impact of CCR on the equilibrium strategies in a netted portfolio in both unilateral and bilateral cases. We consider a portfolio consisting of three Bermudan options on the same underlying asset, with strike prices $K_{11} = 120$, $K_{21} = 100$, and $K_{22} = 90$ as outlined in Table 3.1. It is worth noting that the findings from this analysis can be extrapolated to other portfolios.

3.3.1 Unilateral risk

Figure 3.3 provides a detailed visual presentation of the effects of changing λ_2, when $\lambda_1 = 0$, and $\rho_2 = 0$, on the exercise strategy of the option with a strike of K_{11}. This figure is divided into three sections, each focusing on the change of λ_2 for different parameter values while keeping the others at their benchmark levels. Panels a-c and Panels d-f of the figure illustrates this effect for different values of K_{11}, and K_{21}, respectively, while panels g-i show this effect for various levels of σ. In a similar way, we present a corresponding illustration for the effect of varying ρ_2, when $\lambda_1 = 0$, and $\lambda_2 = 0.3$ on the exercise strategy of the option with a strike of K_{11} in Figure 3.4.

Upon examination of the figures, it is clear that an increase in the risk of loss stemming from counterparty default, either due to a higher default risk or a lower counterparty recovery rate, results in exercising the put option at higher prices of the underlying asset (earlier). This occurrence can be primarily attributed to the fact that an increase in the counterparty default risk or a decrease in the recovery rate leads to a reduction in the holding value of the portfolio $W_m(x,b)$, as defined in the preceding chapter. Consequently, the option holder C_1 is motivated to exercise the option when the exercise payoff is lower, corresponding to higher prices of the underlying asset.

Our exploration suggests that the impact of increasing default risk on the exercise



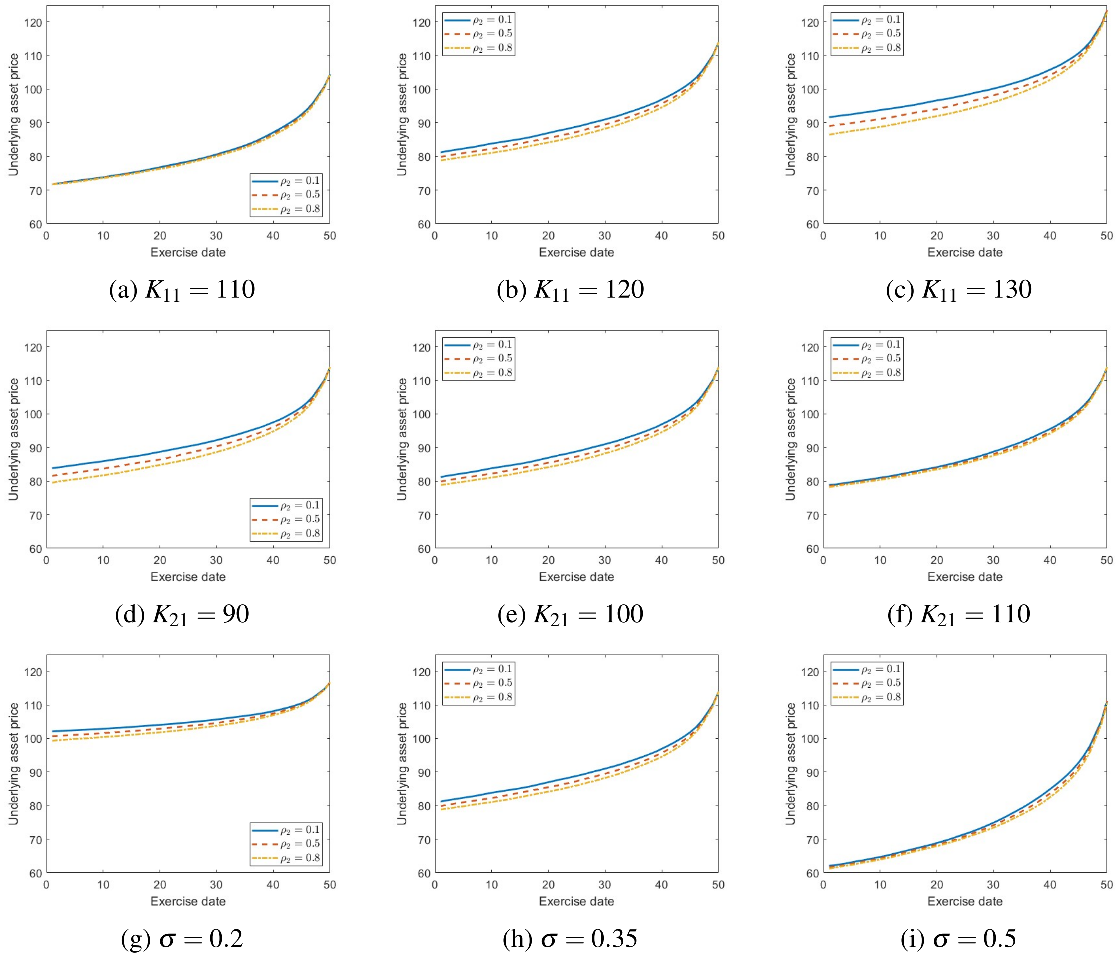
(a) $K_{11} = 110$ (b) $K_{11} = 120$ (c) $K_{11} = 130$

(d) $K_{21} = 90$ (e) $K_{21} = 100$ (f) $K_{21} = 110$

(g) $\sigma = 0.2$ (h) $\sigma = 0.35$ (i) $\sigma = 0.5$

Figure 3.3: Effect of λ_2 variations on the exercise boundary of the option with a strike price of K_{11} in the netted portfolio containing two Bermudan options on the same underlying asset when $\lambda_1 = 0$, and $\rho_2 = 0$. Benchmark parameter values are $K_{11} = 120$, $K_{21} = 100$, and $K_{22} = 90$, and others as reported in 3.1. The figure illustrates the effect of varying K_{11} (panels a-c), K_{21} (panels d-f), and σ (panels g-i) while keeping the other values constant at their benchmark levels.

barrier is heightened with a rise in K_{11}. We also observe that this impact increases with a decrease in K_{21}. The reason is that both these adjustments elevate the value of the portfolio for C_1, increasing C_1's exposure to default risk and prompting C_1 to exercise its option earlier. However, the influence of σ on the increasing default risk's effect on the exercise



(a) $K_{11} = 110$ (b) $K_{11} = 120$ (c) $K_{11} = 130$

(d) $K_{21} = 90$ (e) $K_{21} = 100$ (f) $K_{21} = 110$

(g) $\sigma = 0.2$ (h) $\sigma = 0.35$ (i) $\sigma = 0.5$

Figure 3.4: Effect of ρ_2 variations on the exercise boundary of the option with a strike price of K_{11} in the netted portfolio containing two Bermudan options on the same underlying asset when $\lambda_1 = 0$, and $\lambda_2 = 0.3$. Benchmark parameter values of $K_{11} = 120$, $K_{21} = 100$, and $K_{22} = 90$, and others as reported in 3.1. The figure illustrates the effect of varying K_{11} (panels a-c), K_{21} (panels d-f), and σ (panels g-i) while keeping the other values constant at their benchmark levels.

barrier is not straightforward. An increase in σ affects all options in the portfolio. As the volatility of the underlying asset changes, it impacts both the overall exposure and the value of individual options, potentially in contrasting manners. Thus, predicting the effect of a σ change on the impact of the counterparty's default risk on each option's exercise boundary is not possible.

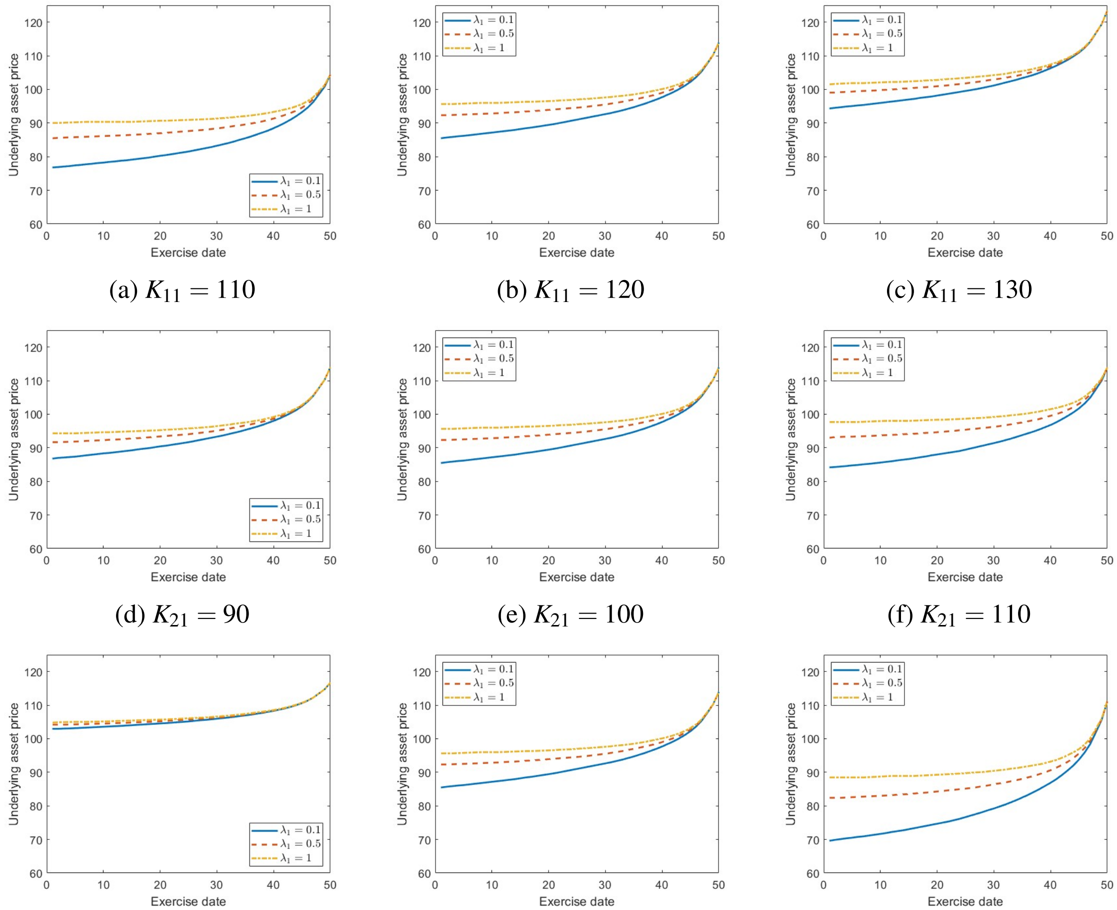
It is important to note that this example illustrates not only how the parameters of an individual option can influence its exercise decision but also how parameters associated with other options in the portfolio can significantly impact the exercise decision of an option within the netted portfolio.

3.3.2 Bilateral case

In the bilateral case, we illustrate the effect of changing λ_1 with conditions $\lambda_2 = 0.3$, and $\rho_1 = \rho_2 = 0$ on the exercise strategy of the option with a strike of K_{11} in Figure 5. Similarly, we show the impact of changing ρ_1 when $\lambda_2 = 0.3$, $\lambda_1 = 0.2$, and $\rho_2 = 0$ on the exercise strategy of the option with a strike of K_{11} in Figure 6.

The interesting result from both figures is the observation that an increase in the risk of loss from the option holder (C_1), either due to a higher default risk or a lower recovery rate, results in earlier option exercise. In essence, combining the results from the unilateral case, it is clear that no matter whose risk increases, a rise in the default risk of both parties can lead to earlier exercise of the options in the netted portfolio. The reason is as follows: an increase in the risk of loss from C_1's default leads to an earlier exercise of the option held by C_2, as explained in the previous section. Consequently, the benefit of C_1 from netting mitigation could disappear or be significantly reduced for a larger portfolio. To avoid this, C_1 is also encouraged to exercise the option in the netted portfolio sooner.

Expanding on our findings, as K_{21} rises or K_{11} falls, the influence of an increase in C_1's default risk on its option's exercise boundary becomes more pronounced. These changes increase the portfolio's value for C_2, heightening C_2's default risk exposure and leading C_2 to exercise its options earlier. Consequently, C_1 is also inclined to exercise its options sooner.

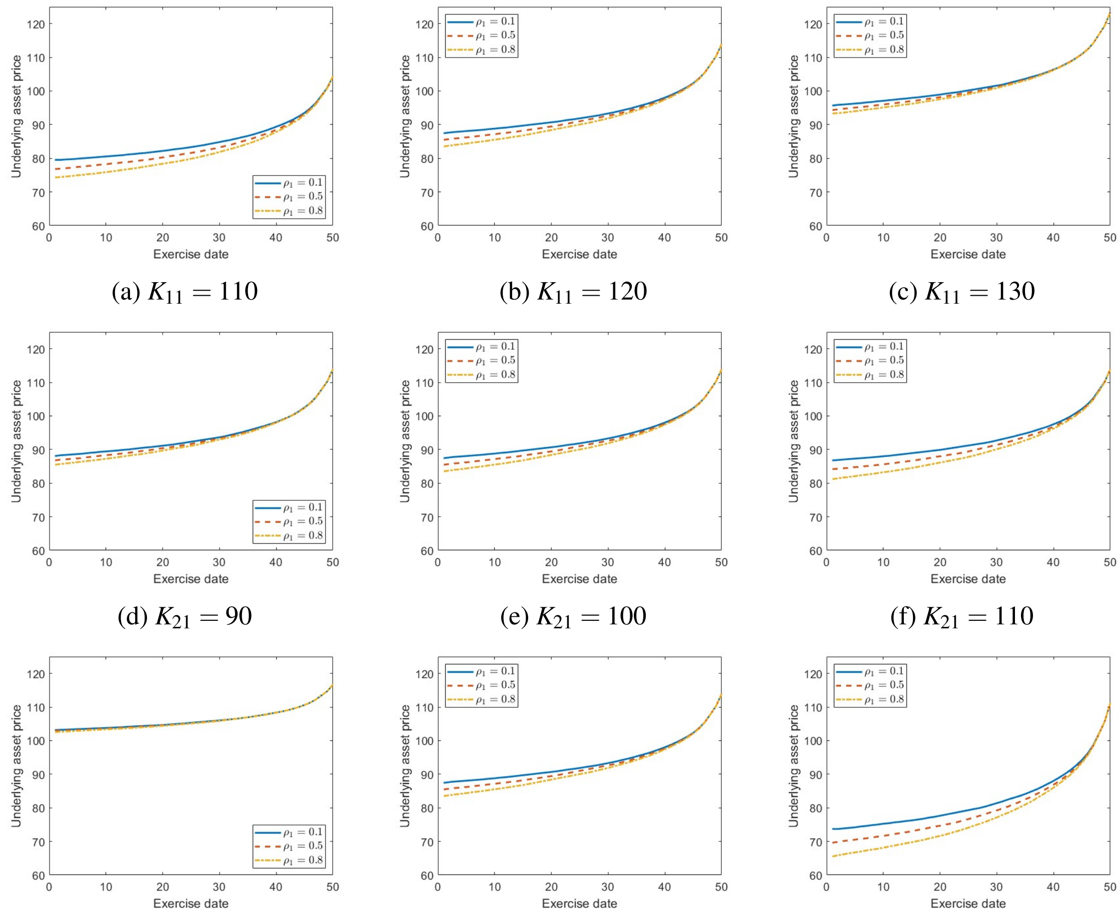


(a) $K_{11} = 110$ (b) $K_{11} = 120$ (c) $K_{11} = 130$

(d) $K_{21} = 90$ (e) $K_{21} = 100$ (f) $K_{21} = 110$

(g) $\sigma = 0.2$ (h) $\sigma = 0.35$ (i) $\sigma = 0.5$

Figure 3.5: Effect of λ_1 variations on the exercise boundary of the option with a strike price of K_{11} in the netted portfolio containing two Bermudan options on the same underlying asset when $\lambda_2 = 0.3$, and $\rho_1 = \rho_2 = 0$. Benchmark parameter values of $K_{11} = 120$, $K_{21} = 100$, and $K_{22} = 90$, and others as reported in 3.1. The figure illustrates the effect of varying K_{11} (panels a-c), K_{21} (panels d-f), and σ (panels g-i) while keeping the other values constant at their benchmark levels.



(a) $K_{11} = 110$ (b) $K_{11} = 120$ (c) $K_{11} = 130$

(d) $K_{21} = 90$ (e) $K_{21} = 100$ (f) $K_{21} = 110$

(g) $\sigma = 0.2$ (h) $\sigma = 0.35$ (i) $\sigma = 0.5$

Figure 3.6: Effect of ρ_1 variations on the exercise boundary of the option with a strike price of K_{11} in the netted portfolio containing two Bermudan options on the same underlying asset when $\lambda_2 = 0.3$, $\lambda_1 = 0.2$ and $\rho_2 = 0$. Benchmark parameter values of $K_{11} = 120$, $K_{21} = 100$, and $K_{22} = 90$, and others as reported in 3.1. The figure illustrates the effect of varying K_{11} (panels a-c), K_{21} (panels d-f), and σ (panels g-i) while keeping the other values constant at their benchmark levels.

3.4 A comparative analysis: Netting impact on exercise boundaries

In this section, the focus is on understanding the exercise boundaries in contexts of both unilateral and bilateral default risk, with and without a netting agreement. We set exercise boundaries under risk-free conditions as a baseline, and then conduct a comparative analysis to highlight the impacts of default risk and netting on the exercise boundaries of options. The studied portfolio includes four Bermudan put options with strike prices of $K_{11} = 110$, $K_{12} = 100$, $K_{21} = 105$, and $K_{22} = 103$, written on the underlying asset as delineated in Table 3.1, and insights drawn from this analysis are applicable to other portfolios as well.

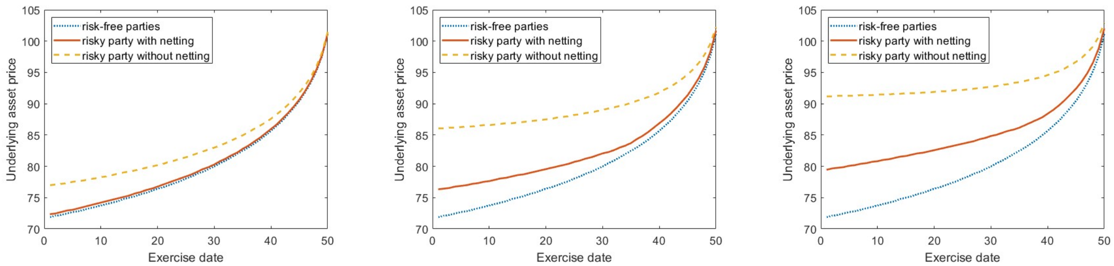
3.4.1 Unilateral case

Figure 3.7 compares the exercise boundaries for C_1 under risk-free conditions and unilateral risk conditions, both with and without a netting agreement. Panels a-c display results for the option with a strike price of K_{11}, and panels d-f present results for the option with a strike price of K_{12} at various λ_2 levels (0.1, 0.5, and 1).

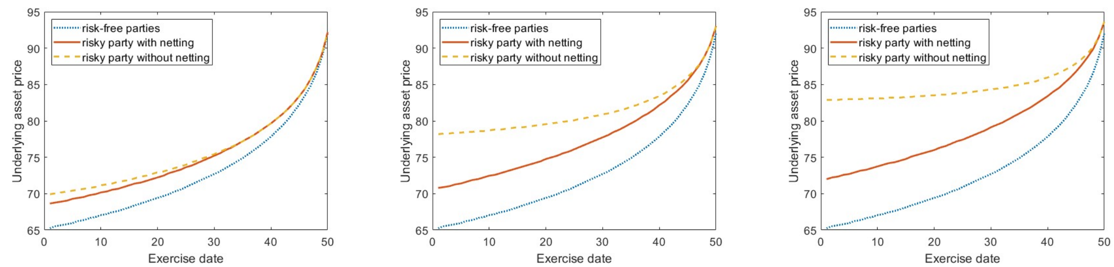
The figure indicates that as counterparty risk increases, C_1 tends to exercise options earlier, which aligns with previous discussions. Notably, without a netting agreement and when counterparty risk is present, C_1 exercises the option sooner than it would with a netting agreement. Essentially, the exercise boundary in a netted portfolio resides between—or matches—the risk-free and risky exercise boundaries in isolation. This is a general rule, linked to the risk mitigation offered by the netting agreement. When counterparty risk increases, the exercise boundary shifts upward, but a netting agreement can moderate this shift.

Figure 3.8 extends the comparative analysis to options with strike prices of K_{21} and K_{22}, held by C_2. Initially, and as anticipated, when C_2 is not exposed to counterparty risk, the exercise boundary of the options, in the absence of a netting agreement, coincides exactly with the risk-free boundary.

The inclusion of a netting agreement changes this, potentially triggering a rise in the



(a) $\lambda_2 = 0.1$ (b) $\lambda_2 = 0.5$ (c) $\lambda_2 = 1$

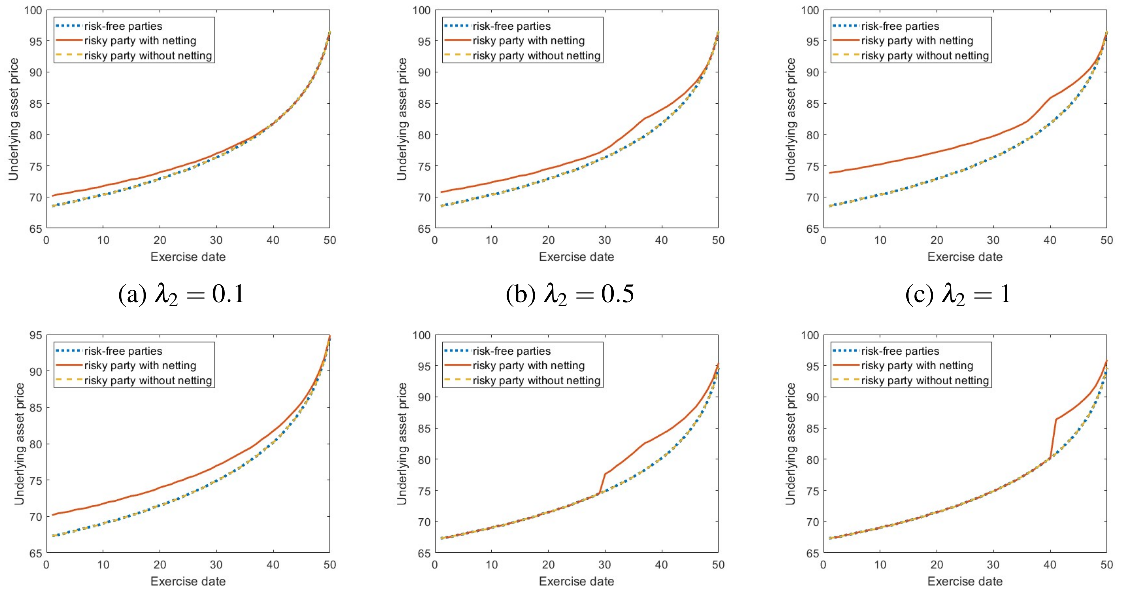


(d) $\lambda_2 = 0.1$ (e) $\lambda_2 = 0.5$ (f) $\lambda_2 = 1$

Figure 3.7: Comparison of the exercise boundaries for two options held by C_1, under risk-free conditions, and in the presence of unilateral default risk with and without netting. The portfolio consists of four Bermudan puts with strike prices $K_{11} = 110$, $K_{12} = 100$, $K_{21} = 105$, and $K_{22} = 103$, on the underlying asset specified in Table 3.1 when $\lambda_1 = 0$. panels a-c show the results for the option with the strike price of K_{11}, while panels d-f show the results for the option with the strike price of K_{12}.

exercise boundary. This can be linked to the mutual decision-making necessitated by a netting agreement. Specifically, exercise decisions are not made independently but are impacted by the other party's actions. As C_1 tends to exercise earlier, C_2 may also be motivated to exercise earlier than in a risk-free situation, to counter C_1's strategies, who aims to utilize netting benefits.

It is important to highlight that, as shown in Figure 3.8, the exercise boundary with netting, derived using the conservative strategy, might display non-smooth behavior. This stems from the intricate nature of the represented game between the parties and the lack of a Nash equilibrium in pure strategies in certain states, causing possible abrupt decision



(a) $\lambda_2 = 0.1$ (b) $\lambda_2 = 0.5$ (c) $\lambda_2 = 1$

(d) $\lambda_2 = 0.1$ (e) $\lambda_2 = 0.5$ (f) $\lambda_2 = 1$

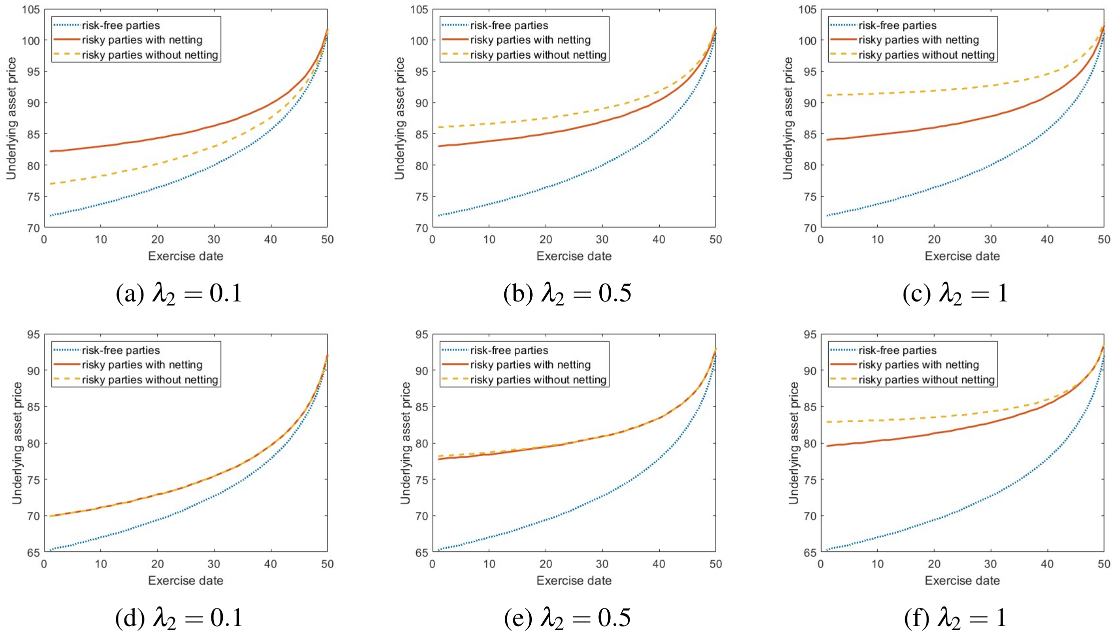
Figure 3.8: Comparison of the exercise boundaries for two options held by C_2, under risk-free conditions, and in the presence of unilateral default risk with and without netting. The portfolio consists of four Bermudan puts with strike prices $K_{11} = 110$, $K_{12} = 100$, $K_{21} = 105$, and $K_{22} = 103$, on the underlying asset specified in Table 3.1 when $\lambda_1 = 0$. panels a-c show the results for the option with the strike price of K_{21}, while panels d-f show the results for the option with the strike price of K_{22}.

changes by the parties and non-smoothness in the exercise boundary.

3.4.2 Bilateral case

In the following analysis, where bilateral default risk is considered and $\lambda_1 = 0.3$, Figures 3.9 and 3.10 present a comparative analysis of exercise boundaries for options held by C_1 and C_2 respectively, exploring both risk-free and risky scenarios, with and without netting, across varying levels of λ_2 (0.1, 0.5, and 1).

Combining results from the unilateral case and various numerical tests, we observe: As a party's default risk increases, the other party's option exercise boundary rises, regardless



(a) $\lambda_2 = 0.1$ (b) $\lambda_2 = 0.5$ (c) $\lambda_2 = 1$

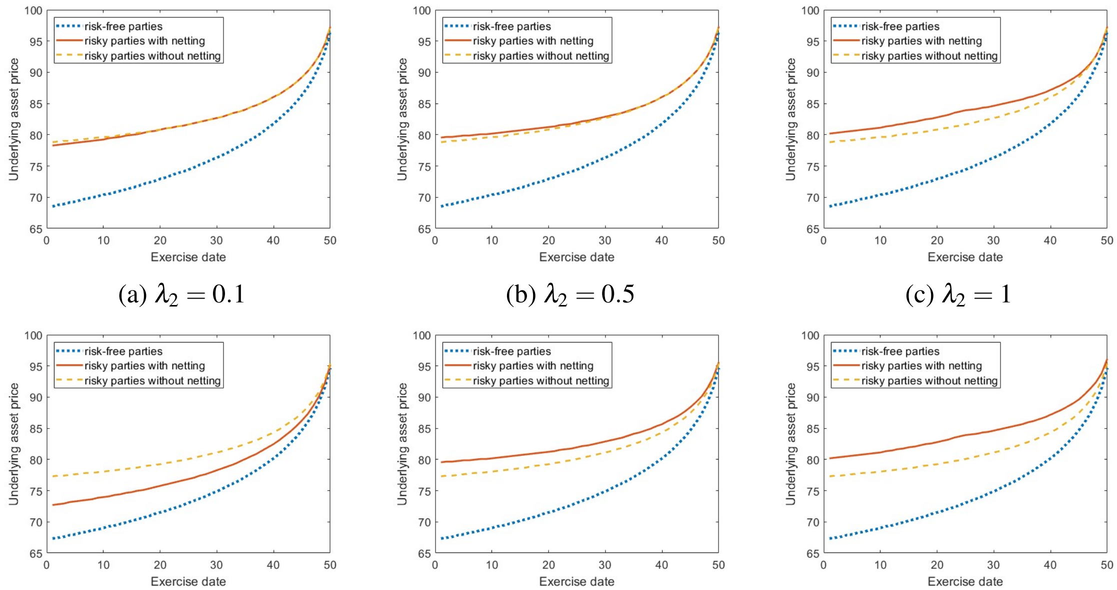
(d) $\lambda_2 = 0.1$ (e) $\lambda_2 = 0.5$ (f) $\lambda_2 = 1$

Figure 3.9: Comparison of the exercise boundaries for two options held by C_1, under risk-free conditions, and in the presence of Bilateral default risk ($\lambda_1 = 0.3$) with and without netting. The portfolio consists of four Bermudan puts with strike prices $K_{11} = 110$, $K_{12} = 100$, $K_{21} = 105$, and $K_{22} = 103$, on the underlying asset specified in Table 3.1. panels a-c show the results for the option with the strike price of K_{11}, while panels d-f show the results for the option with the strike price of K_{12}.

of netting. Within a netted portfolio, the exercise boundary sits between the risk-free boundary and the risk-adjusted one when the option holder's default risk is lower than the counterparty's. This is attributable to the risk-mitigating role of netting.

For example, in Panels b-c and e-f of Figure 3.9, with $\lambda_1 < \lambda_2$, the exercise boundary for C_1's options falls between risk-free and risky scenarios without netting. Similarly, in Panels a and d of Figure 3.10 for C_2, where $\lambda_2 < \lambda_1$, a similar pattern emerges.

When the option holder's default risk exceeds the counterparty's, the exercise boundary for the riskier party's options aligns with or surpasses the risk-adjusted boundary for options without netting. This is visible for C_1 in Panels a and d of Figure 3.9, where $\lambda_2 < \lambda_1$,



(a) $\lambda_2 = 0.1$ (b) $\lambda_2 = 0.5$ (c) $\lambda_2 = 1$

(d) $\lambda_2 = 0.1$ (e) $\lambda_2 = 0.5$ (f) $\lambda_2 = 1$

Figure 3.10: Comparison of the exercise boundaries for two options held by C_2, under risk-free conditions, and in the presence of Bilateral default risk ($\lambda_1 = 0.3$) with and without netting. The portfolio consists of four Bermudan puts with strike prices $K_{11} = 110$, $K_{12} = 100$, $K_{21} = 105$, and $K_{22} = 103$, on the underlying asset specified in Table 3.1. panels a-c show the results for the option with the strike price of K_{21}, while panels d-f show the results for the option with the strike price of K_{22}.

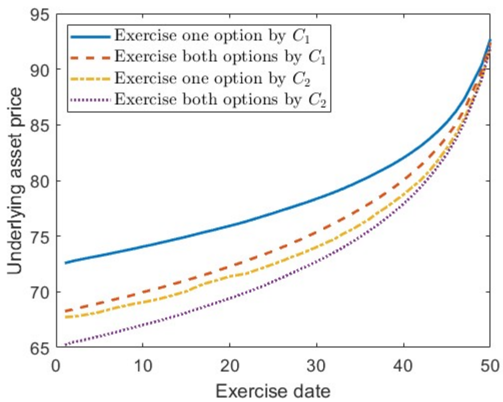
and for C_2 in Panels b-c and e-f of Figure 3.10, where $\lambda_1 < \lambda_2$. The exercise boundary of an option without netting is solely determined by counterparty risk and remains unaffected by the option holder's default risk. Yet, under a netting agreement, it can be influenced by the holder's default risk. The less risky party's earlier exercise decision in a netting scenario can potentially reduce the riskier party's netting mitigation benefits, prompting the latter to exercise options within the netted portfolio earlier than when without netting.

To view it differently: when Condition (2.33) is met, implying that both counterparties exhibit an identical default risk, the netting has no impact and the equilibrium exercise barrier coincides with the risk-adjusted optimal one. Examination of Equation (2.32) shows that netting benefits the party less likely to default first (potentially adjusted for recovery). As a result, for vulnerable (put) options under netting, the exercise barrier is lower (or higher) than its risk-adjusted counterpart for the counterparty with a correspondingly lower (or higher) default probability.

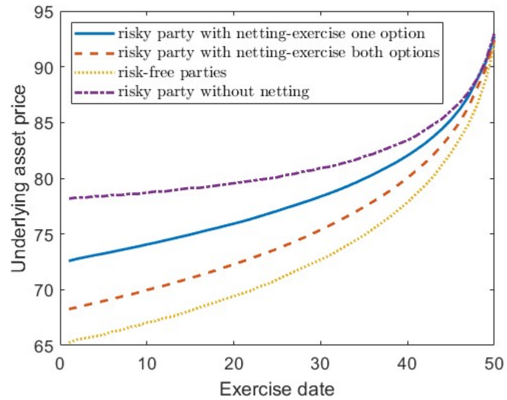
Under a netting agreement, it may be optimal not to exercise identical options simultaneously, since the exercise decision depends on the portfolio composition. Figure 3.11 demonstrates this in a unilateral case ($\lambda_2 = 0.5$) with a portfolio of four Bermudan put options, all with the same strike price ($K_{11} = K_{12} = K_{21} = K_{22} = 100$), and written on the same underlying asset as specified in Table 3.1. Panel (a) contrasts the exercise boundaries of options with identical strike prices within the netted portfolio. Panels b and c offer a comparative view of exercise boundaries for options held by C_1 and C_2 respectively.

Under netting, a region exists where both options are exercised and another where only one option is exercised, applicable to parties both exposed and not exposed to counterparty credit risk (CCR). For C_1, exposed to CCR, both exercise barriers are higher (indicating earlier exercise) than in the risk-free case, yet lower than without a netting agreement. Conversely, C_2, not exposed to CCR, will exercise one of its options at a higher price (earlier) than the risk-free barrier when in a netting agreement, while the exercise barrier for the second option aligns with the risk-free barrier.

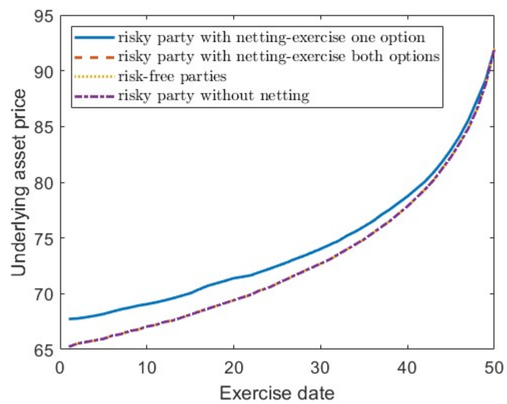
Our study reveals that options will be exercised according to their intrinsic value (or equivalently, their expected loss upon default). This occurs because, in a bilateral netting agreement, all vulnerable contracts held by one party share the same risk level. However, multiple solutions may arise when deciding about an exercise strategy, for instance when a portfolio holds many identical options. This does not mean that identical options should be exercised together, but rather that any subset of those identical options can be exercised in the corresponding region.



(a) Comparison of exercise boundaries of options for the case of the risky party with netting



(b) Comparison of exercise policies for options held by C_1



(c) Comparison of exercise policies for options held by C_2

Figure 3.11: Illustration of the effect of netting and counterparty risk on the exercise boundaries of identical options with the same strike price of 100 on the same underlying asset as characterized in Table 3.1 when $\lambda_1 = 0$, $\lambda_2 = 0.5$, and $\rho_2 = 0$. Panel a compares the exercise strategies of options for the case of the risky party under the netting agreement. Panels b and c show the comparison of the exercise boundaries for options held by C_1 and C_2, respectively, under risk-free conditions, and in the presence of default risk with and without netting

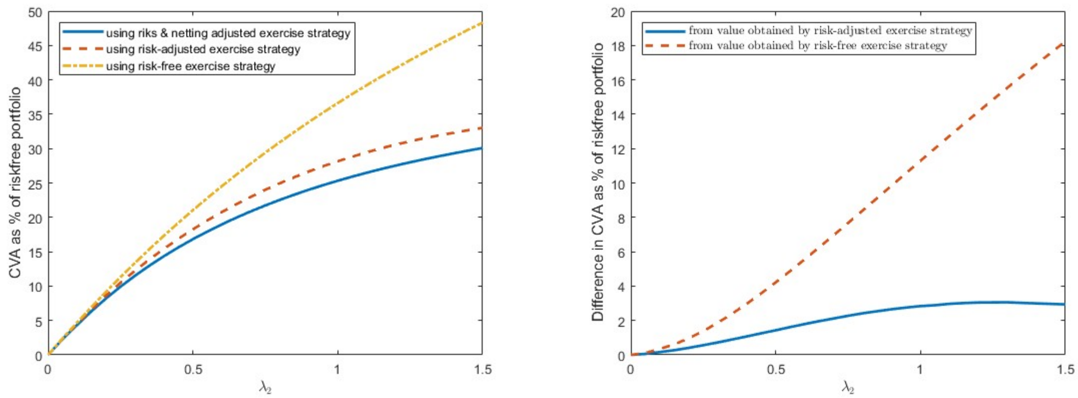
3.5 Methodological choices

The conventional methodology used for assessing the risk adjustment value of derivatives with early-exercise features typically employs two successive steps. The exercise strategy is initially obtained for all derivatives, disregarding counterparty default risk. Subsequently, a Monte Carlo simulation of the default process and of the market risk factors is conducted in order to estimate the expected loss corresponding to the risk-free exercise strategy. This approach, however, works under the assumption that counterparty risk does not influence the exercise mechanism.

As shown in Breton and Marzouk (2018), this two-step technique, while widely used in the industry, can lead to significant inaccuracies in the estimation of the CVA of individual derivative products. The numerical experiments presented in previous sections show that netting can further modify the exercise strategy of options, leading to unpredictable effects on the value of the CVA or BVA.

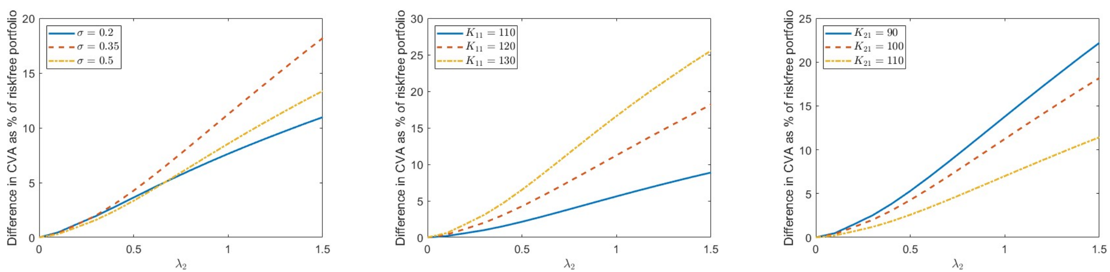
At the portfolio level, our method calculates the portfolio's risk adjustment and the exercise policy of the claims concurrently. This method adjusts the obtained exercise strategy for both default risk and netting. Our next set of experiments illustrates the misestimation that can arise when disregarding the impact of CCR and netting on the exercise strategies of derivatives. We use the simplest example of a portfolio of two Bermudan options with $K_{11} = 120$ and $K_{21} = 100$, written on the same underlying asset, described in Table 3.1, under the assumption that the parties can use mixed strategies

Figure 3.12 compares the CVA at the inception obtained using the risk-free, risk-adjusted, and netting-adjusted exercise strategies. This is presented as a function of λ_2, under the condition that $\rho_2 = 0$ for the unilateral case. Panel a compares the CVA as a percentage of the risk-free portfolio value, while Panel b reveals the potential CVA overestimation, expressed as a percentage of risk-free portfolio value if an exercise strategy,



(a) CVA value (b) Difference

Figure 3.12: Impact of using risk-free and risk-adjusted (without netting) exercise strategy on the CVA at inception as a function of λ_2: illustrated for a portfolio of two Bermudan put options with $K_{11} = 120$ and $K_{21} = 100$ given $\rho_2 = 0$. Other parameter values are reported in Table 3.1



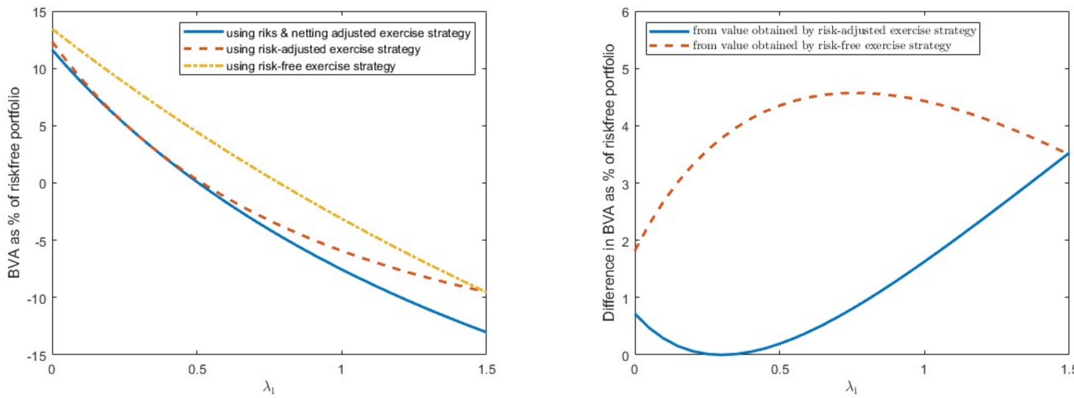
(a) Changes in regard to σ (b) Changes in regard to K_{11} (c) Changes in regard to K_{21}

Figure 3.13: Impact of using risk-free exercise strategy instead of risk and netting adjusted one on the CVA at inception as a function of λ_2: Illustrated for variations in σ, X_0, and K_{21} in a portfolio of two Bermudan options with benchmark parameter values of $K_{11} = 120$, $K_{21} = 100$, under the condition $\rho_2 = 0$. Additional benchmark parameter values are detailed in Table 3.1.

adjusted for both netting and counterparty risk, is not utilized.

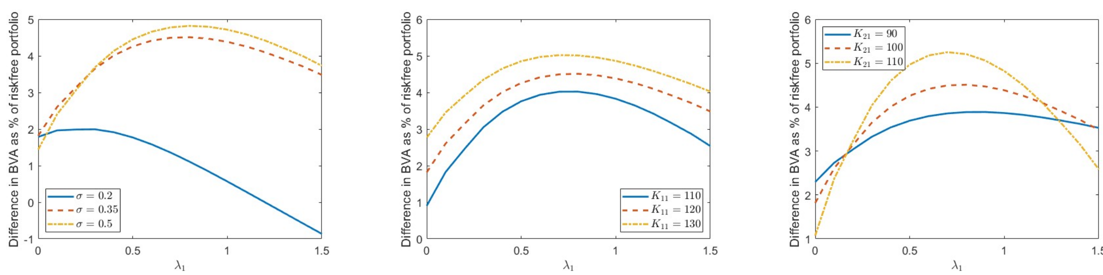
Figure 3.13 further investigates the implications of using a risk-free exercise strategy instead of the one that adjusted for both risk and netting on the CVA at the inception date. This is examined as a function of λ_2 across varying levels of σ, K_{11}, and K_{21}.

Similarly, for the bilateral case, Figure 3.14 and 3.15 show the impact of using risk-free



(a) CVA value (b) Difference

Figure 3.14: Impact of using risk-free and risk-adjusted (without netting) exercise strategy on the BVA at inception as a function of λ_1: illustrated for a portfolio of two Bermudan put options with $K_{11} = 120$ and $K_{21} = 100$ given $\lambda_2 = 0.3$ and $\rho_1 = \rho_2 = 0$. Other parameter values are reported in Table 3.1



(a) Changes in regard to σ (b) Changes in regard to K_{11} (c) Changes in regard to K_{21}

Figure 3.15: Impact of using risk-free exercise strategy instead of risk and netting adjusted one on the BVA at inception as a function of λ_1: Illustrated for variations in σ, X_0, and K_{21} in a portfolio of two Bermudan options with benchmark parameter values of $K_{11} = 120$, $K_{21} = 100$, under the conditions $\lambda_2 = 0.3$ and $\rho_1 = \rho_2 = 0$. additional benchmark parameter values are detailed in Table 3.1.

and risk-adjusted exercise strategies, with and without netting, on the BVA at inception, presented as a function of λ_1, given $\lambda_2 = 0.3$ and $\rho_1 = \rho_2 = 0$.

Note that calculating the risk adjustment value of the portfolio using either the risk-free or risk-adjusted strategy singularly utilizes the model introduced in the preceding chapter, with a key difference: decision-making is not necessitated in each state of the model, and known exercise strategies are used instead.

In the unilateral-risk case, using either the risk-free or the risk-adjusted strategy without accounting for the netting impact always leads to an overestimation of the CVA. This is because both strategies are not optimal compared to the optimal one derived from the dynamic game model, leading to an underestimated portfolio value and consequently, an overestimated risk adjustment. For the portfolio we studied, this overestimation can even reach about 20% of its risk-free value. However, in the bilateral case, the error in the estimation of the BVA can be positive or negative, depending on the relative parties' exposure to CCR, as can be observed from Panel a of Figure 3.15.

3.6 Conclusion

The introduction of a netting agreement fundamentally changes the decision-making process related to the exercise strategies of options in a portfolio. This is because the decision to exercise an option becomes closely tied to the counterparty's actions, rather than being an isolated action. In this chapter, we explore the impact of a netting agreement on the exercise strategy of defaultable parties for Bermudan options.

Our research indicates that a netting agreement can modify the exercise strategy, even for a party not exposed to counterparty default risk. This is because the decision-making process is influenced by the other party's decisions and, consequently, its own default probability. We demonstrate that changes in the default probability and recovery rate can significantly affect the exercise strategy of options within a netted portfolio. In both unilateral and bilateral cases, an increased counterparty default risk or a decreased recovery rate motivates the earlier exercise of the options in the netted portfolio. This effect can be intensified with changes in the portfolio value. Moreover, our study highlights that the parameters of individual options and those associated with other options in the portfolio can significantly influence the exercise decision of an option within the netted portfolio.

Our study also reveals that the exercise boundary within the netted portfolio lies between the risk-free and risk-adjusted boundaries in the absence of netting, assuming the option holder's default risk is less than that of the counterparty. This is due to the risk-mitigating role of netting. Additionally, our research emphasizes the dynamic nature of a netted portfolio, where the exercise of each option alters the portfolio's condition, thereby affecting the exercise boundary of the remaining options.

Lastly, our findings challenge traditional methodologies used to assess the risk adjustment value of netted portfolios. These methods often neglect the impact of counterparty risk and netting on the exercise mechanism. We demonstrate that even when netting is incorporated into the calculation, the industry's prevalent two-step methods, which employ either risk-free or risk-adjusted exercise strategies without considering netting, tend to inaccurately estimate the CVA or BVA of the netted portfolio. This misestimation, which can reach up to 20% of the risk-free portfolio value in a simple scenario, is due to the sub-optimality of these strategies compared to the strategy derived from the dynamic game model.

Chapter 4

CVA Variability in Netted Portfolios

4.1 Introduction

The 2008 financial crisis highlighted the importance of managing CVA variability to maintain banking sector stability. Reflecting on the crisis, Basel III noted that approximately two-thirds of the losses related to counterparty credit risk came from changes in CVA, exceeding losses from actual defaults. Consequently, Basel III introduced a requirement for banks to hold regulatory capital against CVA variability. This aims to ensure banks have enough capital reserves to absorb potential future CVA changes, strengthening the financial system's resilience.

Effectively determining the appropriate capital charge relies on assessing the future probability distribution of CVA at relevant risk horizons. This enables computing analytical risk metrics like CVA Value-at-Risk (VaR) to quantify inherent volatility. Adequate capital can then be determined based on these volatility-dependent risk measures that quantify exposure to adverse CVA movements. Overall, this CVA capital charge provides an additional layer of protection against the systemic risk associated with unmanaged

CVA variability.

For CVA variability estimation, the financial industry often uses nested Monte Carlo simulations, especially for exotic derivatives. This involves two simulation levels: the outer loop generates market scenarios, while the inner evaluates counterparty credit risk exposure in each scenario over a timeframe. However, this approach relies on conducting a large number of simulations, demanding extensive computational resources and time.

Estimating CVA VaR using nested simulations becomes more challenging for a portfolio in the presence of a netting agreement. It must simulate numerous underlying risk factors impacting the exposure to counterparty default. This complexity is further compounded in portfolios where both parties in a netting set have opportunities for early exercise. Including these strategic decisions that can substantially alter the netted set's exposure, adds another layer of complexity. This amplifies the problem's dimensionality and, consequently, the computational demands, sometimes to the extent that nested simulation may not be a feasible choice for accurately estimating CVA risk.

Building on the CVA evaluation method for netted portfolios introduced in the previous chapter, this chapter presents an alternative approach for calculating the CVA VaR for such portfolios with potential early exercise rights. This method is inspired by the approach of Breton and Marzouk (2019) for individual contracts, adapted here to address the netted portfolios.

In this chapter, through a series of numerical experiments, we investigate the influence of netting agreements on the distribution of CVA and its tail risk. We examine how factors such as early exercise rights and the length of the risk horizon can change the movement of CVA. These explorations provide deeper insights into the complexities of netted portfolio management and counterparty risk assessment.

This chapter continues as follows: Section 4.2 outlines the methodology for calculating the CVA VaR. In Section 4.3, we present various numerical examples for illustration. The chapter concludes with Section 4.4.

4.2 Estimating the Distribution of CVA Changes for Netted Portfolios

To assess the risk of adverse CVA changes over a time horizon, we must analyze the potential CVA distribution at the horizon's end. This involves generating scenarios for risk factors affecting net counterparty credit exposure. By simulating these scenarios, we can then calculate CVA values at the horizon close under each one. With sufficient sample sizes, we can approximate the full CVA distribution and compute associated risk measures like CVA VaR.

The previously introduced dynamic game algorithm provides a function approximating CVA across market conditions and dates for each portfolio subset. Requiring only a single run, it eliminates the need for nested simulations. We utilize this CVA function to evaluate CVA changes across a risk horizon for the netted portfolio.

Specifically, given an initial portfolio b_0 under a netting agreement at the inception, our goal is to determine the CVA VaR on a time horizon H. To achieve this, we use the dynamic game method to derive the CVA as a function of risk factors x and portfolio composition binary vector $b \leq b_0$, for each valuation date in set $\mathcal{T}$.

We start by observing the risk factor values X_0 at inception, and then simulate their evolution over the interval $[0,H]$ to obtain trajectories starting from X_0 and ending with X_H at the horizon date H.

In scenarios where no options within the portfolio can be exercised before date H, the process is straightforward: we simply simulate the risk factors up to date H to observe X_H, and then determine the corresponding CVA value for the initial portfolio composition b_0.

However, if the portfolio contains options with exercise opportunities before date H, the situation becomes more complex. We must account for potential changes in the portfolio composition along each simulated path of risk factors. At each decision point, we compare the simulated path against the exercise strategy for each option in the alive portfolio to determine whether the option is exercised. If an option is exercised, the portfolio composition changes and this new composition informs subsequent exercise strategy of remaining options along the path. This process continues until date H, allowing us to determine which options remain alive and the final composition of the portfolio, denoted b_H, at the end of the risk factor path.

Once we have determined the portfolio composition at date H for each simulated scenario, we can calculate the corresponding CVA using the function $\text{CVA}_H(x,b)$, which is derived from our dynamic programming algorithm. The change in CVA is obtained by subtracting the CVA value at inception, $\text{CVA}_0(X_0,b_0)$, from the CVA values at date H, $\text{CVA}_H(X_H,b_H)$. This yields a distribution of potential CVA changes across all simulated paths.

The algorithm to calculate the CVA VaR at a future date H for netted portfolios:

1. Execute the dynamic programming algorithm to derive function $\text{CVA}_m(x,b)$ for all $x \in \mathbb{R}^n$ and for all subsets of the initial portfolio on each evaluation date t_m in set $\mathcal{T}$.

2. On the initial date, observe the current risk factors X_0. Simulate vector process X_t over the interval $(0,H]$ to generate $[X_0,X_H]$.

3. Determine the final portfolio composition b_H for each simulated path based on process X_t over the interval $(0,H]$.

4. For each simulation resulting in X_H and b_H, apply the CVA function to obtain $\text{CVA}_H(X_H,b_H)$. Determine the CVA changes by $\Delta\text{CVA}_H = \text{CVA}_H(X_H,b_H) - \text{CVA}_0(X_0,b_0)$.

5. Compile all the $\Delta\mathrm{CVA}_H$ values to construct a distribution of potential CVA changes. Estimate the CVA VaR from the quantiles of this distribution, corresponding to the desired confidence level.

It is important to note that based on recent research by Breton and Marzouk (2019), the transition between the real-world probability measure $\mathbb{P}$ and the risk-neutral measure $\mathbb{Q}$ does not exert a substantial impact on the tail risk associated with CVA. Their analysis shows that the CVA VaR estimates exhibit remarkable similarity under either measure. This phenomenon is attributed to the fact that the measure change solely influences the drift terms in the process dynamics. The influence exerted by drift on generating extreme values of risk factors is generally less pronounced in comparison to volatility. Consequently, while alterations in drift may induce minor adjustments to the distribution's shape, they are less likely to significantly affect the likelihood of extreme events that contribute to tail risk. As a result, we perform both the CVA calculation in the first step of the algorithm and the simulation in the second step of the algorithm under the risk-neutral measure $\mathbb{Q}$.

4.3 Numerical experiments

In this section, we aim to explore the influence of netting agreements, the existence of early exercise opportunities, and other parameter changes on the distribution of CVA and CVA VaR. This exploration builds on the notational framework established in earlier chapters.

For each set of experiments, we focus on a portfolio comprising Bermudan put options, each on the same underlying asset but with potentially different strike prices. These options share a common maturity of $T = 1$ and offer Ne equally spaced exercise opportunities. In our benchmark model, we set $Ne = 12$, indicating that the options can be exercised monthly.

We model the underlying asset price following a GBM. Under the risk-neutral measure $\mathbb{Q}$, the asset price dynamics are represented as:

$$dX_t = rX_t dt + \sigma X_t dB_t, \tag{4.1}$$

where B indicates a standard Brownian motion under $\mathbb{Q}$. The benchmark parameters for our numerical experiments are outlined in Table 4.1.

We consider the unilateral case, where the counterparty default follows an intensity model with a hazard rate of λ_2. In all experiments, the recovery rate is assumed to be zero, and the CVA VaR is estimated at a 99% confidence level.

Parameters	Underlying asset			T	Ne
	r	X_0	σ		
Base value	0.05	100	0.2	1	12

Table 4.1: Benchmark values for the numerical experiments.

4.3.1 Netting impact on CVA variability

This section examines the effects of netting agreements and counterparty risk levels on CVA changes through a series of illustrative examples. We conduct a comparative analysis, represented in Figures 4.1 and 4.3, which show CVA distributions over a 10-day risk horizon. The figures illustrate the influence of netting under different scenarios: varying strike prices (K_{11}) and counterparty hazard rates (λ_2) for two four-option Portfolios I and II. The strike prices of the options comprising these portfolios are outlined in Table 4.2.

Figures 4.2 and 4.4 present the effects of netting agreements on CVA VaR for Portfolios I and II. These figures demonstrate potential deviations in CVA VaR calculations for

Holder	C_1		C_2	
Strike price	K_{11}	K_{12}	K_{21}	K_{22}
Portfolio I	95 – 100 – 105	95	110	110
Portfolio II	95 – 100 – 105	95	90	90
Portfolio III	115	110	100	100
Portfolio IV	115	110	90	90

Table 4.2: Strike prices of put options within Portfolio I-IV for the numerical experiments. All options in portfolios are written on the same underlying asset, with parameter values detailed in Table 4.1.

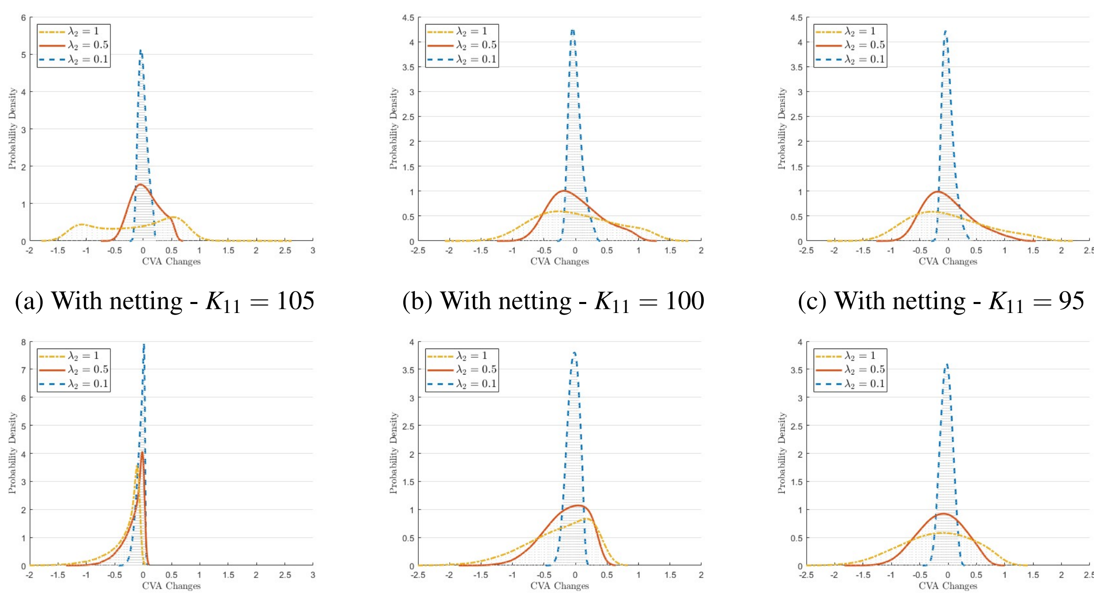
the netted portfolios, illustrating overestimations or underestimations when netting is not incorporated into the valuation. Note that in these analyses due to the options' monthly exercisability and the 10-day risk horizon, exercise opportunities do not arise before the horizon. Therefore, for the estimation of CVA VaR, we consider the entire portfolio as remaining active up to H.

Our analysis of the CVA distribution plots shows that as counterparty default risk rises, the frequency of data around the modal CVA value noticeably declines, regardless of netting agreements. This decreasing density indicates a reduced probability that CVA will remain near the initial CVA (CVA_0). In turn, this pattern signals greater uncertainty in expected losses. The diminishing frequency suggests potential CVA movements may vary over a wider range, reflecting heightened unpredictable credit risk. Overall, the changing shape of the distribution demonstrates that rising default risk translates to increased unpredictability in credit valuation adjustments.

Contrary to our initial expectations, we observe that the presence of a netting agreement does not necessarily decrease tail risk – indicated by extreme values at the upper end of the distribution – or CVA VaR, although it can mitigate CVA in general. This trend is particularly noticeable for Portfolio I. Our observations suggest that this counterintuitive risk increase may be tied to the impact of netting on exercise opportunities of the options within portfolios.

In Portfolio I, with C_2 holding options that are more in-the-money (ITM) than C_1, there is an increased likelihood of earlier exercise by C_2. As discussed in the previous chapter, the presence of netting can encourage earlier exercise for the party not exposed to default risk, in this case C_2. As a result, netting may accelerate the exercise of options with strikes K_{21} and K_{22} held by C_2. This, in turn, could reduce the effectiveness of netting in reducing CCR, thereby potentially elevating the CVA tail risk within the risk horizon.

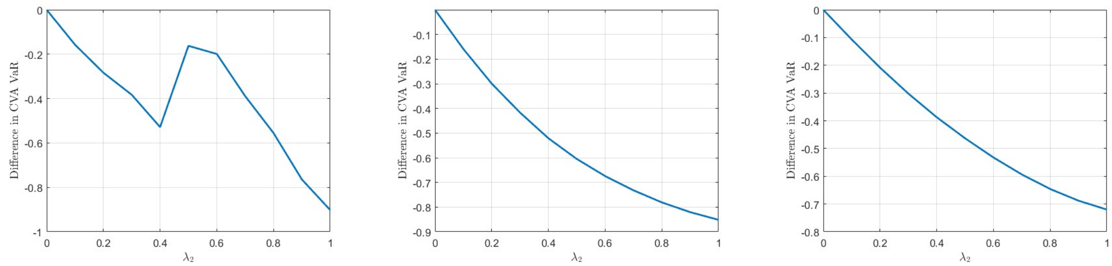
In contrast, for Portfolio II, where C_1 holds options that are more ITM compared to C_2, netting might lead to a higher CVA VaR compared to non-netted scenarios. This effect could be also explained by alterations in exercise strategy, albeit in a different manner. Options held by C_1 are more likely to be exercised earlier. As we highlighted in a previous chapter, without netting agreements, options tend to be exercised sooner, which reduces



(a) With netting - $K_{11} = 105$ (b) With netting - $K_{11} = 100$ (c) With netting - $K_{11} = 95$

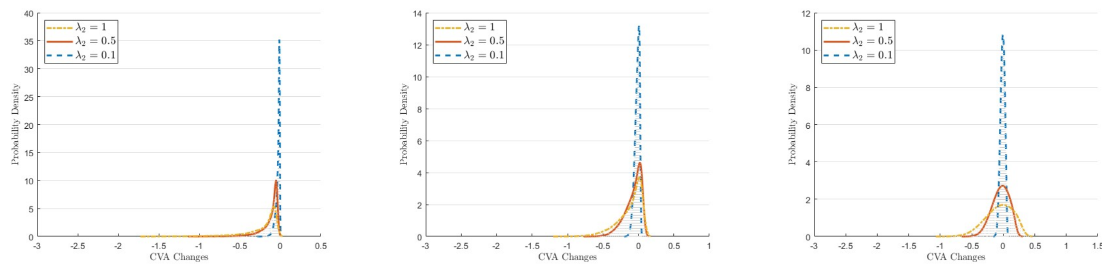
(d) Without netting - $K_{11} = 105$ (e) Without netting - $K_{11} = 100$ (f) Without netting - $K_{11} = 95$

Figure 4.1: CVA movements distribution for Portfolio I, on a risk horizon of $H = 10$ days. The scenarios depicted include those with (Panels a - c) and without (Panels d - f) netting agreements.

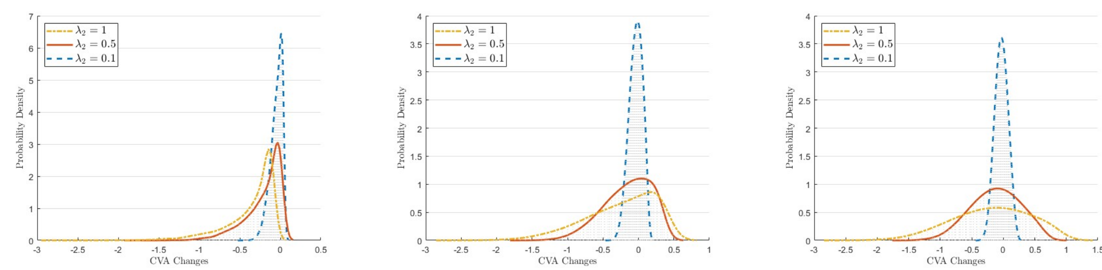


(a) $K_{11} = 105$ (b) $K_{11} = 100$ (c) $K_{11} = 95$

Figure 4.2: Netting Impact on CVA VaR for Portfolio I: The figures show the netting effect over a 10-day risk horizon by subtracting the portfolio's CVA VaR without netting from that with netting for various levels of K_{11}.



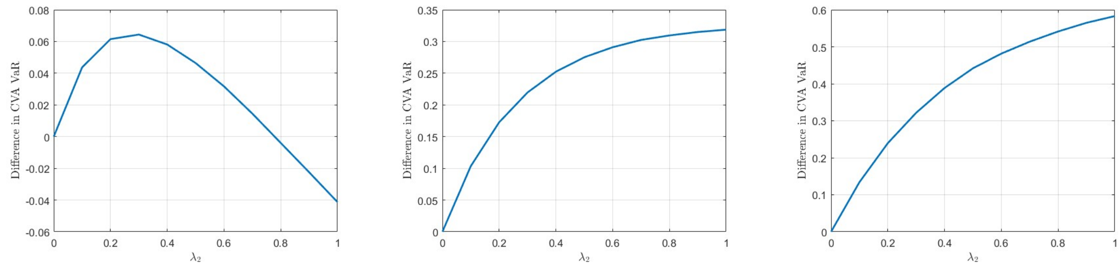
(a) With netting - $K_{11} = 105$ (b) With netting - $K_{11} = 100$ (c) With netting - $K_{11} = 95$



(d) Without netting - $K_{11} = 105$ (e) Without netting - $K_{11} = 100$ (f) Without netting - $K_{11} = 95$

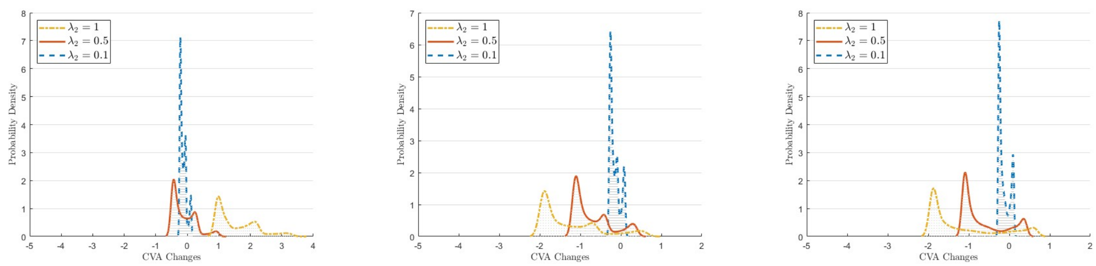
Figure 4.3: CVA movements distribution for Portfolio II, on a risk horizon of $H = 10$ days. The scenarios depicted include those with (Panels a - c) and without (Panels d - f) netting agreements.

exposure to potential default events. Therefore, under rising default risk, these options in a non-netted portfolio might be exercised earlier than those in a netted portfolio, thus potentially reducing credit exposure for those particular positions. Hence the absence of a netting agreement might, in some cases, result in a lower CVA VaR.

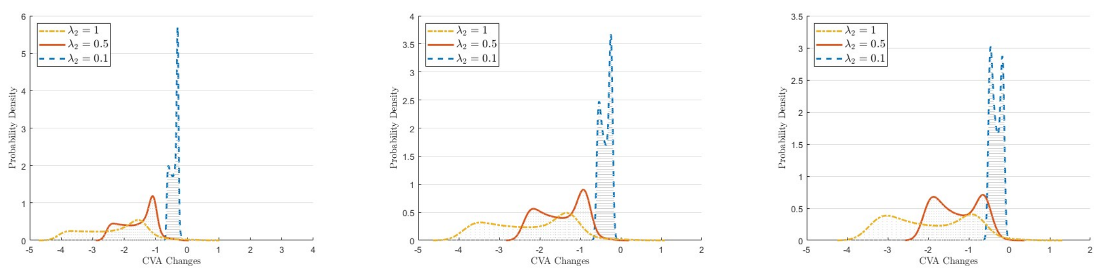


(a) $K_{11} = 105$ (b) $K_{11} = 100$ (c) $K_{11} = 95$

Figure 4.4: Netting Impact on CVA VaR for Portfolio II: The figures show the netting effect over a 10-day risk horizon by subtracting the portfolio's CVA VaR without netting from that with netting for various levels of K_{11}.



(a) With netting - $K_{11} = 105$ (b) With netting - $K_{11} = 100$ (c) With netting - $K_{11} = 95$

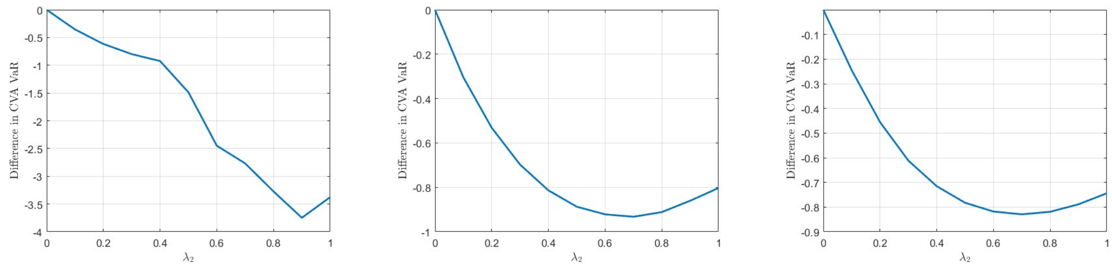


(d) Without netting - $K_{11} = 105$ (e) Without netting - $K_{11} = 100$ (f) Without netting - $K_{11} = 95$

Figure 4.5: CVA movements distribution for Portfolio I, on a risk horizon of $H = 180$ days. The scenarios depicted include those with (Panels a - c) and without (Panels d - f) netting agreements.

The potential for early exercise before the risk horizon can significantly alter the CVA distribution structure. Such opportunities introduce the potential to modify the composition of the portfolio, which can lead to a fundamental change in the exposure to CCR.

This is demonstrated in Figures 4.5 and 4.7, showing CVA movements for Portfolios

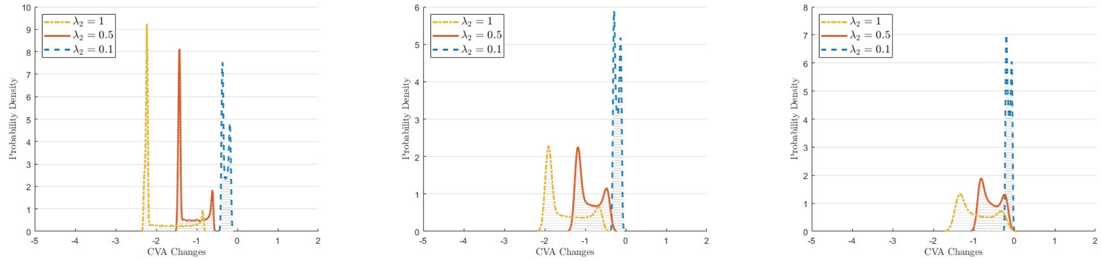


(a) $K_{11} = 105$ (b) $K_{11} = 100$ (c) $K_{11} = 95$

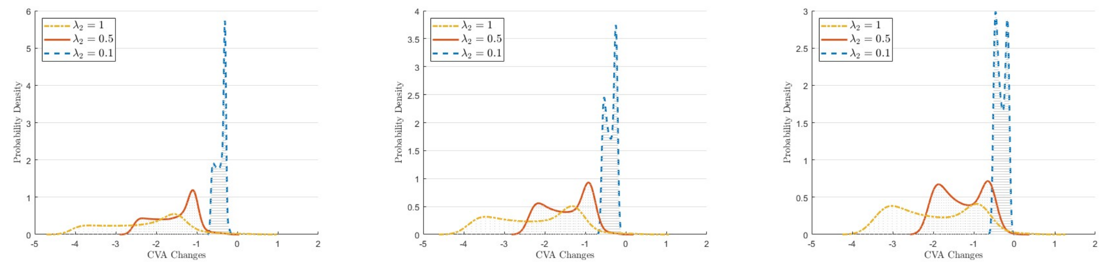
Figure 4.6: Netting Impact on CVA VaR for Portfolio I: The figures show the netting effect over a 180-day risk horizon by subtracting the portfolio's CVA VaR without netting from that with netting for various levels of K_{11}.

I and II with and without netting across a 180-day risk horizon. Furthermore, Figures 4.6 and 4.8 highlight the impact of netting on CVA VaR for these portfolios. As shown, netting may amplify CVA risk versus non-netted cases. Given the potential for up to 5 early exercises within the 180-day timeframe, the effects of netting on CVA VaR are shown to be highly variable and challenging to predict. We explore the effects of exercise opportunities on CVA distributions further in the next section.

The impact of the volatility parameter σ on the CVA VaR exhibits a complex and unpredictable nature. This intricate behavior is exemplified in Figure 4.9, which depicts the CVA VaR for a portfolio comprising four Bermudan put options with strike prices $K_{11} = 120$, $K_{12} = 110$, $K_{21} = 100$, and $K_{22} = 90$, under the condition $\lambda_2 = 0.5$. The underlying reason for this lack of unpredictability is similar to the previously discussed effect of σ on the CVA itself. While an increase in σ generally amplifies the potential for extreme market values, consequently increasing the exposure and potentially leading to an elevation in the CVA VaR for an individual option, the portfolio containing both long and short positions does not exhibit the same straightforward relationship. As σ increases, the exposure associated with each option within the portfolio is increased differently, rendering the net impact on the overall portfolio exposure indeterminate. Consequently, it becomes

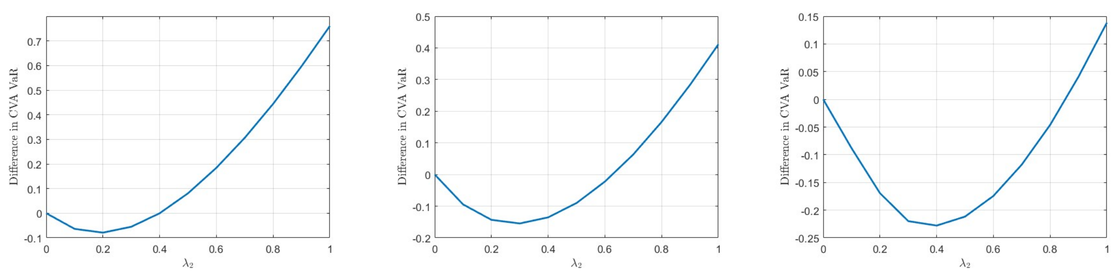


(a) With netting - $K_{11} = 105$ (b) With netting - $K_{11} = 100$ (c) With netting - $K_{11} = 95$



(d) Without netting - $K_{11} = 105$ (e) Without netting - $K_{11} = 100$ (f) Without netting - $K_{11} = 95$

Figure 4.7: CVA movements distribution for Portfolio II, on a risk horizon of $H = 180$ days. The scenarios depicted include those with (Panels a - c) and without (Panels d - f) netting agreements.



(a) $K_{11} = 105$ (b) $K_{11} = 100$ (c) $K_{11} = 95$

Figure 4.8: Netting Impact on CVA VaR for Portfolio II: The figures show the netting effect over a 180-day risk horizon by subtracting the portfolio's CVA VaR without netting from that with netting for various levels of K_{11}.

challenging to predict the cumulative effect on the portfolio's CVA VaR.

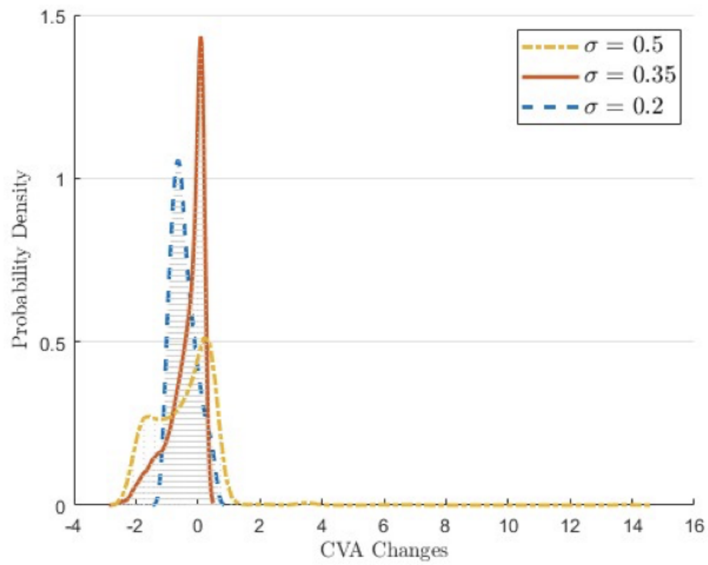


Figure 4.9: Impact of σ on the CVA movements distribution for a portfolio of four Bermudan put options with $K_{11} = 120$, $K_{12} = 110$, $K_{21} = 100$, and $K_{22} = 90$, when $\lambda_2 = 0.5$, and $H = 10$ days. Other parameter values are reported in Table 4.1.

4.3.2 European vs. Bermudan vs. American: A tail risk comparison

In this section, we examine the impact of varying the exercise frequencies of options within a netted portfolio on that portfolio's CVA distribution. Figure 4.10 illustrates the distribution of CVA movements across a 10-day risk horizon for Portfolios III and IV. The strike prices of these portfolios are outlined in Table 4.2. This analysis considers scenarios where all options within these portfolios are European, monthly exercisable Bermudan, or American, under different counterparty hazard rates λ_2 of 0.1, 0.5, and 1. Table 4.3 provides further details on the CVA values and 99% CVA Value-at-Risk estimates observed in Portfolios III and IV for these λ_2 levels over the same 10-day horizon.

Our findings indicate that the presence of early exercise opportunities can markedly influence the distribution of CVA changes. This impact is particularly pronounced for American-style options, which allow for exercise before the risk horizon, leading to changes in portfolio composition. As the default risk increases, the impact on portfolio composition, and consequently on CVA distribution, can become increasingly significant.

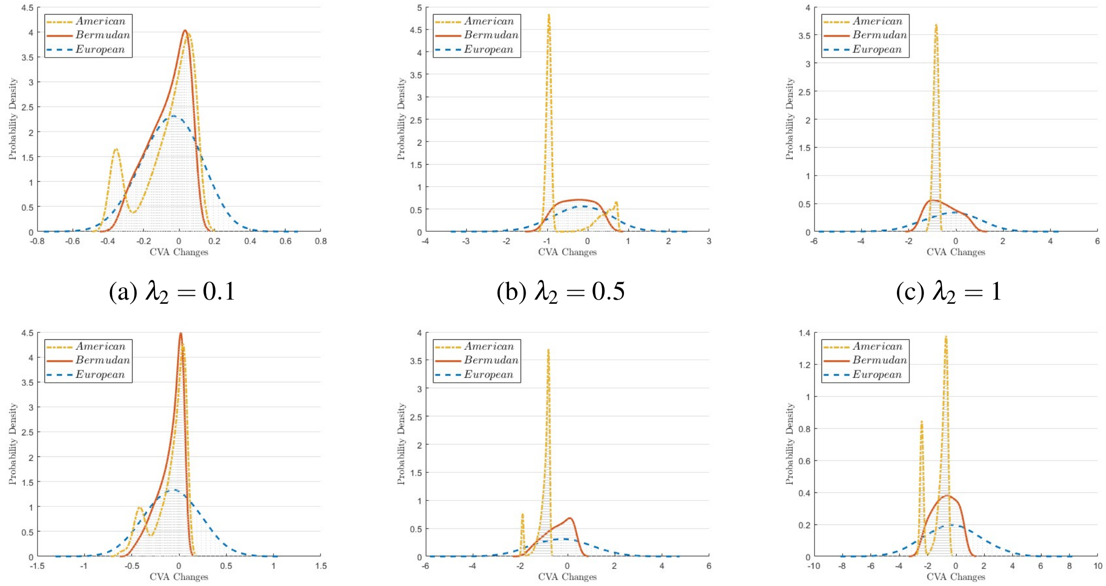
In portfolios where options held by C_1 are significantly ITM compared to those held by C_2, we generally observe a less extreme tail risk in American and Bermudan compared to European portfolios for a given risk horizon. The flexibility to adjust American and Bermudan portfolios in response to market changes tends to mitigate extreme CVA variations. In contrast, European portfolios, where the options remain active until maturity, show a wider distribution of CVA changes. This wider spread suggests higher potential risk due to sustained exposure to default risk throughout the option's lifespan.

Interestingly, the comparison between American and Bermudan portfolios is not straightforward. Figure 4.10, Panel b, shows that Bermudan portfolios, despite fewer exercise opportunities compared to American ones, can have lower tail risk and CVA VaR, yet might report higher CVA. This paradoxical behavior stems from the fact that, in portfolios containing American options, the frequent exercise of these options might lead to a portfolio composition that inadvertently amplifies CCR exposure.

Note that these observations, primarily applicable to portfolios with deeper ITM options for C_1 over C_2, do not consistently extend across all portfolio types. Our analysis of various portfolios reveals diverse responses to netting and exercise frequency, underscoring the absence of a universal pattern in the effects on CVA VaR. Each portfolio's distinct characteristics necessitate a customized evaluation to accurately assess the influence of netting and option exercisability on CVA variability.

4.4 Conclusion

In this chapter, we demonstrated the practical application of the CVA pricing model introduced previously, illustrating its capability to estimate the CVA distribution for netted



(a) $\lambda_2 = 0.1$ (b) $\lambda_2 = 0.5$ (c) $\lambda_2 = 1$

(d) $\lambda_2 = 0.1$ (e) $\lambda_2 = 0.5$ (f) $\lambda_2 = 1$

Figure 4.10: CVA movements distribution in Portfolio III (Panels a - c) and Portfolio IV (Panels d - f) with European, Bermudan, and American options over $H = 10$ days risk horizon.

Options type:		European		Bermudan		American	
Portfolio	λ_2	CVA	CVA VaR	CVA	CVA VaR	CVA	CVA VaR
III	0.1	1.2639	0.3042	0.4328	0.0705	0.3841	0.0959
	0.5	5.2298	1.3003	1.5340	0.4175	1.0648	0.7917
	1	8.4084	2.1994	2.3865	0.6830	1.1893	-0.6807
IV	0.1	1.8782	0.5723	0.6763	0.0451	0.6131	0.0729
	0.5	7.7715	2.5174	2.4557	0.3250	1.9001	-0.7735
	1	12.4948	4.1990	3.8209	0.5498	2.4136	-0.6077

Table 4.3: CVA and 99% CVA VaR Dynamics in Portfolios III and IV with European, Bermudan, and American Options Over $H = 10$ days risk horizon

portfolios with early exercise rights. The model's backward incursion approach allows it to compute CVA as a function of risk factors, portfolio composition, and the evaluation date. This functionality enables efficient CVA VaR estimation across risk horizons for netted portfolios, overcoming computational challenges of nested Monte Carlo methods, while also providing insights into parameters influencing CVA risk.

We examine the impacts of netting agreements on CVA variability across different portfolio constructs and counterparty risk levels. Contrary to expectations, the presence of netting does not uniformly decrease tail risk or CVA VaR. While netting can reduce credit exposures, its effects on CVA risk depend greatly on the portfolio structure and potential for early exercise. We show that netting may amplify tail risk compared to non-netted portfolios, attributed to differences in early exercise behavior altering exposures.

The ability to exercise options before the risk horizon can change portfolio composition, significantly impacting CVA tail risk in complex, unpredictable ways across different portfolios. As substantiated by our illustrative examples and comparative figures, the interplay between netting, early exercise frequencies, and counterparty hazard rates introduces complex portfolio-specific dynamics. This precludes general conclusions about the universal effects of netting agreements on CVA risk.

Chapter 5

General Conclusion

This thesis explored how netting agreements between counterparties affect the pricing of counterparty credit risk (CCR), specifically when the contracts contain early exercise rights. It is widely accepted that netting serves as an effective tool for reducing risk in dealings involving multiple contracts between two parties. However, its impact on CCR pricing has not been studied thoroughly in the context of portfolios including contracts with optional rights.

We demonstrated that the presence of netting agreements fundamentally alters the decision-making process for exercising contracts within netted portfolios. This change stems from the fact that exercise strategies become interdependent under netting, directly affecting the expected payoffs for both counterparties. This interlinked payoff structure converts the exercise decisions into a stochastic zero-sum game.

In the first essay, we introduced a dynamic programming model to recursively determine the value of this game, enabling the valuation of the market price of CCR for netted portfolios. We numerically implemented this recursive algorithm to facilitate assessment across diverse scenarios. Our approach provides a comprehensive assessment of Credit Valuation Adjustment (CVA) or Bilateral Valuation Adjustment(BVA), considering all possible underlying market conditions and portfolio compositions until portfolio maturity.

In the second essay, through extensive numerical experiments, we demonstrated how a netting agreement could change the exercise strategies of both parties involved in the portfolio, even when only one party is exposed to default risk. We also illustrated the considerable impacts that shift in default probabilities and other factors influencing exposure have on exercise decisions within a netted portfolio.

Our findings challenge the conventional methods employed for evaluating the risk adjustment value of netted portfolios. We show these approaches fail to capture how netting and counterparty risk affect decision-making, causing significant CVA and BVA miscalculations. Essentially, overlooking these effects when managing optional rights reduces the expected value of netted portfolios. Therefore, our investigation offers valuable insights into optimally exercising options under netting to maximize the risk-adjusted payoffs.

Our model enables efficient computing of CVA as a function of risk factors, portfolio composition, and date using backward recursion. In the third essay, we leverage this capability to estimate CVA value-at-risk (VaR) across risk horizons for netted portfolios. Contrary to expectations, we find that netting introduction does not reliably reduce CVA tail risk. While netting can decrease CVA, it may also increase CVA volatility versus non-netted portfolios in certain cases. This stems from different early exercise behaviors under netting agreements affecting counterparty credit exposures. The interactions between netting, early exercise frequencies, and default probabilities create highly case-specific dynamics. Given this complexity, we cannot generalize netting's impact on CVA VaR; instead, accurate assessment requires portfolio-by-portfolio analysis.

Expanding upon the findings of this thesis, one potential research direction involves addressing the computational challenges of backward recursion for larger netted portfolios. Increasing portfolio size can exacerbate the well-known 'curse of dimensionality' - the exponential rise in computational and memory loads as problem dimensions grow. Addressing this can enhance the efficiency of our model for extensive portfolios with large state and action spaces. A promising avenue to investigate is the application of Multi-agent Reinforcement Learning (RL). Unlike backward recursion, RL uses forward-looking learning based on simulated interactions, bypassing the need to exhaustively compute all potential state values. This presents a pathway to help mitigate the curse of dimensionality.

However, adopting RL introduces trade-offs in solution accuracy. While substantially reducing computational burden, RL provides approximate solutions, unlike the precise outcomes achievable with backward recursion in dynamic games. Still, as existing literature shows, we can bound errors in these RL solutions, offering a balanced approach between computational efficiency and solution accuracy. Future research should examine RL's potential for netted portfolio evaluation, providing robust, near-optimal solutions. This highlights a promising direction for advancing methodologies to extend counterparty risk pricing models to bigger, more intricate netted portfolios.

Bibliography

Andersson, Kristoffer and Cornelis W Oosterlee (2020). “Deep learning for CVA computations of large portfolios of financial derivatives”. In: *arXiv preprint arXiv:2010.13843*.

Antonelli, Fabio, Alessandro Ramponi, and Sergio Scarlatti (2022). “Approximate value adjustments for European claims”. In: *European Journal of Operational Research* 300.3, pp. 1149–1161.

Ballotta, Laura, Gianluca Fusai, and Daniele Marazzina (2019). “Integrated structural approach to credit value adjustment”. In: *European Journal of Operational Research* 272.3, pp. 1143–1157.

Basel III: A global regulatory framework for more resilient banks and banking systems (2011). Basel Committee on Banking Supervision (BCBS). URL: www.bis.org.

Ben-Ameur, Hatem, Michèle Breton, and Pascal François (2006). “A dynamic programming approach to price installment options”. In: *European Journal of Operational Research* 169.2, pp. 667–676.

Ben-Ameur, Hatem, Michèle Breton, Lotfi Karoui, et al. (2007). “A dynamic programming approach for pricing options embedded in bonds”. In: *Journal of Economic Dynamics and Control* 31.7, pp. 2212–2233.

Ben-Ameur, Hatem, Michèle Breton, and Pierre L'Ecuyer (2002). "A dynamic programming procedure for pricing American-style Asian options". In: *Management Science* 48.5, pp. 625–643.

Bertsekas, Dimitri (2012). *Dynamic programming and optimal control: Volume I*. Vol. 4. Athena scientific.

Bielecki, Tomasz R and Marek Rutkowski (2004). *Credit risk: modeling, valuation and hedging*. Springer Science & Business Media.

Breton, Michèle and Javier de Frutos (2012). "Approximation of dynamic programs". In: *Handbook of Computational Finance*. Springer, pp. 633–649.

Breton, Michèle and Oussama Marzouk (2018). "Evaluation of counterparty risk for derivatives with early-exercise features". In: *Journal of Economic Dynamics and Control* 88, pp. 1–20.

— (2019). "Counterparty risk: credit valuation adjustment variability and value-at-risk". In: *Journal of Risk* 21.5.

Brigo, Damiano and Aurélien Alfonsi (2005). "Credit default swap calibration and derivatives pricing with the SSRD stochastic intensity model". In: *Finance and stochastics* 9.1, pp. 29–42.

Brigo, Damiano, Cristin Buescu, and Massimo Morini (2011). "Impact of the first to default time on Bilateral CVA". In: *arXiv preprint arXiv:1106.3496*.

Brigo, Damiano and Agostino Capponi (2008). "Bilateral counterparty risk valuation with stochastic dynamical models and application to Credit Default Swaps". In: *arXiv preprint arXiv:0812.3705*.

Brigo, Damiano, Marco Francischello, and Andrea Pallavicini (2019). "Nonlinear valuation under credit, funding, and margins: Existence, uniqueness, invariance, and disentanglement". In: *European Journal of Operational Research* 274.2, pp. 788–805.

Brigo, Damiano and Massimo Masetti (2005). "Risk neutral pricing of counterparty risk". In.

Brigo, Damiano, Massimo Morini, and Andrea Pallavicini (2013). *Counterparty credit risk, collateral and funding: with pricing cases for all asset classes*. Vol. 478. John Wiley & Sons.

Brigo, Damiano and Andrea Pallavicini (2007). "Counterparty risk pricing under correlation between default and interest rates". In: *Numerical methods for finance*. Chapman and Hall/CRC, pp. 79–98.

Brigo, Damiano, Andrea Pallavicini, and Vasileios Papatheodorou (2009). "Bilateral counterparty risk valuation for interest-rate products: impact of volatilities and correlations". In: *arXiv preprint arXiv:0911.3331*.

Brigo, Damiano and Frédéric Vrins (2018). "Disentangling wrong-way risk: pricing credit valuation adjustment via change of measures". In: *European Journal of Operational Research* 269.3, pp. 1154–1164.

Broadie, Mark, Paul Glasserman, et al. (2004). "A stochastic mesh method for pricing high-dimensional American options". In: *Journal of Computational Finance* 7, pp. 35–72.

Burgard, Christoph and Mats Kjaer (2017). "Derivatives funding, netting and accounting". In: *Netting and Accounting (March 20, 2017)*.

Cesari, Giovanni et al. (2009). *Modelling, pricing, and hedging counterparty credit exposure: A technical guide*. Springer Science & Business Media.

Duffie, Darrell and Kenneth J Singleton (1999). "Modeling term structures of defaultable bonds". In: *The review of financial studies* 12.4, pp. 687–720.

— (2003). *Credit risk: pricing, measurement, and management*. Princeton university press.

Durand, Cyril (2010). "Valuation of contracts with counterparty risk under netting and collateral agreements". In: *Masters by Research Thesis, University of New South Wales*.

Fudenberg, Drew and Jean Tirole (1991). *Game theory*. MIT press.

Glasserman, Paul (2004). *Monte Carlo methods in financial engineering*. Vol. 53. Springer.

Gregory, Jon (2012). *Counterparty credit risk and credit value adjustment: A continuing challenge for global financial markets*. John Wiley & Sons.

— (2017). *Being two-faced over counterparty credit risk*. SSRN.

Haurie, Alain, Jacek B Krawczyk, and Georges Zaccour (2012). *Games and dynamic games*. Vol. 1. World Scientific Publishing Company.

Hull, John and Alan White (1995). "The impact of default risk on the prices of options and other derivative securities". In: *Journal of Banking & Finance* 19.2, pp. 299–322.

Jarrow, Robert A and Stuart M Turnbull (1995). "Pricing derivatives on financial securities subject to credit risk". In: *The journal of finance* 50.1, pp. 53–85.

Kim, Jinbeom and Tim Leung (2016). "Pricing derivatives with counterparty risk and collateralization: A fixed point approach". In: *European Journal of Operational Research* 249.2, pp. 525–539.

Klein, Peter (1996). "Pricing Black-Scholes options with correlated credit risk". In: *Journal of Banking & Finance* 20.7, pp. 1211–1229.

Klein, Peter and Jun Yang (2013). "Counterparty Credit Risk and AmericanOptions". In: *The Journal of Derivatives* 20.4, pp. 7–21.

Lando, David (1998). "On Cox processes and credit risky securities". In: *Review of Derivatives research* 2.2, pp. 99–120.

Leland, Hayne E (1998). "Agency costs, risk management, and capital structure". In: *The Journal of Finance* 53.4, pp. 1213–1243.

Longstaff, Francis A and Eduardo S Schwartz (2001). "Valuing American options by simulation: a simple least-squares approach". In: *The review of financial studies* 14.1, pp. 113–147.

Merton, Robert C (1974). “On the pricing of corporate debt: The risk structure of interest rates”. In: *The Journal of finance* 29.2, pp. 449–470.

OTC derivatives statistics at end-June 2022 (Nov. 2022). Bank of International Settlements. URL: `www.bis.org`.

Tsitsiklis, John N and Benjamin Van Roy (2001). “Regression methods for pricing complex American-style options”. In: *IEEE Transactions on Neural Networks* 12.4, pp. 694–703.

Zhou, Chunsheng (2001). “An analysis of default correlations and multiple defaults”. In: *The Review of Financial Studies* 14.2, pp. 555–576.