HEC MONTRÉAL
École affiliée à l'Université de Montréal

Essays on Mean Field Games :
Risk-Sensitive Major-Minor and Infinite-Dimensional Systems

par
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Essays on Mean Field Games :
Risk-Sensitive Major-Minor and Infinite-Dimensional Systems

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Résumé

Cette thèse est une contribution à la théorie des jeux à champ moyen de type linéaire-quadratique (LQ) en abordant deux généralisations distinctes : la prise en compte de la sensibilité au risque dans les systèmes comportant un agent majeur, et les dynamiques en dimension infinie.

Le premier essai de cette thèse est consacré aux jeux LQ à champ moyen où les agents (majeur et mineurs) sont sensibles (averses) au risque. À cette fin, nous commençons par développer une approche variationnelle pour traiter les problèmes de contrôle optimal sensibles au risque, c'est-à-dire impliquant un seul agent ayant une fonction coût de type exponentielle d'une intégrale, dans un cadre linéaire-quadratique-gaussien (LQG). Cette approche apporte de nouvelles perspectives sur ce type de problèmes. Notre analyse conduit à la dérivation d'une condition nécessaire et suffisante d'optimalité, de nature non linéaire, exprimée en termes de processus martingales. Sous certaines conditions, nous obtenons une mesure neutre au risque équivalente, sous laquelle la commande optimale admet une forme linéaire de l'état. Nous montrons ensuite que la commande ainsi obtenue reste optimale sous la mesure d'origine. À partir de ce développement, nous proposons un cadre variationnel pour les jeux à champ moyen LQG sensibles au risque en général, et étendons cette théorie en y intégrant un agent majeur. Nous caractérisons les stratégies de meilleure réponse en boucle fermée de type markovien dans le cas limite où le nombre d'agents tend vers l'infini. Nous établissons que l'ensemble des stratégies obtenues constitue un équilibre de Nash dans le cas limite, et un ε-équilibre de Nash dans le cas à N joueurs.

Le deuxième essai de cette thèse présente une étude approfondie des jeux à champ moyen de type LQ dans les espaces de Hilbert, généralisant la théorie classique aux scénarios où les dynamiques des agents sont décrites par des équations stochastiques en dimension infinie. Dans ce cadre, les processus d'état et de contrôle de chacun des agents prennent leurs valeurs dans des espaces de Hilbert séparables. Les agents sont couplés à travers la moyenne des états de la population, qui intervient dans leur dynamique et dans leur fonction de coût. Plus précisément, la dynamique de chaque agent comprend un bruit en dimension infinie, représenté par un processus de Wiener, ainsi qu'un opérateur non borné. Le coefficient de diffusion est stochastique, dépendant de l'état, du contrôle et de la moyenne des états. Nous étudions d'abord le caractère bien posé d'un système d'équations d'évolution stochastiques semi-linéaires en dimension infinie couplant plusieurs agents, posant ainsi les bases des jeux à champ moyen dans les espaces de Hilbert. Nous nous spécialisons ensuite dans le cas linéaire-quadratique et étudions le comportement asymptotique lorsque le nombre d'agents tend vers l'infini. Nous développons une version en dimension infinie du principe d'équivalence de certitude de Nash, et caractérisons un équilibre de Nash unique pour le jeu limite. Enfin, nous étudions la connexion entre le jeu à N joueurs et le jeu limite, en montrant que la moyenne empirique des états converge vers le champ moyen, et que les stratégies de meilleure réponse limites forment un ε-équilibre de Nash pour le jeu à N joueurs dans les espaces de Hilbert.

Dans le troisième essai de cette thèse, nous étudions les jeux à champ moyen en espace de Hilbert en présence d'un bruit commun. Le bruit commun modélise une incertitude partagée entre tous les agents ; il induit une corrélation, rend le champ moyen stochastique, et revêt une importance particulière car les sources d'incertitude partagées sont fréquentes dans les systèmes réels. Dans ce contexte, le terme de décalage dans les stratégies de meilleure réponse ainsi que le champ moyen sont décrits par des équations stochastiques en dimension infinie (tous deux étant déterministes en l'absence de bruit commun). Par conséquent, les conditions de cohérence du champ moyen prennent la forme d'un système d'équations différentielles stochastiques avant-arrière. Nous établissons l'existence et l'unicité des solutions à ce système, et démontrons la propriété de ε-équilibre de Nash. Enfin, nous abordons le cas où le modèle inclut des coefficients aléatoires à valeurs opératorielles.

Keywords

Jeux à champ moyen, systèmes linéaire-quadratique, analyse variationnelle, agent majeur, agent mineur, sensibilité au risque, coût exponentiel, équations stochastiques dans les espaces de Hilbert, jeux à champ moyen en dimension infinie, , processus de Wiener, bruit commun.

Abstract

This thesis develops the theory of linear-quadratic mean field games, focusing on two distinct generalizations: risk-sensitive major-minor systems and infinite-dimensional frameworks. In the first part of the thesis, we study linear-quadratic risk-sensitive mean field games with a major agent. To this purpose, we first develop a variational approach to address risk-sensitive optimal control problems with an exponential-of-integral cost functional in a general linear-quadratic-Gaussian (LQG) single-agent setup, offering new insights into such problems. Our analysis leads to the derivation of a nonlinear necessary and sufficient condition of optimality, expressed in terms of martingale processes. Subject to specific conditions, we find an equivalent risk-neutral measure, under which a linear state feedback form can be obtained for the optimal control. It is then shown that the obtained feedback control is consistent with the imposed condition and remains optimal under the original measure. Building upon this development, we (i) propose a variational framework for general LQG risk-sensitive mean-field games (MFGs) and (ii) advance the LQG risk-sensitive MFG theory by incorporating a major agent in the framework. We derive the Markovian closed-loop best-response strategies of agents in the limiting case where the number of agents goes to infinity. We establish that the set of obtained best-response strategies yields a Nash equilibrium in the limiting case and an ε-Nash equilibrium in the finite-player case.

We then present a comprehensive study of linear-quadratic (LQ) MFGs in Hilbert spaces, generalizing the classic LQ MFG theory to scenarios involving N agents with dynamics governed by infinite-dimensional stochastic equations. In this framework, both state and control processes of each agent take values in separable Hilbert spaces. All agents are coupled through the average state of the population which appears in their linear dynamics and quadratic cost functional. Specifically, the dynamics of each agent incorporates an infinite-dimensional noise, namely a Q-Wiener process, and an unbounded operator. The diffusion coefficient of each agent is stochastic involving the state, control, and average state processes. We first study the well-posedness of a system of N coupled semilinear infinite-dimensional stochastic evolution equations establishing the foundation of MFGs in Hilbert spaces. We then specialize to N-player LQ games described above and study the asymptotic behaviour as the number of agents, N, approaches infinity. We develop an infinite-dimensional variant of the Nash Certainty Equivalence principle and characterize a unique Nash equilibrium for the limiting MFG. Finally, we study the connections between the N-player game and the limiting MFG, demonstrating that the empirical average state converges to the mean field and that the resulting limiting best-response strategies form an ε-Nash equilibrium for the N-player game in Hilbert spaces.

In the final part of this thesis, we address MFGs in Hilbert spaces with common noise. Common noise models randomness shared by all agents; it introduces correlation, makes the mean field stochastic, and is important because shared sources of uncertainty are common in real-world scenarios. In this setting, the offset term in best-response strategies and the mean field are governed by infinite-dimensional stochastic equations (both of which are deterministic in the absence of common noise). Consequently, the mean field consistency equations take the form of a system of forward-backward stochastic equations. We establish the solvability of this system and demonstrate the ε-Nash property. Finally, we discuss the case where the model incorporates operator-valued random coefficients.

Keywords

Mean field games, linear-quadratic systems, variational analysis, major agent, minor agent, risk sensitivity, exponential cost functional, stochastic equations in Hilbert spaces, infinite-dimensional mean field games, infinite-dimensional analysis, Q-Wiener processes, common noise in Hilbert spaces.

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Introduction

Mean field game (MFG) theory concerns the study and analysis of dynamic games involving a large number of indistinguishable agents who are asymptotically negligible. In such games, each agent is weakly coupled with other agents through the empirical distribution of their states or control inputs. The mathematical limit of this distribution, as the number of agents approaches infinity, is referred to as the mean field distribution. In these games, the behavior of agents in large populations, along with the resulting equilibrium, may be approximated by the solution of corresponding limiting games (see, e.g., (Huang et al., 2006, 2007; Lasry and Lions, 2007; Carmona et al., 2018a; Bensoussan et al., 2013; Cardaliaguet et al., 2019)).

MFGs find relevance in various domains, and particularly within financial markets, they emerge as natural modeling choices for addressing a wide array of issues. Notably, applications have been proposed in systemic risk ((Carmona et al., 2015; Garnier et al., 2013; Bo and Capponi, 2015; Ren and Firoozi, 2024; Chang et al., 2025)), price impact and optimal execution ((Casgrain and Jaimungal, 2020; Firoozi and Caines, 2017; Cardaliaguet and Lehalle, 2018; Carmona and Lacker, 2015; Huang et al., 2019)), cryptocurrencies ((Li et al., 2023b)), portfolio trading ((Lehalle and Mouzouni, 2019)), equillibrium pricing ((Shrivats et al., 2022; Gomes and Saúde, 2021; Fujii and Takahashi, 2022)), and market design ((Shrivats et al., 2021)).

Among the various types of MFGs, linear-quadratic mean field games stand out for their analytical tractability and widespread applicability. In linear-quadratic mean field games, the dynamics of each agent are governed by linear stochastic differential equations, while their cost functionals are quadratic in the state and control variables. This structure allows for explicit solutions in many cases, leveraging tools such as Riccati equations, fixed-point analysis, and stochastic calculus.

The classic linear-quadratic MFGs have been extensively studied in the literature (Bensoussan et al., 2016; Huang et al., 2007; Huang, 2010; Firoozi et al., 2020; Liu et al., 2025; Firoozi and

Caines, 2020; Firoozi, 2022; Huang, 2021; Toumi et al., 2024; Li et al., 2023a). The classic linear-quadratic mean field games primarily focus on risk-neutral structures for the cases where a major agent is present and are formulated in finite-dimensional settings, typically within $\mathbb{R}^n$. Therefore, this thesis will consider more advanced linear-quadratic mean field games, aiming to generalize and enrich the existing framework by incorporating risk-sensitive major-minor and infinite-dimensional settings.

In Chapter 1, we consider the case where the cost functional of each agent is exponential-quadratic, representing risk-sensitivity. This consideration is especially pertinent in many economic and financial contexts as risk sensitivity, and its disparity among players, needs to be accounted for when characterizing equilibrium strategies. This is also the case in the area of mean-field games, where recent developments were proposed to address risk-sensitive MFGs (Tembine et al. (2013); Saldi et al. (2018, 2022); Moon and Başar (2016, 2019)). In addition, we allow the model to have a major agent. Unlike minor agents, whose impact decreases as the number of agents increases, the impact of a major agent is not negligible and does not collapse when the size of the population tends to infinity (Huang (2010); Firoozi et al. (2020); Carmona and Zhu (2016)). More specifically, the contributions of this chapter, which is also forthcoming as a published article[1], are summarized as below:

- We develop a variational approach to solve risk-sensitive optimal control problems. This approach offers new perspectives on the inherent nature of risk-sensitive problems, distinct from existing methodologies. Specifically: (i) it demonstrates how the nonlinearity of exponential risk-sensitive cost functionals can be translated into a necessary and sufficient condition for optimality involving a quotient of martingale processes; (ii) by establishing an equivalent risk-neutral measure in terms of the state and control processes, it explains the connection between the model and its counterpart under the risk-neutral measure. This technique enables the derivation of explicit solutions, even in the presence of the nonlinear term within the necessary and sufficient condition of optimality that involves the state and control processes; (iii) it allows a deeper understanding of risk-sensitivity's implications by effectively tracing the impact of risk-sensitivity on the propagation of policy perturbations throughout

[1] Liu, Hanchao, Dena Firoozi, and Michèle Breton. LQG Risk-Sensitive Single-Agent and Major-Minor Mean Field Game Systems: A Variational Framework. *SIAM Journal on Control and Optimization*, vol. 63, no. 4, pp. 2251-2281, 2025.

the system; and (iv) it extends the applicability of variational analysis to the context of risk-sensitive problems with exponential cost functionals and facilitates the characterization of optimal strategies for complex or nonclassical setups.

- We advance the theory of LQG risk-sensitive MFGs by incorporating a non-negligible major agent, whose impact on the system remains significant even as the number of minor agents goes to infinity. To the best of our knowledge, the literature has not yet explored such MFGs. In particular, our work stands apart from Chen, Yan et al. (2023), where a risk-sensitive MFG involving a group of asymptotically negligible major agents is considered and the average state of this group appears in the model. Unlike that scenario, our model features the significant influence of the major agent in the limiting model.

- Building upon the the aforementioned developments, we propose a variational framework for general LQG risk-sensitive MFGs with a major agent. Through this framework, we gain valuable insights into the interplay among risk sensitivity, the major agent and minor agents within the context of LQG MFGs. Specifically, it illustrates: (i) the impact of the major agent's risk sensitivity on the propagation of its policy perturbations throughout the system (i.e. on individual minor agents), and hence the formation of the aggregate effect of minor agents (i.e. the mean field) and the equilibrium, and (ii) the impact of a representative minor agent's risk sensitivity on the propagation of its policy perturbations across the system (i.e. on other minor agents and the major agent) and hence the formation of the the mean field and the equilibrium. More precisely, using this variational approach, we derive the Nash equilibrium under the limiting game setup. Then, we establish that the equilibrium strategies for the limiting game case lead to an ε-Nash equilibrium for the finite-population game. Our proof of the ε-Nash property differs from existing ones in the literature on MFGs with a major agent (Huang (2010); Carmona and Zhu (2016)) and on risk-sensitive MFGs (Moon and Başar (2019, 2016)), leveraging the specific conditions of the model under investigation. We further investigate and provide insights into the impact of agents' risk sensitivity on the equilibrium by conducting a comparative analysis with the risk-neutral case.

Furthermore, MFGs are originally developed in finite-dimensional spaces. However, there are scenarios where Euclidean spaces do not adequately capture the essence of problems such as those involving non-Markovian systems. Beyond practical motivations, investigating MFGs in infinite-dimensional spaces offers an interesting mathematical perspective due to the distinctive treatment required compared to Euclidean spaces. In such spaces, the evolution of a stochastic process is governed by an infinite-dimensional stochastic equation (see e.g. Da Prato and Zabczyk (2014); Gawarecki and Mandrekar (2010)). These equations, also termed stochastic partial differential equations (SPDEs), form a powerful mathematical framework for modeling dynamical systems with infinite-dimensional states and noises. Therefore, the goal of Chapter 2 is to present a comprehensive study of linear-quadratic MFGs in Hilbert spaces, where the state equation of each agent is modeled by an infinite-dimensional stochastic equation. More specifically, the contributions of this chapter, which is also forthcoming as a published article[2], are summarized as below:

- We study a general N-player LQ game. In particular, the state equation of each agent is influenced by the average state in the drift coefficient and incorporates Hilbert space-valued Q-Wiener processes and an unbounded system operator. Additionally, the volatility of each agent involves the state, control, and average state processes, resulting in stochastic volatility and multiplicative noise.

- To ensure the well-posedness of the Hilbert space-valued N-player game described above, we establish regularity results for a system of N coupled semilinear stochastic evolution equations in Hilbert spaces. The dynamics of the N-player game studied falls as a special case within this system.

- We study the limiting problem where the number of agents goes to infinity. Establishing a Nash equilibrium for this model involves identifying a unique fixed point within an appropriate function space. To achieve this, we develop an infinite-dimensional variant of the Nash Certainty Equivalence. The model studied in Federico et al. (2024b) can be viewed as a special case of the limiting MFG model addressed here (see Section 2.4.2). However, due to the different methodologies used, our required conditions and results regarding the existence and uniqueness of a fixed point differ from those established in Federico et al. (2024b).

[2] Liu, Hanchao, and Dena Firoozi. Hilbert Space-Valued LQ Mean Field Games: An Infinite-Dimensional Analysis, (to appear) *SIAM Journal on Control and Optimization*, 2025.

- We study the connections between the N-player game and the limiting mean field game, demonstrating that the Nash equilibrium strategies obtained in the limiting case form an ε-Nash equilibrium for the N-player game in Hilbert spaces. This property is established by showing that the mean field approximates the empirical average state within $\varepsilon = o(\frac{1}{\sqrt{N}})$ when agents follow these strategies.

In Chapter 3, we incorporate a common infinite-dimensional noise to the framework developed in Chapter 2. In many practical scenarios, agents in MFGs are not only influenced by their own individual uncertainties but also by a shared source of randomness, referred to as common noise. This type of noise arises when external factors affect all agents simultaneously, creating dependencies between their actions and dynamics. Common noise can represent various real-world phenomena, such as systemic risks, environmental shocks, or economic fluctuations, where the environment introduces a collective uncertainty that agents must account for. The presence of common noise introduces additional technical challenges in addressing MFGs. Specifically, the mean field system is formulated as a system of forward-backward stochastic differential equations (SDEs), while in cases without common noise, it is described using forward-backward ordinary differential equations (ODEs). The contributions of this chapter are summarized as below:

- We study a general N-player linear-quadratic (LQ) game with common noise. The state dynamics of each agent are influenced by the average state through the drift term and are driven by both idiosyncratic and common noises, modeled as Q-Wiener processes with different covariance operators. The volatility of each agent is stochastic involving the state and average state processes.

- To ensure the well-posedness of the Hilbert space-valued N-player game with common noise, we first establish regularity results for a system of N coupled semilinear stochastic evolution equations impacted by both idiosyncratic and common noises in Hilbert spaces.

- We then study the limiting problem as the number of agents tends to infinity. In this case, the mean field consistency equations are formulated as a system of forward-backward stochastic evolution equations in Hilbert spaces. Furthermore, the ε-Nash property is established for the obtained limiting best-response strategies.

- Finally, we study the scenario where the model operators are themselves operator-valued stochastic processes adapted to the filtration generated by the common noise. We show that, under appropriate assumptions, the structure and solvability of the mean field game remain analogous to the case with non-random operators.

Chapter 1

LQG Risk-Sensitive Single-Agent and Major-Minor Mean-Field Game Systems

Abstract

We develop a variational approach to address risk-sensitive optimal control problems with an exponential-of-integral cost functional in a general linear-quadratic-Gaussian (LQG) single-agent setup, offering new insights into such problems. Our analysis leads to the derivation of a nonlinear necessary and sufficient condition of optimality, expressed in terms of martingale processes. Subject to specific conditions, we find an equivalent risk-neutral measure, under which a linear state feedback form can be obtained for the optimal control. It is then shown that the obtained feedback control is consistent with the imposed condition and remains optimal under the original measure. Building upon this development, we (i) propose a variational framework for general LQG risk-sensitive mean-field games (MFGs) and (ii) advance the LQG risk-sensitive MFG theory by incorporating a major agent in the framework. The major agent interacts with a large number of minor agents, and unlike the minor agents, its influence on the system remains significant even with an increasing number of minor agents. We derive the Markovian closed-loop best-response strategies of agents in the limiting case where the number of agents goes to infinity. We establish that the set of obtained best-response strategies yields a Nash equilibrium in the limiting case and an ε-Nash equilibrium in the finite-player case.

1.1 Introduction

Risk-sensitive optimal control serves as the foundation for the development of risk-sensitive mean field games. The concept of risk-sensitive optimal control was introduced in Jacobson (1973) within a linear-quadratic-Gaussian (LQG) framework. In these problems, the agent's utility is described by an exponential function of the total cost it incurs over time. Since their inception, risk-sensitive control problems have captured considerable interest in the literature. Notably, the theory has been extended to encompass nonlinear risk-sensitive problems (Kumar and Van Schuppen (1981) and Nagai (1996)) and imperfect information (Pan and Başar (1996)), leading to a broader understanding of these systems. Furthermore, different methodologies have been developed, each offering unique insights to address such problems (see, for example, Duncan (2013) and Lim and Zhou (2005)). Başar (2021) provides an extensive overview of the literature on this topic.

The study of risk-sensitive models is crucial as risk-neutral models often fall short in capturing all the behaviors observed in reality. This consideration is especially pertinent in many economic and financial contexts as risk sensitivity, and its disparity among players, needs to be accounted for when characterizing equilibrium strategies. In economics and finance, it is well recognized that attitude toward risk, or *risk sensitivity*, plays an important role in the determination of agents' optimal decisions or strategies (Bielecki et al. (2000); Bielecki and Pliska (2003); Fleming and Sheu (2000)). This is also the case in the area of mean-field games, where recent developments were proposed to address risk-sensitive MFGs (Tembine et al. (2013); Saldi et al. (2018, 2022); Moon and Başar (2016, 2019)).

Another notable advancement of MFG theory involves the integration of the so-called *major* agents within the established framework. Unlike minor agents, whose impact decreases as the number of agents increases, the impact of a major agent is not negligible and does not collapse when the size of the population tends to infinity (Huang (2010); Firoozi et al. (2020); Carmona and Zhu (2016)). Various interpretations of such systems have been proposed. In the area of investment finance, for instance, one can consider that institutional and private investors' decisions do not have a commensurable impact on the market.

In this chapter, we develop a variational approach to address risk-sensitive optimal control problems with an exponential-of-integral cost functional in a general LQG single-agent setup, drawing inspiration from Firoozi et al. (2020). Our analysis leads to the derivation of a nonlinear necessary and sufficient condition of optimality, expressed in terms of martingale processes. Subject to specific conditions we find an equivalent risk-neutral measure, under which a linear state feedback form can be obtained for the optimal control. It is then shown that the obtained feedback control is consistent with the imposed condition and remains optimal under the original measure[1].

This chapter is organized as follows. Section 1.2 reviews the relevant literature. Section 1.3 outlines the variational approach developed to solve LQG risk-sensitive single-agent optimal control problems, which will be used to characterize the best response of MFG agents in the subsequent section. Section 1.4 employs the developed variational analysis to the LQG risk-sensitive MFGs with major and minor agents, in order to obtain the Markovian closed-loop Nash equilibrium of limiting game MFGs, and it shows that this Nash equilibrium provides an approximate equilibrium for the finite-population game. Section 1.5 provides a brief conclusion.

1.2 Literature Review

1.2.1 Risk-Sensitive Single-Agent and Mean-Field Game Systems

Risk-sensitive optimal control problems were introduced in Jacobson (1973) in a finite horizon LQG setting, where the agent's utility is an exponential function of its total cost over time. The author derived the explicit optimal control and the associated Riccati equations for such systems by either finding the limit of the optimal control of their corresponding discrete systems or by solving the generalized Hamilton–Jacobi–Bellman (HJB) equations. Later, Pan and Başar (1996) studied linear singularly perturbed systems with long-term time-average exponential quadratic costs under perfect and noisy state measurements. They obtained a time-scale decomposition, which breaks down the full-order problem into two appropriate slow and fast subproblems, and investigated the performance of the combined optimal controllers. A generalization to the case where costs are

[1]This chapter is a published article: Liu, Hanchao, Dena Firoozi, and Michèle Breton. LQG Risk-Sensitive Single-Agent and Major-Minor Mean Field Game Systems: A Variational Framework. *SIAM Journal on Control and Optimization*, vol. 63, no. 4, pp. 2251-2281, 2025

not quadratic functions was proposed in Kumar and Van Schuppen (1981) and Nagai (1996), and infinite horizon risk-sensitive optimal control problems in discrete-time were addressed in Whittle (1981). More recently, an alternative approach using the first principles was suggested in Duncan (2013), providing additional insights into the solution of such systems. This approach was then used in Duncan (2015) to characterize the Nash equilibrium of a two-player noncooperative stochastic differential game. A novel maximum principle for risk-sensitive control was established in Lim and Zhou (2005), where the authors used a logarithmic transformation and the relationship between the adjoint variables and the value function to address the case where the diffusion term depends on the control action.

Mean-field games have recently been extended to a risk-sensitive context. A general setup for risk-sensitive MFGs was proposed in Tembine et al. (2013). The authors demonstrated that, when utilizing an exponential integral cost functional, the mean-field value aligns with a value function that satisfies an HJB equation, accompanied by an additional quadratic term. Later, Moon and Başar (2016) studied a MFG involving heterogeneous agents with linear dynamics and an exponential quadratic integral cost. By employing the Nash certainty equivalence (NCE) method (see Huang et al. (2006)) via fixed-point analysis, the authors showed that the approximated mass behavior is in fact the best estimate of the actual mass behavior. Recently, Moon and Başar (2019) developed a stochastic maximum principle for risk-sensitive MFGs over a finite horizon. Similar results were obtained for the discrete-time setup with perfect and partial observations by Saldi et al. (2018) and Saldi et al. (2022), respectively. A risk-sensitive MFG involving two large-population groups of agents with different impacts on the system is studied in Chen, Yan et al. (2023), where the average state of each group appears in the model.

Other research avenues that are closely related to risk-sensitive MFGs include robust MFGs (see e.g. Huang and Huang (2017); Huang and Jaimungal (2017)), where model ambiguity is incorporated into the optimization problem, and quantalized MFGs (Tchuendom et al. (2019, 2025)), which focus on targeting a specific quantile of the population distribution.

1.2.2 MFGs with Major and Minor Agents

An advancement of the MFG theory incorporates the interaction between major and minor agents, and studies (approximate) Nash equilibria between them. To address major-minor MFGs, various approaches have been proposed, including NCE (Huang (2010), Nourian and Caines (2013)), probabilistic approaches involving the solution of forward-backward stochastic differential equations (FBSDE) (Carmona and Zhu (2016), Carmona and Wang (2017)), asymptotic solvability (Huang and Yang (2019)), master equations (Lasry and Lions (2018), Cardaliaguet et al. (2019)), and convex analysis (Firoozi et al. (2020)), the last of which being developed under the LQG setup. These approaches have been shown to yield equivalent Markovian closed-loop solutions (Huang (2021), Firoozi (2022)) in the limit. Another line of research characterizes a Stackelberg equilibrium between the major agent and the minor agents; see e.g. Bensoussan et al. (2017); Moon and Başar (2018).

In the next section, we develop a variational analysis for LQG risk-sensitive optimal control problems. This analysis will serve as the foundation for addressing major-minor MFG systems, which will be discussed in detail in the subsequent section.

1.3 Variational Approach to LQG Risk-Sensitive Optimal Control Problems

We begin by examining (single-agent) risk-sensitive optimal control problems under a general linear-quadratic-Gaussian framework. This approach will allow us to streamline the notation and enhance the clarity of the subsequent expositions. Building on the variational analysis method proposed in Firoozi et al. (2020), we extend the methodology by using a change of measure technique to derive the optimal control actions for LQG risk-sensitive problems with exponential cost functionals. These results will then be used in the subsequent section to determine the best-response strategies of both the major agent and a representative minor agent. We consider a general LQG risk-sensitive model with dynamics given by

$$dx_t = (Ax_t + Bu_t + b(t))dt + \sigma(t)dw_t, \tag{1.1}$$

where $x_t \in \mathbb{R}^n$ and $u_t \in \mathbb{R}^m$ are respectively the state and control vectors at t and $w_t \in \mathbb{R}^r$ is a standard r-dimensional Wiener process defined on a given filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in\mathfrak{T}}, \mathbb{P})$, $\mathfrak{T} = [0,T]$ with $T > 0$ fixed. Moreover, we define A and B to be constant matrices with compatible dimensions, and $b(t)$ and $\sigma(t)$ to be deterministic continuous functions on $\mathfrak{T}$.

The risk-sensitive cost functional $J(u)$ to be minimized is given by

$$J(u) = \mathbb{E}\left[\exp\left(\delta\Lambda_T(u)\right)\right], \tag{1.2}$$

where

$$\Lambda_T(u) := \frac{1}{2}\int_0^T \Big(\langle Qx_s, x_s\rangle + 2\langle Su_s, x_s\rangle + \langle Ru_s, u_s\rangle - 2\langle \eta, x_s\rangle - 2\langle \zeta, u_s\rangle \Big) ds + \frac{1}{2}\left\langle \widehat{Q}X_T, X_T\right\rangle. \tag{1.3}$$

All the parameters $(Q, S, R, \eta, \zeta, \widehat{Q})$ in the above cost functional are vectors or matrices of an appropriate dimension. The positive scalar constant δ represents the degree of risk sensitivity, with $0 < \delta < \infty$ modeling a risk-averse behavior. It is worth noting that the risk-neutral cost functional $\mathbb{E}[\Lambda_T(u)]$ can be seen as the limit of the risk-sensitive cost functionals $\frac{1}{\delta}\log\mathbb{E}\left[\exp\left(\delta\Lambda_T(u)\right)\right]$ or $\frac{1}{\delta}\mathbb{E}\left[\exp\left(\delta\Lambda_T(u)\right) - 1\right]$ when $\delta \to 0$. Both of these risk-sensitive cost functionals yield the same optimal control action as the cost functional (1.2)-(1.3), due to the strictly increasing property of the logarithm and linear functions, respectively. For notational convenience, our analysis will focus on the cost functional (1.2)-(1.3).

We make the following assumption regarding parameters of the cost functional.

A1.3.1. *$R > 0$, $\widehat{Q} \geq 0$, and $Q - SR^{-1}S^{\intercal} \geq 0$.*

The filtration $\mathscr{F} = (\mathscr{F}_t)_{t\in\mathfrak{T}}$, with $\mathscr{F}_t := \sigma(x_s; 0 \leq s \leq t)$, which is the σ-algebra generated by the process x_t, constitutes the information set of the agent. Subsequently, the admissible set $\mathscr{U}$ of control actions is the Hilbert space consisting of all $\mathscr{F}$-adapted $\mathbb{R}^m$-valued processes such that $\mathbb{E}\left[\int_0^T \|u_t\|^2 dt\right] < \infty$, where $\|.\|$ is the corresponding Euclidean norm.

Proposition 1. *Suppose A1.3.1 holds. Then the cost functional* (1.2)-(1.3) *is strictly convex.*

Proof. By applying the standard method of completing the square to $\Lambda_T(u)$, given by (1.3), we obtain

$$\begin{aligned}\Lambda_T(u) := \frac{1}{2}\int_0^T \Big(& \left\langle \tilde{Q}x_s, x_s\right\rangle + \left\langle R(u_s - R^{-1}S^{\intercal}x_s), u_s - R^{-1}S^{\intercal}x_s\right\rangle - 2\left\langle \zeta, u_s - R^{-1}S^{\intercal}x_s\right\rangle \\ & - 2\langle \eta, x_s\rangle + 2\left\langle \zeta, R^{-1}S^{\intercal}x_s)\right\rangle \Big) ds + \frac{1}{2}\left\langle \widehat{Q}X_T, X_T\right\rangle.\end{aligned} \tag{1.4}$$

Let $\widetilde{u}$ and $\widehat{u}$ be two different elements in $\mathscr{U}$ such that

$$\mathbb{E}\left[\int_0^T \|\widetilde{u}_t - \widehat{u}_t\|^2 dt\right] > 0. \tag{1.5}$$

Then we have

$$\mathbb{E}\left[\int_0^T \left\|\widetilde{u}_t - \widehat{u}_t - R^{-1}S^\intercal(\widetilde{x}_s - \widehat{x}_s)\right\|^2 dt\right] > 0. \tag{1.6}$$

To verify the validity of the above statement, we proceed by proof by contraposition. Suppose (1.6) does not hold, i.e, $\mathbb{E}\left[\int_0^T \left\|\widetilde{u}_t - \widehat{u}_t - R^{-1}S^\intercal(\widetilde{x}_s - \widehat{x}_s)\right\|^2 dt\right] = 0$, or equivalently $\widetilde{u}_t - R^{-1}S^\intercal\widetilde{x}_t = \widehat{u}_t - R^{-1}S^\intercal\widehat{x}_t$, $\mathbb{P}\times\mu$-almost everywhere, where μ stands for the Lebesgue measure. In this case, from (1.1), $\widetilde{x}_t - \widehat{x}_t$ satisfies

$$d(\widetilde{x}_t - \widehat{x}_t) = [(A + BR^{-1}S^\intercal)(\widetilde{x}_t - \widehat{x}_t)]dt, \tag{1.7}$$

with $\widetilde{x}_0 - \widehat{x}_0 = 0$. Thus, $\widetilde{x}_t = \widehat{x}_t$, $\mathbb{P}\times\mu$-almost everywhere, which leads to

$$\mathbb{E}\left[\int_0^T \left\|\widetilde{u}_t - \widehat{u}_t - R^{-1}S^\intercal(\widetilde{x}_s - \widehat{x}_s)\right\|^2 dt\right] = \mathbb{E}\left[\int_0^T \|\widetilde{u}_t - \widehat{u}_t\|^2 dt\right] = 0. \tag{1.8}$$

Thus, (1.6) holds for any $\widetilde{u} \in \mathscr{U}$ and $\widehat{u} \in \mathscr{U}$ satisfying (1.5). We then define the following sets

$$\begin{aligned}
E_1 &:= \left\{\omega \in \Omega : \int_0^T (\|\widetilde{x}_t\|^2 + \|\widehat{x}_t\|^2 + \|\widetilde{u}_t\|^2 + \|\widehat{u}_t\|^2)dt < \infty\right\},\\
E_2 &:= \left\{\omega \in \Omega : \int_0^T \left\|\widetilde{u}_t - \widehat{u}_t - R^{-1}S^\intercal(\widetilde{x}_s - \widehat{x}_s)\right\|^2 dt > 0\right\},\\
E &:= E_1 \cap E_2.
\end{aligned}$$

Since $P(E_1) = 1$ and $P(E_2) > 0$, it follows that $P(E) > 0$. Hence, $\forall\omega \in E$, we have $\widetilde{u}_t - R^{-1}S^\intercal\widetilde{x}_t \neq \widehat{u}_t - R^{-1}S^\intercal\widehat{x}_t$ on a set within the Borel σ-algebra $\mathscr{B}(\mathfrak{T})$ that has a positive Lebesgue measure. Subsequently, given that $R > 0$, we can easily show that $\forall\omega \in E$

$$\begin{aligned}
&\int_0^T \lambda\left\langle R(\widetilde{u}_t - R^{-1}S^\intercal\widetilde{x}_t), \widetilde{u}_t - R^{-1}S^\intercal\widetilde{x}_t\right\rangle + (1-\lambda)\left\langle R(\widehat{u}_t - R^{-1}S^\intercal\widehat{x}_t), \widehat{u}_t - R^{-1}S^\intercal\widehat{x}_t\right\rangle dt \qquad (1.9)\\
&> \int_0^T \Big\langle R(\lambda(\widetilde{u}_t - R^{-1}S^\intercal\widetilde{x}_t) + (1-\lambda)(\widehat{u}_t - R^{-1}S^\intercal\widehat{x}_t)), \lambda(\widetilde{u}_t - R^{-1}S^\intercal\widetilde{x}_t) + (1-\lambda)(\widehat{u}_t - R^{-1}S^\intercal\widehat{x}_t)\Big\rangle dt\\
&= \int_0^T \Big\langle R(\lambda\widetilde{u}_t + (1-\lambda)\widehat{u}_t - R^{-1}S^\intercal(\lambda\widetilde{x}_t + (1-\lambda)\widehat{x}_t)), \lambda\widetilde{u}_t + (1-\lambda)\widehat{u}_t - R^{-1}S^\intercal(\lambda\widetilde{x}_t + (1-\lambda)\widehat{x}_t)\Big\rangle dt.
\end{aligned}$$

Moreover, A1.3.1 guarantees that all other terms in (1.4) are convex (not necessarily strictly convex). Hence, $\forall\omega \in E$, we can conclude that

$$\lambda\Lambda_T(\widetilde{u}) + (1-\lambda)\Lambda_T(\widehat{u}) > \Lambda_T(\lambda\widetilde{u} + (1-\lambda)\widehat{u}). \tag{1.10}$$

Furthermore, the exponential function is both strictly increasing and strictly convex. By leveraging this property in conjunction with (1.10), $\forall\omega \in E$, we obtain

$$\lambda\exp(\delta\Lambda_T(\widetilde{u})) + (1-\lambda)\exp(\delta\Lambda_T(\widehat{u})) > \exp\delta\Lambda_T(\lambda\widetilde{u} + (1-\lambda)\widehat{u}). \tag{1.11}$$

Therefore, we have

$$\mathbb{E}\left[(\lambda \exp(\delta\Lambda_T(\widetilde{u})) + (1-\lambda)\exp(\delta\Lambda_T(\widehat{u})))\mathbb{I}_E\right] > \mathbb{E}\left[\exp \delta\Lambda_T(\lambda\widetilde{u} + (1-\lambda)\widehat{u})\mathbb{I}_E\right], \tag{1.12}$$

where $\mathbb{I}_E$ denotes an indicator function defined on E. Moreover, it is evident that, due to the convexity of Λ_T, we have $\forall\omega \in E^c$

$$\lambda \exp(\delta\Lambda_T(\widetilde{u})) + (1-\lambda)\exp(\delta\Lambda_T(\widehat{u})) \geq \exp \delta\Lambda_T(\lambda\widetilde{u} + (1-\lambda)\widehat{u}). \tag{1.13}$$

Hence, we obtain

$$\mathbb{E}\left[(\lambda \exp(\delta\Lambda_T(\widetilde{u})) + (1-\lambda)\exp(\delta\Lambda_T(\widehat{u})))\mathbb{I}_{E^c}\right] \geq \mathbb{E}\left[\exp \delta\Lambda_T(\lambda\widetilde{u} + (1-\lambda)\widehat{u})\mathbb{I}_{E^c}\right]. \tag{1.14}$$

Finally, the strict convexity of the cost functional given by (1.2)-(1.3) is established by adding together (1.12) and (1.14). □

The following theorem provides the Gâteaux derivative of the cost functional for the system described by (1.1)-(1.3), which will later be used to derive the corresponding optimality condition.

Theorem 2. *The Gâteaux derivative of the cost functional* (1.2)-(1.3) *in an arbitrary direction* $\omega \in \mathscr{U}$ *is given by*

$$\begin{aligned}\langle \mathscr{D}J(u), \omega\rangle &= \lim_{\varepsilon\to 0}\frac{J(u+\varepsilon\omega) - J(u)}{\varepsilon} \\ &= \delta\mathbb{E}\left[\int_0^T \omega^\intercal \Big[B^\intercal e^{-A^\intercal t}M_{2,t} + M_{1,t}(Ru_t + S^\intercal x_t - \zeta \right. \\ &\qquad \left. + B^\intercal \int_0^t e^{A^\intercal(s-t)}(Qx_s + Su_s - \eta)ds)\Big]dt\right],\end{aligned} \tag{1.15}$$

where

$$M_{1,t}(u) = \mathbb{E}\left[e^{\delta\Lambda_T(u)}\Big|\mathscr{F}_t\right], \tag{1.16}$$

$$M_{2,t}(u) = \mathbb{E}\left[e^{\delta\Lambda_T(u)}(e^{A^\intercal T}\widehat{Q}x_t + \int_0^T e^{A^\intercal s}(Qx_s + Su_s - \eta)ds)\Big|\mathscr{F}_t\right]. \tag{1.17}$$

Proof. First, the strong solution x_t to the SDE (1.1) under the control action u_t is given by

$$x_t = e^{At}x_0 + \int_0^t e^{A(t-s)}(Bu_s + b(s))ds + \int_0^t e^{A(t-s)}\sigma(s)dw_s. \tag{1.18}$$

Subsequently, the solution x_t^ε under the perturbed control action $u_t^\varepsilon := u_t + \varepsilon\omega_t$ is given by

$$x_t^\varepsilon = e^{At}x_0 + \int_0^t e^{A(t-s)}(Bu_s + b(s))ds + \int_0^t e^{A(t-s)}\sigma(s)dw_s + \varepsilon\int_0^t e^{A(t-s)}B\omega_s ds. \tag{1.19}$$

From (1.18)-(1.19), we have

$$x_t^\varepsilon = x_t + \varepsilon\int_0^t e^{A(t-s)}B\omega_s ds. \tag{1.20}$$

By a direct computation, we obtain

$$\begin{aligned}\left\langle \widehat{Q}x_T^\varepsilon, x_T^\varepsilon\right\rangle - \left\langle \widehat{Q}x_T, x_T\right\rangle &= 2\varepsilon\left\langle \widehat{Q}x_T, \int_0^T e^{A(T-s)}B\omega_s ds\right\rangle + \varepsilon^2\left\langle \widehat{Q}\int_0^T e^{A(T-s)}B\omega_s ds, \int_0^T e^{A(T-s)}B\omega_s ds\right\rangle \\ &= 2\varepsilon\int_0^T \left\langle B^\intercal e^{A^\intercal(T-s)}\widehat{Q}x_T, \omega_s\right\rangle ds + \varepsilon^2\left\langle \widehat{Q}\int_0^T e^{A(T-s)}B\omega_s ds, \int_0^T e^{A(T-s)}B\omega_s ds\right\rangle.\end{aligned} \tag{1.21}$$

Finally, from (1.3) and (1.18)-(1.21), we can compute the difference

$$\begin{aligned}\Lambda_T(u+\varepsilon\omega) - \Lambda_T(u) = \varepsilon\Bigg[\int_0^T \Bigg\{ \left(\int_0^s e^{A(s-t)}B\omega_t dt\right)^\intercal (Qx_s + Su_s - \eta)\, ds + \Big(\langle R\omega_s, u_s\rangle \\ + \langle S\omega_s, x_s\rangle - \langle \zeta, \omega_s\rangle + \left\langle B^\intercal e^{A^\intercal(T-s)}\widehat{Q}x_T, \omega_s\right\rangle\Big) ds\Bigg\}\Bigg] + \varepsilon^2\Bigg[\int_0^T \Bigg\{ \left(\int_0^s e^{A(s-t)}B\omega_t dt\right)^\intercal \\ \times \left(Q\int_0^s e^{A(s-t)}B\omega_t dt + S\omega_s\right) + \langle R\omega_s, \omega_s\rangle \Bigg\} ds + \left\langle \widehat{Q}\int_0^T e^{A(T-s)}B\omega_s ds, \int_0^T e^{A(T-s)}B\omega_s ds\right\rangle\Bigg],\end{aligned} \tag{1.22}$$

which equivalently may be expressed as

$$\Lambda_T(u+\varepsilon\omega) - \Lambda_T(u) = \varepsilon\iota_1 + \varepsilon^2\iota_2, \tag{1.23}$$

where the random variables ι_1, ι_2 do not depend on ε and are $\mathbb{P}$-almost surely finite. Therefore, we have

$$\langle \mathscr{D}J(u), \omega\rangle = \lim_{\varepsilon\to 0}\frac{J(u+\varepsilon\omega) - J(u)}{\varepsilon} = \lim_{\varepsilon\to 0}\mathbb{E}\left[e^{\delta\Lambda_T(u)}\left(\frac{e^{\delta\Lambda_T(u+\varepsilon\omega)-\delta\Lambda_T(u)} - 1}{\varepsilon}\right)\right]. \tag{1.24}$$

By assuming the interchangeability of the limit and the expectation[2], and applying L'Hôpital's rule as in

$$\lim_{\varepsilon\to 0}\frac{e^{\delta\Lambda_T(u+\varepsilon\omega)-\delta\Lambda_T(u)} - 1}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{e^{\delta\varepsilon\iota_1 + \delta\varepsilon^2\iota_2} - 1}{\varepsilon} = \delta\iota_1, \tag{1.25}$$

[2] See Remark 3.

we obtain

$$\langle \mathscr{D}J(u),\omega\rangle = \delta\mathbb{E}[e^{\delta\Lambda_T(u)}\iota_1] = \delta\mathbb{E}e^{\delta\Lambda_T(u)}\int_0^T\left[\left\{\int_0^s \omega_t^\intercal B^\intercal e^{A^\intercal(s-t)}(Qx_s+Su_s-\eta)dt\right.\right. \tag{1.26}$$
$$\left.\left. +\,\omega_s^\intercal\left(B^\intercal e^{A^\intercal(T-s)}\widehat{Q}x_T+Ru_s+S^\intercal x_s-\zeta\right)\right\}ds\right].$$

After changing the order of the double integral using Fubini's theorem, (1.26) is equivalent to

$$\langle \mathscr{D}J(u),\omega\rangle = \delta\mathbb{E}\left[\int_0^T \omega_t^\intercal e^{\delta\Lambda_T(u)}\Big[B^\intercal e^{A^\intercal(T-t)}\widehat{Q}x_T+Ru_t+S^\intercal x_t-\zeta\right. \tag{1.27}$$
$$\left. +\,B^\intercal\int_t^T e^{A^\intercal(s-t)}(Qx_s+Su_s-\eta)ds\Big]dt\right]$$
$$=\delta\mathbb{E}\left[\int_0^T \omega_t^\intercal\Big[B^\intercal e^{\delta\Lambda_T(u)}(e^{A^\intercal(T-t)}\widehat{Q}x_T+\int_0^T e^{A^\intercal(s-t)}(Qx_s+Su_s-\eta)ds)\right.$$
$$\left. +\,e^{\Lambda_T(u)}(Ru_t+S^\intercal x_t-\zeta)-B^\intercal e^{\Lambda_T(u)}\int_0^t e^{A^\intercal(s-t)}(Qx_s+Su_s-\eta)ds\Big]dt\right].$$

Finally, using the smoothing property of conditional expectations, (1.27) can be written as in (1.15), where $M_{1,t}(u)$ and $M_{2,t}(u)$ are defined by (1.16) and (1.17). □

Since $J(u)$ is strictly convex, u is the unique minimizer of the risk-sensitive cost functional if $\langle \mathscr{D}J(u),\omega\rangle = 0$ for all $\omega\in\mathscr{U}$ (see Ciarlet (2013)). We observe that (1.15) takes the form of an inner product. Note that the control action u_t° satisfying

$$u_t^\circ = -R^{-1}\left[S^\intercal x_t-\zeta+B^\intercal\left(e^{-A^\intercal t}\frac{M_{2,t}(u^\circ)}{M_{1,t}(u^\circ)}-\int_0^t e^{A^\intercal(s-t)}(Qx_s+Su_s^\circ-\eta)ds\right)\right], \tag{1.28}$$

ensures that $\langle \mathscr{D}J(u^\circ),\omega\rangle = 0$, $\mathbb{P}$-a.s., where the martingale $M_{1,t}(u^\circ)$ characterized by (1.16) is almost surely positive. Therefore, $J(u)$ is Gâteaux differentiable at u° and since $\langle \mathscr{D}J(u^\circ),\omega\rangle = 0$ for all $\omega\in\mathscr{U}$, u° is the unique minimizer of $J(u)$. In other words, (1.28) represents the necessary and sufficient condition of optimality for the risk-sensitive optimal control problem described by (1.1)-(1.3). However, the current characterization of u° is not practical for implementation purposes. Due to the presence of the term $\frac{M_{2,t}(u^\circ)}{M_{1,t}(u^\circ)}$, obtaining a state feedback form directly from (1.28) is challenging. To address this issue, we employ a change of probability measure by applying the Girsanov theorem under certain conditions, as stated in the following lemma.

Before presenting the lemma, we introduce some notations. We denote by $\mathscr{S}^n$ the set of all $n\times n$ symmetric matrices and by $C(\mathfrak{T},\mathscr{S}^n)$ the set all continuous $\mathscr{S}^n$ valued functions on $\mathfrak{T}$. Similarly, we denote by $C(\mathfrak{T},\mathbb{R}^n)$ the set all continuous $\mathbb{R}^n$ valued functions.

Lemma 3. *Consider the control action given by*

$$u^*(t) = -R^{-1}\left[S^\intercal x_t - \zeta + B^\intercal\left(\Pi(t)x_t + s(t)\right)\right], \tag{1.29}$$

where $\Pi\in C(\mathfrak{T},\mathscr{S}^n)$ *and* $s\in C(\mathfrak{T},\mathbb{R}^n)$ *satisfy the ODEs*

$$\begin{aligned}&\dot{\Pi}(t)+\Pi(t)A+A^\intercal\Pi(t)-(\Pi(t)B+S)R^{-1}(B^\intercal\Pi(t)+S^\intercal) \\ &\qquad\qquad +Q+\delta\Pi(t)\sigma(t)\sigma^\intercal(t)\Pi(t)=0, \quad \Pi(T)=\widehat{Q},\end{aligned} \tag{1.30}$$

$$\begin{aligned}&\dot{s}(t)+\left[A^\intercal-\Pi(t)BR^{-1}B^\intercal-SR^{-1}B^\intercal+\delta\Pi(t)\sigma(t)\sigma^\intercal(t)\right]s(t) \\ &\qquad\qquad +\Pi(t)(b(t)+BR^{-1}\zeta)+SR^{-1}\zeta-\eta=0, \quad s(T)=0.\end{aligned} \tag{1.31}$$

Moreover, let C_T^* *be defined by*

$$\begin{aligned}C_T^* =&\frac{\delta}{2}\int_0^T\left(2\langle s(t),b(t)\rangle-\left\langle R^{-1}(B^\intercal s(t)-\zeta),B^\intercal s(t)-\zeta\right\rangle+\mathrm{tr}(\Pi(t)\sigma(t)\sigma^\intercal(t))\right)dt \\ &+\frac{\delta^2}{2}\int_0^T\|\sigma^\intercal(t)s(t)\|^2dt+\frac{\delta}{2}\langle\Pi(0)x_0,x_0\rangle+\delta\langle s(0),x_0\rangle.\end{aligned} \tag{1.32}$$

Then, the random variable $\exp(\delta\Lambda_T(u^*)-C_T^*)$*, where* $\Lambda_T(u^*)$ *is given by* (1.3) *under the control* (1.29)*, is a Radon-Nikodym derivative*

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}}=\exp\left(\delta\Lambda_T(u^*)-C^*(T)\right), \tag{1.33}$$

defining a probability measure $\widehat{\mathbb{P}}$ *equivalent to* $\mathbb{P}$. *Further, the Radon-Nikodym derivative can be represented by*

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}}:=\exp\left(-\frac{\delta^2}{2}\int_0^T\|\gamma_t\|^2dt+\delta\int_0^T\gamma_t^\intercal dw_t\right), \tag{1.34}$$

$$\gamma_t=\sigma^\intercal(\Pi(t)x_t+s(t)). \tag{1.35}$$

Proof. We first define the following quadratic functional of the state process

$$\frac{\delta}{2}\langle\Pi(t)x_t,x_t\rangle+\delta\langle s(t),x_t\rangle, \tag{1.36}$$

with deterministic coefficients $\Pi(t)$ and $s(t)$, drawing inspiration from Duncan (2013). We then apply Itô's lemma to (1.36) and integrate both sides from 0 to T to get

$$\begin{aligned}\frac{\delta}{2}\langle\Pi(T)x_T,x_T\rangle+\delta\langle s(T),x_T\rangle-\frac{\delta}{2}\langle\Pi(0)x_0,x_0\rangle-\delta\langle s(0),x_0\rangle\\ =\frac{\delta}{2}\int_0^T\Big(\big\langle\dot{\Pi}(t)x_t,x_t\big\rangle+\langle(\Pi(t)A+A^{\intercal}\Pi(t))x_t,x_t\rangle+2\langle B^{\intercal}\Pi(t)x_t,u_t\rangle\\ +2\langle\Pi(t)x_t,b(t)\rangle+2\langle\dot{s}(t),x_t\rangle\rangle+2\langle s(t),Ax_t+Bu_t+b(t)\rangle\\ +\operatorname{tr}(\Pi(t)\sigma(t)\sigma(t)^{\intercal})\Big)dt+\frac{\delta}{2}\int_0^T 2\sigma(t)^{\intercal}(\Pi(t)x_t+s(t))dw_t.\end{aligned}\tag{1.37}$$

Next, we add $\delta\Lambda_T(u)$ to both sides of (1.37), yielding

$$\begin{aligned}\delta\Lambda_T(u)-\frac{\delta}{2}\langle\Pi(0)x_0,x_0\rangle-\delta\langle s(0),x_0\rangle=\frac{\delta}{2}\int_0^T\Big(\langle QX_t,X_t\rangle+2\langle Su_t,x_t\rangle+\langle Ru_t,u_t\rangle-2\langle\eta,x_t\rangle\\ -2\langle\zeta,u_t\rangle\Big)dt+\frac{\delta}{2}\Big\langle\widehat{Q}X_T,X_T\Big\rangle-\frac{\delta}{2}\langle\Pi(T)x_T,x_T\rangle-\delta\langle s(T),x_T\rangle+\frac{\delta}{2}\int_0^T\Big(\big\langle\dot{\Pi}(t)x_t,x_t\big\rangle\\ +\langle(\Pi(t)A+A^{\intercal}\Pi(t))x_t,x_t\rangle+2\langle B^{\intercal}\Pi(t)x_t,u_t\rangle+2\langle\Pi(t)x_t,b(t)\rangle+2\langle\dot{s}(t),x_t\rangle\\ +2\langle s(t),Ax_t+Bu_t+b(t)\rangle+\operatorname{tr}(\Pi(t)\sigma(t)\sigma(t)^{\intercal})\Big)dt+\frac{\delta}{2}\int_0^T 2\sigma(t)^{\intercal}(\Pi(t)x_t+s(t))dw_t.\end{aligned}\tag{1.38}$$

Subsequently, we substitute (1.29) in (1.38) and reorganize the terms to represent $\delta\Lambda_T(u)$ as

$$\delta\Lambda_T(u)=-\frac{\delta^2}{2}\int_0^T\|\gamma_t\|^2dt+\delta\int_0^T\gamma_t^{\intercal}dw_t+C_T^*+\frac{1}{2}\Phi_T,\tag{1.39}$$

where γ_t and C_T^* are, respectively, given by (1.35) and (1.32), and Φ_T is a random variable defined by

$$\begin{aligned}\Phi_T=\delta\int_0^T\Big\langle\Big(\dot{\Pi}(t)+Q+A^{\intercal}\Pi(t)+\Pi(t)A-(\Pi(t)B+S)R^{-1}(B^{\intercal}\Pi(t)+S^{\intercal})\\ +\delta\Pi(t)\sigma(t)\sigma^{\intercal}(t)\Pi(t)\Big)x_t,x_t\Big\rangle+\Big\langle\dot{s}(t)+(A^{\intercal}-\Pi(t)BR^{-1}B^{\intercal}-SR^{-1}B^{\intercal}+\delta\Pi(t)\sigma(t)\sigma^{\intercal}(t))s(t)\\ +\Pi(t)(b(t)+BR^{-1}\zeta)+SR^{-1}\zeta-\eta,x_t\Big\rangle dt+\delta\Big\langle\widehat{Q}x_T,x_T\Big\rangle-\delta\langle\Pi(T)x_T,x_T\rangle-2\delta\langle s(T),x_T\rangle.\end{aligned}\tag{1.40}$$

From (1.39), we observe that $\exp(\delta\Lambda_T(u)-C_T^*)$ follows the structure of a Radon-Nikodym derivative if $\Phi_T=0,\mathbb{P}-a.s.$ The above condition is fulfilled if $\Pi(t)$ and $s(t)$, respectively, satisfy (1.30) and (1.31), and if (1.29) holds. Note that, after applying (1.29), the dynamics (1.1) under the $\widehat{\mathbb{P}}$ measure become

$$dx_t=\Big(\widehat{A}(t)x_t+\widehat{b}(t)\Big)dt+\sigma(t)d\widehat{w}_t,\tag{1.41}$$

where $\widehat{A}(t) := A - BR^{-1}(S^\intercal + B^\intercal \Pi(t)) + \delta\sigma(t)\sigma^\intercal(t)\Pi(t)$ and $\widehat{b}(t) := -BR^{-1}(B^\intercal s(t) - \zeta) + b(t) + \delta\sigma(t)\sigma^\intercal(t)s(t)$. The strong solution of the linear SDE (1.41) is given by

$$x_t = \Upsilon_{\widehat{A}}(t)x_0 + \Upsilon_{\widehat{A}}(t)\int_0^t \Upsilon_{\widehat{A}}^{-1}(s)\widehat{b}(t)ds + \Upsilon_{\widehat{A}}(t)\int_0^t \Upsilon_{\widehat{A}}^{-1}(s)\sigma(s)d\widehat{w}_s, \tag{1.42}$$

where $\Upsilon_{\widehat{A}}(t)$ and $\Upsilon_{\widehat{A}}^{-1}(t)$ are, respectively, the solutions of the following matrix-valued ODEs

$$\begin{aligned} \dot{\Upsilon}_{\widehat{A}}(t) &= \widehat{A}(t)\Upsilon_{\widehat{A}}(t)dt, \quad \Upsilon_{\widehat{A}}(0) = I, \\ \dot{\Upsilon}_{\widehat{A}}^{-1}(t) &= -\Upsilon_{\widehat{A}}^{-1}(t)\widehat{A}(t)dt, \quad \Upsilon_{\widehat{A}}^{-1}(0) = I, \end{aligned} \tag{1.43}$$

and satisfy $\Upsilon_{\widehat{A}}(t)\Upsilon_{\widehat{A}}^{-1}(t) = I$ (Yong and Zhou (1999)). Then, one can verify that the affine structure $\gamma_t = \sigma^\intercal(\Pi(t)x_t + s(t))$ ensures that the expression

$$\exp(\delta\Lambda_T(u^*) - C_T^*) = \exp(-\frac{\delta^2}{2}\int_0^T \|\gamma_t\|^2 dt + \delta\int_0^T \gamma_t^\intercal dw_t),$$

represents a Radon-Nikodym derivative (see (Karatzas and Shreve, 1991, Corollary 5.14, page 199)). Hence, $\exp(\delta\Lambda_T(u) - C_T^*)$, defines the probability measure $\widehat{\mathbb{P}}$ equivalent to $\mathbb{P}$. □

From Lemma 3, under the control action u^*, the process $\widehat{M}_t(u^*) := \frac{M_{2,t}(u^*)}{M_{1,t}(u^*)}$, where $M_{1,t}(u^*)$ and $M_{2,t}(u^*)$ are, respectively, given by (1.16) and (1.17), is a $\widehat{\mathbb{P}}$-martingale represented by

$$\widehat{M}_t(u^*) = \widehat{\mathbb{E}}\left[e^{A^\intercal T}\widehat{Q}x_T + \int_0^T e^{A^\intercal s}(Qx_s + Su_s^* - \eta)ds \middle| \mathscr{F}_t\right], \tag{1.44}$$

which is obtained through the corresponding change of measure.

Proposition 4. *Let u_t^*, $\Pi(t)$ and $s(t)$ be as defined in Lemma 3, then*

$$\Pi(t)x_t + s(t) = e^{-A^\intercal t}\widehat{M}_t(u^*) - \int_0^t e^{A^\intercal(s-t)}(Qx_s + Su_s^* - \eta)ds, \quad \widehat{\mathbb{P}} - a.s. \tag{1.45}$$

Proof. By inspection, (1.45) holds at the terminal time T. More specifically, substituting $\widehat{M}_T(u^*)$ from (1.44) in the right-hand side of (1.45) results in

$$\widehat{Q}x_T = e^{-A^\intercal T}\widehat{M}_T(u^*) - \int_0^T e^{A^\intercal(s-T)}(Qx_s + Su_s^* - \eta)ds, \quad \widehat{\mathbb{P}} - a.s., \tag{1.46}$$

which is $\widehat{\mathbb{P}} - a.s.$ equal to $\Pi(T)x_t + s(T)$ according to (1.30) and (1.31). Hence, in order to establish the validity of (1.45) for all $t \in \mathfrak{T}$, it suffices to demonstrate that the infinitesimal variations of both sides of the equation are $\widehat{\mathbb{P}}$-almost surely equal.

By the martingale representation theorem, $\widehat{M}_t(u)$ may be expressed as

$$\widehat{M}_t(u^*) = \widehat{M_0} + \int_0^t Z(s)d\widehat{w}_s, \quad \widehat{\mathbb{P}} - a.s. \tag{1.47}$$

We apply Itô's lemma to both sides of (1.45) and substitute (1.1), (1.29), (1.47) as required. For the resulting drift and diffusion coefficients to be equal on both sides, the equations

$$\Big(\dot{\Pi}(t) + Q + A^{\intercal}\Pi(t) + \Pi(t)A - (\Pi(t)B + S)R^{-1}(B^{\intercal}\Pi(t) + S^{\intercal}) + \delta\Pi(t)\sigma(t)\sigma^{\intercal}(t)\Pi(t)\Big)x_t + \Big(\dot{s}(t) \tag{1.48}$$

$$+ (A^{\intercal} - \Pi(t)BR^{-1}B^{\intercal} - SR^{-1}B^{\intercal} + \delta\Pi(t)\sigma(t)\sigma^{\intercal}(t))s(t) + \Pi(t)(b(t) + BR^{-1}\zeta) + SR^{-1}\zeta - \eta\Big) = 0,$$

and

$$\Pi(t)\sigma(t) = e^{-A^{\intercal}t}Z(t), \quad \widehat{\mathbb{P}} - a.s. \tag{1.49}$$

must hold for all $t \in \mathfrak{T}$. It is evident that (1.48) holds if (1.30) and (1.31) hold. It remains to demonstrate that (1.49) holds subsequently for all $t \in \mathfrak{T}$.

To determine $Z(t)$, we substitute (1.29) and (1.42) in (1.44) and equate the stochastic components of the resulting equation with those of (1.47). This leads to

$$\int_0^t Z(s)d\widehat{w}_s = \widehat{\mathbb{E}}\Bigg[\underbrace{e^{A^{\intercal}T}\widetilde{Q}\Upsilon_{\widehat{A}}(T)\int_0^T \Upsilon_{\widehat{A}}^{-1}(s)\sigma(s)dw_s}_{D_1} + \underbrace{\int_0^T e^{A^{\intercal}s}\widetilde{Q}(s)\Upsilon_{\widehat{A}}(s)\int_0^s \Upsilon_{\widehat{A}}^{-1}(r)\sigma(r)dw_r ds}_{D_2}\Bigg|\mathscr{F}_t\Bigg], \tag{1.50}$$

where $\widetilde{Q}(t) = Q - SR^{-1}(S^{\intercal} + B^{\intercal}\Pi(t))$. Further, let $\mathfrak{L}_1(t) = e^{A^{\intercal}t}\widetilde{Q}(t)\Upsilon_{\widehat{A}}(t)$, $\mathfrak{L}_2(t) = \Upsilon_{\widehat{A}}^{-1}(t)\sigma(t)$, and rewrite $\widehat{\mathbb{E}}[D_2|\mathscr{F}_t]$ in (1.50) as

$$\begin{aligned}
\widehat{\mathbb{E}}[D_2|\mathscr{F}_t] &= \widehat{\mathbb{E}}\left[\int_0^T \mathfrak{L}_1(s)\int_0^s \mathfrak{L}_2(r)dw_r ds\middle|\mathscr{F}_t\right] = \int_0^T \mathfrak{L}_1(s)\widehat{\mathbb{E}}\left[\int_0^s \mathfrak{L}_2(r)d\widehat{w}_r\middle|\mathscr{F}_t\right]ds \\
&= \int_0^t \mathfrak{L}_1(s)\widehat{\mathbb{E}}\left[\int_0^s \mathfrak{L}_2(r)d\widehat{w}_r\middle|\mathscr{F}_t\right]ds + \int_t^T \mathfrak{L}_1(s)\widehat{\mathbb{E}}\left[\int_0^s \mathfrak{L}_2(r)d\widehat{w}_r\middle|\mathscr{F}_t\right]ds \\
&= \int_0^t \mathfrak{L}_1(s)\left[\int_0^s \mathfrak{L}_2(r)d\widehat{w}_r\right]ds + \int_t^T \mathfrak{L}_1(s)\left[\int_0^t \mathfrak{L}_2(r)d\widehat{w}_r\right]ds \\
&= \int_0^t\int_r^t \mathfrak{L}_1(s)ds\mathfrak{L}_2(r)d\widehat{w}_r + \int_0^t\int_t^T \mathfrak{L}_1(s)ds\mathfrak{L}_2(r)d\widehat{w}_r \qquad \text{(change order of integration)} \\
&= \int_0^t\int_r^T \mathfrak{L}_1(s)ds\mathfrak{L}_2(r)d\widehat{w}_r,
\end{aligned} \tag{1.51}$$

where the fourth equality holds due to the measurability and martingale property of $\int_0^s \mathfrak{L}_2(r)d\widehat{w}_r$, $s \in \mathfrak{T}$. Moreover, due to the martingale property, we can rewrite $\widehat{\mathbb{E}}[D_1|\mathscr{F}_t]$ in (1.50) as

$$\widehat{\mathbb{E}}[D_1|\mathscr{F}_t] = \widehat{\mathbb{E}}\left[e^{A^\intercal T}\widehat{Q}\Upsilon_{\widehat{A}}(T)\int_0^T \Upsilon_{\widehat{A}}^{-1}(s)\sigma(s)dw_s\middle|\mathscr{F}_t\right] = e^{A^\intercal T}\widehat{Q}\Upsilon_{\widehat{A}}(T)\int_0^t \mathfrak{L}_2(s)dw_s. \tag{1.52}$$

Since the martingale representation theorem ensures the uniqueness of the expression $Z(t)$, from (1.50)-(1.52), we conclude that

$$Z(t) = \int_t^T \mathfrak{L}_1(r)dr\mathfrak{L}_2(t) + e^{A^\intercal T}\widehat{Q}\Upsilon_{\widehat{A}}(T)\mathfrak{L}_2(t), \quad \forall t \in \mathfrak{T}. \tag{1.53}$$

We proceed by using the representation $e^{-A^\intercal t}Z(t) = \widetilde{Z}(t)\sigma(t)$, where

$$\widetilde{Z}(t) := e^{-A^\intercal t}(\int_t^T \mathfrak{L}_1(s)ds\Upsilon_{\widehat{A}}^{-1}(t) + e^{A^\intercal T}\widehat{Q}\Upsilon_{\widehat{A}}(T)\Upsilon_{\widehat{A}}^{-1}(t)) = \Pi(t), \quad \widehat{\mathbb{P}} - a.s. \tag{1.54}$$

for all $t \in \mathfrak{T}$. We will now demonstrate that $\widetilde{Z}(t) = \Pi(t)$, which verifies equation (1.49). Since $\widetilde{Z}(T) = \widehat{Q} = \Pi(T)$, it is enough to show that $\widetilde{Z}(t)$ and $\Pi(t)$ satisfy the same ODE. From (1.54), we have

$$\dot{\widetilde{Z}}(t) = -A^\intercal\widetilde{Z}(t) + e^{-A^\intercal t}\left(-\mathfrak{L}_1(t)\Upsilon_{\widehat{A}}^{-1}(t) - \int_t^T \mathfrak{L}_1(s)ds\Upsilon_{\widehat{A}}^{-1}(t)\widehat{A}(t) - e^{A^\intercal T}\widehat{Q}\Upsilon_{\widehat{A}}(T)\Upsilon_{\widehat{A}}^{-1}(t)\widehat{A}(t)\right) \tag{1.55}$$

$$\begin{aligned}
&= -A^\intercal\widetilde{Z}(t) - \widetilde{Q}(t) - \widetilde{Z}(t)\widehat{A}(t) \\
&= -A^\intercal\widetilde{Z}(t) - Q + SR^{-1}(S^\intercal + B^\intercal\Pi(t)) - \widetilde{Z}(t)(A - BR^{-1}(S^\intercal + B^\intercal\Pi(t)) - \delta\sigma(t)\sigma^\intercal(t)\Pi(t)) \\
&= -A^\intercal\widetilde{Z}(t) - \widetilde{Z}(t)A - Q + (\widetilde{Z}(t)B + S)R^{-1}(B^\intercal\Pi(t) + S^\intercal) - \delta\widetilde{Z}(t)\sigma(t)\sigma^\intercal(t)\Pi(t),
\end{aligned}$$

which is a linear ODE of the Sylvester type with $\Pi(t)$ fixed as the solution of (1.30). This ODE admits a unique solution that coincides with $\Pi(t)$ (see, e.g., Behr et al. (2019).). □

The following theorem demonstrates that the control action u^* given by (1.29)-(1.31) is indeed the optimal control action for the LQG risk-sensitive system under $\mathbb{P}$.

Theorem 5. *Suppose a1.3.1 holds. Then u^* given by* (1.29)-(1.31) *is the unique optimal control action for the LQG risk-sensitive optimal control problem described by* (1.1)-(1.3).

Proof. We first show that the control action u^* given by (1.29)-(1.31) satisfies the necessary and sufficient optimality condition given by (1.28). We have

$$u_t^* = -R^{-1}\left[S^\intercal x_t - \zeta + B^\intercal\left(\Pi(t)x_t + s(t)\right)\right] \tag{1.56}$$

$$= -R^{-1}\left[S^\intercal x_t - \zeta + B^\intercal \left(e^{-A^\intercal t}\widehat{M}_t(u^*) - \int_0^t e^{A^\intercal(s-t)}(Qx_s + Su_s^* - \eta)ds\right)\right] \text{(by Proposition 4)}$$
$$= -R^{-1}\left[S^\intercal x_t - \zeta + B^\intercal \left(e^{-A^\intercal t}\frac{M_{2,t}(u^*)}{M_{1,t}(u^*)} - \int_0^t e^{A^\intercal(s-t)}(Qx_s + Su_s^* - \eta)ds\right)\right],\ \widehat{\mathbb{P}} - a.s.,$$

where the last equality is a direct result of Lemma 3. Finally, due to the equivalence of $\widehat{\mathbb{P}}$ and $\mathbb{P}$, we have

$$u_t^* = -R^{-1}\left[S^\intercal x_t - \zeta + B^\intercal \left(e^{-A^\intercal t}\frac{M_{2,t}(u^*)}{M_{1,t}(u^*)} - \int_0^t e^{A^\intercal(s-t)}(Qx_s + Su_s^* - \eta)ds\right)\right],\ \mathbb{P} - a.s. \quad (1.57)$$

Hence, u^* is an optimal control action for the system described by (1.1)-(1.3). The uniqueness of u^* as the optimal control action is established due to the strict convexity of the cost functional (1.2)-(1.3), as demonstrated in Proposition 1. □

Remark 1. *(Risk-Neutral Probability Measure) The probability measure $\widehat{\mathbb{P}}$ may be termed risk-neutral because, under this measure, the necessary and sufficient optimality condition for risk-sensitive LQG optimal control problems described by* (1.1)-(1.3) *is expressed as*

$$u_t^* = -R^{-1}\left[S^\intercal x_t - \zeta + B^\intercal \left(e^{-A^\intercal t}\widehat{M}_t(u^*) - \int_0^t e^{A^\intercal(s-t)}(Qx_s + Su_s^* - \eta)ds\right)\right], \quad (1.58)$$

with $\widehat{M}_t$ given by (1.44)*, which is similar to the optimality condition of risk-neutral LQG optimal control problems as detailed in (Firoozi et al., 2020, eq (24)). More specifically, under this measure, the optimality condition of the risk-sensitive optimal control problems described by* (1.1)-(1.3) *has the same structure as that of risk-neutral optimal control problems described by*

$$dx_t = (Ax_t + Bu_t + b(t))dt + \sigma dw_t, \quad (1.59)$$

$$J(u) = \mathbb{E}\left[\Lambda_T(u)\right], \quad (1.60)$$

where $\Lambda_T(u)$ is given by (1.3).

The results obtained in this section can be readily extended to the case where the system matrix is time varying. The following remark provides a summary of this extension, which will be used in Section 1.4.

Remark 2. *(Time-varying system matrix $A(t)$) Consider the system described by the dynamics*

$$dx_t = (A(t)x_t + Bu_t + b(t))dt + \sigma(t)dw_t, \quad (1.61)$$

where $A(t)$ is a continuous function on $\mathfrak{T}$, and the cost functional is given by (1.2)-(1.3)*. With some slight modifications, the Gâteaux derivative of the cost functional is given by*

$$\langle \mathscr{D}J(u),\omega\rangle = \delta\mathbb{E}\Bigg[\int_0^T \omega^\intercal[B^\intercal(\Upsilon_A^{-1}(t))^\intercal M_{2,t}(u)+M_{1,t}(u)(Ru_t+S^\intercal x_t-\zeta \\ +B^\intercal(\Upsilon_A^{-1}(t))^\intercal\int_0^t(\Upsilon_A(s))^\intercal(Qx_s+Su_s-\eta)ds)]dt\Bigg], \tag{1.62}$$

where $\Upsilon_A(t)$ and $\Upsilon_A^{-1}(t)$ are defined in the same way as in (1.43)*. The martingale term $M_{1,t}(u)$ is given by* (1.16) *and $M_{2,t}(u)$ by*

$$M_{2,t}(u) = \mathbb{E}\left[e^{\frac{\delta}{2}\Lambda_T(u)}((\Upsilon_A(T))^\intercal\widehat{Q}x_t+\int_0^T(\Upsilon_A(s))^\intercal(Qx_s+Su_s-\eta)ds)\Big|\mathscr{F}_t\right]. \tag{1.63}$$

Subsequently, the necessary and sufficient optimality condition for the control action u_t° is given by

$$u_t^\circ = -R^{-1}\left[S^\intercal x_t-\zeta+B^\intercal\left((\Upsilon_A^{-1}(t))^\intercal\frac{M_{2,t}(u^\circ)}{M_{1,t}(u^\circ)}-(\Upsilon_A^{-1}(t))^\intercal\int_0^t(\Upsilon_A(s))^\intercal(Qx_s+Su_s^\circ-\eta)ds\right)\right]. \tag{1.64}$$

By applying adapted versions of lemma 3, proposition 4, and theorem 5, it can be shown that the optimal control action is given by (1.29)-(1.31)*, where A is replaced with $A(t)$ in* (1.30)-(1.31)*.*

Remark 3 (Technical Comparison with Existing Methodologies)**.***The work Lim and Zhou (2005) develops a risk-sensitive maximum principle requiring that the running and terminal cost functionals in the exponent be uniformly bounded and Lipschitz continuous (see (Lim and Zhou, 2005, Assumption B2).). This condition is not automatically met for LQG risk-sensitive models with quadratic terminal and running costs in the exponent, unless the state and control spaces are restricted to compact sets. In Duncan (2013), a combination of completing the square and a Radon-Nikodym derivative is used to determine an optimal control for an LQG risk-sensitive problem. To verify the optimality of a candidate control, a perturbation process (see (Duncan, 2013, eq. (14))) is introduced over a specific subset of the time interval. This perturbation of the control action is a bounded process, although the admissible control set includes L^2 processes. The work Jacobson (1973) uses dynamic programming to obtain solutions to continuous-time risk-sensitive optimal control problems, where no verification theorem is presented. Similarly, Fleming and Soner (2006) employs a dynamic programming approach to address such risk-sensitive problems. However, the verification theorem in this work assumes that both the state and control spaces are bounded (See*

(Fleming and Soner, 2006, Thm. 8.2 & eq. (3.12))). The works Moon et al. (2018), Başar and Olsder (1998), and Moon and Başar (2019), which respectively study risk-sensitive two-player games and mean-field games, employ methodologies and impose conditions similar to those in Lim and Zhou (2005). They require uniform boundedness and Lipschitz continuity for terminal and running costs in the exponent. Furthermore, they state that restricting state and control spaces to sufficiently large compact subsets of Euclidean spaces is necessary for applying the methodology to LQG counterpart models (see (Moon et al., 2018, Sec. VII, footnote 8), (Başar and Olsder, 1998, Chap. 6), (Moon and Başar, 2019, Example 1).). Our variational approach takes advantage of the fact that the Gâteaux derivative may be computed explicitly for LQG risk-sensitive models. Given that the cost functional is strictly convex, this allows us to obtain the necessary and sufficient condition of optimality by setting the Gâteaux derivative to zero. However, the interchangeability of the limit and expectation is required to enable the calculation of the Gâteaux derivative (see (1.25)-(1.26) *in the proof of Theorem 2). Although boundedness of state and control processes, as assumed in the literature, provides a sufficient condition for this interchangeability, it is not a necessary condition.*

The variational analysis developed above will be employed in the next section to obtain the best-response strategies of major and minor agents in MFG systems.

1.4 Risk-Sensitive Major-Minor LQG Mean-Field Game Systems

1.4.1 Finite-Population Game

We consider a system that contains one major agent, who has a significant impact on other agents, and N minor agents, who individually have an asymptotically negligible impact on the system. Minor agents form K subpopulations, such that the agents in each subpopulation share the same model parameters. We define the index set $\mathscr{I}_k = \{i : \theta_i = \theta^{(k)}\}, k \in \mathscr{K} := \{1,\dots,K\}$, where $\theta^{(k)}$ denotes the model parameters of subpopulation k that will be introduced throughout this section. Moreover, we denote the empirical distribution of the parameters $(\theta^{(1)},\dots,\theta^{(K)})$ by $\pi^{(N)} = (\pi_1^{(N)}....\pi_K^{(N)})$, where $\pi_k^{(N)} = \frac{|\mathscr{I}_k|}{N}$ and $|\mathscr{I}_k|$ is the counting measure of $\mathscr{I}_k$.

The dynamics of the major agent and of a representative minor agent indexed by i in subpopulation k are, respectively, given by

$$dx_t^0 = (A_0 x_t^0 + F_0 x_t^{(N)} + B_0 u_t^0 + b_0(t))dt + \sigma_0(t) dw_t^0 \tag{1.65}$$

$$dx_t^i = (A_k x_t^i + F_k x_t^{(N)} + G_k x_t^0 + B_k u_t^i + b_k(t))dt + \sigma_k(t) dw_t^i \tag{1.66}$$

where $i \in \mathcal{N} = \{1, \dots, N\}$, $k \in \mathcal{K}$, and $t \in \mathcal{T}$. The state and the control action are denoted, respectively, by $x_t^i \in \mathbb{R}^n$ and $u_t^i \in \mathbb{R}^m$, $i \in \mathcal{N}_0 = \{0, 1, \dots, N\}$. Moreover, the processes $\{w^i \in \mathbb{R}^r, i \in \mathcal{N}_0\}$, are $(N+1)$ standard r-dimensional Wiener processes defined on the filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t^{(N)}\}_{t\in\mathcal{T}}, \mathbb{P}\right)$, where $\mathcal{F}_t^{(N)} := \sigma(x_0^i, w_s^i, i \in \mathcal{N}_0, s \le t)$. Finally, the volatility processes $\sigma_0(t), \sigma_k(t) \in \mathbb{R}^{n\times r}$ and the offset processes $b_0(t), b_k(t) \in \mathbb{R}^n$ are deterministic functions of time, while all other parameters (A_0, F_0, B_0), (A_k, F_k, B_k) are constants of an appropriate dimension.

The empirical average state $x_t^{(N)}$ of minor agents is defined by

$$x_t^{(N)} := \frac{1}{N} \sum_{i\in\mathcal{N}} x_t^i \tag{1.67}$$

where the same weight is assigned to each minor agent's state, implying that minor agents have a uniform impact on the system. Denoting $u^{-0} := (u^1, \dots, u^N)$ and $u^{-i} := (u^0, \dots, u^{i-1}, u^{i+1}, \dots, u^N)$, the major agent's cost functional is given by

$$\begin{aligned} J_0^{(N)}(u^0, u^{-0}) =& \mathbb{E}\bigg[\exp\bigg(\frac{\delta_0}{2}\left\langle \widehat{Q}_0(x_T^0 - \Phi_T^{(N)}), x_T^0 - \Phi_T^{(N)}\right\rangle + \frac{\delta_0}{2}\int_0^T \Big(\left\langle Q_0(x_t^0 - \Phi_t^{(N)}), x_t^0 - \Phi_t^{(N)}\right\rangle \\ &+ 2\left\langle S_0 u_t^0, x_t^0 - \Phi_t^{(N)}\right\rangle + \left\langle R_0 u_t^0, u_t^0\right\rangle\Big) dt\bigg)\bigg], \end{aligned} \tag{1.68}$$

where $\Phi_t^{(N)} := H_0 x_t^{(N)} + \eta_0$ and all parameters ($\widehat{Q}_0$, Q_0, S_0, H_0, $\widehat{H}_0$, R_0, η_0) are of an appropriate dimension.

A1.4.1. *$R_0 > 0$, $\widehat{Q}_0 \ge 0$, $Q_0 - S_0 R_0^{-1} S_0^\intercal \ge 0$, and $\delta_0 \in [0, \infty)$.*

For the representative minor agent i in subpopulation k, the cost functional is given by

$$\begin{aligned} J_i^{(N)}(u^i, u^{-i}) =& \mathbb{E}\bigg[\exp\bigg(\frac{\delta_k}{2}\left\langle \widehat{Q}_k(x_T^i - \Psi_T^{(N)}), x_T^i - \Psi_T^{(N)}\right\rangle + \frac{\delta_k}{2}\int_0^T \Big(\left\langle Q_k(x_t^i - \Psi_t^{(N)}), x_t^i - \Psi_t^{(N)}\right\rangle \\ &+ 2\left\langle S_k u_t^i, x_t^i - \Psi_T^{(N)}\right\rangle + \left\langle R_k u_t^i, u_t^i\right\rangle\Big) dt\bigg)\bigg] \end{aligned} \tag{1.69}$$

where $\Psi_t^{(N)} := H_k x_t^0 + \widehat{H}_k x_t^{(N)} + \eta_k$ and all parameters ($\widehat{Q}_k$, Q_k, S_k, H_k, $\widehat{H}_k$, R_k, η_k) are of an appropriate dimension.

A1.4.2. $R_k > 0$, $\widehat{Q}_k \geq 0$, $Q_k - S_k R_k^{-1} S_k^\intercal \geq 0$, *and* $\delta_k \in [0,\infty)$, $\forall k \in \mathscr{K}$.

Under Assumptions 1.4.1–1.4.2, (1.68) and (1.69) are strictly convex.

From (1.65)-(1.69), the dynamics and cost functionals of both the major agent and the representative minor agent-i are influenced by the empirical average state $x_t^{(N)}$. Moreover, the representative minor agent's model is also influenced by the major agent's state x_t^0.

For both the major agent and the representative minor agent-i, an admissible set $\mathscr{U}^g$ of control actions consists of all $\mathbb{R}^m$-valued $\mathscr{F}_t^{(N)}$-adapted processes u_t^i, $i \in \mathscr{N}_0$, such that $\mathbb{E}\left[\int_0^T \left\|u_t^i\right\|^2 dt\right] < \infty$.

In general, solving the N-player differential game described in this section becomes challenging, even for moderate values of N. The interactions between agents lead to a high-dimensional optimization problem, where each agent needs to observe the states of all other interacting agents. To address the dimensionality and the information restriction, we investigate the limiting problem as the number of agents N tends to infinity. In this limiting model, the average behavior of the agents, known as the mean field, can be mathematically characterized, simplifying the problem. Specifically, in the limiting case, the major agent interacts with the mean field, while a representative minor agent interacts with both the major agent and the mean field. In the next sections, we derive a Markovian closed-loop Nash equilibrium for the limiting game and show that it yields an ε-Nash equilibrium for the original finite-player model.

1.4.2 Limiting Game

In order to derive the limiting model, we begin by imposing the following assumption.

A1.4.3. *There exists a vector of probabilities* π *such that* $\lim_{N\to\infty} \pi^{(N)} = \pi$.

Mean Field: We first characterize the average state of minor agents in the limiting case. The average state of subpopulation k is defined by

$$x_t^{(N_k)} = \frac{1}{N_k} \sum_{i \in \mathscr{I}_k} x_t^i. \tag{1.70}$$

Let $(x^{[N]})^\intercal = [(x^{(N_1)})^\intercal, (x^{(N_2)})^\intercal, \ldots, (x^{(N_K)})^\intercal]$. If it exists, the pointwise in time limit (in quadratic mean) of $x_t^{(N)}$ is called the *state mean field* of the system and denoted by $\bar{x}^\intercal = [(\bar{x}^1)^\intercal, \ldots, (\bar{x}^K)^\intercal]$. Equivalently, in the limiting case, the representation $\bar{x}_t^k = \mathbb{E}[x^{\cdot k} | \mathscr{F}_t^0]$ may be used, where $x^{\cdot k}$ denotes the state of a representative agent in subpopulation k (Nourian and Caines (2013); Carmona and Wang (2017)).

In a similar manner, we define the vector $(u^{[N]})^\intercal = [(u^{(N_1)})^\intercal, (u^{(N_2)})^\intercal, \ldots, (u^{(N_K)})^\intercal]$, the pointwise in time limit (in quadratic mean) of which, if it exists, is called the *control mean field* of the system and denoted by $\bar{u}^\intercal = [(\bar{u}^1)^\intercal, ..., (\bar{u}^K)^\intercal]$. We can obtain the SDE satisfied by the state mean field $\bar{x}^k$ of subpopulation k by taking the average of the solution x_t^i to (1.66) for all agents in subpopulation k (i.e., $\forall i \in \mathscr{I}_k$), and then taking its L^2 limit as $N_k \to \infty$. This SDE is given by

$$d\bar{x}_t^k = \Big[(A_k \mathbf{e}_k + F_k^\pi)\,\bar{x}_t + G_k x_t^0 + B_k \bar{u}_t^k + b_k(t)\Big]\,dt, \tag{1.71}$$

where $F_k^\pi = \pi \otimes F_k := [\pi_1 F_k, \ldots, \pi_K F_k]$, and $\mathbf{e}_k = [0_{n\times n}, ..., 0_{n\times n}, \mathbb{I}_n, 0_{n\times n}, ..., 0_{n\times n}]$, where the $n \times n$ identity matrix $\mathbb{I}_n$ appears in the kth block, and the $n \times n$ zero matrix appears in all other blocks. The dynamics of the mean-field vector $(\bar{x}_t)^\intercal := [(\bar{x}_t^1)^\intercal, \ldots, (\bar{x}_t^K)^\intercal]$, referred to as the *mean-field equation*, are then given by

$$d\bar{x}_t = \big(\breve{A}\,\bar{x}_t + \breve{G} x_t^0 + \breve{B}\,\bar{u}_t + \breve{m}(t)\big)\,dt, \tag{1.72}$$

where

$$\breve{A} = \begin{bmatrix} A_1\mathbf{e}_1 + F_1^\pi \\ \vdots \\ A_K\mathbf{e}_K + F_K^\pi \end{bmatrix}, \quad \breve{G} = \begin{bmatrix} G_1 \\ \vdots \\ G_K \end{bmatrix}, \quad \breve{B} = \begin{bmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_K \end{bmatrix}, \quad \breve{m}(t) = \begin{bmatrix} b_1(t) \\ \vdots \\ b_K(t) \end{bmatrix}. \tag{1.73}$$

<u>Major Agent</u>: In the limiting case, the dynamics of the major agent are given by

$$dx_t^0 = [A_0 x_t^0 + F_0^\pi \bar{x}_t + B_0 u_t^0 + b_0(t)]dt + \sigma_0(t) dw_t^0, \tag{1.74}$$

where $F_0^\pi := \pi \otimes F_0^\pi = [\pi_1 F_0, \ldots, \pi_K F_0]$ and the empirical state average is replaced by the state mean field. Following Huang (2010), in order to make the major agent's model Markovian, we form the extended state $(X_t^0)^\intercal := \big[(x_t^0)^\intercal, (\bar{x}_t)^\intercal\big]$ satisfying

$$dX_t^0 = \Big(\widetilde{A}_0 X_t^0 + \mathbb{B}_0 u_t^0 + \widetilde{B}_0 \bar{u}_t + \widetilde{M}_0(t)\Big)\,dt + \Sigma_0 dW_t^0, \tag{1.75}$$

where

$$\widetilde{A}_0 = \begin{bmatrix} A_0 & F_0^\pi \\ \breve{G} & \breve{A} \end{bmatrix}, \quad \mathbb{B}_0 = \begin{bmatrix} B_0 \\ 0 \end{bmatrix}, \quad \widetilde{B}_0 = \begin{bmatrix} 0 \\ \breve{B} \end{bmatrix},$$

$$\widetilde{M}_0(t) = \begin{bmatrix} b_0(t) \\ \breve{m}(t) \end{bmatrix}, \quad \Sigma_0 = \begin{bmatrix} \sigma_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad W_t^0 = \begin{bmatrix} w_t^0 \\ 0 \end{bmatrix}. \tag{1.76}$$

The cost functional $J_0^\infty(\cdot)$ of the major agent under this framework is given by

$$J_0^\infty(u^0) = \mathbb{E}\Bigg[\exp(\frac{\delta_0}{2}\left\langle \mathbb{G}_0 X_t^0, X_t^0\right\rangle + \frac{\delta_0}{2}\int_0^T \left\langle \mathbb{Q}_0 X_s^0, X_s^0\right\rangle + 2\left\langle \mathbb{S}_0 u_s^0, X_s^0\right\rangle$$

$$+\left\langle R_0 u_s^0, u_s^0\right\rangle - 2\left\langle X_s^0, \bar{\eta}_0\right\rangle - 2\left\langle u_s^0, \bar{n}_0\right\rangle dt)\Bigg] \tag{1.77}$$

$$\mathbb{G}_0 = [\mathbb{I}_n, -H_0^\pi]^\intercal \widehat{Q}_0 [\mathbb{I}_n, -H_0^\pi], \quad \mathbb{Q}_0 = [\mathbb{I}_n, -H_0^\pi]^\intercal Q_0 [\mathbb{I}_n, -H_0^\pi], \quad \mathbb{S}_0 = [\mathbb{I}_n, -H_0^\pi]^\intercal S_0,$$

$$\bar{\eta}_0 = [\mathbb{I}_n, -H_0^\pi]^\intercal Q_0 \eta_0, \quad \bar{n}_0 = S_0^\intercal \eta_0, \quad H_0^\pi = [\pi_1 H_0, \ldots, \pi_K H_0]. \tag{1.78}$$

Minor Agent: The limiting dynamics of the representative minor agent i in subpopulation k are given by

$$dx_t^i = [A_k x_t^i + F_k^\pi \bar{x}_t + G_k x_t^0 + B_k u_t^i + b_k(t)]dt + \sigma_k dw_t^i, \tag{1.79}$$

where $F_k^\pi := \pi \otimes F_k^\pi = [\pi_1 F_k, \ldots, \pi_K F_k]$. As for the major agent, we form the representative minor agent's extended state $(X_t^i)^\intercal := \left[(x_t^i)^\intercal, (x_t^0)^\intercal, (\bar{x}_t)^\intercal\right]$ in order to make the model Markovian. The extended dynamics are given by

$$dX_t^i = (\widetilde{A}_k X_t^i + \mathbb{B}_k u_t^i + \widetilde{\mathbb{B}}_0 u_t^0 + \widetilde{B}\bar{u}_t + \widetilde{M}_k(t))dt + \Sigma_k dW_t^i, \tag{1.80}$$

where (1.75) and (1.79) are used, and

$$\widetilde{A}_k = \begin{bmatrix} A_k & [G_k \ F_k^\pi] \\ 0 & \widetilde{A}_0 \end{bmatrix}, \quad \mathbb{B}_k = \begin{bmatrix} B_k \\ 0 \end{bmatrix}, \quad \widetilde{\mathbb{B}}_0 = \begin{bmatrix} 0 \\ \mathbb{B}_0 \end{bmatrix}, \quad \widetilde{B} = \begin{bmatrix} 0 \\ \widetilde{B}_0 \end{bmatrix}$$

$$\widetilde{M}_k(t) = \begin{bmatrix} b_k(t) \\ \widetilde{M}_0(t) \end{bmatrix}, \quad \Sigma_k = \begin{bmatrix} \sigma_k & 0 \\ 0 & \Sigma_0 \end{bmatrix}, \quad W_t^i = \begin{bmatrix} w_t^i \\ W_t^0 \end{bmatrix}. \tag{1.81}$$

The cost functional for minor agent i, expressed in terms of its extended state, can be reformulated as

$$J_i^\infty(u^i) = \mathbb{E}\Big[\exp\Big(\frac{\delta_k}{2}\left\langle \mathbb{G}_k X_t^i, X_t^i\right\rangle + \frac{\delta_k}{2}\int_0^T \left\langle \mathbb{Q}_k X_s^i, X_s^i\right\rangle + 2\left\langle \mathbb{S}_k u_s^i, X_s^i\right\rangle$$

$$+\left\langle R_k u_s^i, u_s^i\right\rangle - 2\left\langle X_s^i, \bar{\eta}_k\right\rangle - 2\left\langle u_s^i, \bar{n}_k\right\rangle dt\Big)\Big], \tag{1.82}$$

where

$$\mathbb{G}_k = [\mathbb{I}_n, -H_k, -\widehat{H}_k^\pi]^\intercal \widehat{Q}_k [\mathbb{I}_n, -H_k, -\widehat{H}_k^\pi], \quad \mathbb{Q}_k = [\mathbb{I}_n, -H_k, -\widehat{H}_k^\pi]^\intercal Q_k [\mathbb{I}_n, -H_k, -\widehat{H}_k^\pi],$$

$$\mathbb{S}_k = [\mathbb{I}_n, -H_k, -\widehat{H}_k^\pi]^\intercal S_k, \quad \bar{\eta}_k = [\mathbb{I}_n, -H_k, \widehat{H}_k^\pi]^\intercal Q_k \eta_k, \quad \bar{n}_k = S_k^\intercal \eta_k, \quad \widehat{H}_k^\pi = \left[\pi_1 \widehat{H}_k, \ldots, \pi_K \widehat{H}_k\right]. \tag{1.83}$$

Finally, for the limiting system, we define (i) the major agent's information set $\mathscr{F}^0 := (\mathscr{F}_t^0)_{t\in\mathscr{T}}$ as the filtration generated by $(w_t^0)_{t\in\mathscr{T}}$, and (ii) a generic minor agent i's information set $\mathscr{F}^i := (\mathscr{F}_t^i)_{t\in\mathscr{T}}$ as the filtration generated by $(w_t^i, w_t^0)_{t\in\mathscr{T}}$.

Nash Equilibria

The limiting system described in Section 1.4.2 is a stochastic differential game involving the major agent, the mean field, and the representative minor agent. Our goal is to find the Markovian closed-loop Nash equilibria for this game. We define the admissible set of Markovian closed-loop strategies according to the following assumption.

A1.4.4. *(Admissible Controls) (i) For the major agent, the set of admissible control inputs $\mathscr{U}^0$ is defined to be the collection of Markovian linear closed-loop control laws $u^0 := (u_t^0)_{t\in\mathscr{T}}$ such that $\mathbb{E}[\int_0^T u_t^{0\intercal} u_t^0 \, dt] < \infty$. More specifically, $u_t^0 = \ell_0^0(t) + \ell_0^1(t)x_t^0 + \ell_0^2(t)\bar{x}_t$ for some deterministic functions $\ell_0^0(t), \ell_0^1(t)$, and $\ell_0^2(t)$. (ii) For each minor agent $i\, i \in \mathscr{N}$, the set of admissible control inputs $\mathscr{U}^i$ is defined to be the collection of Markovian linear closed-loop control laws $u^i := (u_t^i)_{t\in\mathscr{T}}$ such that $\mathbb{E}[\int_0^T u_t^{i\intercal} u_t^i \, dt] < \infty$. More specifically, $u_t^i = \ell_k^0(t) + \ell_k^1(t)x_t^i + \ell_k^2(t)x_t^0 + \ell_k^{3,\pi}(t)\bar{x}_t$ for some deterministic functions $\ell_k^0(t), \ell_k^1(t), \ell_k^2(t)$ and $\ell_k^{3,\pi}(t)$.*

From (1.75)–(1.78) and (1.80)–(1.83), the major agent's problem involves $\bar{u}$, whereas the representative minor agent's problem involves u^0 and $\bar{u}$. Therefore, solving these individual limiting problems requires a fixed-point condition in terms of $\bar{u}$. To obtain the Nash equilibria, we employ a fixed-point approach, outlined as follows:

(i) Fix $\bar{u}$ as an $\mathscr{F}^0$-adapted process and solve the differential game given by (1.75)–(1.78) and (1.80)–(1.83) to obtain the best-response strategies u^0 and u^i, respectively, for the major agent and a representative minor agent i.

(ii) Impose the consistency condition $u_t^{(N)} = \frac{1}{N}\sum_{i\in\mathscr{N}} u_t^i \to \bar{u}_t$ as $N \to \infty$. To derive the best-response strategies in (i) we use the variational analysis presented in Section 1.3. The following theorem summarizes our results.

Theorem 6. *[Nash Equilibrium] Suppose Assumptions 1.4.1–1.4.4 hold. The set of control laws* $\{u^{0,*}, u^{i,*}, i \in \mathcal{N}\}$*, where* $u^{0,*}$ *and* $u^{i,*}$ *are respectively given by*

$$u_t^{0,*} = -R_0^{-1}\left[\mathbb{S}_0^{\intercal} X_t^0 - \bar{n}_0 + \mathbb{B}_0^{\intercal}\big(\Pi_0(t) X_t^0 + s_0(t)\big)\right] \tag{1.84}$$

$$u_t^{i,*} = -R_k^{-1}\left[\mathbb{S}_k^{\intercal} X_t^i - \bar{n}_k + \mathbb{B}_k^{\intercal}\big(\Pi_k(t) X_t^i + s_k(t)\big)\right] \tag{1.85}$$

forms a unique Markovian closed-loop Nash equilibrium for the limiting system (1.75)-(1.78) *and* (1.80)-(1.83) *subject to the following consistency equations*

$$\begin{cases} -\dot{\Pi}_0 = \Pi_0\mathbb{A}_0 + \mathbb{A}_0^{\intercal}\Pi_0 - (\Pi_0\mathbb{B}_0 + \mathbb{S}_0)R_0^{-1}(\mathbb{B}_0^{\intercal}\Pi_0 + \mathbb{S}_0^{\intercal}) + \mathbb{Q}_0 + \delta_0\Pi_0\Sigma_0\Sigma_0^{\intercal}\Pi_0, \quad \Pi_0(T) = \mathbb{G}_0 \\ -\dot{\Pi}_k = \Pi_k\mathbb{A}_k + \mathbb{A}_k^{\intercal}\Pi_k - (\Pi_k\mathbb{B}_k + \mathbb{S}_k)R_k^{-1}(\mathbb{B}_k^{\intercal}\Pi_k + \mathbb{S}_k^{\intercal}) + \mathbb{Q}_k + \delta_k\Pi_k\Sigma_k\Sigma_k^{\intercal}\Pi_k, \quad \Pi_k(T) = \mathbb{G}_k \\ \bar{A}_k = \left[A_k - B_kR_k^{-1}(\mathbb{S}_{k,11}^{\intercal} + B_k^{\intercal}\Pi_{k,11})\right]\mathbf{e}_k + F_k^{\pi} - B_kR_k^{-1}(\mathbb{S}_{k,31}^{\intercal} + B_k^{\intercal}\Pi_{k,13}), \\ \bar{G}_k = G_k - B_kR_k^{-1}(\mathbb{S}_{k,21}^{\intercal} + B_k^{\intercal}\Pi_{k,12}), \end{cases} \tag{1.86}$$

$$\begin{cases} -\dot{s}_0 = [(\mathbb{A}_0 - \mathbb{B}_0R_0^{-1}\mathbb{S}_0^{\intercal})^{\intercal} - \Pi_0\mathbb{B}_0R_0^{-1}\mathbb{B}_0^{\intercal}]s_0 + \Pi_0(\mathbb{M}_0 + \mathbb{B}_0R_0^{-1}\bar{n}_0) + \mathbb{S}_0R_0^{-1}\bar{n}_0 \\ \qquad\qquad - \bar{\eta}_0 + \delta_0\Pi_0\Sigma_0\Sigma_0^{\intercal}s_0, \qquad s_0(T) = 0 \\ -\dot{s}_k = [(\mathbb{A}_k - \mathbb{B}_kR_k^{-1}\mathbb{S}_k^{\intercal})^{\intercal} - \Pi_k\mathbb{B}_kR_k^{-1}\mathbb{B}_k^{\intercal}]s_k + \Pi_k(\mathbb{M}_k + \mathbb{B}_kR_k^{-1}\bar{n}_k) + \mathbb{S}_kR_k^{-1}\bar{n}_k \\ \qquad\qquad - \bar{\eta}_k + \delta_k\Pi_k\Sigma_k\Sigma_k^{\intercal}s_k, \qquad s_k(T) = 0 \\ \bar{m}_k = b_k + B_kR_k^{-1}\bar{n}_k - B_kR_k^{-1}B_k^{\intercal}s_{k,11}. \end{cases} \tag{1.87}$$

where, for Π_k *and* $\mathbb{S}_k$*, we use the representation*

$$\Pi_k = \begin{bmatrix} \Pi_{k,11} & \Pi_{k,12} & \Pi_{k,13} \\ \Pi_{k,21} & \Pi_{k,22} & \Pi_{k,23} \\ \Pi_{k,31} & \Pi_{k,32} & \Pi_{k,33} \end{bmatrix}, \quad \mathbb{S}_k = \begin{bmatrix} \mathbb{S}_{k,11} \\ \mathbb{S}_{k,21} \\ \mathbb{S}_{k,31} \end{bmatrix}, \quad s_k = \begin{bmatrix} s_{k,11} \\ s_{k,21} \\ s_{k,31} \end{bmatrix} \tag{1.88}$$

with $\Pi_{k,11}, \Pi_{k,22} \in \mathbb{R}^{n\times n}$, $\Pi_{k,33} \in \mathbb{R}^{nK\times nK}$, $\mathbb{S}_{k,11}, \mathbb{S}_{k,21} \in \mathbb{R}^{n\times m}$, $\mathbb{S}_{k,31} \in \mathbb{R}^{nK\times m}$, $s_{k,11}, s_{k,21} \in \mathbb{R}^n$, $s_{k,31} \in \mathbb{R}^{nK}$*, and*

$$\bar{A} = \begin{bmatrix} \bar{A}_1 \\ \vdots \\ \bar{A}_K \end{bmatrix}, \quad \bar{G} = \begin{bmatrix} \bar{G}_1 \\ \vdots \\ \bar{G}_K \end{bmatrix}, \quad \bar{m} = \begin{bmatrix} \bar{m}_1 \\ \vdots \\ \bar{m}_K \end{bmatrix}, \tag{1.89}$$

$$\mathbb{A}_0 = \begin{bmatrix} A_0 & F_0^{\pi} \\ \bar{G} & \bar{A} \end{bmatrix}, \; \mathbb{A}_k = \begin{bmatrix} A_k & [G_k \;\; F_k^{\pi}] \\ 0 & \mathbb{A}_0 - \mathbb{B}_0 R_0^{-1}(\mathbb{S}_0^{\intercal} + \mathbb{B}_0^{\intercal}\Pi_0) \end{bmatrix}, \; \mathbb{M}_0 = \begin{bmatrix} b_0 \\ \bar{m} \end{bmatrix}, \; \mathbb{M}_k = \begin{bmatrix} b_k \\ \mathbb{M}_0 - \mathbb{B}_0 R_0^{-1}\mathbb{B}_0^{\intercal} s_0 \end{bmatrix}. \tag{1.90}$$

In addition, the mean field $\bar{x}_t$ satisfies

$$d\bar{x}_t = \left(\bar{A}\,\bar{x}_t + \bar{G}x_t^0 + \bar{m}\right)dt. \tag{1.91}$$

Proof. Under Assumption 1.4.4, the mean field of control actions $\bar{u}_t$ may be expressed as

$$\bar{u}_t = \Xi(t)X_t^0 + \varsigma(t). \tag{1.92}$$

where the matrix $\Xi(t)$ and the vector $\varsigma(t)$ are deterministic functions of appropriate dimensions.We begin by examining the major agent's system (1.75)–(1.78). Using the representation (1.92), the major agent's extended dynamics may be rewritten as

$$dX_t^0 = \Big((\widetilde{A}_0 + \widetilde{B}_0\Xi)\,X_t^0 + \mathbb{B}_0\,u_t^0 + \widetilde{M}_0 + \widetilde{B}_0\varsigma(t)\Big)\,dt + \Sigma_0 dW_t^0. \tag{1.93}$$

Subsequently, the optimal control problem faced by the major agent reduces to a single-agent optimization problem. We use the methodology presented in Section 1.3 to solve this resulting optimal control problem for the major agent's extended problem. According to Theorem 2, the major agent's best-response strategy is given by

$$\begin{aligned} u_t^{0,*} = -R_0^{-1}\Bigg[\mathbb{S}_0^{\intercal}X_t^0 - \bar{n}_0 + \mathbb{B}_0^{\intercal}\Big((\Upsilon^{-1}_{\widetilde{A}_0+\widetilde{B}_0\Xi}(t))^{\intercal}\frac{M^0_{2,t}(u^{0,*})}{M^0_{1,t}(u^{0,*})} \\ - (\Upsilon^{-1}_{\widetilde{A}_0+\widetilde{B}_0\Xi}(t))^{\intercal}\int_0^t(\Upsilon_{\widetilde{A}_0+\widetilde{B}_0\Xi}(s))^{\intercal}(\mathbb{Q}_0X_s^0 + \mathbb{S}_0u_s^{0,*} - \bar{\eta}_0)ds\Big)\Bigg] \end{aligned} \tag{1.94}$$

where $\Upsilon_{\widetilde{A}_0+\widetilde{B}_0\Xi}(t)$ and $\Upsilon^{-1}_{\widetilde{A}_0+\widetilde{B}_0\Xi}(t)$ are defined as in (1.43), and the martingale terms $M^0_{1,t}(u^{0,*})$ and $M^0_{2,t}(u^{0,*})$ are defined as in (1.16) and (1.17), respectively. From Lemma 3, Theorem 4, and Corollary 5, we can show that (1.94) admits a unique feedback representation

$$u_t^{0,*} = -R_0^{-1}\left[\mathbb{S}_0^{\intercal}X_t^0 - \bar{n}_0 + \mathbb{B}_0^{\intercal}\left(\Pi_0(t)X_t^0 + s_0(t)\right)\right] \tag{1.95}$$

where $\Pi_0(t)$ and $s_0(t)$ satisfy

$$\begin{aligned} \Big(\dot{\Pi}_0(t) + \mathbb{Q}_0 + (\widetilde{A}_0 + \widetilde{B}_0\Xi(t))^{\intercal}\Pi_0(t) + \Pi_0(t)(\widetilde{A}_0 + \widetilde{B}_0\Xi(t)) - (\Pi_0(t)\mathbb{B}_0 + \mathbb{S}_0)R_0^{-1}(\mathbb{B}_0^{\intercal}\Pi_0(t) + \mathbb{S}_0^{\intercal}) \\ + \delta_0\Pi_0(t)\Sigma_0\Sigma_0^{\intercal}\Pi_0(t)\Big)X_t^0 + \Big(\dot{s}_0(t) + ((\widetilde{A}_0 + \widetilde{B}_0\Xi(t))^{\intercal} - \Pi_0(t)\mathbb{B}_0R_0^{-1}\mathbb{B}_0^{\intercal} - \mathbb{S}_0R_0^{-1}\mathbb{B}_0^{\intercal} \\ + \delta_0\Pi_0(t)\Sigma_0\Sigma_0^{\intercal})s_0(t) + \Pi_0(t)\mathbb{B}_0R_0^{-1}\bar{n}_0 + \Pi_0(t)(\widetilde{M}_0(t) + \widetilde{B}_0\varsigma(t)) + \mathbb{S}_0R_0^{-1}\bar{n}_0 - \bar{\eta}_0\Big) = 0. \end{aligned} \tag{1.96}$$

This linear-state feedback form is obtained through a change of measure to $\widehat{\mathbb{P}}^0$, defined by $\frac{d\widehat{\mathbb{P}}^0}{d\mathbb{P}} = \exp(-\frac{\delta_0^2}{2}\int_0^T \left\|\gamma_t^0\right\|^2 dt + \delta_0 \int_0^T (\gamma_t^0)^\intercal dW_t^0)$, where $\gamma_t^0 = \delta_0 \Sigma_0^\intercal(\Pi_0(t)X_t^0 + s_0(t))$. Under the equivalent measure $\widehat{\mathbb{P}}^0$, the process

$$\frac{M_{2,t}^0(u^{0,*})}{M_{1,t}^0(u^{0,*})} = \widehat{M}_t^0(u^{0,*}) \tag{1.97}$$

is a martingale, and we have

$$(\Upsilon^{-1}_{\widetilde{A}_0+\widetilde{B}_0\Xi}(t))^\intercal \widehat{M}_t^0(u^{0,*}) - (\Upsilon^{-1}_{\widetilde{A}_0+\widetilde{B}_0\Xi}(t))^\intercal \int_0^t (\Upsilon_{\widetilde{A}_0+\widetilde{B}_0\Xi}(s))^\intercal (\mathbb{Q}_0 X_s^0 + \mathbb{S}_0 u_s^{0,*} - \bar{\eta}_0) ds = \Pi_0(t) X_t^0 + s_0(t), \tag{1.98}$$

where $\Upsilon_{\widetilde{A}_0+\widetilde{B}_0\Xi}(t)$ and $\Upsilon^{-1}_{\widetilde{A}_0+\widetilde{B}_0\Xi}(t)$ are defined as in (1.43). By applying Ito's lemma to both sides of the above equation and equating the resulting SDEs, we obtain (1.96). However, we cannot proceed any further at this point since Ξ and ς are not yet characterized. We hence turn to the problem of a representative minor agent. Using the mean-field representation (1.92) and the major agent's best-response strategy (1.95), the extended dynamics of minor agent i are given by

$$dX_t^i = (\grave{A}_k(t) X_t^i + \mathbb{B}_k u_t^i + \grave{M}_k(t))dt + \Sigma_k dW_t^i \tag{1.99}$$

where

$$\grave{A}_k(t) = \begin{bmatrix} A_k & [G_k \ F_k^\pi] \\ 0 & \widetilde{A}_0 + \widetilde{B}_0 \Xi - \mathbb{B}_0 R_0^{-1}(\mathbb{S}_0 - \mathbb{B}_0^\intercal \Pi_0(t)) \end{bmatrix} \tag{1.100}$$

$$\grave{M}_k(t) = \begin{bmatrix} b_k(t) \\ \widetilde{M}_0(t) - \mathbb{B}_0 R_0^{-1} \mathbb{B}_0^\intercal s_0(t) + \widetilde{B}_0 \varsigma(t) \end{bmatrix}. \tag{1.101}$$

From Theorem 2, the best-response of minor agent i having a cost functional (1.82) is given by

$$u_t^{i,*} = -R_k^{-1}\left[\mathbb{S}_k^\intercal X_t^k - \bar{n}_k + \mathbb{B}_k^\intercal \left((\Upsilon_{\grave{A}}^{-1}(t))^\intercal \frac{M_{2,t}^i(u^{i,*})}{M_{1,t}^i(u^{i,*})} - (\Upsilon_{\grave{A}}^{-1}(t))^\intercal \int_0^t (\Upsilon_{\grave{A}_k}(s))^\intercal (\mathbb{Q}_k X_s^i + \mathbb{S}_k u_s^{i,*} - \bar{\eta}_k) ds \right)\right], \tag{1.102}$$

where $\Upsilon_{\grave{A}}(t)$ satisfies (1.43), and the martingale terms $M_{1,t}^i(u^{i,*})$ and $M_{2,t}^i(u^{i,*})$ are defined as in (1.16) and (1.17), respectively. Similarly, according to Lemma 3, Theorem 4, and Corollary 5, (1.102) admits the unique feedback form

$$u_t^{i,*} = -R_k^{-1}\left[\mathbb{S}_k^\intercal X_t^{i,*} - \bar{n}_k + \mathbb{B}_k^\intercal \left(\Pi_k(t) X_t^{i,*} + s_k(t) \right)\right], \tag{1.103}$$

where

$$\Big(\dot{\Pi}_k(t) + \mathbb{Q}_k + \grave{A}_k(t)^\intercal \Pi_k(t) + \Pi_k(t)\grave{A}_k(t) - (\Pi_k(t)\mathbb{B}_k + \mathbb{S}_k)R_k^{-1}(\mathbb{B}_k^\intercal \Pi_k(t) + \mathbb{S}_k^\intercal)$$
$$+ \delta_k \Pi_k(t)\Sigma_k\Sigma_k^\intercal \Pi_k(t)\Big)X_t^i + \Big(\dot{s}_k(t) + (\grave{A}_k(t)^\intercal - \Pi_k(t)\mathbb{B}_k R_k^{-1}\mathbb{B}_k^\intercal - \mathbb{S}_k R_k^{-1}\mathbb{B}_k^\intercal + \delta_k \Pi_k(t)\Sigma_k\Sigma_k^\intercal)s_k(t)$$
$$+ \Pi_k(t)\mathbb{B}_k R_k^{-1}\bar{n}_k + \Pi_k(t)\grave{M}_k(t) + \mathbb{S}_k R_k^{-1}\bar{n}_k - \bar{\eta}_k\Big) = 0. \quad (1.104)$$

The state feedback form (1.103) is obtained through a change of measure to $\widehat{\mathbb{P}}^i$, defined by $\frac{d\widehat{\mathbb{P}}^i}{d\mathbb{P}} = \exp(-\frac{\delta_k^2}{2}\int_0^T \left\|\gamma_t^i\right\|^2 dt + \delta_k \int_0^T (\gamma_t^i)^\intercal dW_t^i)$, with $\gamma_t^i = \delta_k \Sigma_k^\intercal(\Pi_k(t)X_t^i + s_k(t))$, such that the process

$$\frac{M_{2,t}^i(u^{i,*})}{M_{1,t}^i(u^{i,*})} = M_t^i(u^{i,*}) \quad (1.105)$$

is a $\widehat{\mathbb{P}}^i$-martingale. To continue our analysis, we then characterize $\bar{u}$ by applying the consistency condition (ii). To this end, we represent Π_k and $\mathbb{S}_k$ in (1.103) as in (1.88). From (1.103) and (1.88), the average control action of a minor agent in subpopulation k is given by

$$u_t^{(N_k)} = -R_k^{-1}\left(\begin{bmatrix}\mathbb{S}_{k,11}^\intercal + B^\intercal \Pi_{k,11} & \mathbb{S}_{k,21}^\intercal + B^\intercal \Pi_{k,12} & \mathbb{S}_{k,31}^\intercal + B^\intercal \Pi_{k,13}\end{bmatrix}\begin{bmatrix} x_t^{(N_k)} \\ x_t^0 \\ \bar{x}_t \end{bmatrix} - \bar{n}_k + B_k^\intercal s_{k,11}\right). \quad (1.106)$$

In the limit, as $N_k \to \infty$, $u_t^{(N_k)}$ converges in quadratic mean to (see e.g. Caines and Kizilkale (2017))

$$\bar{u}_t^k = -R_k^{-1}\left(\begin{bmatrix}\mathbb{S}_{k,11}^\intercal + B^\intercal \Pi_{k,11} & \mathbb{S}_{k,21}^\intercal + B^\intercal \Pi_{k,12} & \mathbb{S}_{k,31}^\intercal + B^\intercal \Pi_{k,13}\end{bmatrix}\begin{bmatrix} \bar{x}_t^k \\ x_t^0 \\ \bar{x}_t \end{bmatrix} - \bar{n}_k + B_k^\intercal s_{k,11}\right). \quad (1.107)$$

We observe that the expression in (1.107) has the same structure as (1.92). Hence, by comparing these two equations, we specify Ξ and ς in terms of Π_k and s_k, $k \in \mathfrak{K}$. We then substitute the obtained expressions for Ξ and ς in (1.72), (1.96), and (1.104) to obtain (1.91), (1.86) and (1.87). □

Remark 4. *(Comparison of equilibria in the risk-sensitive and risk-neutral cases) In risk-neutral MFGs with a major agent, neither the volatility of the major agent nor that of the minor agents explicitly affects the Nash equilibrium. This is not the case for the corresponding risk-sensitive MFGs, where we observe the following:*

- *The mean field is influenced by the volatility of the major agent and the volatility of all K types of minor agents.*

- *The equilibrium control action of the major agent explicitly depends on its own volatility.*

- *The equilibrium control action of a representative minor agent explicitly depends on its own volatility as well as on the volatility of the major agent.*

- *The equilibrium control actions of the major agent and of a representative minor agent are impacted by the volatility of the K types of minor agents through the mean-field equation.*

Furthermore, in the risk-neutral case, only the first block rows of Π_k *and* s_k *impact the equilibrium control, as shown in Firoozi et al. (2020). However, in the risk-sensitive case, all the blocks of* Π_k *and* s_k *have an impact on the equilibrium control actions.*

Numerical Solution of Consistency Equations

In this section, we discuss the solvability of the set of mean field consistency equations given by (1.86)-(1.87). We note that (1.86) represents a set of coupled Riccati equations, which may be solved independently from (1.87). It is challenging to analytically show the existence and uniqueness of a solution to (1.86). However, given a solution to (1.86), (1.87) may be viewed as a system of coupled first order linear ODEs, for which a unique analytical solution is guaranteed.

Here, we use a numerical scheme to solve the set of consistency equations, (1.86)-(1.87), for a specific system instance. To this end, we adapt the iterative method used in Huang (2010). In particular, in our case $\bar{A}$ and $\bar{G}$ are time-dependent functions defined on $\mathscr{T}$. Our algorithm initializes with arbitrary trajectories for $\bar{A}$ and $\bar{G}$ and iterates until the corresponding trajectories of two consecutive iterations converge, where the error in iteration j relative to iteration $j-1$ is defined by

$$\text{error}^{(j)} = \left|\bar{A}^{(j)} - \bar{A}^{(j-1)}\right|_\infty + \left|\bar{G}^{(j)} - \bar{G}^{(j-1)}\right|_\infty, \quad j = 1,2,\dots \tag{1.108}$$

with $|\cdot|_\infty$ denoting the supremum norm and the superscript $(\cdot)$ indicating the iteration number.

We illustrate a particular case of the system described by equations (1.65)-(1.69) where the dynamics and cost functionals are given by

$$dx_t^0 = (-2.5x_t^0 + 2.5x_t^{(N)} + u_t^0)dt + 0.5dw_t^0,$$
$$dx_t^i = (-5x_t^i + 2.5x_t^{(N)} + 2.5x_t^0 + u_t^i)dt + 0.5dw_t^i,$$

$$J_0^{(N)}(u^0, u^{-0}) = \mathbb{E}\left[\exp\left(\int_0^T \left(10\left(x_t^0 - x_t^{(N)}\right)^2 + (u_t^0)^2\right)dt\right)\right],$$
$$J_i^{(N)}(u^i, u^{-i}) = \mathbb{E}\left[\exp\left(\int_0^T \left(7\left(x_t^i - 0.5x_t^{(N)} - 0.5x_t^0\right)^2 + \left(u_t^i\right)^2\right)dt\right)\right].$$

Fig. 1.1 and Fig. 1.2 illustrate, respectively, the trajectories of $\bar{A}^{(j)}$ and $\bar{G}^{(j)}$ over iterations $j = 1, 2, \ldots, 7$. The trajectories show convergence after just a few iterations. Specifically, the iterative error is reduced to 0.7×10^{-14} after ten iterations, starting from the initial trajectories $\bar{A}^{(0)}(t) = \bar{G}^{(0)}(t) = 0$, $\forall t \in \mathscr{T}$. Moreover, for the completeness of numerical solutions, Fig. 1.3 depicts the associated trajectories for $\bar{m}^{(j)}$, obtained using a similar iterative method.

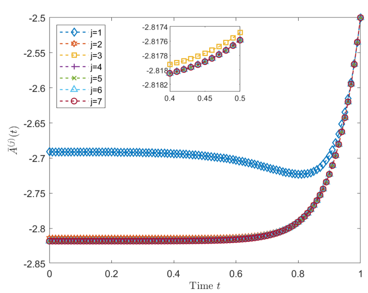

Figure 1.1: Trajectories of $\bar{A}^{(j)}$ for $j = 1, \ldots, 7$.

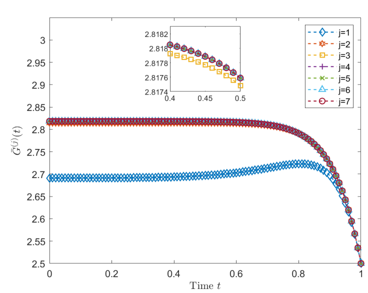

Figure 1.2: Trajectories of $\bar{G}^{(j)}$ for $j = 1, \ldots, 7$.

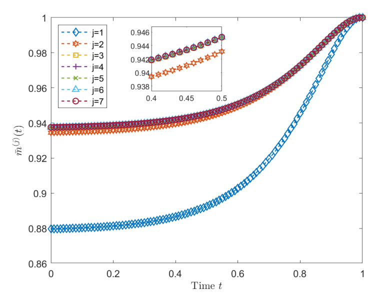

Figure 1.3: Trajectories of $\bar{m}^{(j)}$ for $j = 1, \ldots, 7$.

1.4.3 ε-Nash Property

In this section, we show that the control laws defined in the previous section yield an ε-Nash equilibrium for the finite-population game described in Section 1.4.1 under certain conditions. Due to the fact that linear-quadratic (LQ) exponential cost functional do not admit the boundedness or Liptschitz continuous properties, a proof of the ε-Nash property without imposing further conditions on the system is still an open question. To the best of our knowledge, the only way to establish the ε-Nash property is to find a relationship between linear-quadratic risk-sensitive and risk-neutral cost functionals (see also Moon and Başar (2019)). To be more precise, we first represent the risk-sensitive cost functional as

$$J(u) = \mathbb{E}\left[\exp\left(g(x_T, v_T) + \int_0^T f(x_t, u_t, v_t)dt\right)\right] \tag{1.109}$$

$$f(x_t, u_t, v_t) = \frac{\delta}{2}\left(\langle Q(x_t - v_t), x_t - v_t\rangle + 2\langle Su_t, x_t - v_t\rangle + \langle Ru_t, u_t\rangle\right) \tag{1.110}$$

$$g(x_T, v_T) = \frac{\delta}{2}\left\langle \widehat{Q}(x_T - v_T), x_T - v_T\right\rangle, \tag{1.111}$$

where v_t is a square integrable process and we drop the index i for notational brevity. We also assume that $R > 0$, $\widehat{Q} \geq 0$, and $Q - SR^{-1}S^{\intercal} \geq 0$. The desired relationship can be built if the following inequalities hold

$$\begin{aligned}
&|J(u_1) - J(u_2)| \\
&\leq \mathbb{E}\left[\left|\exp(g(x_T^1, v_T^1) + \int_0^T f(x_t^1, u_t^1, v_t^1)dt) - \exp(g(x_T^2, v_T^2) + \int_0^T f(x_t^2, u_t^2, v_t^2)dt)\right|\right] \\
&\leq C\mathbb{E}\left[\left|g(x_T^1, v_T^1) - g(x_T^2, v_T^2) + \int_0^T f(x_t^1, u_t^1, v_t^1) - f(x_t^2, u_t^2, v_t^2)dt\right|\right],
\end{aligned} \tag{1.112}$$

where C is a constant that does not depend on a particular sample path of the stochastic processes involved. However, without any further assumptions, we cannot claim (1.112) since the exponential function and quadratic functions are not globally Lipschitz continuous or bounded. A compromise to address this issue is to confine the control and state vectors within a sufficiently large (unattainable) compact set in Euclidean space without affecting the optimization outcome. This restriction is customary in the context of LQG risk-sensitive problems and appears in one way or another in different methodologies. For more details on this restriction, we refer the reader to Moon et al. (2018); Moon and Başar (2019), Chapter 6 of Başar and Olsder (1998), and Lim and Zhou (2005), among others.

From (1.112), the proof of the ε-Nash property for LQG risk-sensitive MFGs may be reduced to establishing the same property for LQG risk-neutral MFGs. We refer to Huang et al. (2006) for a related proof for LQG risk-neutral MFGs. In this chapter, we leverage the imposed restriction to present an alternative proof. It is easy to see that, under this restriction, we further have

$$\mathbb{E}\left[\left|g(x_T^1,v_T^1)-g(x_T^2,v_T^2)+\int_0^T f(x_t^1,u_t^1,v_t^1)-f(x_t^2,u_t^2,v_t^2)dt\right|\right]$$
$$\leq C\mathbb{E}\left[\left\|x_T^1-x_T^2\right\|_1+\left\|v_T^1-v_T^2\right\|_1+\int_0^T\left(\left\|x_t^1-x_t^2\right\|_1+\left\|u_t^1-u_t^2\right\|_1+\left\|v_t^1-v_t^2\right\|_1\right)dt\right], \quad (1.113)$$

where $\|.\|_1$ is the L^1 vector norm defined as $\|x\|_1=\sum_{j=1}^n\left|x_j\right|$, with $x=[x_1\ x_2\ \dots\ x_n]^\intercal$. The proof of the ε-Nash property can then be established by using the basic approximation of linear SDEs.

Theorem 7. *Suppose that Assumptions 1.4.1–1.4.4 hold, and that the control and state vectors are restricted within sufficiently large compact sets. Further, suppose that there exists a sequence of real numbers $\{\tau_N, N\in\{1,2,\dots\}\}$ such that $\tau_N\to 0$ and $\left|\frac{N_k}{N}-\pi_k\right|=o(\tau_N)$, for all $k\in\mathscr{K}$. The set of control laws $\{u^{0,*},u^{i,*},i\in\mathscr{N}\}$ where $u^{0,*}$ and $u^{i,*}$ are respectively given by* (1.84)–(1.90)*, forms an ε-Nash equilibrium for the finite-population system described by* (1.65)–(1.69)*. That is, for any alternative control action $u^i\in\mathscr{U}^g, i\in\mathscr{N}_0$, there is a sequence of nonnegative numbers $\{\varepsilon_N, N\in\{1,2,\dots\}\}$ converging to zero, such that*

$$J_i^{(N)}(u^{i,*},u^{-i,*})\leq J_i^{(N)}(u^i,u^{-i,*})+\varepsilon_N,\quad i\in\mathscr{N}_0 \quad (1.114)$$

where $\varepsilon_N=o(\frac{1}{\sqrt{N}})+o(\tau_N)$.

Proof. Without loss of generality, to streamline the notation and facilitate the understanding of the approach, we provide a proof for a simplified scenario, where processes are scalar and where $H_0=\delta_0=1$ and $H_k=\widehat{H}_k=\delta_k=1\forall k\in\mathscr{K}$. The proof may be readily extended to the general system considered in Section 1.4.1. Additionally, we use the notation $A\lesssim B$ to indicate that $A\leq CB$ for some constant C.

We establish the ε-Nash property for the major agent and for a representative minor agent.

(I) ε-Nash property for the major agent: Under the assumption that all minor agents follow the Nash equilibrium strategies $\{u^{i,*},i\in\mathscr{N}\}$ given in Theorem 6 and that the major agent adopts an arbitrary strategy u^0, we introduce the following finite-population and limiting game systems for the major agent. These systems share the same initial conditions.

- *limiting game System:* The major agent's state and the mean field are represented, respectively, by x_t^0 and $(\bar{x}_t)^\intercal := [(\bar{x}_t^1)^\intercal, \dots, (\bar{x}_t^K)^\intercal]$. The dynamics are given by

$$\begin{cases} dx_t^0 = & [A_0 x_t^0 + F_0^\pi \bar{x}_t + B_0 u_t^0]dt + \sigma_0(t)\, dw_t^0 \\ d\bar{x}_t^k = & [(A_k - R_k^{-1}(B_k)^2(\Pi_{k,11}(t) + \mathbb{S}_{k,11}))\bar{x}_t^k - R_k^{-1}(B_k)^2(\Pi_{k,13}(t) + \mathbb{S}_{k,13}^\intercal)\bar{x}_t \\ & + \bar{G}_k x_t^0 + F_k^\pi \bar{x}_t + \bar{m}_k(t)]dt, \qquad k \in \mathscr{K}, \end{cases} \tag{1.115}$$

where the coefficients $\bar{G}_k$ and $\bar{m}_k$ are given, respectively, by (1.86) and (1.87). This system contains $K+1$ equations. We represent it in expanded form to facilitate the analysis. For this case, the major agent's cost functional is given by

$$\begin{aligned} &J_0^\infty(u^0, u^{-0,*}) = \mathbb{E}\left[\exp\left(\int_0^T f^0(x_t^0, u_t^0, \pi\bar{x}_t + \eta_0)dt + g^0(x_T^0, \pi\bar{x}_T + \eta_0)\right)\right] \\ &f^0(x_t, u_t, v_t) = \frac{1}{2}Q_0(x_t - v_t)^2 + S_0 u_t(x_t - v_t) + \frac{1}{2}R_0 u_t^2 \\ &g^0(x_T, v_T) = \widehat{Q}_0(x_T - v_T)^2. \end{aligned} \tag{1.116}$$

- *Finite-Population System:* This system consists of one major agent and N minor agents. We represent the major agent's state by $x_t^{0,(N)}$ and the vector of average states by $(x^{[N]})^\intercal = [(x^{(N_1)})^\intercal, (x^{(N_2)})^\intercal, \dots, (x^{(N_K)})^\intercal]$ (as defined in (1.70)). Furthermore, we introduce the vector process $(\bar{x}^{(N)})^\intercal = [\bar{x}_t^{1,(N)}, \bar{x}_t^{2,(N)}, \dots, \bar{x}_t^{K,(N)}]$, which is calculated using the mean-field equation (1.72), where the vector of average states $x_t^{[N]}$ is used. These processes satisfy

$$\begin{cases} dx_t^{0,(N)} & = [A_0 x_t^{0,(N)} + F_0^{\pi^{(N)}} x_t^{[N]} + B_0 u_t^0]dt + \sigma_0(t) dw_t^0 \\ dx_t^{(N_k)} & = [(A_k - R_k^{-1}(B_k)^2(\Pi_{k,11}(t) + \mathbb{S}_{k,11}))x_t^{(N_k)} - R_k^{-1}(B_k)^2(\Pi_{k,13}(t) + \mathbb{S}_{k,13}^\intercal)\bar{x}_t^{(N)} \\ & \qquad + \bar{G}_k x_t^{0,(N)} + F_k^{\pi^{(N)}} x_t^{[N]} + \bar{m}_k(t)]dt + \frac{1}{N_k}\sum_{i \in \mathscr{I}_k} \sigma_k(t) dw_t^i, \qquad k \in \mathscr{K} \\ d\bar{x}_t^{k,(N)} & = [(A_k - R_k^{-1}(B_k)^2(\Pi_{k,11}(t) + \mathbb{S}_{k,11}))\bar{x}_t^{k,(N)} - R_k^{-1}(B_k)^2(\Pi_{k,13}(t) + \mathbb{S}_{k,13}^\intercal)\bar{x}_t^{(N)} \\ & \qquad + \bar{G}_k x_t^{0,(N)} + F_k^\pi \bar{x}_t^{(N)} + \bar{m}_k(t)]dt, \qquad k \in \mathscr{K}, \end{cases} \tag{1.117}$$

where the dynamics of $x^{(N_k)}$ are obtained using (1.66), and the Nash equilibrium strategies using $x_t^{i,(N)}$, $x_t^{0,(N)}$, and $\bar{x}_t^{(N)}$ instead of, respectively, x_t^i, x_t^0, and $\bar{x}_t$.

The major agent's cost functional is then given by

$$J_0^{(N)}(u^0, u^{-0,*}) = \mathbb{E}\left[\exp\left(\int_0^T f^0(x_t^{0,(N)}, u_t^0, \pi^{(N)} x_t^{[N]} + \eta_0)dt + g^0(x_T^{0,(N)}, \pi^{(N)} x_T^{[N]} + \eta_0)\right)\right]. \tag{1.118}$$

From (1.113), we have

$$\begin{aligned}
&\left|J_0^{(N)}(u^0, u^{-0,*}) - J_0^\infty(u^0, u^{-0,*})\right| \\
&\lesssim \int_0^T \mathbb{E}\left[\left|x_t^0 - x_t^{0,(N)}\right| + \left|\pi\bar{x}_t - \pi^{(N)} x_t^{[N]}\right|\right] dt + \mathbb{E}\left|\pi\bar{x}_T - \pi^{(N)} x_T^{[N]}\right| + \mathbb{E}\left|x_T^0 - x_T^{0,(N)}\right|.
\end{aligned} \tag{1.119}$$

Moreover, we can write

$$\left|\pi\bar{x}_t - \pi^{(N)} x_t^{[N]}\right| \leq \left|\pi\bar{x}_t - \pi x_t^{[N]}\right| + \left|\pi x_t^{[N]} - \pi^{(N)} x_t^{[N]}\right| \leq \left\|\bar{x}_t - x_t^{[N]}\right\|_1 + C\tau_N \tag{1.120}$$

for all $t \in [0, T]$, where $\tau_N := \sup_{1 \leq i \leq K}\left|\pi_k^{(N)} - \pi_k\right|$ is a sequence converging to zero. Using (1.120), (1.119) may be written as

$$\begin{aligned}
&\left|J_0^{(N)}(u^0, u^{-0,*}) - J_0^\infty(u^0, u^{-0,*})\right| \\
&\lesssim \int_0^T \mathbb{E}\left[\left|x_t^0 - x_t^{0,(N)}\right| + \left\|\bar{x}_t - x_t^{[N]}\right\|_1 + \tau_N\right] dt + \mathbb{E}\left\|\bar{x}_T - x_T^{[N]}\right\|_1 + \mathbb{E}\left|x_T^0 - x_T^{0,(N)}\right| + \tau_N \\
&\lesssim \int_0^T \xi_N^0(t)dt + \xi_N^0(T),
\end{aligned} \tag{1.121}$$

where

$$\xi_N^0(t) := \mathbb{E}\left|x_t^0 - x_t^{0,(N)}\right| + \mathbb{E}\left\|\bar{x}_t - \bar{x}_t^{(N)}\right\|_1 + \mathbb{E}\left\|\bar{x}_t^{(N)} - x_t^{[N]}\right\|_1 + \tau_N \tag{1.122}$$

and $\xi_N^0(0) = \tau_N$. We aim to find an upper bound for $\xi_N^0(t)$. From the second and third equations in (1.117), we have

$$\begin{aligned}
\mathbb{E}\left\|\bar{x}_t^{(N)} - x_t^{[N]}\right\|_1 &= \sum_{k=1}^K \mathbb{E}\left|\bar{x}_t^{k,(N)} - x_t^{(N_k)}\right| \\
&\leq \int_0^t \left[\sum_{k=1}^K \mathbb{E}\left|(A_k - R_k^{-1}(B_k)^2(\Pi_{k,11}(s) + \mathbb{S}_{k,11}))(\bar{x}_s^{k,(N)} - x_s^{(N_k)})\right|\right. \\
&\quad \left. + \sum_{k=1}^K \mathbb{E}\left|F_k^\pi \bar{x}_s^{(N)} - F_k^{\pi^{(N)}} x_s^{[N]}\right|\right] ds + \sum_{k=1}^K \frac{1}{N_k}\mathbb{E}\left|\sum_{i\in\mathscr{I}_k}\int_0^t \sigma_k(s)dw_s^i\right|.
\end{aligned} \tag{1.123}$$

Since the coefficients are bounded, we have

$$\sum_{k=1}^K \mathbb{E}\left|(A_k - R_k^{-1}(B_k)^2(\Pi_{k,11}(t) + \mathbb{S}_{k,11}))(\bar{x}_t^{k,(N)} - x_t^{(N_k)})\right| \lesssim \mathbb{E}\left\|\bar{x}_t^{(N)} - x_t^{[N]}\right\|_1. \tag{1.124}$$

Moreover, following the same approach as in (1.120), we obtain

$$\sum_{k=1}^{K} \mathbb{E}\left|F_k^{\pi}\bar{x}_t^{(N)} - F_k^{\pi^{(N)}} x_t^{[N]}\right| \lesssim \mathbb{E}\left\|\bar{x}_t^{(N)} - x_t^{[N]}\right\|_1 + \tau_N. \tag{1.125}$$

Therefore, we have

$$\begin{aligned}\mathbb{E}\left\|\bar{x}_t^{(N)} - x_t^{[N]}\right\|_1 &\leq C\int_0^t \left[\mathbb{E}\left\|\bar{x}_s^{(N)} - x_s^{[N]}\right\|_1 + \tau_N\right] ds + \sum_{k=1}^{K} \frac{1}{N_k}\mathbb{E}\left|\sum_{i\in\mathscr{I}_k}\int_0^t \sigma_k(s)dw_s^i\right| \\ &\leq C\int_0^t \xi_N^0(s)ds + \sum_{k=1}^{K}\frac{1}{N_k}\mathbb{E}\left|\sum_{i\in\mathscr{I}_k}\int_0^t \sigma_k(s)dw_s^i.\right| \end{aligned} \tag{1.126}$$

By following similar steps as in (1.123)–(1.126) for $\mathbb{E}\left|x_t^0 - x_t^{0,(N)}\right|$ and $\mathbb{E}\left\|\bar{x}_t - \bar{x}_t^{(N)}\right\|_1$, we obtain the inequality

$$\xi_N^0(t) \lesssim \kappa_N^0(t) + \int_0^t \xi_N^0(s)ds \tag{1.127}$$

where

$$\kappa_N^0(t) = \tau_N + \sum_{k=1}^{K}\frac{1}{N_k}\mathbb{E}\left|\sum_{i\in\mathscr{I}_k}\int_0^t \sigma_k(s)dw_s^i\right|. \tag{1.128}$$

Note that $\sum_{i\in\mathscr{I}_k}\int_0^t \sigma_k(s)dw_s^i \sim \mathscr{N}(0, N_k\int_0^t \sigma_k^2(s)ds)$ and hence $\left|\sum_{i\in\mathscr{I}_k}\int_0^t \sigma_k(s)dw_s^i\right|$ follows a folded normal distribution. We then have

$$\frac{1}{N_k}\mathbb{E}\left|\sum_{i\in\mathscr{I}_k}\int_0^t \sigma_k(s)dw_s^i\right| \lesssim \frac{1}{\sqrt{N_k}} \lesssim \frac{1}{\sqrt{N}}, \tag{1.129}$$

where the second inequality holds due to the boundedness of $\frac{1}{\sqrt{\pi_k^{(N)}}}$, so that

$$\kappa_N^0(t) \lesssim \tau_N + \frac{1}{\sqrt{N}}, \qquad \forall t \in \mathscr{T}. \tag{1.130}$$

We use Grönwall's inequality to write (1.127) as

$$\xi_N^0(t) \leq C\kappa_N^0(t) + Ce^{Ct}\int_0^t e^{-Cs}\kappa_N^0(s)ds. \tag{1.131}$$

The above expression indicates that $\{\xi_N^0(t), N = 1, 2, \dots\}$ forms a sequence converging to 0 as $N \to \infty$. Therefore, from (1.121), we have

$$J_0^{(N)}(u^0, u^{-0,*}) \leq J_0^{\infty}(u^0, u^{-0,*}) + \varepsilon_N, \tag{1.132}$$

where $\varepsilon_N = o(\frac{1}{\sqrt{N}}) + o(\tau_N)$. Note that (1.132) holds for both the optimal strategy $u^{0,*}$ and an arbitrary strategy u^0 for the major agent. Furthermore, we have

$$\begin{aligned} J_0^{(N)}(u^{0,*}, u^{-0,*}) &\le J_0^{\infty}(u^{0,*}, u^{-0,*}) + \varepsilon_N \\ &\le J_0^{\infty}(u^{0}, u^{-0,*}) + \varepsilon_N \\ &\le J_0^{(N)}(u^{0}, u^{-0,*}) + 2\varepsilon_N, \end{aligned} \tag{1.133}$$

where the second inequality holds since $u^{0,*}$ represents the Nash equilibrium strategy for the limiting game system, and the third inequality follows from (1.121). This concludes the proof of the ε-Nash property for the major agent.

(II) ε-Nash property for the representative minor agent i: Assuming that the major agent and all minor agents, except minor agent i in subpopulation 1, follow the Nash equilibrium strategies $\{u^{0,*}, u^{1,*}, \ldots, u^{i-1,*}, u^{i+1,*}, \ldots, u^{i+1,*}\}$ as outlined in Theorem 6, while minor agent i adopts an arbitrary strategy u^i, we introduce the following finite-population and limiting game systems for minor agent i. These systems share the same initial conditions.

- *limiting game System:* The minor agent's state, the major agent's state and the mean field are represented, respectively, by x_t^i, x_t^0 and $(\bar{x}_t)^\intercal := [(\bar{x}_t^1)^\intercal, \ldots, (\bar{x}_t^K)^\intercal]$. The dynamics are given by

$$\begin{cases} dx_t^i &= [A_1 x_t^i + F_1^\pi \bar{x}_t + G_1 x_t^0 + B_1 u_t^i]dt + \sigma_1(t) dw_t^i \\ dx_t^0 &= [(A_0 - R_0^{-1} B_0^2(\Pi_{0,11}(t) + \mathbb{S}_{0,11}))x_t^0 - R_0^{-1} B_0^2(\Pi_{0,12}(t) + \mathbb{S}_{0,12}^\intercal)\bar{x}_t \\ &\qquad + F_0^\pi \bar{x}_t + b_0 + B_0 R_k^{-1} \bar{n}_0 - R_0^{-1}(B_0)^2 s_0]dt + \sigma_0(t) dw_t^0 \\ d\bar{x}_t^k &= [(A_k - R_k^{-1}(B_k)^2(\Pi_{k,11}(t) + \mathbb{S}_{k,11}))\bar{x}_t^k - R_k^{-1}(B_k)^2(\Pi_{k,13}(t) + \mathbb{S}_{k,13}^\intercal)\bar{x}_t \\ &\qquad + \bar{G}_k x_t^0 + F_k^\pi \bar{x}_t + \bar{m}_k(t)]dt, \end{cases} \tag{1.134}$$

  where the coefficients $\bar{G}_k$ and $\bar{m}_k$ are, respectively, given by (1.86) and (1.87). The cost functional of minor agent i belonging to subpopulation 1 is given by

$$\begin{aligned} &J_i^\infty(u^i, u^{-i,*}) = \mathbb{E}[\exp(\int_0^T f^i(x_t^i, u_t^i, \pi\bar{x}_t + x_t^0 + \eta_1)dt + g^i(x_T^i, \pi\bar{x}_T + x_T^0 + \eta_1))] \\ &f^i(x_t^i, u_t^i, v_t) = \frac{1}{2}Q_1(x_t^i - v_t)^2 + S_1 u_t^i(x_t^i - v_t) + \frac{1}{2}R_1(u_t^i)^2 \\ &g^i(x_T^i, v_T) = \widehat{Q}_1(x_T^i - v_T)^2. \end{aligned} \tag{1.135}$$

- *Finite-Population System:* This system consists of one major agent and N minor agents. We represent the state of minor agent i belonging to subpopulation 1 by $x_t^{i,(N)}$, the major agent's state by $x_t^{0,(N)}$, the average state of subpopulation 1 by $x^{(N_1)}$, and the average state of subpopulation $k \neq 1$ by $x^{(N_k)}$ (as defined in (1.70)). Furthermore, we introduce the vector process $(\bar{x}^{(N)})^\intercal = [\bar{x}_t^{1,(N)}, \bar{x}_t^{2,(N)}, \dots, \bar{x}_t^{K,(N)}]$, which is calculated using the mean-field equation (1.72), using the vector of average states $(x^{[N]})^\intercal = [(x^{(N_1)})^\intercal, (x^{(N_2)})^\intercal, \dots, (x^{(N_K)})^\intercal]$. These processes satisfy

$$\begin{cases} dx_t^{i,(N)} &= [A_1 x_t^{(N_1)} + F_1^{\pi^{(N)}} \bar{x}_t^{(N)} + G_1 x_t^{0,(N)} + B_1 u_t^i] + \sigma_1(t) dw_t^i \\ dx_t^{0,(N)} &= [(A_0 - R_0^{-1} B_0^2 (\Pi_{0,11}(t) + \mathbb{S}_{0,11})) x_t^{0,(N)} - R_0^{-1} B_0^2 (\Pi_{0,12}(t) + \mathbb{S}_{0,12}^\intercal) \bar{x}_t^{(N)} \\ &\qquad + F_0^{\pi^{(N)}} x_t^{[N]} + \bar{m}_0(t)] dt + \sigma_0(t) dw_t^0 \\ dx_t^{(N_1)} &= [(A_1 - R_1^{-1} B_1^2 (\Pi_{1,11}(t) + \mathbb{S}_{1,11})) x_t^{(N_1)} - R_1^{-1} B_1^2 (\Pi_{1,13}(t) + \mathbb{S}_{1,13}^\intercal) \bar{x}_t^{(N)} \\ &\qquad + \bar{G}_1 x_t^{0,(N)} + F_1^{\pi^{(N)}} x_t^{[N]} + \bar{m}_1(t)] dt + \frac{1}{N_1} \sum_{i \in \mathscr{I}_1} \sigma_1(t) dw_t^i + e_t^{u^{i,*}, u^i} dt \\ dx_t^{(N_k)} &= [(A_k - R_k^{-1} (B_k)^2 (\Pi_{k,11}(t) + \mathbb{S}_{k,11})) x_t^{(N_k)} - R_k^{-1} (B_k)^2 (\Pi_{k,13}(t) + \mathbb{S}_{k,13}^\intercal) \bar{x}_t^{(N)} \\ &\qquad + \bar{G}_k x_t^{0,(N)} + F_k^{\pi^{(N)}} x_t^{[N]} + \bar{m}_k(t)] dt + \frac{1}{N_k} \sum_{i \in \mathscr{I}_k} \sigma_k(t) dw_t^i, \qquad k \neq 1 \\ d\bar{x}_t^{k,(N)} &= [(A_k - R_k^{-1} (B_k)^2 (\Pi_{k,11}(t) + \mathbb{S}_{k,11})) \bar{x}_t^{k,(N)} - R_k^{-1} (B_k)^2 (\Pi_{k,13}(t) + \mathbb{S}_{k,13}^\intercal) \bar{x}_t^{(N)} \\ &\qquad + \bar{G}_k x_t^{0,(N)} + F_k^{\pi} \bar{x}_t^{(N)} + \bar{m}_k(t)] dt, \end{cases} \tag{1.136}$$

where the average state $x_t^{(N_1)}$ of subpopulation 1 involves the term

$$e_t^{u^{i,*}, u^i} := -\frac{B_1}{N_1} (u_t^{i,*} - u_t^i), \tag{1.137}$$

where $u^{i,*}$ represents the Nash equilibrium strategy of agent i, as given by (1.103) for $k = 1$. The inclusion of this term accounts for the arbitrary strategy adopted by minor agent i. Notably, this term becomes zero when the agent chooses to follow the Nash equilibrium strategy. The cost functional of minor agent i is given by

$$J_i^{(N)}(u^i, u^{-i,*}) = \mathbb{E}\Big[\exp\Big(\int_0^T f^i(x_t^{i,(N)}, u_t^i, \pi^{(N)} x_t^{[N]} + x_t^{0,(N)} + \eta_1) dt \\ + g^i(x_T^{i,(N)}, \pi^{(N)} x_T^{[N]} + x_T^{0,(N)} + \eta_1)\Big)\Big]. \tag{1.138}$$

Following a similar approach as for the major agent's problem, we obtain

$$\begin{aligned}\left|J_i^{(N)}(u^i,u^{-i,*})-J_0^{\infty}(u^i,u^{-i,*})\right| &\lesssim \int_0^T \mathbb{E}\left[\left|x_t^i-x_t^{i,(N)}\right|+\left|x_t^0-x_t^{0,(N)}\right|+\left|\pi\bar{x}_t-\pi^{(N)}x_t^{[N]}\right|\right]dt\\ &\quad+\mathbb{E}\left|x_T^i-x_T^{i,(N)}\right|+\mathbb{E}\left|\pi\bar{x}_T-\pi^{(N)}x_T^{[N]}\right|+\mathbb{E}\left|x_T^0-x_T^{0,(N)}\right|\\ &\lesssim \int_0^T \xi_N^i(t)dt+\xi_N^i(T), \end{aligned} \tag{1.139}$$

where

$$\xi_N^i(t):=\mathbb{E}\left|x_t^i-x_t^{i,(N)}\right|+\mathbb{E}\left|x_t^0-x_t^{0,(N)}\right|+\mathbb{E}\left\|\bar{x}_t-\bar{x}_t^{(N)}\right\|_1+\mathbb{E}\left\|\bar{x}_t^{(N)}-x_t^{[N]}\right\|_1+\tau_N. \tag{1.140}$$

Again, as in the case of the major agent, we have

$$\xi_N^i(t)\le C\kappa_N^i(t)+Ce^{Ct}\int_0^t e^{-Cs}\kappa_N^i(s)ds, \tag{1.141}$$

where

$$\kappa_N^i(t)=\tau_N+\sum_{k=1}^K\frac{1}{N_k}\mathbb{E}\left|\sum_{i\in\mathscr{I}_k}\int_0^t\sigma_k(s)dw_s^i\right|+\mathbb{E}\left|e_t^{u^{i,*},u^i}\right|. \tag{1.142}$$

Note that, according to our assumptions,

$$\mathbb{E}\left|e_t^{u^{i,*},u^i}\right|\lesssim\frac{1}{N}. \tag{1.143}$$

Thus, $\kappa_N^i(t)\to 0$ as $N\to\infty$, $\forall t\in[0,T]$. Then, following similar steps as in the major agent's problem, we can show that $J_i^{(N)}(u^{i,*},u^{-i,*})\le J_i^{(N)}(u^i,u^{-i,*})+\varepsilon_N$, where $\varepsilon_N=o(\frac{1}{\sqrt{N}})+o(\tau_N)$. □

1.5 Conclusions

In this chapter, we began by developing a variational analysis framework for solving risk-sensitive optimal control problems. Subsequently, we extended our investigation to risk-sensitive MFGs that involve both a major agent and a substantial number of minor agents. In particular, we derive an ε-Nash equilibrium for LQG models, incorporating risk sensitivity through exponential-of-integral-cost formulations. The variational analysis developed in this work is applied to obtain the equilibrium strategies. Our study emphasizes the significance of incorporating risk sensitivity, particularly in economic and financial models. The obtained results contribute to a deeper understanding of the implications of risk-sensitive decision-making.

Chapter 2

Hilbert Space-Valued LQ Mean Field Games

Abstract

This chapter presents a comprehensive study of linear-quadratic (LQ) mean field games (MFGs) in Hilbert spaces, generalizing the classic LQ MFG theory to scenarios involving N agents with dynamics governed by infinite-dimensional stochastic equations. In this framework, both state and control processes of each agent take values in separable Hilbert spaces. All agents are coupled through the average state of the population which appears in their linear dynamics and quadratic cost functional. Specifically, the dynamics of each agent incorporates an infinite-dimensional noise, namely a Q-Wiener process, and an unbounded operator. The diffusion coefficient of each agent is stochastic involving the state, control, and average state processes. We first study the well-posedness of a system of N coupled semilinear infinite-dimensional stochastic evolution equations establishing the foundation of MFGs in Hilbert spaces. We then specialize to N-player LQ games described above and study the asymptotic behaviour as the number of agents, N, approaches infinity. We develop an infinite-dimensional variant of the Nash Certainty Equivalence principle and characterize a unique Nash equilibrium for the limiting MFG. Finally, we study the connections between the N-player game and the limiting MFG, demonstrating that the empirical average state converges to the mean field and that the resulting limiting best-response strategies form an ε-Nash equilibrium for the N-player game in Hilbert spaces.

2.1 Introduction

MFGs are originally developed in $\mathbb{R}^n$ spaces, which are considered as finite-dimensional. However, there are scenarios where Euclidean spaces do not adequately capture the essence of a problem such as non-Markovian systems. A clear and intuitive example is systems involving time delays. For instance, consider the interbank market model initially introduced in Carmona et al. (2015), where the logarithmic monetary reserve (state) of each bank is driven by its rate of borrowing or lending (control action). An extension of this model, studied in Fouque and Zhang (2018), incorporates a scenario where each bank must make a repayment after a specific period, drawing inspirations from Carmona et al. (2018b). This modification introduces delayed control actions into the state dynamics. Consequently, the state process is lifted to an infinite-dimensional function space (for Markovian lifting of stochastic delay equations see e.g. Da Prato and Zabczyk (2014)). However, due to a gap in the literature on infinite-dimensional MFGs, Fouque and Zhang (2018) merely assumes the existence of the mean field limit.

Beyond practical motivations, investigating MFGs in infinite-dimensional spaces offers an interesting mathematical perspective due to the distinctive treatment required compared to Euclidean spaces. In such spaces, the evolution of a stochastic process is governed by an infinite-dimensional stochastic equation (see e.g. Da Prato and Zabczyk (2014); Gawarecki and Mandrekar (2010)). These equations, also termed stochastic partial differential equations (SPDEs), form a powerful mathematical framework for modeling dynamical systems with infinite-dimensional states and noises. In other words, these equations describe the evolution of random processes in infinite-dimensional Hilbert spaces. The extension to infinite dimensions becomes essential when dealing with phenomena that exhibit spatial or temporal complexities at various scales, such as fluid dynamics and quantum field theory.

Single-agent optimal control problems in Hilbert spaces have been well-studied in the past (see, e.g., Ichikawa (1979); Hu and Tang (2022); Tessitore (1992); Nurbekyan (2012); Gomes and Nurbekyan (2015) for the LQ setting.). Recent works Dunyak and Caines (2022, 2024) develop a framework in which optimal control problems over large-size networks are approximated by infinite-dimensional stochastic equations driven by Q-noise processes in $L^2[0,1]$. Moreover, McKean-Vlasov control problems in Hilbert spaces have recently been studied in Cosso et al.

(2023). To the best of our knowledge, there are only a few works related to mean field games in Hilbert spaces. Besides Fouque and Zhang (2018) (where the noise processes are real Brownian motions), the contemporaneous work Federico et al. (2024b) studies a limiting mean field game system involving a constant volatility and a cylindrical Wiener process in Hilbert spaces, where the existence and uniqueness of the solution on both small and arbitrary time intervals are discussed.

The goal of this Chapter is to present a comprehensive study of LQ MFGs in Hilbert spaces, where the state equation of each agent is modeled by an infinite-dimensional stochastic equation (for classic LQ MFGs with $\mathbb{R}^n$-valued state and control processes extensively studied in the literature we refer to Bensoussan et al. (2016); Huang et al. (2007); Huang (2010); Firoozi et al. (2020); Liu et al. (2025); Firoozi and Caines (2020); Firoozi (2022); Huang (2021); Toumi et al. (2024); Li et al. (2023a); Firoozi and Jaimungal (2022); Firoozi et al. (2022).). Specifically, we consider an N-player game where the dynamics of agents are modeled by coupled linear stochastic evolution equations in a Hilbert space, with coupling occurring through the empirical average of the states. Each agent aims to minimize a quadratic cost functional, which is also affected by the empirical average of states[1].

The organization of this chapter is as follows. Section 2.2 introduces the notations and some preliminaries in infinite-dimensional stochastic calculus to ensure the chapter is self-contained and accessible. Section 2.3 presents the regularity results for coupled stochastic evolution equations in Hilbert spaces. Section 2.4 addresses MFGs in Hilbert spaces including optimal control in the limiting case, the fixed-point argument, and Nash equilibria. The ε-Nash property is addressed in Section 2.4.3. Finally, Section 2.5 examines a toy model inspired by the model presented in Fouque and Zhang (2018) and discusses potential extensions.

[1] This chapter is forthcoming as a published article: Liu, Hanchao, and Dena Firoozi. Hilbert Space-Valued LQ Mean Field Games: An Infinite-Dimensional Analysis, (to appear) *SIAM Journal on Control and Optimization*, 2025, arXiv:2403.01012.

2.2 Preliminaries in Infinite-Dimensional Stochastic Calculus

2.2.1 Notations and Basic Definitions

In this section, we introduces the notations and some preliminaries in infinite-dimensional stochastic calculus for both Chapter 2 and Chapter 3. We denote by $(H, \langle\cdot\rangle_H)$ and $(V, \langle\cdot\rangle_V)$ two separable Hilbert spaces (we drop the letter subscripts in the notation when they are clear from the context). By convention, we use $|\cdot|_.$ to denote the norm in usual normed spaces and $\|\cdot\|_.$ to denote the operator norm in operator spaces. Moreover, we denote the space of all bounded linear operators from V to H by $\mathscr{L}(V,H)$, which is a Banach space equipped with the norm $\|\mathrm{T}\|_{\mathscr{L}(V,H)} = \sup_{|x|_V=1} |\mathrm{T}x|_H$. Let $\{e_i\}_{i\in\mathbb{N}}$ denote an orthonormal basis of V, where $\mathbb{N}$ denotes the set of natural numbers. The space of Hilbert–Schmidt operators from V to H, denoted $\mathscr{L}_2(V,H)$, is defined as $\mathscr{L}_2(V,H) := \left\{\mathrm{T} \in \mathscr{L}(V,H) : \sum_{i\in\mathbb{N}} |\mathrm{T}e_i|_H^2 < \infty\right\}$, where $|\mathrm{T}e_i|_H = \sqrt{\langle \mathrm{T}e_i, \mathrm{T}e_i\rangle_H}$. Note that $\mathscr{L}_2(V,H)$ is a separable Hilbert space equipped with the inner product $\langle \mathrm{T}, \mathrm{S}\rangle_2 := \sum_{i\in\mathbb{N}} \langle \mathrm{T}e_i, \mathrm{S}e_i\rangle_H$ for all $\mathrm{T}, \mathrm{S} \in \mathscr{L}_2(V,H)$.. This inner product does not depend on the choice of the basis. Moreover, an operator $\mathrm{T} \in \mathscr{L}(V,H)$ is called trace class if T admits the representation $\mathrm{T}x = \sum_{i\in\mathbb{N}} b_i \langle a_i, x\rangle_V$, where $\{a_i\}_{i\in\mathbb{N}}$ and $\{b_i\}_{i\in\mathbb{N}}$ are two sequences in V and H, respectively, such that $\sum_{i\in\mathbb{N}} |a_i|_V |b_i|_H < \infty$. We denote the set of trace class operators from V to H by $\mathscr{L}_1(V,H)$, which is a separable Banach space equipped with the norm $\|\cdot\|_{\mathscr{L}_1(V,H)}$ defined as

$$\|\mathrm{T}\|_{\mathscr{L}_1(V,H)} := \inf\left\{\sum_{i\in\mathbb{N}} |a_i|_V |b_i|_H : \{a_i\}_{i\in\mathbb{N}} \in V, \{b_i\}_{i\in\mathbb{N}} \in H \text{ and } \mathrm{T}x = \sum_{i\in\mathbb{N}} b_i \langle a_i, x\rangle_V, \forall x \in V\right\}.$$

Moreover, $\mathscr{L}_1(V)$ denotes the space of operators acting on H, which may be equivalently expressed as $\mathscr{L}_1(V,V)$. For an operator $Q \in \mathscr{L}_1(V)$, the trace of Q is defined as

$$\mathrm{tr}(Q) = \sum_{i\in\mathbb{N}} \langle Qe_i, e_i\rangle .$$

The series converges absolutely, i.e., $\sum_{i\in\mathbb{N}} |\langle Qe_i, e_i\rangle| < \infty$. Furthermore, it can be shown that $|\mathrm{tr}(Q)| \leq \|Q\|_{\mathscr{L}_1(V)}$. For more details on Hilbert–Schmidt and trace class operators, we refer the reader to Peszat and Zabczyk (2007), Da Prato and Zabczyk (2014) and Fabbri et al. (2017).

We use T^* to denote an adjoint operator which is the unique operator that satisfies $\langle \mathrm{T}x, y\rangle = \langle x, \mathrm{T}^* y\rangle$ for a linear operator T and all vectors x and y in the appropriate spaces.

Let $(\mathcal{S},\mathcal{A},\mu)$ be a measure space and $(\mathscr{X},|\cdot|_{\mathscr{X}})$ be a Banach space. We denote by $L^p(\mathcal{S};\mathscr{X}), 1 \leq p \leq \infty$, the corresponding Bochner spaces, which generalize the classic $L^p(\mathcal{S};\mathbb{R})$ spaces. For details on Bochner spaces, we refer to Hytönen et al. (2016). We fix the time interval $\mathfrak{T} = [0,T]$ with $T > 0$. We denote $C(\mathfrak{T};H)$ as the set of all continuous mappings $h : \mathfrak{T} \to H$, a Banach space equipped with the supremum norm, and $C_s(\mathfrak{T};\mathscr{L}(H))$ as the set of all strongly continuous mappings $f : \mathfrak{T} \to \mathscr{L}(H)$.

Definition 1 (Q-Wiener Process Da Prato and Zabczyk (2014))**.** *Let $(\Omega,\mathfrak{F},\mathbb{P})$ be a fixed complete probability space. Additionally, $Q \in \mathscr{L}_1(V)$ and is a positive operator, i.e. Q is self-adjoint and $\langle Qx,x\rangle \geq 0, \forall x \in V$. Then, a V-valued stochastic process $W = \{W(t) : t \in \mathfrak{T}\}$ is called a Q-Wiener process if*

(i) $W(0) = 0, \mathbb{P} - a.s.$,

(ii) *W has continuous trajectories,*

(iii) *W has independent increments,*

(iv) *$\forall s,t \in \mathfrak{T}$ such that $0 < s < t$, the increment $W(t) - W(s)$ is normally distributed. More specifically, $W(t) - W(s) \sim \mathscr{N}(0,(t-s)Q)$.*

A V-valued Q-Wiener process W may be constructed as

$$W(t) = \sum_{j\in\mathbb{N}} \sqrt{\lambda_j}\beta_j(t)e_j, \quad t \in \mathfrak{T}, \tag{2.1}$$

where $\{\beta_j\}_{j\in\mathbb{N}}$ is a sequence of mutually independent real-valued Brownian motions defined on a given filtered probability space. Moreover, $\{e_j\}_{j\in\mathbb{N}}$ is a complete orthonormal basis of V and $\{\lambda_i\}_{i\in\mathbb{N}}$ is a sequence of positive numbers that diagonalize the operator Q. Moreover, $\{\lambda_i\}_{i\in\mathbb{N}}$ is a sequence of positive numbers, and $\{e_j\}_{j\in\mathbb{N}}$ is a complete orthonormal basis of V that, together, diagonalize the operator Q. In other words, for each $j \in \mathbb{N}$, λ_j is an eigenvalue of Q corresponding to the eigenvector e_j such that

$$Qe_j = \lambda_j e_j.$$

Note that in this case, we have $\mathrm{tr}(Q) = \|Q\|_{\mathscr{L}_1(V)}$ since Q is a positive operator. Moreover, the series in (2.1) converges in $L^2(\Omega, C(\mathfrak{T},V))$ (Gawarecki and Mandrekar, 2010).

Consider the probability space $(\Omega, \mathfrak{F}, \mathscr{F}, \mathbb{P})$ where the filtration $\mathscr{F} = \{\mathscr{F}_t : t \in \mathfrak{T}\}$ satisfies the usual condition. Similarly to the literature (see e.g. Da Prato and Zabczyk (2014); Gawarecki and Mandrekar (2010); Fabbri et al. (2017)), we assume that W, defined in (2.1), is a Q-Wiener process with respect to $\mathscr{F}$, i.e. $W(t)$ is $\mathscr{F}_t$-measurable, and $W(t+h) - W(t)$ is independent of $\mathscr{F}_t$, $\forall h \geq 0$, $\forall t, t+h \in \mathfrak{T}$.

We denote by $V_Q = Q^{\frac{1}{2}} V$ the separable Hilbert space endowed with the inner product

$$\langle u, v \rangle_{V_Q} = \sum_{j \in \mathbb{N}} \frac{1}{\lambda_j} \left\langle u, e_j \right\rangle_V \left\langle v, e_j \right\rangle_V, \quad u, v \in V_Q. \tag{2.2}$$

Note that

$$\begin{aligned} &\mathscr{L}(V, H) \subseteq \mathscr{L}_2(V_Q, H) \\ &\|\mathrm{T}\|^2_{\mathscr{L}_2(V_Q, H)} \leq \mathrm{tr}(Q) \, \|\mathrm{T}\|^2_{\mathscr{L}(V, H)}, \quad \forall \mathrm{T} \in \mathscr{L}(V, H) \end{aligned} \tag{2.3}$$

Below, we introduce certain spaces of stochastic processes defined on a filtered probability space $\left(\Omega, \mathfrak{F}, \mathscr{G} = \{\mathscr{G}_t : t \in \mathfrak{T}\}, \mathbb{P}\right)$ with values in a Banach space $(\mathscr{X}, |\cdot|_{\mathscr{X}})$.

- $\mathscr{M}^2_{\mathscr{G}}(\mathfrak{T}; \mathscr{X})$ denotes the Banach space of all $\mathscr{X}$-valued and $\mathscr{G}$-progressively measurable processes $x(t)$ satisfying

$$|x|_{\mathscr{M}^2_{\mathscr{G}}(\mathfrak{T}; \mathscr{X})} := \left(\mathbb{E} \int_0^T |x(t)|^2_{\mathscr{X}} \, dt \right)^{\frac{1}{2}} < \infty. \tag{2.4}$$

- $\mathscr{H}^2_{\mathscr{G}}(\mathfrak{T}; \mathscr{X})$ denotes the Banach space of all $\mathscr{X}$-valued and $\mathscr{G}$-progressively measurable processes $x(t)$ satisfying

$$|x|_{\mathscr{H}^2_{\mathscr{G}}(\mathfrak{T}; \mathscr{X})} = \left(\sup_{t \in \mathfrak{T}} \mathbb{E} \, |x(t)|^2_{\mathscr{X}} \right)^{\frac{1}{2}} < \infty. \tag{2.5}$$

Obviously, $\mathscr{H}^2_{\mathscr{G}}(\mathfrak{T}; \mathscr{X}) \subseteq \mathscr{M}^2_{\mathscr{G}}(\mathfrak{T}; \mathscr{X})$. We omit the filtration subscript for the two spaces when no possible confusion arises. Furthermore, $\mathscr{B}(\mathscr{X})$ denotes the Borel sigma-algebra on the space $\mathscr{X}$.

2.2.2 Controlled Infinite-Dimensional Linear SDEs

We denote by H, U, and V three real separable Hilbert spaces. We then introduce a controlled infinite-dimensional stochastic differential equation (SDE) as

$$\begin{aligned} dx(t) &= (Ax(t)+Bu(t)+m(t))dt+(Dx(t)+Eu(t)+v(t))dW(t), \\ x(0) &= \xi, \end{aligned} \tag{2.6}$$

where $\xi \in L^2(\Omega;H)$. Moreover, $x(t) \in H$ denotes the state and $u(t) \in U$ the control action at time t. The control process $u = \{u(t) : t \in \mathfrak{T}\}$ is assumed to be in $\mathscr{M}^2(\mathfrak{T};U)$. The Q-Wiener process W is defined on a filtered probability space $\left(\Omega, \mathfrak{F}, \{\mathscr{F}_t\}_{t\in\mathfrak{T}}, \mathbb{P}\right)$ and takes values in V. The unbounded linear operator A, with domain $\mathscr{D}(A)$, is an infinitesimal generator of a C_0-semigroup $S(t) \in \mathscr{L}(H), t \in \mathfrak{T}$. Moreover, there exists a constant M_T such that

$$\|S(t)\|_{\mathscr{L}(H)} \leq M_T, \quad \forall t \in \mathfrak{T}. \tag{2.7}$$

where $M_T := M_A e^{\alpha T}$, with $M_A \geq 1$ and $\alpha \geq 0$ Goldstein (2017). The choices of M_A and α are independent of T. Furthermore, $B \in \mathscr{L}(U,H)$, $D \in \mathscr{L}(H,\mathscr{L}(V,H))$, $E \in \mathscr{L}(U,\mathscr{L}(V,H))$, the process $m \in C(\mathfrak{T};H)$, and the process $v \in L^\infty(\mathfrak{T};\mathscr{L}(V,H))$. We focus on the mild solution of (2.6).

Definition 2 (Mild Solution of an Infinite-Dimensional SDE Da Prato and Zabczyk (2014))**.** *A mild solution of* (2.6) *is a process $x \in \mathscr{H}^2(\mathfrak{T};H)$ such that $\forall t \in \mathfrak{T}$, we have*

$$\begin{aligned} &x(t) = S(t)\xi + \int_0^t S(t-r)(Bu(r)+m(r))dr + \int_0^t S(t-r)(Dx(r)+Eu(r)+v(r))dW(r), \\ &\mathbb{P}-a.s. \end{aligned} \tag{2.8}$$

For the results on the existence and uniqueness of a mild solution to (2.6), we refer to (Fabbri et al., 2017, Section 1.4).

Remark 5. *((Da Prato and Zabczyk, 2014, Chapter 6)) Since the diffusion term in* (2.6) *takes the form of multiplicative noise, the mild solution can be equivalently expressed as*

$$x(t) = S(t)\xi + \int_0^t S(t-r)(Bu(r)+m(r))dr + \int_0^t \sum_{j=1}^{\infty} S(t-r)(D_j x(r)+E_j u(r)+v(r))d\beta_j(r), \tag{2.9}$$

where the bounded linear operators $D_j \in \mathscr{L}(H)$ and $E_j \in \mathscr{L}(U,H)$ are defined as

$$D_j x := \sqrt{\lambda_j}\,(Dx)\,e_j, \quad E_j u := \sqrt{\lambda_j}\,(Eu)\,e_j, \quad x \in H,\ u \in U. \tag{2.10}$$

Moreover, the operators $D \in \mathscr{L}(H,\mathscr{L}(V,H))$ and $E \in \mathscr{L}(U,\mathscr{L}(V,H))$ may be expressed as

$$(Dx)v = \sum_{j=1}^{\infty} \frac{1}{\sqrt{\lambda_j}} \langle v, e_j \rangle D_j x, \quad (Eu)v = \sum_{j=1}^{\infty} \frac{1}{\sqrt{\lambda_j}} \langle v, e_j \rangle E_j u, \quad v \in V. \tag{2.11}$$

Remark 6. *In general, stochastic integrals with respect to a Q-Wiener process are constructed for suitable processes which take values in $\mathscr{L}_2(V_Q,H)$, see e.g. Da Prato and Zabczyk (2014) and Gawarecki and Mandrekar (2010). However, in this chapter, as well as in many works concerning infinite-dimensional control problems, the integrand processes can only be $\mathscr{L}(V,H)$-valued. We refer to Curtain and Falb (1970) and Ichikawa (1982) for the construction of stochastic integrals for suitable $\mathscr{L}(V,H)$-valued processes. Such constructions are special cases of those presented in Da Prato and Zabczyk (2014) and Gawarecki and Mandrekar (2010).*

2.3 Coupled Controlled Stochastic Evolution Equations in Hilbert Space

In classical mean field games, the dynamics of the relevant N-player game is modeled as a system of finite-dimensional SDEs, the regularities of which are well-studied in the literature. However, in this chapter, the dynamics of the N-player game will be modeled as N coupled infinite-dimensional stochastic equations. To be more specific, the state of each agent satisfies an infinite-dimensional stochastic equation which is involved with the states of all agents. The well-posedness of such a system has not been rigorously established in the literature. Thus, we aim to address it in this section. For this purpose, we first discuss the existence of a sequence of independent Q-Wiener processes. Next, we prove the existence and uniqueness of the solution to a system of N coupled infinite-dimensional stochastic equations.

More precisely, in the classic setup of MFGs, the individual idiosyncratic noises form a sequence of independent real-valued Brownian motions. In the current context, however, we require a sequence of independent Q-Wiener processes. The following proposition examines the existence of such a sequence.

Proposition 8. *Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space and Q be a positive trace class operator on the separable Hilbert space V. Then, there exists a sequence of independent V-valued Q-Wiener processes $\{W_i\}_{i \in \mathbb{N}}$ defined on the given probability space.*

Proof. Let $W, \{\beta_j\}_{j\in\mathbb{N}}$ be the processes defined in (2.1), and the corresponding natural filtrations be defined as $\mathscr{F}^W = \{\mathscr{F}_t^W : t \in \mathfrak{T}\}$ and $\mathscr{F}^\beta = \{\mathscr{F}_t^\beta : t \in \mathfrak{T}\}$, where $\mathscr{F}_t^W = \sigma(W(s), 0 \le s \le t)$, $\mathscr{F}_t^\beta = \sigma(\beta_j(s), 0 \le s \le t, j \in \mathbb{N}) = \sigma(\bigcup_{j\in\mathbb{N}} \sigma(\beta_j(s), 0 \le s \le t))$. Subsequently, the augmented filtrations $\bar{\mathscr{F}}^W = \{\bar{\mathscr{F}}_t^W : t \in \mathfrak{T}\}$ and $\bar{\mathscr{F}}^\beta = \{\bar{\mathscr{F}}_t^\beta : t \in \mathfrak{T}\}$ consist of $\bar{\mathscr{F}}_t^W = \sigma(\mathscr{F}_t^W \cup \mathscr{N})$ and $\bar{\mathscr{F}}_t^\beta = \sigma(\mathscr{F}_t^\beta \cup \mathscr{N})$. It is evident that $\bar{\mathscr{F}}_t^W = \bar{\mathscr{F}}_t^\beta$. By applying the enumeration of $\mathbb{N} \times \mathbb{N}$ to the sequence of mutually independent Brownian motions $\{\beta_j\}_{j\in\mathbb{N}}$, we can obtain infinitely many distinct sequences of Brownian motions $\{\beta_j^i\}_{j\in\mathbb{N}} = \{\beta_1^i, \beta_2^i, \ldots, \beta_j^i, \ldots\}$, each sequence indexed by i. The real-valued Brownian motions β_j^i are mutually independent for all indices $i, j \in \mathbb{N}$. Now we construct a sequence of Q-Wiener processes $\{W_i\}_{i\in\mathbb{N}}$, where $W_i(t)$ is defined by

$$W_i(t) = \sum_{j\in\mathbb{N}} \sqrt{\lambda_j} \beta_j^i(t) e_j, \quad t \in \mathfrak{T}. \tag{2.12}$$

For our purpose, it is enough to show that the augmented filtrations $\{\bar{\mathscr{F}}^{\beta^i}\}_{i\in\mathbb{N}}$ are independent. Recall that $\mathscr{F}^{\beta^i} = \{\mathscr{F}_t^{\beta^i} : t \in \mathfrak{T}\}$, where $\mathscr{F}_t^{\beta^i} = \sigma(\bigcup_{j\in\mathbb{N}} \sigma(\beta_j^i(s), 0 \le s \le t))$ and $\{\beta_j^i\}_{j,i\in\mathbb{N}}$ are independent Brownian motions. Then the independence of $\{\mathscr{F}_t^{\beta^i}\}_{i\in\mathbb{N}}, \forall t \in \mathfrak{T}$, follows from the standard results in measure theory (see for instance (Cohn, 2013, Proposition 10.1.7)), and hence $\{\bar{\mathscr{F}}_t^{\beta^i}\}_{i\in\mathbb{N}}, \forall t \in \mathfrak{T}$ are also independent. □

It is straightforward to verify that the sequence of processes $\{W_i\}_{i\in\mathbb{N}}$, constructed in Proposition 8, are (mutually independent) Q-Wiener processes with respect to $\mathscr{F}$. This measurability arises because each W_i is constructed using a subsequence of real-valued Brownian motions generating the original Q-Wiener process W given by (2.1). Usually, this “universal" filtration $\mathscr{F}$ is larger than necessary. Below, we construct a reduced filtration.

Reduced Filtration $\mathscr{F}^{[N]} = \{\mathscr{F}_t^{[N]}, t \in \mathfrak{T}\}$: *Consider a set $\mathscr{N} = \{1, 2, ..., N\}$ and let $\{W_i\}_{i\in\mathscr{N}}$ be N independent Q-Wiener processes constructed in Proposition 8. A reduced filtration $\mathscr{F}^{[N]}$ may be constructed under which these processes are independent Q-Wiener processes. Note that the processes $\{W_i\}_{i\in\mathscr{N}}$ are constructed as described in* (2.12) *using N sequences of mutually independent Brownian motions $\{\beta_j^i\}_{j\in\mathbb{N}, i\in\mathscr{N}}$. These N sequences may be combined to form a new sequence of mutually independent Brownian motions. We then construct a new Q-Wiener process W^N using this resulting sequence as in* (2.1) *and define $\mathscr{F}^{[N]}$ as the normal filtration that makes W^N a Q-*

Wiener process. Clearly, this filtration only makes the processes $\{W_i\}_{i\in\mathscr{N}}$ independent Q-Wiener processes and can be smaller than $\mathscr{F}$.

We are now ready to introduce a system of coupled infinite-dimensional stochastic equations defined on $\left(\Omega, \mathfrak{F}, \mathscr{F}^{[N]}, \mathbb{P}\right)$ describing the temporal evolution of the vector process $\mathbf{x} = \{\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_N(t)), t \in \mathfrak{T}\}$. Note that $\mathbf{x}$ is an H^N-valued stochastic process, where H^N denotes the N-product space of H, equipped with the product norm $|\mathbf{x}(t)|_{H^N} = \left(\sum_{i\in\mathscr{N}} |x_i(t)|_H^2\right)^{\frac{1}{2}}$. Subsequently, $\mathscr{M}^2(\mathfrak{T}; H^N)$ and $\mathscr{H}^2(\mathfrak{T}; H^N)$ are defined as the spaces of all H^N-valued progressively measurable processes $\mathbf{x}$, respectively, satisfying $|\mathbf{x}|_{\mathscr{M}^2(\mathfrak{T}; H^N)} < \infty$ and $|\mathbf{x}|_{\mathscr{H}^2(\mathfrak{T}; H^N)} < \infty$.

The differential form of a system of coupled infinite-dimensional stochastic equations can be represented by

$$dx_i(t) = (Ax_i(t) + F_i(t, \mathbf{x}(t), u_i(t)))dt + B_i(t, \mathbf{x}(t), u_i(t))dW_i(t), \tag{2.13}$$

with $x_i(0) = \xi_i$, and where, as defined in (2.6), A is a C_0-semigroup generator. Moreover, the control action $u_i, i \in \mathscr{N}$, is a U-valued progressively measurable process, and the initial condition $\xi_i, i \in \mathscr{N}$, is H-valued and $\mathscr{F}_0^{[N]}$-measurable. Moreover, $\{W_i\}_{i\in\mathscr{N}}$ is a set of mutually independent Q-Wiener processes, each constructed as in Proposition 8 and applied to the filtration $\mathscr{F}^{[N]}$. Furthermore, the family of maps $F_i : \mathfrak{T} \times H^N \times U \to H$ and $B_i : \mathfrak{T} \times H^N \times U \to \mathscr{L}_2(V_Q, H), \forall i \in \mathscr{N}$, are defined for all $i \in \mathscr{N}$.

A2.3.1. *For each $i \in \mathscr{N}$, the initial condition ξ_i belongs to $L^2(\Omega; H)$ and is $\mathscr{F}_0^{[N]}$-measurable.*

We focus on the solution of (2.13) in a mild sense, which is defined below.

Definition 3. *(Mild Solution of Coupled Infinite-Dimensional SDEs) A process $\boldsymbol{x} \in \mathscr{H}^2(\mathfrak{T}; H^N)$, where $\boldsymbol{x} = \{\boldsymbol{x}(t) = (x_1(t), x_2(t), \ldots, x_N(t)), t \in \mathfrak{T}\}$, is said to be a mild solution of* (2.13) *if, for each $i \in \mathscr{N}$, the process x_i is defined $\mathbb{P}$-almost surely by the integral equation*

$$x_i(t) = S(t)\xi_i + \int_0^t S(t-r)F_i(r, \boldsymbol{x}(r), u_i(r))dr + \int_0^t S(t-r)B_i(r, \boldsymbol{x}(r), u_i(r))dW_i(r), \ \forall t \in \mathfrak{T}, \tag{2.14}$$

where $S(t)$ is the C_0-semigroup generated by A.

We make the following assumptions on the system of coupled stochastic evolution systems described by (2.13) for every $i \in \mathscr{N}$.

A2.3.2. $u_i \in \mathscr{M}^2(\mathfrak{T};U)$.

A2.3.3. *The mapping* $F_i : \mathfrak{T} \times H^N \times U \to H$ *is* $\mathscr{B}(\mathfrak{T}) \otimes \mathscr{B}(H^N) \otimes \mathscr{B}(U)/\mathscr{B}(H)$*-measurable.*

A2.3.4. *The mapping* $B_i : \mathfrak{T} \times H^N \times U \to \mathscr{L}_2(V_Q, H)$ *is* $\mathscr{B}(\mathfrak{T}) \otimes \mathscr{B}(H^N) \otimes \mathscr{B}(U)/\mathscr{B}(\mathscr{L}_2(V_Q, H))$*-measurable, where the Hilbert space* V_Q *is as defined in Section 2.2.*

A2.3.5. *There exists a constant* C *such that, for every* $t \in \mathfrak{T}$*,* $u \in U$ *and* $\boldsymbol{x}, \boldsymbol{y} \in H^N$*, we have*

$$|F_i(t,\boldsymbol{x},u) - F_i(t,\boldsymbol{y},u)|_H + \|B_i(t,\boldsymbol{x},u) - B_i(t,\boldsymbol{y},u)\|_{\mathscr{L}_2} \le C\,|\boldsymbol{x} - \boldsymbol{y}|_{H^N},$$

$$|F_i(t,\boldsymbol{x},u)|_H^2 + \|B_i(t,\boldsymbol{x},u)\|_{\mathscr{L}_2}^2 \le C^2 \left(1 + |\boldsymbol{x}|_{H^N}^2 + |u|_U^2\right).$$

The following theorem establishes the existence and uniqueness of a mild solution to the coupled stochastic evolution equations given by (2.13). This result extends Theorem 7.2 in Da Prato and Zabczyk (2014), which addresses the existence and uniqueness of a mild solution for a single stochastic evolution equation without coupling.

Theorem 9. *(Existence and Uniqueness of a Mild Solution) Under A2.3.2-A2.3.5, the set of coupled stochastic evolution equations given by* (2.13) *admits a unique mild solution in the space* $\mathscr{H}^2(\mathfrak{T};H^N)$.

Proof. The existence and uniqueness of a mild solution can be established through the classic fixed-point argument for a mapping from $\mathscr{H}^2(\mathfrak{T};H^N)$ to $\mathscr{H}^2(\mathfrak{T};H^N)$. To this end, for any given element $\mathbf{x} \in \mathscr{H}^2(\mathfrak{T};H^N)$, the operator Γ is defined component-wise as

$$\Gamma\mathbf{x}(t) = (\Gamma_1\mathbf{x}(t), \Gamma_2\mathbf{x}(t), \ldots, \Gamma_N\mathbf{x}(t)),$$

where each component $\Gamma_i\mathbf{x}(t)$ is represented by the integral equation

$$\Gamma_i\mathbf{x}(t) = S(t)\xi_i + \int_0^t S(t-r)F_i(r,\mathbf{x}(r),u_i(r))dr + \int_0^t S(t-r)B_i(r,\mathbf{x}(r),u_i(r))dW_i(r). \tag{2.15}$$

We show that Γ indeed maps $\mathscr{H}^2(\mathfrak{T};H^N)$ into itself. The measurability of (2.15) as an H-valued process is established using the standard argument found in Da Prato and Zabczyk (2014) and Gawarecki and Mandrekar (2010), based on our assumptions. This is because F_i and B_i are progressively measurable processes, valued in H and $\mathscr{L}_2(V_Q,H)$, respectively. We use the inequality

$|a+b+c|^2 \leq 3|a|^2+3|b|^2+3|c|^2$, for each $i \in \mathscr{N}$ and $t \in \mathfrak{T}$, to get

$$\mathbb{E}\,|\Gamma_i \mathbf{x}(t)|_H^2 \leq 3\mathbb{E}\,|S(t)\xi_i|_H^2 + 3\mathbb{E}\left[\left|\int_0^t S(t-r)F_i(r,\mathbf{x}(r),u_i(r))dr\right|_H^2\right] + 3\mathbb{E}\left[\left|\int_0^t S(t-r)B_i(r,\mathbf{x}(r),u_i(r))dW_i(r)\right|_H^2\right]. \quad (2.16)$$

For the Bochner integral in (2.16), we have

$$\begin{aligned}\mathbb{E}\left[\left|\int_0^t S(t-r)F_i(r,\mathbf{x}(r),u_i(r))dr\right|_H^2\right] &\leq T\mathbb{E}\left[\int_0^t |S(t-r)F_i(r,\mathbf{x}(r),u_i(r))|_H^2\,dr\right] \\ &\leq T\mathbb{E}\left[\int_0^t \|S(t-r)\|_{\mathscr{L}(H)}^2\,|F_i(r,\mathbf{x}(r),u_i(r))|_H^2\,dr\right] \\ &\leq TM_T^2C^2\mathbb{E}\left[\int_0^t \left(|(\mathbf{x}(r)|_{H^N}^2+|u_i(r)|_U^2+1\right)dr\right], \quad (2.17)\end{aligned}$$

where the first inequality results from the Cauchy–Schwarz inequality.

For the stochastic integral in (2.16), we have

$$\begin{aligned}\mathbb{E}\left[\left|\int_0^t S(t-r)B_i(r,\mathbf{x}(r),u_i(r))dW_i(r)\right|_H^2\right] &\leq C^{'}\mathbb{E}\left[\int_0^t \|B_i(r,\mathbf{x}(r),u_i(r))\|_{\mathscr{L}_2}^2\,dr\right] \\ &\leq C^2C^{'}\mathbb{E}\left[\int_0^t \left(|\mathbf{x}(r)|_{H^N}^2+|u_i(r)|_U^2+1\right)dr\right], \quad (2.18)\end{aligned}$$

where the first inequality is obtained by the standard approximation technique for stochastic convolutions (see e.g. (Da Prato and Zabczyk, 2014, Theorem 4.36) and (Gawarecki and Mandrekar, 2010, Corollary 3.2)) in the current context for every $t \in \mathfrak{T}$ and $i \in \mathscr{N}$.

Note that the constant $C^{'}$ only depends on T and M_T (defined in (2.7)). Substituting (2.17) and (2.18) in (2.16), we obtain

$$\begin{aligned}\mathbb{E}\,|\Gamma_i\mathbf{x}(t)|_H^2 &\leq 3\mathbb{E}\,|S(t)\xi_i|_H^2 + 3(C^2C^{'}+TM_T^2C^2)\mathbb{E}\left[\int_0^t \left(|\mathbf{x}(r)|_{H^N}^2+|u_i(r)|_U^2+1\right)dr\right] \\ &\leq 3\mathbb{E}\,|S(t)\xi_i|_H^2 + 3(C^2C^{'}+TM_T^2C^2)\mathbb{E}\left[\int_0^T \left(|\mathbf{x}(r)|_{H^N}^2+|u_i(r)|_U^2+1\right)dr\right], \ \forall t \in \mathfrak{T}.\end{aligned} \quad (2.19)$$

Hence,

$$\begin{aligned}\sum_{i\in\mathscr{N}}\mathbb{E}\,|\Gamma_i\mathbf{x}(t)|_H^2 \leq{}& 3M_T^2\sum_{i\in\mathscr{N}}\mathbb{E}\,|\xi_i|_H^2 + 3(C^2C^{'}+TM_T^2C^2)\sum_{i\in\mathscr{N}}\mathbb{E}\left[\int_0^T \left(|u_i(r)|_U^2\right)dr\right] \\ &+3N(C^2C^{'}+TM_T^2C^2)\mathbb{E}\left[\int_0^T \left(|\mathbf{x}(r)|_{H^N}^2+1\right)dr\right], \quad \forall t \in \mathfrak{T}. \quad (2.20)\end{aligned}$$

From (2.20), we have

$$|\Gamma \mathbf{x}|_{\mathscr{H}^2(\mathfrak{T};H^N)} = \left(\sup_{t\in\mathfrak{T}} \sum_{i\in\mathscr{N}} \mathbb{E}\,|\Gamma_i \mathbf{x}(t)|_H^2\right)^{\frac{1}{2}} < \infty.$$

Thus, the transformation Γ is well-defined and maps $\mathscr{H}^2(\mathfrak{T};H^N)$ into itself. The remaining part of the proof is to show that the mapping Γ is a contraction, that is, for any two elements $\mathbf{x},\mathbf{y} \in \mathscr{H}^2(\mathfrak{T};H^N)$, it holds that

$$|\Gamma \mathbf{y} - \Gamma \mathbf{x}|_{\mathscr{H}^2(\mathfrak{T};H^N)} < |\mathbf{y} - \mathbf{x}|_{\mathscr{H}^2(\mathfrak{T};H^N)}.$$

Using the inequlity $|a+b|^2 \leq 2|a|^2 + 2|b|^2$, for each $i \in \mathscr{N}$ and $t \in \mathfrak{T}$, we can write

$$\begin{aligned}\mathbb{E}\,|\Gamma_i \mathbf{y}(t) - \Gamma_i \mathbf{x}(t)|_H^2 \leq & 2\mathbb{E}\left[\left|\int_0^t S(t-r)(F_i(r,\mathbf{x}(r),u_i(r)) - F_i(r,\mathbf{y}(r),u_i(r)))dr\right|_H^2\right] \\ & + 2\mathbb{E}\left[\left|\int_0^t S(t-r)(B_i(r,\mathbf{x}(r),u_i(r)) - B_i(r,\mathbf{y}(r),u_i(r)))dW_i(r)\right|_H^2\right]. \quad (2.21)\end{aligned}$$

For the first term on the RHS of (2.21), $\forall t \in \mathfrak{T}$, we obtain

$$\begin{aligned}\mathbb{E}\left[\left|\int_0^t S(t-r)\left(F_i(r,\mathbf{x}(r),u_i(r)) - F_i(r,\mathbf{y}(r),u_i(r))\right)dr\right|_H^2\right] & \leq C^2 M_T^2 T\,\mathbb{E}\left[\int_0^T |\mathbf{x}(t) - \mathbf{y}(t)|_{H^N}^2\,dt\right] \\ & \leq C^2 M_T^2 T^2 \sup_{t\in\mathfrak{T}} \mathbb{E}\,|\mathbf{x}(t) - \mathbf{y}(t)|_{H^N}^2 \\ & \leq C^2 M_T^2 T^2\,|\mathbf{x} - \mathbf{y}|_{\mathscr{H}^2(\mathfrak{T};H^N)}^2. \quad (2.22)\end{aligned}$$

Similarly, for the second term on the RHS of (2.21) we have

$$\begin{aligned}& \mathbb{E}\left[\left|\int_0^t S(t-r)(B_i(r,\mathbf{x}(r),u_i(r)) - B_i(r,\mathbf{y}(r),u_i(r)))dW_i(r)\right|_H^2\right] \\ & \leq C^{'}\mathbb{E}\left[\int_0^T \|B_i(r,\mathbf{x}(r),u_i(r)) - B_i(r,\mathbf{y}(r),u_i(r))\|_{\mathscr{L}_2}^2\,dr\right] \\ & \leq M_T^2 C^{'} C^2 T\,(\sup_{t\in\mathfrak{T}} \mathbb{E}\,|\mathbf{x}(t) - \mathbf{y}(t)|_{H^N}^2) \\ & \leq M_T^2 C^{'} C^2 T\,|\mathbf{x} - \mathbf{y}|_{\mathscr{H}^2(\mathfrak{T};H^N)}^2, \quad \forall t \in \mathfrak{T}. \quad (2.23)\end{aligned}$$

Based on (2.21) to (2.23), we obtain

$$\sum_{i\in\mathscr{N}} \mathbb{E}\,|\Gamma_i \mathbf{y}(t) - \Gamma_i \mathbf{x}(t)|_H^2 \leq 2NM_T^2 C^2 T(C^{'} + T)\,|\mathbf{x} - \mathbf{y}|_{\mathscr{H}^2(\mathfrak{T};H^N)}^2, \quad \forall t \in \mathfrak{T},$$

and subsequently

$$|\Gamma \mathbf{y} - \Gamma \mathbf{x}|^2_{\mathscr{H}^2(\mathfrak{T};H^N)} \leq 2NM_T^2 C^2 T(C' + T)\,|\mathbf{x} - \mathbf{y}|^2_{\mathscr{H}^2(\mathfrak{T};H^N)}. \tag{2.24}$$

By employing the same argument as in Theorem 7.2 of Da Prato and Zabczyk (2014), if T is sufficiently small, then the mapping Γ is a contraction. We apply this reasoning on the intervals $\big[0,\widetilde{T}\big], \big[\widetilde{T},2\widetilde{T}\big], \cdots, \big[(n-1)\widetilde{T},T\big]$, where $\widetilde{T}$ satisfies (2.24) and $n\widetilde{T} = T$. □

Remark 7. *An alternative approach to prove Theorem 9 could involve formulating the set of N stochastic evolution equations as a single H^N-valued equation. This requires defining appropriate operators between the associated spaces and verifying that a valid H^N-valued Wiener process can be constructed using $\{W_i\}_{i\in\mathscr{N}}$. Such a reformulation also involves technical considerations. Following this, the existence and uniqueness of the solution may be established by adapting existing results related to single stochastic evolution equations.*

2.4 Hilbert Space-Valued LQ Mean Field Games

2.4.1 *N*-Player Game

We consider a differential game in Hilbert spaces defined on $\big(\Omega,\mathfrak{F},\mathscr{F}^{[N]},\mathbb{P}\big)$, where $\mathscr{F}^{[N]}$ is constructed in Section 2.3. This game involves N asymptotically negligible (minor) agents, whose dynamics are governed by a system of coupled stochastic evolution equations, each given by the linear form of (2.13). More precisely, the dynamics of a representative agent indexed by i, $i \in \mathscr{N}$, are given by

$$dx_i(t) = (Ax_i(t) + Bu_i(t) + F_1 x^{(N)}(t))dt + (Dx_i(t) + Eu_i(t) + F_2 x^{(N)}(t) + \sigma)dW_i(t), \tag{2.25}$$
$$x_i(0) = \xi_i,$$

where the agents are coupled through the term $x^{(N)}(t) := \frac{1}{N}(\sum_{i\in\mathscr{N}} x_i(t))$, which represents the average state of the N agents. We assume that all agents are homogeneous and share the same operators. Specifically, $F_1 \in \mathscr{L}(H)$, $F_2 \in \mathscr{L}(H;\mathscr{L}(V;H))$ and $\sigma \in \mathscr{L}(V;H)$ and all other operators are as defined in (2.6). In addition to A2.3.1, we impose the following assumption for the initial conditions.

A2.4.1. *The initial conditions* $\{\xi_i\}_{i\in\mathscr{N}}$ *are i.i.d. with* $\mathbb{E}[\xi_i]=\bar{\xi}$.

A2.4.2. *(Filtration & Admissible Control) The filtration available to agent i,* $i\in\mathscr{N}$*, is* $\mathscr{F}^{[N]}$*. Subsequently, the set of admissible control actions for agent i, denoted by* $\mathscr{U}^{[N]}$*, is defined as the collection of* $\mathscr{F}^{[N]}$*-adapted control laws* u^i *that belong to* $\mathscr{M}^2(\mathfrak{T};U)$.

Clearly, the system described by (2.25) satisfies the assumptions A2.3.2-A2.3.5 and its well-posedness is ensured by Theorem 9.

Moreover, agent i, $i\in\mathscr{N}$, aims to minimize the cost functional

$$J_i^{[N]}(u_i,u_{-i})=\mathbb{E}\int_0^T\left(\left|M^{\frac{1}{2}}\left(x_i(t)-\widehat{F}_1x^{(N)}(t)\right)\right|^2+|u_i(t)|^2\right)dt+\mathbb{E}\left|G^{\frac{1}{2}}\left(x_i(T)-\widehat{F}_2x^{(N)}(T)\right)\right|^2, \tag{2.26}$$

where M and G are positive operators on H, and $\widehat{F}_1,\widehat{F}_2\in\mathscr{L}(H)$. We note that a positive operator can be added to the quadratic term associated with the control process in the cost functional. However, this operator must satisfy specific conditions to ensure that its inverse is also a well-defined positive operator. To avoid further complexity in the notation, we use the identity operator in this work.

In general, solving the N-player differential game described in this section becomes challenging, even for moderate values of N and for finite-dimensional cases. The interactions between agents lead to a large-scale optimization problem, where each agent needs to observe the states of all other interacting agents. To address the dimensionality and the information restriction, following the classical MFG methodology, we investigate the limiting problem as the number of agents N tends to infinity. In this limiting model, the average behavior of the agents, known as the mean field, can be mathematically characterized, simplifying the problem. Specifically, in the limiting case, a generic agent interacts with the mean field, rather than a large number of agents. Furthermore, the mean field happens to coincide with the mean state of the agent. In the subsequent sections, we develop the Nash Certainty Equivalence principle and characterize a Nash equilibrium for the limiting game in Hilbert spaces. We then demonstrate that this equilibrium yields an ε-Nash equilibrium for the original N-player game.

2.4.2 Limiting Game

In this section we present the limiting game which reflects the scenario where, in system (2.25)-(2.26), the number of agents N tends to infinity. In this case, the optimization problem faced by a representative agent i is described as follows. Specifically, the dynamics of a representative agent, indexed by i, is given by

$$
\begin{aligned}
dx_i(t) &= (Ax_i(t) + Bu_i(t) + F_1\bar{x}(t))dt + (Dx_i(t) + Eu_i(t) + F_2\bar{x}(t) + \sigma)dW_i(t), \\
x_i(0) &= \xi_i,
\end{aligned}
\tag{2.27}
$$

and the cost functional to be minimized by agent i by

$$
J_i^\infty(u_i) = \mathbb{E}\int_0^T \left(\left| M^{\frac{1}{2}} \left(x_i(t) - \widehat{F}_1\bar{x}(t) \right) \right|^2 + |u_i(t)|^2 \right) dt + \mathbb{E}\left| G^{\frac{1}{2}} \left(x_i(T) - \widehat{F}_2\bar{x}(T) \right) \right|^2, \tag{2.28}
$$

where $\bar{x}(t)$ represents the coupling term in the limit and is termed the mean field. In this context, on the one hand, a Nash equilibrium for the system consists of the best response strategies of the agents to the mean field $\bar{x}(t)$. On the other hand, in the equilibrium where all agents follow Nash strategies, together they replicate the mean field, i.e. $\frac{1}{N}\sum_{i\in\mathscr{N}} x_i(t) \xrightarrow{\text{q.m.}} \bar{x}(t)$. We impose the following assumption for the limiting problem.

A2.4.3. *(Filtration & Admissible Control) The filtration $\mathscr{F}^{i,\infty}$ of agent i satisfies the usual conditions and ensures that W_i is a Q-Wiener process and that the initial condition ξ_i is $\mathscr{F}_0^{i,\infty}$-measurable. Subsequently, the set of admissible control actions for agent i, denoted by $\mathscr{U}_i$, is defined as the collection of $\mathscr{F}^{i,\infty}$-adapted control laws u^i that belong to $\mathscr{M}^2(\mathfrak{T};U)$.*

We first, in Section 2.4.2, treat the interaction term as an input $g \in C(\mathfrak{T};H)$ and solve the resulting optimal control problem for a representative agent given by the dynamics

$$
\begin{aligned}
dx_i(t) &= (Ax_i(t) + Bu_i(t) + F_1 g(t))dt + (Dx_i(t) + Eu_i(t) + F_2 g(t) + \sigma)dW_i(t), \\
x_i(0) &= \xi_i,
\end{aligned}
\tag{2.29}
$$

and the cost functional

$$
J(u_i) = \mathbb{E}\int_0^T \left(\left| M^{\frac{1}{2}} \left(x_i(t) - \widehat{F}_1 g(t) \right) \right|^2 + |u_i(t)|^2 \right) dt + \mathbb{E}\left| G^{\frac{1}{2}} \left(x_i(T) - \widehat{F}_2 g(T) \right) \right|^2. \tag{2.30}
$$

Then, in Section 2.4.2, we address the consistency condition described by

$$
\mathbb{E}\left[x_i^\circ(t)\right] = g(t), \quad \forall i \in \mathscr{N}, t \in \mathfrak{T}, \tag{2.31}
$$

where x_i° represents the optimal state of agent i corresponding to the control problem described by (2.29)-(2.30). Finally, in Section 2.4.3, we show that $\frac{1}{N}\sum_{i\in\mathscr{N}} x_i^\circ(t) \xrightarrow{\text{q.m.}} \mathbb{E}[x_i^\circ(t)]$.

Due to the symmetry among all agents, we drop the index i in Section 2.4.2 and Section 2.4.2, where we discuss the optimal control problem of individual agents and the relevant fixed-point problem. However, in Section 2.4.2, where Nash equilibrium is discussed, we use the index i to effectively distinguish between agents by their independent trajectories and respective filtrations. Before addressing the limiting problem in these sections, we will introduce, in the next section, some mappings and their Riesz representations that are essential for the discussions.

Mappings Associated with Riesz Representations

In this section, we introduce multiple mappings and their associated Riesz representations that will be used throughout the remainder of the chapter. These mappings are the same as those defined in Ichikawa (1979). However, since the solution of the limiting problem heavily relies on these mappings, we include more details here. We note that in the special case where the state and control processes do not appear in the diffusion coefficient of each agent, these mappings are not required for the analysis.

Recall that Q is a positive trace class operator on the Hilbert space V. For any $\mathscr{R} \in \mathscr{L}(H)$, it can be easily verified that the following expressions are bounded bilinear functionals on their corresponding product spaces:

$$\begin{aligned}
&\mathrm{tr}((Eu)^*\mathscr{R}(Dx)Q), \quad \forall x \in H, u \in U, \\
&\mathrm{tr}((Dx)^*\mathscr{R}(Dy)Q), \quad \forall x, y \in H, \\
&\mathrm{tr}((Eu)^*\mathscr{R}(Ev)Q), \quad \forall u, v \in U.
\end{aligned}$$

Moreover, for any $\mathscr{P} \in \mathscr{L}(H;V)$, the expressions below are bounded linear functionals on H and U, respectively:

$$\begin{aligned}
&\mathrm{tr}(\mathscr{P}(Dx)Q), \quad \forall x \in H, \\
&\mathrm{tr}(\mathscr{P}(Eu)Q), \quad \forall u \in U.
\end{aligned}$$

Definition 4. *(Riesz Mappings) Using the Riesz representation theorem the mappings* $\Delta_1 : \mathscr{L}(H) \to$

$\mathscr{L}(H;U)$, $\Delta_2 : \mathscr{L}(H) \to \mathscr{L}(H)$ and $\Delta_3 : \mathscr{L}(H) \to \mathscr{L}(U)$ are defined such that

$$\begin{aligned}
\operatorname{tr}((Eu)^*\mathscr{R}(Dx)Q) &= \langle \Delta_1(\mathscr{R})x,u\rangle, \quad \forall x \in H, \forall u \in U, \quad \Delta_1(\mathscr{R}) \in \mathscr{L}(H;U),\\
\operatorname{tr}((Dx)^*\mathscr{R}(Dy)Q) &= \langle \Delta_2(\mathscr{R})x,y\rangle, \quad \forall x,y \in H, \qquad\quad \Delta_2(\mathscr{R}) \in \mathscr{L}(H),\\
\operatorname{tr}((Eu)^*\mathscr{R}(Ev)Q) &= \langle \Delta_3(\mathscr{R})u,v\rangle, \quad \forall u,v \in U, \qquad\quad \Delta_3(\mathscr{R}) \in \mathscr{L}(U).
\end{aligned}$$

Similarly, the mappings $\Gamma_1 : \mathscr{L}(H;V) \to H$ and $\Gamma_2 : \mathscr{L}(H;V) \to U$ are defined such that

$$\begin{aligned}
\operatorname{tr}(\mathscr{P}(Dx)Q) &= \langle \Gamma_1(\mathscr{P}),x\rangle, \quad \forall x \in H, \quad \Gamma_1(\mathscr{P}) \in H,\\
\operatorname{tr}(\mathscr{P}(Eu)Q) &= \langle \Gamma_2(\mathscr{P}),u\rangle, \quad \forall u \in U, \quad \Gamma_2(\mathscr{P}) \in U.
\end{aligned}$$

In the following proposition, we establish the linearity and boundness of the introduced Riesz mappings.

Theorem 10. *The mappings $\Delta_k, k = 1,2,3,$ and $\Gamma_k, k = 1,2,$ are linear and bounded. Specifically, we have*

$$\Gamma_1 \in \mathscr{L}(\mathscr{L}(H);H) \quad \text{and} \quad \|\Gamma_1\| \le R_1 \quad \text{with} \quad R_1 = \operatorname{tr}(Q)\|D\|, \tag{2.32}$$

$$\Gamma_2 \in \mathscr{L}(\mathscr{L}(H);U) \quad \text{and} \quad \|\Gamma_2\| \le R_2 \quad \text{with} \quad R_2 = \operatorname{tr}(Q)\|E\|, \tag{2.33}$$

$$\Delta_1 \in \mathscr{L}(\mathscr{L}(H);\mathscr{L}(H;U)) \quad \text{and} \quad \|\Delta_1\| \le R_3 \quad \text{with} \quad R_3 = \operatorname{tr}(Q)\|D\|\|E\|, \tag{2.34}$$

$$\Delta_2 \in \mathscr{L}(\mathscr{L}(H);\mathscr{L}(H)) \quad \text{and} \quad \|\Delta_2\| \le R_4 \quad \text{with} \quad R_4 = \operatorname{tr}(Q)\|D\|^2, \tag{2.35}$$

$$\Delta_3 \in \mathscr{L}(\mathscr{L}(H);\mathscr{L}(U)) \quad \text{and} \quad \|\Delta_3\| \le R_5 \quad \text{with} \quad R_5 = \operatorname{tr}(Q)\|E\|^2. \tag{2.36}$$

Moreover, for any positive operator $\mathscr{R} \in \mathscr{L}(H)$ we have

$$\left\|(I+\Delta_3(\mathscr{R}))^{-1}(B^*\mathscr{R}+\Delta_1(\mathscr{R}))\right\| \le R_6\|\mathscr{R}\|, \quad \text{with} \quad R_6 = \|B\| + R_3. \tag{2.37}$$

Proof. We present the demonstration only for the Riesz mapping Δ_1 and the demonstrations for other Riesz mappings follow by a similar argument. To verify the linear property, consider $\mathscr{R}_1, \mathscr{R}_2 \in \mathscr{L}(H)$ and $a,b \in \mathbb{R}$. For all $x \in H$ and $u \in U$, it is straightforward to check that

$$\operatorname{tr}\big((Eu)^*(a\mathscr{R}_1+b\mathscr{R}_2)(Dx)Q\big) = a\operatorname{tr}\big((Eu)^*(\mathscr{R}_1)(Dx)Q\big)+b\operatorname{tr}\big((Eu)^*(\mathscr{R}_2)(Dx)Q\big). \tag{2.38}$$

Thus, for all $x \in H$ and $u \in U$, we have

$$\langle \Delta_1(a\mathscr{R}_1+b\mathscr{R}_2)x,u\rangle = \langle (a\Delta_1(\mathscr{R}_1)+b\Delta_1(\mathscr{R}_2))x,u\rangle, \tag{2.39}$$

from which we conclude that $\Delta_1(a\mathscr{R}_1 + b\mathscr{R}_2) = a\Delta_1(\mathscr{R}_1) + b\Delta_1(\mathscr{R}_2)$. Next, by simple calculations, for all $x \in H$, for all $u \in U$, and $\mathscr{R} \in \mathscr{L}(H)$, we have

$$|\mathrm{tr}((Eu)^*\mathscr{R}(Dx)Q)| \leq \|(Eu)^*\mathscr{R}(Dx)\|_{\mathscr{L}(V,H)} \|Q\|_{\mathscr{L}_1(V)} \leq R_3 \|\mathscr{R}\| |x|_H |u|_U . \tag{2.40}$$

Thus, by the Riesz representation theorem, we have

$$\|\Delta_1(\mathscr{R})\| = \sup_{|x|_H=1, |u|_U=1} \left|\mathrm{tr}\big((Eu)^*\mathscr{R}(Dx)Q\big)\right| \leq R_3 \|\mathscr{R}\|, \tag{2.41}$$

which implies that $\|\Delta_1\| \leq R_3$. For the second part, we can easily verify that if $\mathscr{R}$ is a positive operator on H, then $\Delta_3(\mathscr{R})$ is also a positive operator on U. Consequently, it follows that $\left\|(I+\Delta_3(\mathscr{R}))^{-1}(t)\right\| \leq 1, \forall t \in \mathfrak{T}$. Thus, we have

$$\left\|(I+\Delta_3(\mathscr{R}))^{-1}(B^*\mathscr{R} + \Delta_1(\mathscr{R}))\right\| \leq \|B^*\mathscr{R} + \Delta_1(\mathscr{R})\| \leq R_6 \|\mathscr{R}\|. \tag{2.42}$$

□

Optimal Control of Individual Agents

In this section, we address the optimal control problem for a representative agent described by (2.29)-(2.30). Infinite-dimensional LQ optimal control problems have been studied in works such as Ichikawa (1979); Tessitore (1992); Hu and Tang (2022). We address our specific problem by presenting the results in a compact and self-contained manner, relying on the existing literature. Due to the symmetry among all agents, we drop the index i.

Theorem 11 (Optimal Control Law)**.** *Consider the mappings $\Delta_k, k = 1,2,3$, and $\Gamma_k, k = 1,2$, given in Definition 4, and suppose A2.4.1 holds. Then, the optimal control law u° for the Hilbert-space valued system described by* (2.29)-(2.30) *is given by*

$$u^\circ(t) = -K^{-1}(T-t)\left[L(T-t)x(t) + \Gamma_2\big(p^*(t)\Pi(T-t)\big) + B^*q(T-t)\right], \tag{2.43}$$

where

$$K(t) = I + \Delta_3(\Pi(t)), \quad L(t) = B^*\Pi(t) + \Delta_1(\Pi(t)), \quad p(t) = F_2 g(t) + \sigma, \tag{2.44}$$

with $\Pi \in C_s(\mathfrak{T};\mathscr{L}(H))$, such that $\Pi(t)$ is a positive operator $\forall t \in \mathfrak{T}$, and $q \in C(\mathfrak{T};H)$, each satisfying, respectively, the operator differential Riccati equation and the linear evolution equation, *given by*

$$\frac{d}{dt}\langle\Pi(t)x,x\rangle = 2\langle\Pi(t)x,Ax\rangle - \left\langle L^*(t)K^{-1}(t)L(t)x,x\right\rangle + \langle\Delta_2(\Pi(t))x,x\rangle + \langle Mx,x\rangle,$$
$$\Pi(0) = G, \quad x\in\mathscr{D}(A), \tag{2.45}$$
$$\dot{q}(t) = \left(A^* - L^*(t)K^{-1}(t)B^*\right)q(t) + \Gamma_1\left(p^*(T-t)\Pi(t)\right) - L^*(t)K^{-1}(t)\Gamma_2\left(p^*(T-t)\Pi(t)\right)$$
$$+ (\Pi(t)F_1 - M\widehat{F}_1)g(T-t), \quad q(0) = -G\widehat{F}_2 g(T). \tag{2.46}$$

Proof. Similar to finite-dimensional LQ control problems, the optimal control law involves a Riccati equation but in the operator form, and an offset equation which is an H-valued deterministic evolution equation.

Consider the operator differential Riccati equation given by (2.45). We refer to Hu and Tang (2022) for the existence and uniqueness of the solution $\Pi(t)$ to (2.45), as our problem falls within the framework studied in Hu and Tang (2022). Specifically, from (2.9), the mild solution of (2.29) is in the same form as that of (Hu and Tang, 2022, eq. (2.1)). Note that the deterministic terms in the model (2.29)-(2.30), i.e. $F_1g(t)$, $F_2g(t)+\sigma$, $\widehat{F}_1g(t)$ and $\widehat{F}_2g(T)$, do not affect the Riccati equation (2.45). Moreover, it can be easily verified that D_j and E_j introduced in Remark 5 satisfy the conditions specified by (Hu and Tang, 2022, eq. (2.3)) and (Hu and Tang, 2022, eq. (2.5)), respectively.The solution $\Pi(t)$ is a positive operator on H for each $t\in\mathfrak{T}$, and is strongly continuous on $\mathfrak{T}$. Moreover, it is uniformly bounded over the interval $\mathfrak{T}$, such that $\|\Pi(t)\|_{\mathscr{L}(H)}\leq C$ for all $t\in\mathfrak{T}$. For the case where $E=0$ in (2.29), we refer to works such as Ichikawa (1979), Tessitore (1992) and Da Prato (1984).

Next, consider the (deterministic) linear evolution equation given by (2.46).

Given that $\Pi(t)$ and $F_2g(t)+\sigma$ are bounded on the interval $\mathfrak{T}$, the terms $\Gamma_1((F_2g(T-t)+\sigma)^*\Pi(t))$ and $\Gamma_2((F_2g(T-t)+\sigma)^*\Pi(t))$ are also bounded over $\mathfrak{T}$. Consequently, the existence and uniqueness of a mild solution to (2.46) follow from the established results for linear evolution equations Diagana (2018). Specifically, this solution lies within the space $C(\mathfrak{T};H)$.

Now, we begin to solve the corresponding control problem, described by (2.29)–(2.30). The mild solution of (2.6) is expressed as

$$x(t) = S(t)\xi + \int_0^t S(t-r)(Bu(r)+F_1g(r))dr + \int_0^t S(t-r)(Dx(r)+Eu(r)+F_2g(r)+\sigma)dW(r). \tag{2.47}$$

We introduce a standard approximating sequence given by

$$
\begin{aligned}
dx_n(t) &= (Ax_n(t) + J_n(Bu(t) + F_1 g(t)))dt + J_n(Dx_n(t) + Eu(t) + F_2 g(t) + \sigma)dW(t), \\
x_n(0) &= J_n \xi,
\end{aligned}
\tag{2.48}
$$

where $J_n = nR(n,A)$, with $R(n,A) = (A - nI)^{-1}$ being the resolvent operator of A, is the Yosida approximation of A. For more details on Yosida approximation we refer to Da Prato and Zabczyk (2014); Fabbri et al. (2017).

Then, the following two standard results hold for (2.47) and (2.48) (see e.g. Ichikawa (1982, 1979)). Firstly, the approximating SDE (2.48) admits a strong solution, represented as

$$
x_n(t) = J_n \xi + \int_0^t \big(Ax_n(r) + J_n(Bu(r) + F_1 g(r))\big)dr + \int_0^t J_n(Dx_n(r) + Eu(r) + F_2 g(r) + \sigma)dW(r).
\tag{2.49}
$$

This means that for each n, there exists an adapted process x_n that satisfies the integral form of the approximating SDE almost surely for all t in the interval $\mathfrak{T}$. Secondly, the sequence of solutions $\{x_n\}_{n\in\mathbb{N}}$ converges to the mild solution x of the original SDE in the mean square sense uniformly over the interval $\mathfrak{T}$, i.e.

$$
\lim_{n\to\infty} \sup_{0\le t\le T} \mathbb{E}\,|x_n(t) - x(t)|^2 = 0.
\tag{2.50}
$$

The rest of the proof follows the standard methodology as in Ichikawa (1979), summarized in the following two steps:

(i) Apply Itô's lemma ((Ichikawa, 1979, Theorem 2.1)) to $(\langle \Pi(T-t)x_n(t), x_n(t)\rangle + 2\langle q(T-t), x_n(t)\rangle)$, integrate from 0 to T, and substitute the corresponding terms using (2.45)-(2.46), (2.49) and Definition 4. Then take the expectation of both sides of the resulting equation.

(ii) Take the limit as $n \to \infty$ of both sides of the expression derived in step (i) and use the convergence property (2.50).

Note that, compared to finite-dimensional LQ control problems (see e.g. (Yong and Zhou, 1999, Section 6.6)), we must additionally implement step (ii). This is necessary because, in general, Itô's lemma applies only to the strong solutions of infinite-dimensional stochastic equations.

Finally, by some standard algebraic manipulations, we obtain

$$J(u) = \mathbb{E}\langle \Pi(T)\xi, \xi\rangle + 2\mathbb{E}\langle q(T), \xi\rangle + 2\left\langle G\widehat{F}_2 g(T), \widehat{F}_2 g(T)\right\rangle + \mathbb{E}\Bigg[\int_0^T \Big| K^{\frac{1}{2}}(T-t)\big[u(t)$$

$$+ K^{-1}(T-t)L(T-t)x(t) + K^{-1}(T-t)\left(B^* q(T-t) + \Gamma_2((F_2 g(t)+\sigma)^*\Pi(T-t))\right)\big]\Big|^2 dt\Bigg]$$

$$+ \int_0^T \Bigg[\mathrm{tr}\left((F_2 g(t)+\sigma)^*\Pi(T-t)(F_2 g(t)+\sigma)Q\right) + \left\langle M\widehat{F}_1 g(t), \widehat{F}_1 g(t)\right\rangle + 2\langle q(T-t), F_1 g(t)\rangle$$

$$- \left|K^{-\frac{1}{2}}(T-t)\left(B^* q(T-t) + \Gamma_2((F_2 g(t)+\sigma)^*\Pi(T-t))\right)\right|^2\Bigg] dt.$$

Note that the above equation holds for any initial condition ξ in $L^2(\Omega; H)$. Therefore, the optimal control law is given by (2.43)-(2.44). □

Fixed-Point Problem

In this section, we address the fixed-point problem described in Section 2.4.2. From (2.29) and under the optimal control given by (2.43), the optimal state satisfies

$$x^\circ(t) = S(t)\xi - \int_0^t S(t-r)\big(BK^{-1}(T-r)L(T-r)x^\circ(r) + BK^{-1}(T-r)B^* q(T-r) + \psi(r)\big)\, dr$$
$$+ \int_0^t S(t-r)\big((D - EK^{-1}(T-r)L(T-r))x^\circ(r) - EK^{-1}(T-r)B^* q(T-r) + \phi(r)\big)\, dW(r), \tag{2.51}$$

where

$$\psi(t) = BK^{-1}(T-t)\Gamma_2((F_2 g(t)+\sigma)^*\Pi(T-t)) - F_1 g(t),$$
$$\phi(t) = -EK^{-1}(T-t)\Gamma_2((F_2 g(t)+\sigma)^*\Pi(T-t)) + F_2 g(t) + \sigma. \tag{2.52}$$

By taking the expectation of both sides in (2.51), we obtain the linear evolution equation

$$\mathbb{E}[x^\circ(t)] = S(t)\xi - \int_0^t S(t-r)(BK^{-1}(T-r)L(T-r)\mathbb{E}[x^\circ(r)] + BK^{-1}(T-t)B^* q(T-t) + \psi(r))dr, \tag{2.53}$$

which admits a unique mild solution in $C(\mathfrak{T}; H)$ for any input $g \in C(\mathfrak{T}; H)$, using the same argument as that used for (2.46). Therefore, (2.53) defines the mapping

$$\Upsilon : g \in C(\mathfrak{T}; H) \longrightarrow \mathbb{E}[x^\circ(.)] \in C(\mathfrak{T}; H). \tag{2.54}$$

We show that the mapping Υ admits a unique fixed point. For this purpose, we first establish bounds on $\mathfrak{T}$ for relevant operators and processes that appear in the mapping Υ characterized by (2.53). We start with the operator $\Pi(t)$ which satisfies (2.45). The following lemma establishes a uniform bound for $\Pi(t)$ across $\mathfrak{T}$.

Proposition 12 (Bound of Π). *Let $\Pi \in C_s(\mathfrak{T}; \mathscr{L}(H))$ be the unique solution of the operator differential Riccati equation* (2.45)*, then we have*

$$\|\Pi(t)\|_{\mathscr{L}(H)} \leq C_1, \quad \forall t \in \mathfrak{T}, \tag{2.55}$$

$$C_1 := 2M_T^2 \exp(8TM_T^2 \|D\|^2 \operatorname{tr}(Q)) (\|G\| + T\|M\|). \tag{2.56}$$

Proof. For the purpose of illustration, without loss of generality, we introduce a simpler model for which the optimal control law involves the same operator Riccati differential equation as (2.45). For this specialized model, the dynamics are given by

$$\begin{aligned} dy(t) &= (Ay(t) + Bu(t))dt + (Dy(t) + Eu(t))dW(t), \\ y(0) &= \theta \in \mathscr{D}(A), \end{aligned} \tag{2.57}$$

and the cost functional by

$$\mathbb{E}\int_0^T \left(\left| M^{\frac{1}{2}} y(t) \right|^2 + |u(t)|^2 \right) dt + \mathbb{E} \left| G^{\frac{1}{2}} y(T) \right|^2,$$

where all the operators are as defined in (2.29)-(2.30). The strong solution of the corresponding approximating sequence is given by

$$y_n(t) = \theta + \int_0^t \left(Ay_n(r) + J_n(Bu(r)) \right) dr + \int_0^t J_n(Dy_n(r) + Eu(r)) dW(r).$$

Applying Itô's lemma to $\langle \Pi(t-r) y_n(r), y_n(r) \rangle$, for $t \in \mathfrak{T}$, integrating with respect to r from 0 to t, taking the expectation of both sides of the resulting equation and then taking the limit as $n \to \infty$, we obtain for any admissible control u

$$\begin{aligned} \langle \Pi(t)\theta, \theta \rangle &= \mathbb{E}\int_0^t \left(\left| M^{\frac{1}{2}} y(r) \right|^2 + |u(r)|^2 \right) dr + \mathbb{E} \left| G^{\frac{1}{2}} y(t) \right|^2 - \mathbb{E}\int_0^t \left| u(r) + K^{-1} L(T-r) y(r) \right|^2 dr \\ &\leq \mathbb{E}\int_0^t \left(\left| M^{\frac{1}{2}} y(r) \right|^2 + |u(r)|^2 \right) dr + \mathbb{E} \left| G^{\frac{1}{2}} y(t) \right|^2. \end{aligned}$$

Setting $u(t)=0$, $\forall t\in\mathfrak{T}$, we have

$$\langle\Pi(t)\theta,\theta\rangle\leq\mathbb{E}\int_0^t\left|M^{\frac{1}{2}}y_0(r)\right|^2dr+\mathbb{E}\left|G^{\frac{1}{2}}y_0(t)\right|^2, \tag{2.58}$$

where $y_0(t)$ is the mild solution to (2.57) under $u(t)=0$, satisfying

$$y_0(t)=S(t)\theta+\int_0^t S(t-r)Dy_0(r)dW(r).$$

By performing similar computations as in (2.18), we have

$$\begin{aligned}\mathbb{E}\left|y_0(t)\right|^2&\leq 2M_T^2\left|\theta\right|^2+2\mathbb{E}\left|\int_0^t S(t-r)Dy_0(r)dW(r)\right|^2\leq 2M_T^2\left|\theta\right|^2+8M_T^2\mathbb{E}\int_0^t\left\|Dy_0(r)\right\|_{\mathscr{L}_2}^2dr\\&\leq 2M_T^2\left|\theta\right|^2+8M_T^2\left\|D\right\|^2\operatorname{tr}(Q)\mathbb{E}\int_0^t\left|y_0(r)\right|^2dr.\end{aligned}$$

Then, applying Grönwall's inequality, for every $t\in\mathfrak{T}$, we have

$$\mathbb{E}\left|y_0(t)\right|^2\leq 2M_T^2\left|\theta\right|^2\exp\left(16TM_T^2\left\|D\right\|^2\operatorname{tr}(Q)\right). \tag{2.59}$$

Finally, from (2.58) and (2.59), for every $x\in H$, we obtain

$$\langle\Pi(t)\theta,\theta\rangle\leq\mathbb{E}\int_0^t\left|M^{\frac{1}{2}}y_0(r)\right|^2dr+\mathbb{E}\left|G^{\frac{1}{2}}y_0(t)\right|^2\leq\left|\theta\right|^2C_1,\quad\forall\xi\in H,$$

where C_1 is given by (2.56). Then, the conclusion follows from the spectral property of self-adjoint operators and the fact that $\mathscr{D}(A)$ is dense in H. □

Furthermore, the Riesz mappings $\Delta_k, k=1,2,3$, and $\Gamma_k, k=1,2$, given in Definition 4 and associated with $\Pi(t)$, appear in (2.53). We can easily apply the results of Theorem 10 and Proposition 12 to establish the bounds for these operators. This in turn facilitates the determination of bounds for the operators $K^{-1}(t)$ and $K^{-1}(t)L(t)$, both of which are present in (2.53) and defined in (2.44). For instance, for every $t\in\mathfrak{T}$, we have $\left\|\Delta_1(\Pi(t))\right\|\leq C_1R_3$, where R_3 is given by (2.34), and hence

$$\left\|K^{-1}(t)L(t)\right\|\leq\left\|K^{-1}(t)\right\|\left\|L(t)\right\|\leq\left\|L(t)\right\|\leq C_1R_6, \tag{2.60}$$

where R_6 is given by (2.37).

Now, we establish that the variations of the solution $q\in C(\mathfrak{T};H)$ to the the linear evolution equation given by (2.46) are bounded with respect to the variations in the input $g\in C(\mathfrak{T};H)$.

Lemma 13 (Bounded Variations of $q(t)$ wrt Variations of Input $g(t)$)**.** *Consider the processes* $\Pi \in C_s(\mathfrak{T};\mathscr{L}(H))$ *and* $q \in C(\mathfrak{T};H)$*, respectively, satisfying* (2.45) *and* (2.46)*. Moreover, let* $g_1, g_2 \in C(\mathfrak{T};H)$ *be two processes on* $\mathfrak{T}$*. Then, we have*

$$|q_1 - q_2|_{C(\mathfrak{T};H)} \leq |g_1 - g_2|_{C(\mathfrak{T};H)} M_T (TC_2 + \|G\| \|\widehat{F}_2\|) e^{M_T T C_3}, \tag{2.61}$$

$$C_2 := C_1(R_1 \|F_2\| + C_1 R_6 R_2 \|F_2\| + \|F_1\|) + \|M\| \|\widehat{F}_1\| \tag{2.62}$$

$$C_3 := C_1 R_6 \|B\|, \tag{2.63}$$

where q_1 *and* q_2 *are the corresponding solutions of* (2.46) *to the inputs* $g = g_1 \in C(\mathfrak{T};H)$ *and* $g = g_2 \in C(\mathfrak{T};H)$*, respectively.*

Proof. The mild solutions $q_i, i = 1,2$, of (2.46) subject to the inputs $g = g_i, i = 1,2$, are given by

$$q_i(t) = -S^*(t) G \widehat{F}_2 g_i(T) + \int_0^t S^*(t-r)(-L^*(r)K^{-1}(r)B^* q_i(r) - M\widehat{F}_1 g_i(T-r) + \eta_i(r))dr,$$

where

$$\eta_i(t) = \Gamma_1((F_2 g_i(T-t) + \sigma)^* \Pi(t)) - L^*(t)K^{-1}(t)\Gamma_2((F_2 g_i(T-t) + \sigma)^* \Pi(t)) + \Pi(t) F_1 g_i(T-t).$$

We can show that, $\forall t \in \mathfrak{T}$,

$$\begin{aligned} |\eta_1(t) - \eta_2(t)| \leq & |\Gamma_1((F_2(g_1(T-t) - g_2(T-t)))^* \Pi(t))| + |\Pi(t) F_1(g_1(T-t) - g_2(T-t))| \\ & + \|L^*(t)K^{-1}(t)\| |\Gamma_2((F_2(g_1(T-t) - g_2(T-t)))^* \Pi(t))| \\ \leq & C_1(R_1 \|F_2\| + C_1 R_6 R_2 \|F_2\| + \|F_1\|) |g_1 - g_2|_{C(\mathfrak{T};H)}. \end{aligned} \tag{2.64}$$

Thus, $\forall t \in \mathfrak{T}$, we have,

$$\begin{aligned} |q_1(t) - q_2(t)| \leq & \left| S^*(t) G \widehat{F}_2 (g_1(T) - g_2(T)) \right| + \left| \int_0^t S^*(t-r) L^*(r) K^{-1}(r) B^* (q_1(r) - q_2(r)) dr \right| \\ & + \left| \int_0^t S^*(t-r)(\eta_1(r) - \eta_2(r)) dr \right| + \left| \int_0^t S^*(t-r) M \widehat{F}_1 (g_1(T-r) - g_2(T-r)) dr \right| \\ \leq & M_T (TC_2 + \|G\| \|\widehat{F}_2\|) |g_1 - g_2|_{C(\mathfrak{T};H)} + M_T C_3 \int_0^t |q_1(r) - q_2(r)| dr, \end{aligned} \tag{2.65}$$

where C_2 and C_3 are, respectively, given by (2.62) and (2.63). Finally, by applying Grönwall's inequality to (2.65), we obtain (2.61). □

So far, we have demonstrated that all the operators and deterministic processes appearing in the mapping Υ, characterized by (2.53)-(2.54), are bounded. We may now establish the condition under which this mapping admits a unique fixed point.

Theorem 14 (Contraction Condition)**.** *The mapping* $\Upsilon : g \in C(\mathfrak{T};H) \longrightarrow \mathbb{E}[x^{\circ}(.)] \in C(\mathfrak{T};H)$*, described by* (2.53)*, admits a unique fixed point if*

$$C_4 e^{TM_T\|B\|C_1R_6} < 1, \tag{2.66}$$

where

$$C_4 := TM_T \left(M_T \|B\|^2 \left(TC_2 + \|G\| \left\|\widehat{F_2}\right\| \right) e^{M_T T C_3} + C_1R_2 \|B\| \|F_2\| + \|F_1\| \right). \tag{2.67}$$

Proof. Subject to the inputs $g_1, g_2 \in C(\mathfrak{T};H)$, the optimal control characterized in (2.43) is given by

$$u^{\circ,i}(t) = -K^{-1}(T-t)[L(T-t)x^{\circ,i}(t) + B^* q_i(T-t) + \Gamma_2 \left((F_2 g_i(t) + \sigma)^* \Pi(T-t) \right)],$$

Subsequently, the expectation of the resulting optimal state $\mathbb{E}[x^{\circ,i}(t)]$, $i = 1,2$, satisfies

$$\begin{aligned} \mathbb{E}[x^{\circ,i}(t)] = S(t)\xi - \int_0^t & S(t-r)(BK^{-1}(T-r)L(T-r)\mathbb{E}\left[x^{\circ,i}(r)\right] \\ & + BK^{-1}(T-r)B^* q_i(T-r) + \psi_i(r))dr, \end{aligned} \tag{2.68}$$

where

$$\psi_i(t) = BK^{-1}(T-t)\Gamma_2((F_2 g_i(t) + \sigma)^* \Pi(T-t)) - F_1 g_i(t) \tag{2.69}$$

From (2.69), $\forall t \in \mathfrak{T}$, we have

$$\begin{aligned} |\psi_1(t) - \psi_2(t)| &\leq \|B\| \left\|K^{-1}(T-t)\right\| |\Gamma_2(F_2(g_1(t) - g_2(t))^* \Pi(T-t))| + \|F_1\| |g_1(t) - g_2(t)| \\ &\leq (C_1R_2 \|B\| \|F_2\| + \|F_1\|) |g_1(t) - g_2(t)|. \end{aligned} \tag{2.70}$$

Hence,

$$\begin{aligned} \left| \int_0^t S(t-r)(\psi_1(r) - \psi_2(r))\, dr \right| &\leq M_T (C_1R_2 \|B\| \|F_2\| + \|F_1\|) \int_0^t |\psi_1(r) - \psi_2(r)|\, dr \\ &\leq TM_T (C_1R_2 \|B\| \|F_2\| + \|F_1\|) |g_1 - g_2|_{C(\mathfrak{T};H)}. \end{aligned} \tag{2.71}$$

By applying the result of Lemma 13, $\forall t \in \mathfrak{T}$, we obtain

$$\left| \int_0^t S(t-r)BK^{-1}(T-t)B^* \left(q_1(r) - q_2(r)\right) dr \right| \leq TM_T \|B\|^2 |q_1 - q_2|_{C(\mathfrak{T};H)}$$
$$\leq TM_T^2 \|B\|^2 (TC_2 + \|G\| \left\|\widehat{F}_2\right\|) e^{M_T T C_3} |g_1 - g_2|_{C(\mathfrak{T};H)} . \quad (2.72)$$

Moreover, $\forall t \in \mathfrak{T}$, we have

$$\left| \int_0^t S(t-r)BK^{-1}(T-r)L(T-r) \left(\mathbb{E}[x^{\circ,1}(r)] - \mathbb{E}[x^{\circ,2}(r)]\right) dr \right|$$
$$\leq M_T \|B\| C_1 R_6 \int_0^t \left|\mathbb{E}[x^{\circ,1}(r)] - \mathbb{E}[x^{\circ,2}(r)]\right| dr, \quad (2.73)$$

From (2.71)-(2.73), $\forall t \in \mathfrak{T}$, we obtain

$$\left|\mathbb{E}[x^{\circ,1}(t)] - \mathbb{E}[x^{\circ,2}(t)]\right| \leq C_4 |g_1 - g_2|_{C(\mathfrak{T};H)} + M_T \|B\| C_1 R_6 \int_0^t \left|\mathbb{E}[x^{\circ,1}(r)] - \mathbb{E}[x^{\circ,2}(r)]\right| dr.$$

Finally, we apply Grönwall's inequality to the above inequality to get

$$\left|\mathbb{E}[x^{\circ,1}(.)] - \mathbb{E}[x^{\circ,2}(.)]\right|_{C(\mathfrak{T};H)} \leq C_4 e^{TM_T\|B\|C_1R_6} |g_1 - g_2|_{C(\mathfrak{T};H)} , \quad (2.74)$$

from which the fixed-point condition (2.66) follows. □

We now discuss the feasibility of the contraction condition (2.67). For this purpose, we do not impose additional assumptions on the operators involved in (2.29) and (2.30), and nor on the C_0-semigroup $S(t), t \in \mathfrak{T}$.

Proposition 15 (Contraction Condition Feasibility)**.** *There exists $T > 0$ such that the contraction condition* (2.66) *holds.*

Proof. From (2.7), the C_0-semigroup $S(t) \in \mathscr{L}(H), t \in \mathfrak{T}$, is uniformly bounded by a constant $M_T = M_A e^{\alpha T}$. This constant depends only on T, given fixed values of M_A and α. Hence, we can treat M_T, along with $C_i, i = 1,2,3$, as real-valued functions of T. It is evident that $M_T \downarrow M_A$ as $T \downarrow 0$. In addition, we can easily verify that, as $T \downarrow 0$, each $C_i, i = 1,2,3$, monotonically decreases to a positive constant and that $C_4 \downarrow 0$. Hence, $C_4 e^{TM_T\|B\|C_1R_6} \downarrow 0$ as $T \downarrow 0$. Then from the continuity of the real valued function $C_4 e^{TM_T\|B\|C_1R_6}$ with respect to T, we conclude that there exists $T > 0$ such that the contraction condition (2.67) holds. □

Remark 8 (Contraction Condition Feasibility for Fixed T)**.** *Proposition 15 states that for a sufficiently small T the contraction condition* (2.66) *holds. This result is consistent with the findings in the finite-dimensional case (see e.g. Bensoussan et al. (2016); Huang et al. (2007)). Moreover, for any fixed $T > 0$ the condition* (2.66) *may be satisfied if, for example, F_1, F_2, $\widehat{F}_1$, and $\widehat{F}_2$ are sufficiently small.*

Nash Equilibrium

The following theorem concludes this section.

Theorem 16 (Nash Equilibrium)**.** *Consider the Hilbert space-valued limiting system, described by* (2.27)-(2.28) *for $i \in \mathbb{N}$, and the relevant Riesz mappings $\Delta_k, k = 1,2,3$, $\Gamma_k, k = 1,2$, given in Definition 4. Suppose A2.4.1-A2.4.3, and condition* (2.66) *hold. Then, the set of control laws $\{u_i^\circ\}_{i\in\mathbb{N}}$, where u_i° is given by*

$$u_i^\circ(t) = -K^{-1}(T-t)\left[L(T-t)x_i(t) + \Gamma_2\big((F_2\bar{x}(t)+\sigma)^*\Pi(T-t)\big) + B^*q(T-t)\right], \tag{2.75}$$

$$K(t) = I + \Delta_3(\Pi(t)), \quad L(t) = B^*\Pi(t) + \Delta_1(\Pi(t)), \tag{2.76}$$

forms a unique Nash equilibrium for the limiting system where the mean field $\bar{x}(t) \in H$, the operator $\Pi(t) \in \mathscr{L}(H)$ and the offset term $q(t) \in H$, are characterized by the unique fixed point of the following set of consistency equations

$$\begin{aligned}\bar{x}(t) = S(t)\bar{\xi} - \int_0^t S(t-r)\Big(BK^{-1}(T-r)\Big(L(T-r)\bar{x}(r) + B^*q(T-r)\\ + \Gamma_2\big((F_2\bar{x}(r)+\sigma)^*\Pi(T-r)\big)\Big) - F_1\bar{x}(r)\Big)dr,\end{aligned} \tag{2.77}$$

$$\frac{d}{dt}\langle\Pi(t)x,x\rangle = 2\langle\Pi(t)x,Ax\rangle - \left\langle L^*K^{-1}L(t)x,x\right\rangle + \langle\Delta_2(\Pi(t))x,x\rangle + \langle Mx,x\rangle, \tag{2.78}$$

$$\begin{aligned}\dot{q}(t) = \left(A^* - L^*(t)K^{-1}(t)B^*\right)q(t) + \Gamma_1\big((F_2\bar{x}(T-t)+\sigma)^*\Pi(t)\big)\\ - L^*(t)K^{-1}(t)\Gamma_2\big((F_2\bar{x}(T-t)+\sigma)^*\Pi(t)\big) + \Big(\Pi(t)F_1 - M\widehat{F}_1\Big)\bar{x}(T-t),\end{aligned} \tag{2.79}$$

with $\Pi(0) = G$, $x \in \mathscr{D}(A)$, and $q(0) = -G\widehat{F}_2\bar{x}(T)$.

Proof. According to the demonstrations in Section 2.4.2, if the contraction condition (2.66) holds, then there exist $\Pi \in C_s(\mathfrak{T}, \mathscr{L}(H))$, $q \in C(\mathfrak{T};H)$ and $\bar{x} \in C(\mathfrak{T};H)$, which are the unique solution to the set of consistency equations give by (2.77)-(2.79). In addition, the set of feedback control laws $\{u_i^\circ\}_{i\in\mathbb{N}}$, where u_i° is given by (2.75)-(2.76), forms a unique Nash equilibrium for the limiting system described by (2.27)-(2.28), i.e.

$$J_i^\infty(u_i^\circ, u_{-i}^\circ) = \inf_{u_i\in\mathscr{U}_i} J_i^\infty(u_i, u_{-i}^\circ), \quad \forall i \in \mathbb{N}. \tag{2.80}$$

This is because, in the limit when the number of agents N goes to infinity, the agents get decoupled from each other and hence the high-dimensional optimization problem faced by agent i, $i\in\mathbb{N}$, turns into a single-agent optimal control problem for which there is a unique solution. Hence, agent i cannot improve its cost by deviating from the optimal strategy (2.75)-(2.76) and the set of these strategies yields a Nash equilibrium for the limiting system. In other words, the Nash equilibrium property, as defined in (2.80), holds trivially for the set of strategies $\{u_i^\circ\}_{i\in\mathbb{N}}$ because the limiting cost functional of agent i, given by (2.28), is independent of the strategies of other agents.

Subsequently, the equilibrium state of agent i is given by

$$\begin{aligned} x_i^\circ(t) = S(t)\xi_i &- \int_0^t S(t-r)\left(BK^{-1}(T-r)L(T-r)x_i^\circ(r) + B\tau(r) - F_1\bar{x}(r)\right)dr \\ &+ \int_0^t S(t-r)\Big[(D - EK^{-1}(T-r)L(T-r))x_i^\circ(r) - E\tau(r) + F_2\bar{x}(r) + \sigma\Big]dW_i(r), \end{aligned} \tag{2.81}$$

where $\tau(t) = K^{-1}(T-t)\left[B^* q(T-t) + \Gamma_2((F_2\bar{x}(t)+\sigma)^*\Pi(T-t))\right]$. Moreover, we have $\mathbb{E}\left[x_i^\circ(t)\right] = \bar{x}(t)$, $\forall i \in \mathscr{N}$, $\forall t \in \mathfrak{T}$, where $\bar{x}(t)$ is given by (2.77). We note that (2.77) represents the mild solution to the mean field equation

$$\begin{aligned} d\bar{x}(t) = \Big[A\bar{x}(t) - BK^{-1}(T-r)\Big(L(T-r)\bar{x}(r) + B^*q(T-r) + \Gamma_2\big((F_2\bar{x}(r)+\sigma)^*\Pi(T-r)\big)\Big) \\ + F_1\bar{x}(r)\Big]dt. \end{aligned} \tag{2.82}$$

□

2.4.3 ε-Nash Property

In this section, we establish the ε-Nash property of the set of strategies given by (2.75)-(2.76) for the N-player game described by (2.25)-(2.26). Due to the symmetric properties (exchangeability) of agents, we study the case where agent $i = 1$ deviates from this set of strategies. Specifically, we suppose that any agent i, $i \in \mathscr{N}$ and $i \neq 1$, employs the feedback strategy $u_i^{[N],\circ}$ given by

$$u_i^{[N],\circ}(t) = -K^{-1}(T-t)\left[L(T-t)x_i^{[N]}(t) + \Gamma_2\left((F_2\bar{x}(t)+\sigma)^*\Pi(T-t)\right) + B^*q(T-t)\right], \quad (2.83)$$

at $t \in \mathfrak{T}$ and agent $i = 1$ is allowed to choose an arbitrary control process $u_1^{[N]} \in \mathscr{U}^{[N]}$. Here, for clarity we use the superscript $[N]$ to denote the processes associated with the N-player game. In this context, the dynamics of agent $i = 1$ and agent i, $i \in \mathscr{N}$ and $i \neq 1$, in the N-player game are, respectively, given by

$$\begin{aligned} x_1^{[N]}(t) =& S(t)\xi_1 + \int_0^t S(t-r)\left(Bu_1^{[N]}(r) + F_1 x^{(N)}(r)\right)dr \\ &+ \int_0^t S(t-r)(Dx_1^{[N]}(r) + Eu_1^{[N]}(r) + F_2 x^{(N)}(r) + \sigma)dW_1(r), \quad (2.84) \\ x_i^{[N]}(t) =& S(t)\xi_i + \int_0^t S(t-r)\left(Bu_i^{[N],\circ}(r) + F_1 x^{(N)}(r)\right)dr \\ &+ \int_0^t S(t-r)(Dx_i^{[N]}(r) + Eu_i^{[N],\circ}(r) + F_2 x^{(N)}(r) + \sigma)dW_i(r), \quad (2.85) \end{aligned}$$

where $x^{(N)}(t) := \frac{1}{N}\sum_{i\in\mathscr{N}} x_i^{[N]}(t)$ is the average state of N agents. We note that for any control process $u_1^{[N]} \in \mathscr{U}^{[N]}$, the coupled system described by (2.84)-(2.85) satisfies A2.3.2-A2.3.5. Thus, the well-posedness of the system is ensured. Moreover, the sequence of N-player games described by (2.84) and (2.85), as N ranges over $\mathbb{N}$, is associated with the sequence of control processes $\{u_1^{[N]}\}_{N\in\mathbb{N}}$ employed by agent $i = 1$. The cost functional of agent $i = 1$ in the N-player game is given by

$$J_1^{[N]}(u_1^{[N]}, u_{-1}^{[N],\circ}) := \mathbb{E}\int_0^T\left(\left|M^{\frac{1}{2}}\left(x_1^{[N]}(t) - \widehat{F}_1 x^{(N)}(t)\right)\right|^2 + \left|u_1^{[N]}(t)\right|^2\right)dt + \mathbb{E}\left|G^{\frac{1}{2}}\left(x_1^{[N]}(T) - \widehat{F}_2 x^{(N)}(T)\right)\right|^2. \quad (2.86)$$

At the equilibrium, where agent $i = 1$ employs the strategy $u_1^{[N],\circ}$, we denote its state by $x_1^{[N],\circ}$ and the corresponding average state of N agents by $x^{(N),\circ}$.

The ε-Nash property indicates that

$$J_1^{[N]}(u_1^{[N],\circ}, u_{-1}^{[N],\circ}) \leq \inf_{\left\{u_1^{[N]}\in\mathscr{U}^{[N]}\right\}} J_1^{[N]}(u_1^{[N]}, u_{-1}^{[N],\circ}) + \varepsilon_N, \quad (2.87)$$

where the sequence $\{\varepsilon_N\}_{N\in\mathbb{N}}$ converges to zero. To establish this property, we start by identifying relevant bounds for the systems described by (2.84)-(2.85) for a fixed number of agents N and a given deviating control $u_1^{[N]} \in \mathscr{U}^{[N]}$ for agent $i=1$. These bounds are detailed in Lemma 17, Theorem 18, and Proposition 19, where, by an abuse of notation, the constant $C(u_1^{[N]})$ [2] may vary from one instance to another. We obtain universal bounds for this system in the proof of Theorem 20. Furthermore, we obtain the relevant bounds associated with the equilibrium , denoted by C°, which do not depend on N.

Lemma 17. *Consider the N coupled systems described by* (2.84)-(2.85)*. Then, the property*

$$\mathbb{E}\left[\sum_{i\in\mathcal{N}} \left|x_i^{[N]}(t)\right|_H^2\right] \le C(u_1^{[N]})\,N, \tag{2.88}$$

holds uniformly for all $t \in \mathfrak{T}$. Here, the constant $C(u_1^{[N]})$ depends on the model parameters and $u_1^{[N]}$.

Proof. From (2.85), for agent i, $i \in \mathcal{N}$ and $i \neq 1$, by a simple computation, we have

$$\begin{aligned}
\mathbb{E}\left|x_i^{[N]}(t)\right|^2 &\le C\,\mathbb{E}\left[|\xi_i|^2 + \int_0^t \left(\left|Bu_i^{[N],\circ}(r) + F_1 x^{(N)}(r)\right|^2 + \left\|Dx_i^{[N]}(r) + Eu_i^{[N],\circ}(r) + F_2 x^{(N)}(r) + \sigma\right\|_{\mathscr{L}_2}^2\right) dr\right] \\
&\le C\Big(\int_0^t \mathbb{E}\left|x_i^{[N]}(r)\right|^2 dr + \int_0^t \mathbb{E}\left|x^{(N)}(r)\right|^2 dr + 1\Big) \\
&\le C\Big(\int_0^t \mathbb{E}\left|x_i^{[N]}(r)\right|^2 dr + \frac{1}{N}\int_0^t \mathbb{E}\Big[\sum_{j\in\mathcal{N}} \left|x_j^{[N]}(r)\right|^2\Big] dr + 1\Big).
\end{aligned} \tag{2.89}$$

From (2.84), for agent $i=1$ we have

$$\mathbb{E}\left|x_1^{[N]}(t)\right|^2 \le C\Big(\int_0^t \mathbb{E}\left|x_1^{[N]}(r)\right|^2 dr + \frac{1}{N}\int_0^t \mathbb{E}\Big[\sum_{j\in\mathcal{N}} \left|x_j^{[N]}(r)\right|^2\Big] dr + 1\Big). \tag{2.90}$$

From (2.89) and (2.90), we obtain

$$\mathbb{E}\left[\sum_{i\in\mathcal{N}} \left|x_i^{[N]}(t)\right|^2\right] \le C\left(N + \int_0^t \mathbb{E}\left[\sum_{i\in\mathcal{N}} \left|x_i^{[N]}(r)\right|^2\right] dr\right). \tag{2.91}$$

Applying Grönwall's inequality to the above equation results in (2.88). ☐

Note that Lemma 17 is closely related to a part of Theorem 9 (see (2.20)), and it also demonstrates that the solution of the system (2.84)-(2.85) belongs to $\mathscr{H}^2(\mathfrak{T}; H^N)$. A similar argument is

[2] For the sake of notational simplicity, $u_1^{[N]}$ is omitted from $C(u_1^{[N]})$ in the proofs of Lemma 17, Theorem 18, and Proposition 19.

presented in (Da Prato and Zabczyk, 2014, Theorem 9.1). As direct consequences of Lemma 17, we have

$$\mathbb{E}\left|x^{(N)}(t)\right|^2 \leq C(u_1^{[N]}), \quad \mathbb{E}\left|x_1^{[N]}(t)\right|^2 \leq C(u_1^{[N]}), \quad \forall t \in \mathfrak{T}. \tag{2.92}$$

It is straightforward to verify that, at the equilibrium, we have

$$\mathbb{E}\left|x^{(N),\circ}(t)\right|^2 \leq C^\circ, \quad \mathbb{E}\left|x_1^{[N],\circ}(t)\right|^2 \leq C^\circ, \quad \forall t \in \mathfrak{T}, \tag{2.93}$$

The next theorem demonstrates the convergence of the average state $x^{(N)}(t)$ to the mean field $\bar{x}(t)$.

Theorem 18 (Average State Error Bound)**.** *Suppose the state of any agent i, $i \in \mathscr{N}$ and $i \neq 1$, satisfies* (2.85)*, where the agent employs the strategy $u_i^{[N],\circ}$ given by* (2.83)*. For any control process $u_1^{[N]} \in \mathscr{U}^{[N]}$ that agent $i = 1$ chooses, we have*

$$\sup_{t\in\mathfrak{T}} \mathbb{E}\left|\bar{x}(t) - x^{(N)}(t)\right|_H^2 \leq \frac{C(u_1^{[N]})}{N}. \tag{2.94}$$

Proof. From theorem 16, recall that $\tau(t) = K^{-1}(T-t)[B^*q(T-t) + \Gamma_2((F_2\bar{x}(t)+\sigma)^*\Pi(T-t))]$, and

$$\bar{x}(t) = S(t)\bar{\xi} - \int_0^t S(t-r)(BK^{-1}(T-r)L(T-r)\bar{x}(r) + B\tau(r) - F_1\bar{x}(r))dr. \tag{2.95}$$

Moreover, from (2.85) and (2.84) subject to any control process $u_1^{[N]} \in \mathscr{U}^{[N]}$, we have

$$\begin{aligned} x^{(N)}(t) =& S(t)x^{(N)}(0) - \int_0^t S(t-r)(BK^{-1}(T-r)L(T-r)x^{(N)}(r) + B\tau(r) - F_1 x^{(N)}(r))dr \\ &+ \frac{1}{N}\left[\sum_{i\in\mathscr{N}} \Xi_i(t)\right] + \frac{1}{N}\int_0^t S(t-r)B(u_1^{[N]}(r) + K^{-1}(T-r)L(T-r)x_1^{[N]}(r) + B\tau(r))dr, \end{aligned} \tag{2.96}$$

where the stochastic convolution processes $\Xi_1(t)$ and $\Xi_i(t)$, $i \in \mathscr{N}$ and $i \neq 1$, are, respectively, given by

$$\begin{aligned} \Xi_1(t) &= \int_0^t S(t-r)(Dx_1^{[N]}(r) + Eu_1^{[N]}(r) + F_2 x^{(N)}(r) + \sigma)dW_1(r), \\ \Xi_i(t) &= \int_0^t S(t-r)\Big[(D - EK^{-1}(T-r)L(T-r))x_i^{[N]}(r) - E\tau(r) + F_2 x^{(N)}(r) + \sigma\Big]dW_i(r). \end{aligned} \tag{2.97}$$

Now, define $y(t) := \bar{x}(t) - x^{(N)}(t)$. Then, we have

$$\begin{aligned} y(t) =& S(t)y(0) - \int_0^t S(t-r)(BK^{-1}(T-r)L(T-r) - F_1)y(r)dr - \frac{1}{N}\left[\sum_{i\in\mathscr{N}} \Xi_i(t)\right] \\ &- \frac{1}{N}\int_0^t S(t-r)B(u_1^{[N]}(r) + K^{-1}(T-r)L(T-r)x_1^{[N]}(r) + B\tau(r))dr. \end{aligned} \tag{2.98}$$

Furthermore, from the above equation we obtain

$$\mathbb{E}\,|y(t)|^2 \le C\Bigg(|y(0)|^2 + \int_0^t \mathbb{E}\,|y(r)|^2\,dr + \frac{1}{N^2}\Bigg[\mathbb{E}\Big|\sum_{i\in\mathcal{N}} \Xi_i(t)\Big|^2 + \int_0^t \mathbb{E}\left|u_1^{[N]}(r) + K^{-1}(T-r)L(T-r)x_1^{[N]}(r) + B\tau(r)\right|^2 dr\Bigg]\Bigg). \tag{2.99}$$

Moreover, since $\Xi_i(t)$, $i\in\mathcal{N}$, are driven by independent Q-Wiener processes, we have

$$\mathbb{E}\left|\sum_{i\in\mathcal{N}} \Xi_i(t)\right|^2 = \mathbb{E}\left[\sum_{i\in\mathcal{N}} |\Xi_i(t)|^2\right]. \tag{2.100}$$

More specifically, in the above equation, we use the property that $\mathbb{E}\left\langle \Xi_i(t), \Xi_j(t)\right\rangle_H = 0$ for $i\neq j$ and for all $i,j\in\mathcal{N}$, and for every $t\in\mathfrak{T}$. A straightforward method to verify this property is to demonstrate that it holds for stochastic integrals of elementary processes. This can be achieved by applying the same techniques used to prove the Itô isometry (see, e.g., (Da Prato and Zabczyk, 2014, Proposition 4.20) and (Gawarecki and Mandrekar, 2010, Proposition 2.1)).

From (2.97) and using the standard approximation technique for stochastic convolutions, and given that all operators are uniformly bounded on $\mathfrak{T}$, $\forall i\in\mathcal{N}$ and $i\neq 1$, and $\forall t\in\mathfrak{T}$, we obtain

$$\begin{aligned}\mathbb{E}\,|\Xi_i(t)|^2 &\le C\int_0^t \mathbb{E}\left\|(D - EK^{-1}L(T-r))x_i^{[N]}(r) - E\tau(r) + F_2 x^{(N)}(r) + \sigma\right\|^2 dr\\ &\le C\int_0^t (\mathbb{E}\left|x_i^{[N]}(r)\right|^2 + \mathbb{E}\left|x^{(N)}(r)\right|^2 + 1)dr.\end{aligned} \tag{2.101}$$

Similarly, for $\Xi_1(t), \forall t\in\mathfrak{T}$, we have

$$\mathbb{E}\,|\Xi_1(t)|^2 \le C\int_0^t (\mathbb{E}\left|x_1^{[N]}(r)\right|^2 + \mathbb{E}\left|x^{(N)}(r)\right|^2 + 1)dr. \tag{2.102}$$

Subsequently, we obtain

$$\mathbb{E}\sum_{i\in\mathcal{N}} |\Xi_i(t)|^2 \le C\left(\int_0^t \mathbb{E}\left[\sum_{i\in\mathcal{N}}\left|x_i^{[N]}(r)\right|^2\right]dr + N\int_0^t \mathbb{E}\left|x^{(N)}(r)\right|^2 dr + N\right) \le CN. \tag{2.103}$$

Moreover, for the last term on the RHS of (2.99), we have

$$\int_0^t \mathbb{E}\left|u_1^{[N]}(r) + K^{-1}(T-r)L(T-r)x_1^{[N]}(r) + B\tau(r)\right|^2 dr \le C\int_0^t (\mathbb{E}\left|x_1^{[N]}(r)\right|^2 + 1)dr \le C. \tag{2.104}$$

From (2.99) and (2.103)-(2.104), we conclude that

$$\mathbb{E}\,|y(t)|^2 \le C(\frac{1}{N} + \frac{1}{N^2}) + C\int_0^t \mathbb{E}\,|y(r)|^2\,dr. \tag{2.105}$$

Then, by Grönwall's inequality, the property (2.94) follows. □

Proposition 19 (Error Bounds for Agent $i=1$)**.** *Let $x_1(t)$ and $x_1^{[N]}(t)$, respectively, denote the state of agent $i=1$ in the limiting game and the N-player game satisfying* (2.27) *and* (2.84)*. Moreover, let $J^\infty(u_1^\circ)$ and $J^{[N]}(u_1^\circ, u_{-1}^\circ)$, respectively, denote the cost functional of agent $i=1$ in the limiting game and the N-player game given by* (2.28) *and* (2.86)*.*

(i) If agent $i=1$ employs the control law u_1° given by (2.75)*, we have*

$$\sup_{t\in\mathfrak{T}} \mathbb{E}\left|x_1^\circ(t) - x_1^{[N],\circ}(t)\right|_H^2 \leq \frac{C^\circ}{N}, \tag{2.106}$$

$$\left|J_1^\infty(u_1^\circ) - J_1^{[N]}(u_1^{[N],\circ}, u_{-1}^{[N],\circ})\right| \leq \frac{C^\circ}{\sqrt{N}}. \tag{2.107}$$

(ii) If agent $i=1$ employs any control process $u_1^{[N]} \in \mathscr{U}^{[N]}$, we have

$$\sup_{t\in\mathfrak{T}} \mathbb{E}\left|x_1(t) - x_1^{[N]}(t)\right|_H^2 \leq \frac{C(u_1^{[N]})}{N}, \tag{2.108}$$

$$\left|J_1^\infty(u_1^{[N]}) - J_1^{[N]}(u_1^{[N]}, u_{-1}^{[N],\circ})\right| \leq \frac{C(u_1^{[N]})}{\sqrt{N}}. \tag{2.109}$$

Proof. From (2.27), (2.77) and (2.84), for the case where agent $i=1$ employs the control law u_1° given by (2.75)-(2.76), by direct computation, we have

$$\begin{aligned} x_1^\circ(t) - x_1^{[N],\circ}(t) = &-\int_0^t S(t-r)BK^{-1}(T-r)L(T-r)(x_1^\circ(r) - x_1^{[N],\circ}(r))dr \\ &+ \int_0^t S(t-r)F_1\left(\bar{x}(r) - x^{(N),\circ}(r)\right)dr + \int_0^t S(t-r)F_2\left(\bar{x}(r) - x^{(N),\circ}(r)\right)dW_1(r) \\ &+ \int_0^t S(t-r)(D - EK^{-1}(T-r)L(T-r))(x_1^\circ(r) - x_1^{[N],\circ}(r))dW_1(r). \end{aligned} \tag{2.110}$$

Moreover, for the case where agent $i=1$ employs an arbitrary control law $u_1 \in \mathscr{U}^{[N]}$, we have

$$\begin{aligned} x_1(t) - x^{[N]}(t) = &\int_0^t S(t-r)F_1\left(\bar{x}(r) - x^{(N)}(r)\right)dr + \int_0^t S(t-r)D(x_1(r) - x_1^{[N]}(r))dW_1(r) \\ &+ \int_0^t S(t-r)F_2\left(\bar{x}(r) - x^{(N)}(r)\right)dW_1(r). \end{aligned} \tag{2.111}$$

The rest of the proof for (2.106) and (2.108) follows the method used in Theorem 18 by taking square norms, expectation, applying Grönwall's inequality (see (2.99)), and leveraging the results of Theorem 18.

For the property (2.107), a simple computation shows that

$$\left|J_1^\infty(u_1^\circ) - J_1^{[N]}(u_1^{[N],\circ}, u_{-1}^{[N],\circ})\right| \leq I_1 + I_2, \tag{2.112}$$

$$I_1 = \mathbb{E}\int_0^T \left| \left| M^{\frac{1}{2}}(x_1^\circ(t) - \widehat{F}_1\bar{x}(t)) \right|^2 - \left| M^{\frac{1}{2}}(x_1^{[N],\circ}(t) - \widehat{F}_1 x^{(N),\circ}(t)) \right|^2 \right| dt$$
$$+ \mathbb{E}\left| \left| G^{\frac{1}{2}}(x_1^\circ(T) - \widehat{F}_2\bar{x}(T)) \right|^2 - \left| G^{\frac{1}{2}}(x_1^{[N],\circ}(T) - \widehat{F}_2 x^{(N),\circ}(T)) \right|^2 \right|, \qquad (2.113)$$

$$I_2 = \mathbb{E}\int_0^T \left| \left| K^{-1}(T-t)L(T-t)x_1^\circ(t) + \tau(t) \right|^2 - \left| K^{-1}(T-t)L(T-t)x_1^{[N],\circ}(t) + \tau(t) \right|^2 \right| dt,$$

where $\tau(t) = K^{-1}(T-t)\,[B^* q(T-t) + \Gamma_2((F_2\bar{x}(t)+\sigma)^*\Pi(T-t))]$. For $t \in [0,T)$, we have

$$\begin{aligned}
&\left| \left| M^{\frac{1}{2}}(x_1^\circ(t) - \widehat{F}_1\bar{x}(t)) \right|^2 - \left| M^{\frac{1}{2}}(x_1^{[N],\circ}(t) - \widehat{F}_1 x^{(N),\circ}(t)) \right|^2 \right| \\
&\leq \left\| M^{\frac{1}{2}} \left[(x_1^\circ(t) - \widehat{F}_1\bar{x}(t)) - (x_1^{[N],\circ}(t) - \widehat{F}_1 x^{(N),\circ}(t)) \right] \right\|^2 \\
&\quad + 2\left| M^{\frac{1}{2}}(x_1^\circ(t) - \widehat{F}_1\bar{x}(t)) \right| \left| M^{\frac{1}{2}} \left[(x_1^\circ(t) - \widehat{F}_1\bar{x}(t)) - (x_1^{[N],\circ}(t) - \widehat{F}_1 x^{(N),\circ}(t)) \right] \right| \\
&\leq 2\left| M^{\frac{1}{2}}(x_1^\circ(t) - x_1^{[N],\circ}(t)) \right|^2 + 2\left| M^{\frac{1}{2}}\widehat{F}_1(\bar{x}(t) - x^{(N),\circ}(t)) \right|^2 \\
&\quad + 2\left| M^{\frac{1}{2}}(x_1^\circ(t) - \widehat{F}_1\bar{x}(t)) \right| \left(2\left| M^{\frac{1}{2}}(x_1^\circ(t) - x_1^{[N],\circ}(t)) \right|^2 + 2\left| M^{\frac{1}{2}}\widehat{F}_1(\bar{x}(t) - x^{(N),\circ}(t)) \right|^2 \right)^{\frac{1}{2}}. \qquad (2.114)
\end{aligned}$$

We apply the same method to the terminal condition in I_1. Then, by using the Cauchy–Schwarz inequality, Theorem 18, and Proposition 19, we obtain $I_1 \leq \frac{C}{\sqrt{N}}$. We employ the same method as above for I_2 to obtain $I_2 \leq \frac{C}{\sqrt{N}}$.

For the property (2.109), we have

$$\left| J_1^\infty(u_1^{[N]}) - J_1^{[N]}(u_1^{[N]}, u_{-1}^{[N],\circ}) \right| \leq \mathbb{E}\int_0^T \left| \left| M^{\frac{1}{2}}(x_1(t) - \widehat{F}_1\bar{x}(t)) \right|^2 - \left| M^{\frac{1}{2}}(x_1^{[N]}(t) - \widehat{F}_1 x^{(N)}(t)) \right|^2 \right| dt$$
$$+ \mathbb{E}\left| \left| G^{\frac{1}{2}}(x_1(T) - \widehat{F}_2\bar{x}(T)) \right|^2 - \left| G^{\frac{1}{2}}(x_1^{[N]}(T) - \widehat{F}_2 x^{(N)}(T)) \right|^2 \right|. \qquad (2.115)$$

Then, we repeat the same method as for the property (2.107) to obtain the property (2.109). □

Remark 9. *We note that* $\{C(u_1^{[N]})\}_{N\in\mathbb{N}}$ *is a sequence of real numbers, although* $C(u_1^{[N]})$ *does not explicitly depend on* N*. Therefore, the convergence properties given by* (2.94), (2.108), *and* (2.109) *hold as* $N \to \infty$*, provided the sequence is bounded. For instance, this condition is met if* $\mathbb{E}\left[\int_0^T \left| u_1^{[N]}(t) \right|^2 dt\right]$ *is uniformly bounded across all* $N \in \mathbb{N}$*, or if the system is at equilibrium.*

Now, we establish the ε-Nash property.

Theorem 20. *(ε-Nash Equilibrium) Suppose that A2.4.1 and condition* (2.66) *hold. Then, the set of control laws* $\{u_i^{[N],\circ}\}_{i\in\mathscr{N}}$*, where* $u_i^{[N]\circ}(t)$ *is given by*

$$u_i^{[N],\circ}(t) = -K^{-1}(T-t)\left[L(T-t)x_i^{[N]}(t) + \Gamma_2\left((F_2\bar{x}(t)+\sigma)^*\Pi(T-t)\right) + B^*q(T-t)\right],$$

with $x_i^{[N]}(t)$ *satisfying*

$$\begin{aligned} x_i^{[N]}(t) =& S(t)\xi_i + \int_0^t S(t-r)\left(Bu_i^{[N],\circ}(r) + F_1x^{(N)}(r)\right)dr \\ &+ \int_0^t S(t-r)(Dx_i^{[N]}(r) + Eu_i^{[N],\circ}(r) + F_2x^{(N)}(r) + \sigma)dW_i(r), \end{aligned}$$

forms an ε-Nash equilibrium for the N-player system described by (2.25)–(2.26)*. That is,*

$$J_1^{[N]}(u_1^{[N],\circ}, u_{-1}^{[N],\circ}) \le \inf_{\left\{u_1^{[N]}\in\mathscr{U}^{[N]}\right\}} J_1^{[N]}(u_1^{[N]}, u_{-1}^{[N],\circ}) + \varepsilon_N, \tag{2.116}$$

where $\varepsilon_N = O(\frac{1}{\sqrt{N}})$ *is a sequence of nonnegative numbers* $\{\varepsilon_N\}_{N\in\mathbb{N}}$ *converging to zero.*

Proof. It is evident that the ε-Nash property (2.116) can be equivalently expressed as

$$J_1^{[N]}(u_1^{[N],\circ}, u_{-1}^{[N],\circ}) \le \inf_{\left\{u_1^{[N]}\in\mathscr{A}^{[N]}\right\}} J_1^{[N]}(u_1^{[N]}, u_{-1}^{[N],\circ}) + \varepsilon_N, \tag{2.117}$$

where $\mathscr{A}^{[N]} := \left\{u_1^{[N]} \in \mathscr{U}^{[N]} : J_1^{[N]}(u_1^{[N]}, u_{-1}^{[N],\circ}) \le J_1^{[N]}(u_1^{[N],\circ}, u_{-1}^{[N],\circ})\right\}$ is a non-empty set. We then note that, by (2.93), the equilibrium cost $J_1^{[N]}(u_1^{[N],\circ}, u_{-1}^{[N],\circ})$ is uniformly bounded by a constant, denoted by $C^{\circ\circ}$, for all $N\in\mathbb{N}$. Thus, for any $N\in\mathbb{N}$ and for any $u_1^{[N]}\in\mathscr{A}^{[N]}$, we have

$$\mathbb{E}\left[\int_0^T \left|u_1^{[N]}\right|^2 dt\right] \le J_1^{[N]}(u_1^{[N]}, u_{-1}^{\circ}) \le J_1^{[N]}(u_1^{\circ}, u_{-1}^{\circ}) \le C^{\circ\circ}. \tag{2.118}$$

From (2.118), it is straightforward to verify that the constants $C(u_1^{[N]})$ in Lemma 17-Proposition 19 can be chosen universally for all $N\in\mathbb{N}$ and $u_1^{[N]}\in\mathscr{A}^{[N]}$ in each inequality. Therefore, for (2.109), we have

$$\left|J_1^{\infty}(u_1^{[N]}) - J_1^{[N]}(u_1^{[N]}, u_{-1}^{\circ})\right| \le \frac{C^*}{\sqrt{N}}, \quad \forall N\in\mathbb{N}, \quad \forall u_1^{[N]}\in\mathscr{A}^{[N]}, \tag{2.119}$$

where C^* only depends on the model parameters and the constant $C^{\circ\circ}$. Therefore, we have

$$J_1^{\infty}(u_1^{[N]}) \le J_1^{[N]}(u_1^{[N]}, u_{-1}^{[N],\circ}) + \frac{C^*}{\sqrt{N}}, \quad \forall N\in\mathbb{N}, \quad \forall u_1^{[N]}\in\mathscr{A}^{[N]}, \tag{2.120}$$

and hence

$$J_1^{\infty}(u_1^{[N]}) \le \inf_{\left\{u_1^{[N]} \in \mathscr{A}^{[N]}\right\}} J_1^{[N]}(u_1^{[N]}, u_{-1}^{[N],\circ}) + \frac{C^*}{\sqrt{N}}, \quad \forall N \in \mathbb{N}, \quad \forall u_1^{[N]} \in \mathscr{A}^{[N]}. \tag{2.121}$$

We then recall from Section 2.4.2 and Section 2.4.2 that

$$J_1^{\infty}(u_1^{\circ}) \le J_1^{\infty}(u_1^{[N]}), \quad \forall N \in \mathbb{N}, \quad \forall u_1^{[N]} \in \mathscr{A}^{[N]}. \tag{2.122}$$

Next, from (2.107) and (2.121), we have

$$J_1^{[N]}(u_1^{[N],\circ}, u_{-1}^{[N],\circ}) - \frac{C^{\circ}}{\sqrt{N}} \le J_1^{\infty}(u_1^{\circ}) \le J_1^{\infty}(u_1^{[N]}) \le \inf_{\left\{u_1^{[N]} \in \mathscr{A}^{[N]}\right\}} J_1^{[N]}(u_1^{[N]}, u_{-1}^{[N],\circ}) + \frac{C^*}{\sqrt{N}}, \tag{2.123}$$

which finally gives

$$J_1^{[N]}(u_1^{[N],\circ}, u_{-1}^{[N],\circ}) \le \inf_{\left\{u_1^{[N]} \in \mathscr{A}^{[N]}\right\}} J_1^{[N]}(u_1^{[N]}, u_{-1}^{[N],\circ}) + \frac{C^* + C^{\circ}}{\sqrt{N}}. \tag{2.124}$$

□

2.5 Concluding Remarks

We conclude the chapter by studying a toy model and introducing a slight generalization of our framework that could broaden its applicability.

2.5.1 A Toy Model

We now study a toy model inspired by the model presented in Fouque and Zhang (2018), where the dynamics of a representative agent indexed by $i, i \in \mathscr{N}$, is given by

$$\begin{aligned} dx_i(t) &= (Ax_i(t) + Bu_i(t))dt + \sigma dW_i(t), \\ x_i(0) &= \xi. \end{aligned} \tag{2.125}$$

Moreover, agent i aims to minimize the cost functional

$$J^{[N]}(u_i, u_{-i}) = \mathbb{E}\int_0^T \left(\left| M^{\frac{1}{2}} \left(x_i(t) - x^{(N)}(t) \right) \right|^2 + |u_i(t)|^2 \right) dt + \mathbb{E}\left| G^{\frac{1}{2}} \left(x_i(T) - x^{(N)}(T) \right) \right|^2. \tag{2.126}$$

According to (2.25)-(2.26), we have $F_1 = F_2 = D = E = 0$, and $\widehat{F}_1 = \widehat{F}_2 = I$ for the above model. Applying our results, the contraction condition (2.66) simplifies to

$$TC_6 \exp(4TM_T C_6) < 1,$$

where $C_6 = 2M_T^2 \|B\|^2 (\|G\| + T\|M\|)$. To find a solution for a fixed $T > 0$, we can adjust the parameters G and M to ensure that the contraction condition is satisfied. Then the ε-Nash equilibrium is given by $\{u_i^\circ\}_{i \in \mathscr{N}}$, where

$$u_i^\circ(t) = -B^*(\Pi(T-t)x_i(t) + q(T-t)), \tag{2.127}$$

$$\bar{x}(t) = S(t)\xi - \int_0^t S(t-r)B^*(\Pi(T-t)\bar{x}(t) + q(T-t))dr, \tag{2.128}$$

$$\frac{d}{dt}\langle \Pi(t)x, x\rangle = 2\langle \Pi(t)x, Ax\rangle - \langle \Pi(t)BB^*\Pi(t)x, x\rangle + \langle Mx, x\rangle, \tag{2.129}$$

$$\dot{q}(t) = (A^* - \Pi(t)BB^*)\, q(t) - M\bar{x}(T-t), \tag{2.130}$$

with $\Pi(0) = G$, $x \in \mathscr{D}(A)$ and $q(0) = -G\bar{x}(T)$.

2.5.2 A Slight Generalization

Recall that we have defined $D \in \mathscr{L}(H, \mathscr{L}(V, H))$, $E \in \mathscr{L}(U, \mathscr{L}(V, H))$, $F_2 \in \mathscr{L}(H; \mathscr{L}(V; H))$, and $\sigma \in \mathscr{L}(V; H)$ for the volatility in (2.25). As mentioned in Remark 6, this setting might be more restrictive than necessary. One reason for this conservatism is our desire to align, in particular the derivations of Section 2.4.2, with the foundational literature, notably references such as Ichikawa (1979) and Ichikawa (1982). These settings could potentially be generalized to bounded linear operators from H and U to $\mathscr{L}_2(V_Q, H)$, with σ also set as $\sigma \in \mathscr{L}_2(V_Q, H)$. The conclusions of this chapter could likely be achieved with only minor modifications.

Chapter 3

Hilbert Space-Valued LQ Mean Field Games with Common Noise

Abstract

In this chapter, we extend the results of Chapter 2, which develops the theory of linear-quadratic mean field games in Hilbert spaces. We consider a framework in which the model also includes an infinite-dimensional common noise with a covariance operator different from that of the individual noise. In this setting, the offset term associated with the individual control problem and the mean field evolve as infinite-dimensional stochastic equations, whereas both are deterministic in the absence of common noise. As a result, the mean field consistency conditions take the form of a system of forward-backward stochastic equations in Hilbert spaces. We establish the solvability of this system and verify the ε-Nash property. Finally, we discuss the scenario where the model operators are themselves operator-valued stochastic processes adapted to the filtration generated by the common noise. We show that, under appropriate assumptions, the structure and solvability of the mean field game remain analogous to the case with non-random operators.

3.1 Introduction

A central assumption in classical mean field games (MFGs) is the presence of idiosyncratic noise, typically modeled as independent Brownian motions affecting individual agents' states.

However, many real-world scenarios are also influenced by common sources of randomness. This type of noise arises when external factors affect all agents simultaneously, creating dependencies between their actions and dynamics. In many financial models, macroeconomic factors—such as monetary policy announcements, aggregate demand shocks, or systemic market events—are modeled as common sources of randomness that impact all agents in the system. These influences can be understood as a common noise affecting the collective behavior of market participants. Another typical example of such situations is the presence of a major agent in the system as introduced in Chapter 1. Mathematically, such a common source is often modeled as a Wiener process that appears in the dynamics of all agents. The presence of common noise introduces additional challenges in solving MFGs. Specifically, the mean field consistency equations become a system of forward-backward stochastic differential equations (SDEs), whereas they are deterministic in the absence of common noise.

MFGs with common noise were seemingly first introduced in Carmona et al. (2015) to model systemic risk in interbank markets, and were studied at a theoretical level in Carmona et al. (2016) and its sequel Lacker (2016). In analogy with the theory of stochastic differential equations, these works formulated notions of strong and weak solutions for MFGs and established the existence of weak solutions under broad assumptions. Their approach captures the evolution of the distribution of states as a random measure flow adapted to the filtration generated by the common noise. This direction has been followed by subsequent studies such as Lacker and Webster (2015); Kolokoltsov and Troeva (2019); Lacker and Le Flem (2023).

The linear-quadratic setting is studied in Graber (2016), which explores the connection to linear-quadratic mean field type control. The solution of the mean field game is presented both in terms of a system of forward-backward SDEs and via a pair of Riccati equations. The study highlights the influence of common noise on the conditional mean field and clarifies the distinctions between decentralized games and centralized control problems. Other works on linear-quadratic mean field games with common noise include Ahuja (2016); Li et al. (2023a); Tchuendom (2018); Bensoussan et al. (2021); Ren and Firoozi (2024).

To the best of our knowledge, there are only a few works on mean field games in infinite-dimensional spaces Federico et al. (2024b); Liu and Firoozi (2025); Federico et al. (2024a). However, none of these works incorporate common noise.

The organization of this chapter is as follows. Section 3.2 presents the regularity results for coupled stochastic evolution equations with commcon noise in Hilbert spaces. Section 3.3 addresses MFGs in Hilbert spaces including the optimal control in the limiting case, the fixed-point argument, the Nash equilibrium and the ε-Nash property. Finally, Section 3.4 discusses the case where the model parameters are operator-valued stochastic process.

3.2 Coupled Controlled Stochastic Evolution Equations with Common Noise in Hilbert Space

In this chapter, we study a set of N coupled stochastic evolution equations driven by N idiosyncratic Q-Wiener processes as well as an infinite dimensional common noise W_0, which has a positive trace-class covariance operator Q_0. The eigenvalues and the corresponding eigenvectors of Q_0 are denoted by $\{\lambda_i^0, e_i^0\}_{i\in\mathbb{N}}$ such that $Q_0 e_i^0 = \lambda_i^0 e_i^0$. The common noise can be constructed using the first subsequence (labelled it as $\{\beta_j^0\}_{j\in\mathbb{N}}$) of the Brownian motions introduced in Proposition 8 , together with Q_0, i.e,

$$W_0(t) = \sum_{j\in\mathbb{N}} \sqrt{\lambda_j^0}\beta_j^0(t)e_j^0. \tag{3.1}$$

The system of N coupled stochastic evolution equations in presence of common noise W^0 is given by

$$\begin{aligned} x_i(t) =& S(t)\xi_i + \int_0^t S(t-r)F_i(r,\mathbf{x}(r),u_i(r))dr + \int_0^t S(t-r)B_i(r,\mathbf{x}(r),u_i(r))dW_i(r) \\ & \int_0^t S(t-r)B_0(r,\mathbf{x}(r),u_i(r))dW_0(r), \end{aligned} \tag{3.2}$$

where $i\in\mathscr{N}$ and $t\in\mathfrak{T}$. In (3.2), the vector process $\mathbf{x}$, the C_0-semigroup $S(t)$, $t\in\mathfrak{T}$, the mappings F_i and B_i are as defined in Section 2.3. The mapping B_0 is defined as $B_0 : \mathfrak{T}\times H^N\times U \to \mathscr{L}_2(V_{Q_0}, H)$. The filtration $\mathscr{F}^{[N],0}$ is now the one generated by $\{W_i\}_{i\in\mathscr{N}}$ and W_0. To establish the well-posedness of the above set of coupled stochastic evolution equations, we need to slightly adjust A2.3.2-A2.3.5, more specifically, only A2.3.4 and A2.3.5, as follows.

A3.2.1. $u_i \in \mathscr{M}^2(\mathfrak{T};U)$.

A3.2.2. *The mapping $F_i : \mathfrak{T}\times H^N\times U \to H$ is $\mathscr{B}(\mathfrak{T})\otimes\mathscr{B}(H^N)\otimes\mathscr{B}(U)/\mathscr{B}(H)$-measurable.*

A3.2.3. *The mappings* $B_i : \mathfrak{T} \times H^N \times U \to \mathscr{L}_2(V_Q, H)$ *are* $\mathscr{B}(\mathfrak{T}) \otimes \mathscr{B}(H^N) \otimes \mathscr{B}(U)/\mathscr{B}(\mathscr{L}_2(V_Q, H))$*-measurable and* $B_0 : \mathfrak{T} \times H^N \times U \to \mathscr{L}_2(V_{Q_0}, H)$ *is* $\mathscr{B}(\mathfrak{T}) \otimes \mathscr{B}(H^N) \otimes \mathscr{B}(U)/\mathscr{B}(\mathscr{L}_2(V_{Q_0}, H))$ *, where the Hilbert spaces* V_Q, V_{Q_0} *are as defined in Section 2.2.*

A3.2.4. *There exists a constant* C *such that, for every* $t \in \mathfrak{T}$*,* $u \in U$ *and* $\boldsymbol{x}, \boldsymbol{y} \in H^N$*, we have*

$$|F_i(t,\boldsymbol{x},u) - F_i(t,\boldsymbol{y},u)|_H + \|B_i(t,\boldsymbol{x},u) - B_i(t,\boldsymbol{y},u)\|_{\mathscr{L}_2} + \|B_0(t,\boldsymbol{x},u) - B_0(t,\boldsymbol{y},u)\|_{\mathscr{L}_2} \le C\,|\boldsymbol{x}-\boldsymbol{y}|_{H^N},$$
$$|F_i(t,\boldsymbol{x},u)|_H^2 + \|B_i(t,\boldsymbol{x},u)\|_{\mathscr{L}_2}^2 + \|B_0(t,\boldsymbol{x},u)\|_{\mathscr{L}_2}^2 \le C^2\left(1 + |\boldsymbol{x}|_{H^N}^2 + |u|_U^2\right).$$

Theorem 21. *(Existence and Uniqueness of a Mild Solution for the Case with Common Noise) Under A3.2.1-A3.2.4, the set of coupled stochastic evolution equations given by* (3.2) *admits a unique mild solution in the space* $\mathscr{H}^2(\mathfrak{T}; H^N)$.

Proof. Under assumptions A3.2.1 to A3.2.4, the following theorem can be proven in a similar manner as in Theorem 9. □

3.3 Hilbert Space-Valued LQ Mean Field Games with Common Noise

3.3.1 N-Player Game

In this chapter, we consider a differential game in Hilbert spaces defined on $\left(\Omega, \mathfrak{F}, \mathscr{F}^{[N],0}, \mathbb{P}\right)$, where $\mathscr{F}^{[N],0}$ is generated by $\{W_i\}_{i\in\mathscr{N}}$ and W_0. This game involves N asymptotically negligible (minor) agents, whose dynamics are governed by a system of coupled stochastic evolution equations, each given by the linear form of (3.2). More precisely, the dynamics of a representative agent indexed by i, $i \in \mathscr{N}$, are given by

$$\begin{aligned} x_i(t) =& S(t)\xi + \int_0^t S(t-r)(Bu_i(r) + F_1 x^N(r))\,dr + \int_0^t S(t-r)(Dx_i(t) + F_2 x^N(r) + \sigma)\,dW_i(r) \\ &+ \int_0^t S(t-r)(D_0 x_i(t) + F_0 x^N(r) + \sigma_0)\,dW_0(r), \end{aligned} \tag{3.3}$$

where the first three terms on the right-hand side of (3.3) are as defined in Section 2.2.2 and Section 2.4.1 with a slight (and standard) generalization of operator spaces, as described in Section 2.5.2. Specifically, we now assume that $D, F_2 \in \mathscr{L}(H, \mathscr{L}_2(V_Q, H))$. Moreover, the operators

$D_0, F_0 \in \mathscr{L}(H, \mathscr{L}_2(V_{Q_0}, H))$, and $\sigma_0 \in \mathscr{L}(V, H)$. Moreover, the initial condition is still given by A2.4.1.

A3.3.1. *The set of admissible control actions for agent i, denoted by $\mathscr{U}^{[N],0}$, is defined as the collection of $\mathscr{F}^{[N],0}$-adapted control laws u^i that belong to $\mathscr{M}^2(\mathfrak{T}; U)$*

The existence and uniqueness of the solution to (3.3) is guaranteed by Theorem 21. In addition, agent i aims to minimize the cost functional

$$J_i^{[N]}(u_i, u_{-i}) = \mathbb{E}\int_0^T \left(\left| M^{\frac{1}{2}} \left(x_i(t) - \widehat{F}_1 x^{(N)}(t) \right) \right|^2 + |u_i(t)|^2 \right) dt + \mathbb{E}\left| G^{\frac{1}{2}} \left(x_i(T) - \widehat{F}_2 x^{(N)}(T) \right) \right|^2, \tag{3.4}$$

where M and G are positive operators on H, and $\widehat{F}_1, \widehat{F}_2 \in \mathscr{L}(H)$.

3.3.2 Limiting Game

Under the limiting case, where the number of agents N goes to infinity, the state and cost functional of a representative agent, indexed by i, are, respectively, given by

$$\begin{aligned} x_i(t) =& S(t)\xi_i + \int_0^t S(t-r)(Bu_i(r) + F_1\bar{x}(r))dr \\ &+ \int_0^t S(t-r)(Dx_i(r) + F_2\bar{x}(r) + \sigma)\, dW_i(r) + \int_0^t S(t-r)(D_0 x_i(r) + F_0\bar{x}(r) + \sigma_0)\, dW_0(r), \end{aligned} \tag{3.5}$$

and

$$J_i^{\infty}(u_i) = \mathbb{E}\int_0^T \left(\left| M^{\frac{1}{2}} \left(x_i(t) - \widehat{F}_1 \bar{x}(t) \right) \right|^2 + |u_i(t)|^2 \right) dt + \mathbb{E}\left| G^{\frac{1}{2}} \left(x_i(T) - \widehat{F}_2 \bar{x}(T) \right) \right|^2, \tag{3.6}$$

where $\bar{x}(t)$ represents the coupling term in the limit and is termed the mean field. In this context, on the one hand, a Nash equilibrium for the system consists of the best response strategies of the agents to the mean field $\bar{x}(t)$. On the other hand, in the equilibrium where all agents follow Nash strategies, together they replicate the mean field, i.e. $\frac{1}{N}\sum_{i\in\mathscr{N}} x_i(t) \xrightarrow{\text{q.m.}} \bar{x}(t)$.

First, we denote by $\mathscr{F}_0$ be the filtration generated by the common noise, and define the Banach space $C_{\mathscr{F}_0}(\mathfrak{T}; L^2(\Omega; H)), |\cdot|_\infty)$ as in

$$C_{\mathscr{F}_0}(\mathfrak{T}; L^2(\Omega; H)) := \left\{ g : \mathfrak{T} \to L^2(\Omega; H) \,\middle|\, g \text{ is } \mathscr{F}_0 - \text{ adapted and is continuous in } L^2(\Omega; H) \right\}.$$

We note that every $g \in C_{\mathscr{F}_0}(\mathfrak{T}; L^2(\Omega; H))$ has a progressively measurable modification.

We then proceed with the following steps to solve the described mean field game problem. First, we treat the interaction term as an input $g \in C_{\mathscr{F}_0}(\mathfrak{T}; L^2(\Omega; H))$, and solve the resulting optimal control problem for a representative agent described by the dynamics

$$\begin{aligned} dx_i(t) &= (Ax_i(t) + Bu_i(t) + F_1 g(t))dt + (Dx_i(t) + F_2 g(r) + \sigma)dW_i(t) \\ &\quad + (D_0 x_i(t) + F_0 g(r) + \sigma_0)dW_0(t), \quad x_i(0) = \xi_i, \end{aligned} \tag{3.7}$$

and the cost functional

$$J(u_i) = \mathbb{E}\int_0^T \left(\left| M^{\frac{1}{2}} \left(x_i(t) - \widehat{F}_1 g(t) \right) \right|^2 + |u_i(t)|^2 \right) dt + \mathbb{E}\left| G^{\frac{1}{2}} \left(x_i(T) - \widehat{F}_2 g(T) \right) \right|^2. \tag{3.8}$$

The solution of the above optimal control problem yields the optimal pair (x_i°, u_i°).

Then, we address the consistency condition described by

$$\sup_{t \in \mathfrak{T}} \mathbb{E}\left| g(t) - x^{(N),\circ}(t) \right|_H^2 \to 0, \tag{3.9}$$

where $x^{(N),\circ}(t) := \frac{1}{N}(\sum_{i \in \mathscr{N}} x_i^\circ(t))$ and x_i° represents the optimal state of agent i corresponding to the control problem described by (3.7)-(3.8).

Riesz Mappings

We now (slightly) generalize the mappings introduced in Definition 4 to the present setting, i.e. for $D \in \mathscr{L}_2(V_Q, H)$ and $D_0 \in \mathscr{L}_2(V_{Q_0}, H)$. The continuous embeddings $\mathscr{L}(V, H) \hookrightarrow \mathscr{L}_2(V_Q, H)$ and $\mathscr{L}(V, H) \hookrightarrow \mathscr{L}_2(V_{Q_0}, H)$ play a central role in this context (see (2.3)). Note that if $\mathscr{R} \in \mathscr{L}(H)$, the term

$$\mathrm{Tr}\left(\mathscr{R}\left((Dx)Q^{1/2} \right) \left((Dx)Q^{1/2} \right)^* \right) \tag{3.10}$$

still defines a bounded bilinear functional on H^2, since $(Dx)Q^{1/2} \in \mathscr{L}_2(V, H)$ whenever $D \in \mathscr{L}_2(V_Q, H)$. Therefore, the mapping defined in Section 2.4.2 is still valid in this case, i.e, $\Delta_2 : \mathscr{L}(H) \to \mathscr{L}(H)$ such that

$$\mathrm{Tr}\left(\mathscr{R}\left((Dx)Q^{1/2} \right) \left((Dy)Q^{1/2} \right)^* \right) = \langle \Delta_2(\mathscr{R})x, y \rangle, \quad \forall x, y \in H, \tag{3.11}$$

with $\Delta_2(\mathscr{R}) \in \mathscr{L}(H)$, and

$$\|\Delta_2\| \leq \|D\|^2_{\mathscr{L}(H, \mathscr{L}_2(V_Q, H))}. \tag{3.12}$$

In a similar way, we define $\Delta_2^0 : \mathscr{L}(H) \to \mathscr{L}(H)$ by

$$\operatorname{Tr}\left(\mathscr{R}\left((D_0x)Q_0^{1/2}\right)\left((D_0y)Q_0^{1/2}\right)^*\right) = \langle \Delta_2^0(\mathscr{R})x, y\rangle, \quad \forall x, y \in H, \tag{3.13}$$

where $\Delta_2^0(\mathscr{R}) \in \mathscr{L}(H)$, and

$$\left\|\Delta_2^0\right\| \leq \|D_0\|^2_{\mathscr{L}(H,\mathscr{L}_2(V_Q,H))}. \tag{3.14}$$

Now, the mapping Γ_1 introduced in Definition 4 is generalized to $\Gamma_1 \in \mathscr{L}(\mathscr{L}_2(V_Q;H);H)$ defined by

$$\operatorname{tr}\left(\left((Dx)Q^{1/2}\right)\left(\mathscr{R}Q^{1/2}\right)^*\right) = \langle \Gamma_1(\mathscr{R}), x\rangle, \quad \forall x \in H, \quad \Gamma_1(\mathscr{R}) \in H. \tag{3.15}$$

Similarly $\Gamma_0 \in \mathscr{L}(\mathscr{L}_2(V_{Q_0};H);H)$ is given by

$$\operatorname{tr}\left(\left((D_0x)Q_0^{1/2}\right)\left(\mathscr{R}Q_0^{1/2}\right)^*\right) = \langle \Gamma_0(\mathscr{R}), x\rangle, \quad \forall x \in H, \quad \Gamma_0(\mathscr{R}) \in H, \tag{3.16}$$

with

$$\|\Gamma_1\|_{\mathscr{L}(\mathscr{L}_2(V_Q;H);H)} \leq \|D\|_{\mathscr{L}(H,\mathscr{L}_2(V_Q,H))}, \tag{3.17}$$

$$\|\Gamma_0\|_{\mathscr{L}(\mathscr{L}_2(V_{Q_0};H);H)} \leq \|D_0\|_{\mathscr{L}(H,\mathscr{L}_2(V_{Q_0},H))}. \tag{3.18}$$

Optimal Control Problem of a Representative Agent

Due to the homogeneity of the agents, we drop the index i in this section. Each agent faces a stochastic control problem in Hilbert spaces described by the state evolution equation

$$\begin{aligned} x(t) =& S(t)\xi + \int_0^t S(t-r)(Bu(r) + F_1g(r))\,dr + \int_0^t S(t-r)(Dx(r) + p(r))\,dW(r) \\ &+ \int_0^t S(t-r)(D_0x(r) + p_0(r))\,dW_0(r), \end{aligned} \tag{3.19}$$

where $p(t) = F_2g(t) + \sigma$, $p_0(t) = F_0g(t) + \sigma_0$, and $g \in C_{\mathscr{F}_0}(\mathfrak{T}; L^2(\Omega; H))$, and by the cost functional

$$J(u) = \mathbb{E}\int_0^T \left(\left|M^{\frac{1}{2}}\left(x(t) - \widehat{F}_1g(t)\right)\right|^2 + |u_i(t)|^2\right) dt + \mathbb{E}\left|G^{\frac{1}{2}}\left(x(T) - \widehat{F}_2g(T)\right)\right|^2. \tag{3.20}$$

Remark 10. *For convenience, we rewrite* (3.19) *as an equation driven by a single Q-Wiener process as in*

$$x(t) = S(t)\xi + \int_0^t S(t-r)(Bu(r) + F_1g(r))\,dr + \int_0^t S(t-r)(\bar{D}x(r) + \bar{p}(r))\,d\bar{W}(r), \tag{3.21}$$

where the V^2-valued Q-Wiener process is defined by (W, W_0), with the covariance operator (Q, Q_0). The operator $\overline{D} \in \mathscr{L}(H, \mathscr{L}_2(V_Q \times V_{Q_0}, H))$ is defined as

$$(\bar{D}x)(v_1, v_2) = (Dx)v_1 + (D_0x)v_2, \quad \forall x \in H, v_1, v_2 \in V. \tag{3.22}$$

Obviously, we have

$$\|\bar{D}\|^2_{\mathscr{L}(H,\mathscr{L}_2(V_Q\times V_{Q_0},H))} \leq \|D\|^2_{\mathscr{L}(H,\mathscr{L}_2(V_Q,H))} + \|D_0\|^2_{\mathscr{L}(H,\mathscr{L}_2(V_{Q_0},H))}. \tag{3.23}$$

Similarly, we define $\bar{p}(t) \in \mathscr{L}_2(V_Q \times V_{Q_0}, H)$ as

$$\bar{p}(t)(v_1, v_2) = p(t)v_1 + p_0(t)v_2. \tag{3.24}$$

The next theorem characterizes the optimal control law for the problem described by (3.19) and (3.20).

Theorem 22 (Optimal Control Law for the Case with Common Noise)**.** *The optimal control law u° for the Hilbert-space valued system described by* (3.19)-(3.20) *is given by*

$$u^\circ(t) = -B^* \left(\Pi(t)x(t) - q(t)\right), \tag{3.25}$$

with $\Pi \in C_s(\mathfrak{T}; \mathscr{L}(H))$, such that $\Pi(t)$ is a positive operator $\forall t \in \mathfrak{T}$, and with the pair ($q \in \mathscr{M}^2_{\mathscr{F}_0}(\mathfrak{T}; H), \tilde{q} \in \mathscr{M}^2_{\mathscr{F}_0}(\mathfrak{T}; \mathscr{L}_2(V_{Q_0}, H))$), respectively, satisfying the operator differential Riccati equation

$$\frac{d}{dt}\langle \Pi(t)x, x\rangle = -2\langle \Pi(t)x, Ax\rangle + \langle \Pi(t)BB^*\Pi(t)x, x\rangle - \langle \Delta_2(\Pi(t))x, x\rangle - \left\langle \Delta_2^0(\Pi(t))x, x\right\rangle - \langle Mx, x\rangle, \tag{3.26}$$

and the barkward linear stochastic evolution equation

$$\begin{aligned} q(t) =& S^*(T-t)G\hat{F}_2 g(T) - \int_t^T S^*(r-t)\Big(\Pi(r)BB^*q(r) - M\hat{F}_1 g(r) + \Gamma_1(\Pi(r)p(r)) \\ &+ \Gamma_0(\Pi(r)p_0(r) - \tilde{q}(r)) + \Pi(r)F_1 g(r)\Big)dr - \int_t^T S^*(r-t)\tilde{q}(r)dW_0(r). \end{aligned} \tag{3.27}$$

Proof. We point out that (3.26) is simply the backward form of (2.45) with $E = 0$. This distinction is inconsequential, as the Riccati equation is deterministic. We refer to Hu and Peng (1991); Guatteri and Tessitore (2005) for the existence and uniqueness of the solution to (3.27). The solution is a pair ($q \in \mathscr{M}^2_{\mathscr{F}_0}(\mathfrak{T}; H), \tilde{q} \in \mathscr{M}^2_{\mathscr{F}_0}(\mathfrak{T}; \mathscr{L}_2(V_{Q_0}, H))$) satisfying (3.27). The spaces $\mathscr{M}^2_{\mathscr{F}_0}$ are as defined in (2.4). Since the Riccati equation is deterministic, while the offset equation is stochastic, we treat them differently. First, we apply the same approximation method as in Ichikawa (1979); Liu and Firoozi (2025) to the process $\langle \Pi(t)x(t), x(t)\rangle$ to derive

$$\mathbb{E}\langle \Pi(T)x(T), x(T)\rangle = \mathbb{E}\langle \Pi(0)\xi, \xi\rangle + \mathbb{E}\int_0^T \Big[-\langle Mx(t), Mx(t)\rangle + 2\langle \Pi(t)x(t), Bu(t) + F_1 g(t)\rangle$$
$$+\langle \Pi(t)BB^*\Pi(t)x(t), x(t)\rangle + 2\langle \Gamma(\Pi(r)p(t)) + \Gamma_0(\Pi(r)p_0(t)), x(t)\rangle \Big] dt$$
$$+\mathbb{E}\int_0^T [\mathrm{tr}\left(\Pi(t)\left(p(t)Q^{1/2}\right)\left(p(t)Q^{1/2}\right)^*\right) + \mathrm{tr}\left(\Pi(t)\left(p_0(t)Q_0^{1/2}\right)\left(p_0(t)Q_0^{1/2}\right)^*\right)]dt \qquad (3.28)$$

Now, we introduce an approximation sequence for (3.27) which is a sequence of strong solutions $(q_n(t), \tilde{q}_n(t))$ such that

$$q_n(t) = G\hat{F}_2 g(T) + \int_t^T (A_n^* q_n(r) - \Pi(r)BB^* q_n(r) + M\hat{F}_1 g(r) - \Gamma(\Pi(r)p(r)) \qquad (3.29)$$
$$-\Gamma_0(\Pi(r)p_0(r) - \tilde{q}_n(r)) - \Pi(r)F_1 g(r)))dr - \int_t^T \tilde{q}_n(r)dW_0(r),$$

where $A_n^* = A^* n(A^* - nI)^{-1}$. From (Guatteri and Tessitore, 2005, Theorem 4.4), we have

$$\lim_{n\to\infty} \sup_{0\le t\le T} \mathbb{E}|q_n(t) - q(t)|^2 = 0, \lim_{n\to\infty} \mathbb{E}\int_0^T \|\tilde{q}_n(t) - \tilde{q}(t)\|\, dt = 0. \qquad (3.30)$$

Similarly, for (3.7) we have the approximation sequence given by

$$x_n(t) = \xi + \int_0^t (A_n x_n(t) + Bu(r) + Fg(r))\, dr + \int_0^t (Dx_n(t) + p(t))\, dW_i(r)$$
$$+\int_0^t (D_0 x_n(t) + p_0(t))\, dW_0(r), \qquad (3.31)$$

with

$$\lim_{n\to\infty} \sup_{0\le t\le T} \mathbb{E}|x_n(t) - x(t)|^2 = 0. \qquad (3.32)$$

Then, we apply Itô's formula to the process $\langle q_n(t), x_n(t)\rangle$ over the interval $\mathfrak{T}$, as in Guatteri and Tessitore (2005); Brzeźniak et al. (2008), and take the expectation of both sides to obtain

$$\mathbb{E}[\langle q(T), x(T)\rangle] = \mathbb{E}\langle q(0), \xi\rangle + \mathbb{E}\int_0^T \Big[\Big\langle x(t), \Pi(r)BB^* q(r) - M\hat{F}_1 g(r) + \Gamma_1(\Pi(r)p(r)) \qquad (3.33)$$
$$+\Gamma_0(\Pi(r)p_0(r)) + \Pi(r)F_1 g(r)\Big\rangle + \langle q(t), Bu(t) + F_1 g(t)\rangle$$
$$+\mathrm{tr}\left(\left(\tilde{q}(t)Q_0^{1/2}\right)\left(p_0(t)Q_0^{1/2}\right)^*\right)\Big]dt.$$

From (3.28) and (3.33), we obtain an expression for $\mathbb{E}[\langle \Pi(t)x(t),x(t)\rangle - 2\langle q(t),x(t)\rangle]$, which yields

$$\begin{aligned}J(u) =&\mathbb{E}\langle \Pi(0)\xi,\xi\rangle - 2\mathbb{E}\langle q(0),\xi\rangle + 2\mathbb{E}\langle GF_2g(T),F_2g(T)\rangle + \mathbb{E}\left[\int_0^T \left|u(t)+B^*\Pi(t)x(t)-B^*q(t)\right|^2 dt\right]\\ &+\mathbb{E}\int_0^T [\langle MF_1g(t),F_1g(t)\rangle - 2\mathrm{tr}\left(\left(\tilde{q}(t)Q_0^{1/2}\right)\left(p_0(t)Q_0^{1/2}\right)^*\right)\\ &+\mathrm{tr}\left(\Pi(t)\left(p(t)Q^{1/2}\right)\left(p(t)Q^{1/2}\right)^*\right) + \mathrm{tr}\left(\Pi(t)\left(p_0(t)Q_0^{1/2}\right)\left(p_0(t)Q_0^{1/2}\right)^*\right)\\ &-|B^*q(t)|^2 - 2\langle q(t),Fg(t)\rangle]dt. \end{aligned}\tag{3.34}$$

Finally, we derive the optimal feedback control as given by (3.25). □

We also point out that (3.27) may be understood as the mild solution of the following backward SDE in Hilbert space

$$\begin{aligned}dq(t) =& -\Big(A^* - (\Pi(t)BB^*q(t) - M\hat{F}_1g(t) + \Gamma_1(\Pi(t)p(t)) + \Gamma_0(\Pi(t)p_0(t)-\tilde{q}(t)) + \Pi(t)F_1g(t))\Big)dt\\ &+\tilde{q}(t)dW_0(t)\end{aligned}\tag{3.35}$$

From Theorem 22, the optimal state $x_i^\circ(t)$ for the representative agent i in the limiting case is given by

$$\begin{aligned}x_i^\circ(t) =&S(t)\xi_i - \int_0^t S(t-r)(BB^*\Pi(r)x_i^\circ(r) - BB^*q(r) - F_1g(r))dr\\ &+\int_0^t S(t-r)(Dx_i^\circ(r)+F_2g(r)+\sigma)\,dW_i(r)\\ &+\int_0^t S(t-r)(D_0x_i^\circ(r)+F_0g(r)+\sigma_0)\,dW_0(r).\end{aligned}\tag{3.36}$$

Fixed-Point Problem

We now solve the fixed point problem described by (3.9). We first note that (3.27) represents a mapping $g \mapsto q$ in the space $C_{\mathscr{F}_0}(\mathscr{T};L^2(\Omega;H))$. We present the following proposition which is essential for the satisfaction of the consistency condition (3.9). In the analysis for the rest of the chapter, the constant C may vary from one instance to another.

Proposition 23. *For the optimal state of a representative agent, satisfying* (3.36) *we have*

$$\mathbb{E}|x_i^\circ(t)|^2 \leq C, \quad \forall t \in \mathfrak{T},\tag{3.37}$$

where the constant C only depends on the parameters. Moreover, for a fixed $g \in \mathcal{M}^2_{\mathcal{F}_0}(\mathfrak{T};H)$, we have

$$\sup_{t\in\mathfrak{T}} \mathbb{E}\left|y_g(t) - x^{(N),\circ}(t)\right|^2_H \to 0, \tag{3.38}$$

where

$$\begin{aligned} y_g(t) = &S(t)\bar{\xi} - \int_0^t S(t-r)\,(BB^*\Pi(r)y_g(r) - BB^*q(r) - F_1 g(r))\,dr \\ &+ \int_0^t S(t-r)\,(D_0 y_g(r) + F_0 g(r) + \sigma_0)\,dW_0(r), \end{aligned} \tag{3.39}$$

and $\bar{\xi}$ is defined as in A2.4.1. The index g indicating that y_g is a function of $g \in C_{\mathcal{F}_0}(\mathfrak{T};L^2(\Omega;H))$.

Proof. From (3.36), we obtain

$$\begin{aligned} \mathbb{E}\,|x_i^\circ(t)|^2 \le\, & C\Big(\mathbb{E}\,|\xi_i|^2 + \mathbb{E}\int_0^t |BB^*\Pi(r)x_i^\circ(r) - BB^*q(r) - F_1 g(r)|^2\,dr \\ & + \mathbb{E}\int_0^t |Dx_i^\circ(r) + F_2 g(r) + \sigma|^2\,dr + \mathbb{E}\int_0^t |D_0 x_i^\circ(r) + F_0 g(r) + \sigma_0|^2\,dr\Big) \\ \le\, & C\Big(\mathbb{E}\,|\xi_i|^2 + (\|B\|^2 \mathscr{C}_\Pi(T)^2 + \|D\|^2 + \|D_0\|^2)\mathbb{E}\int_0^t |x_i^\circ(r)|^2\,dr \\ & + (\|F_0\|^2 + \|F_1\|^2 + \|F_2\|^2)\mathbb{E}\int_0^t |g(r)|^2\,dr + \|B\|^2\,\mathbb{E}\int_0^t |q(r)|^2\,dr \\ & + \mathbb{E}\int_0^t (\|\sigma\|^2 + \|\sigma_0\|^2)dr\Big) \\ \le\, & C(1 + \mathbb{E}\int_0^t |x_i^\circ(r)|^2\,dr). \end{aligned} \tag{3.40}$$

Then, by applying Grönwall's inequality, we obtain (3.37). We omit the proof of (3.38), as it follows the same arguments as in Theorem 18 and Theorem 28, which appears later in this chapter. The proof is substantially more straightforward in the current case since all agents are decoupled. □

Note that, for a fixed $g \in C_{\mathcal{F}_0}(\mathscr{T};L^2(\Omega;H))$ (and thus a fixed $q \in C_{\mathcal{F}_0}(\mathscr{T};L^2(\Omega;H))$), equation (3.39) admits a unique solution y_g in $C_{\mathcal{F}_0}(\mathfrak{T};L^2(\Omega;H))$. We seek a fixed point of the mapping $\Upsilon : g \mapsto y_g$ in the space $C_{\mathcal{F}_0}(\mathscr{T};L^2(\Omega;H))$ associated with the mean field consistency condition (3.9).

We now establish bounds on $\mathfrak{T}$ for relevant operators and processes that appear in the fixed-point mapping Υ. By reformulating the state equation as in (3.21) and from Proposition 12, we easily obtain a bound for the operator $\Pi(t)$ as

$$\|\Pi(t)\|_{\mathscr{L}(H)} \leq \mathscr{C}_{\Pi}(T), \quad \forall t \in \mathfrak{T}, \tag{3.41}$$

$$\mathscr{C}_{\Pi}(T) := 2M_T^2 \exp(8TM_T^2(\|D\|^2 + \|D_0\|^2)\,(\|G\| + T\,\|M\|)\,. \tag{3.42}$$

We also impose the following assumption in the remainder of this section.

A3.3.2. *The model parameters are such that* $\alpha(T) := 16M_T^2 T\,\|D_0\|^2 < 1$.

Lemma 24 (Bounded Variations of $q(t)$ wrt Variations of Input $g(t)$ for the Case with Common Noise)**.** *Consider the processes* $\Pi \in C_s(\mathfrak{T};\mathscr{L}(H))$ *and* ($q \in \mathscr{M}^2_{\mathscr{F}_0}(\mathfrak{T};H), \tilde{q} \in \mathscr{M}^2_{\mathscr{F}_0}(\mathfrak{T};\mathscr{L}_2(V_{Q_0},H))$, *respectively, satisfying* (3.26) *and* (3.27). *Moreover, let* $g_1, g_2 \in C_{\mathscr{F}_0}(\mathscr{T};L^2(\Omega;H))$ *be two processes on* $\mathfrak{T}$. *Then, we have*

$$\mathbb{E}\,|q_1(t) - q_2(t)|^2 \leq \mathscr{C}_1(T)\,|g_1(t) - g_2(t)|^2_{\infty}, \forall t \in \mathfrak{T} \tag{3.43}$$

where

$$\begin{aligned}\mathscr{C}_1(T) =& \frac{2M_T^2}{1-\alpha(T)}\Big(16T^2((\|M\|\,\|\hat{F}_1\|)^2 + (\mathscr{C}_{\Pi}(T)|\,\|F_1\|)^2 + \mathscr{C}^2_{\Pi}(T)((\|D\|\,\|F_2\|)^2 + (\|D_0\|\,\|F_0\|)^2)\\ &+ (\|G\|\,\|\hat{F}_2\|)^2)\Big) \times \exp(\frac{8M_T^2}{1-\alpha(T)}\mathscr{C}^2_{\Pi}(T)\,\|B\|^4).\end{aligned} \tag{3.44}$$

q_1 *and* q_2 *are the corresponding solutions of* (3.27) *to the inputs* $g = g_1 \in C_{\mathscr{F}_0}(\mathscr{T};L^2(\Omega;H))$ *and* $g = g_2 \in C_{\mathscr{F}_0}(\mathscr{T};L^2(\Omega;H))$, *respectively.*

Proof. For $g_1, g_2 \in C_{\mathscr{F}_0}(\mathscr{T};L^2(\Omega;H))$, we have

$$\begin{aligned}d(t) =& S^*(T-t)G\hat{F}_2(g_1(T) - g_2(T)) - \int_t^T S^*(r-t)(\Pi(r)BB^*d(r) + \psi(r) - \Gamma_0(\tilde{d}(r)))dr\\ &- \int_t^T S^*(r-t)\tilde{d}(r)dW_0(r)\end{aligned} \tag{3.45}$$

where

$$\begin{aligned}&\psi(t) = (\Pi(t)F_1 - M\hat{F}_1)(g_1(t) - g_2(t)) + \Gamma_1(\Pi(r)F_2(g_1(t) - g_2(t)) + \Gamma_0(\Pi(t)F_0(g_1(t) - g_2(t)),\\ &d(t) = q_1(t) - q_2(t),\\ &\tilde{d}(t) = \tilde{q}_1(t) - \tilde{q}_2(t).\end{aligned} \tag{3.46}$$

See Hu and Peng (1991) for the following result,

$$\begin{aligned}\mathbb{E}\,|d(t)|^2 \leq & 2M_T^2 T\mathbb{E}\int_t^T \left|\Pi(r)BB^*d(r)+\psi(r)-\Gamma_0(\tilde{d}(r)))\right|^2 dr + 2(M_T\,\|G\|\,\|\hat{F}_2\|)^2\mathbb{E}\,|g_1(T)-g_2(T)|^2 \\ & \leq 4M_T^2 T\mathbb{E}\int_t^T \left|\Pi(r)BB^*d(r)+\psi(r)\right|^2 dr + \frac{\alpha(T)}{4}\int_t^T \|\tilde{d}(t)\|^2\,dr \\ & + 2(M_T\,\|G\|\,\|\hat{F}_2\|)^2\mathbb{E}\,|g_1(T)-g_2(T)|^2 \end{aligned} \tag{3.47}$$

and

$$\begin{aligned}\mathbb{E}\int_t^T \|\tilde{d}(t)\|^2\,dt \leq & 8M_T^2 T\mathbb{E}\int_t^T \left|\Pi(r)BB^*d(r)+\psi(r)-\Gamma_0(\tilde{d}(r)))\right|^2 dr \\ & + 8(M_T\,\|G\|\,\|\hat{F}_2\|)^2\mathbb{E}\,|g_1(T)-g_2(T)|^2 \\ & \leq 16M_T^2 T\mathbb{E}\int_t^T \left|\Pi(r)BB^*d(r)+\psi(r)\right|^2 dr + \alpha(T)\int_t^T \|\tilde{d}(t)\|^2\,dr \\ & + 8(M_T\,\|G\|\,\|\hat{F}_2\|)^2\mathbb{E}\,|g_1(T)-g_2(T)|^2 \end{aligned} \tag{3.48}$$

Based on A3.3.2, we further obtain

$$\begin{aligned}\mathbb{E}\int_t^T \|\tilde{d}(t)\|^2\,dt \leq & \frac{1}{1-\alpha(T)}(16M_T^2 T\mathbb{E}\int_t^T \left|\Pi(r)BB^*d(r)+\psi(r)\right|^2 dr \\ & + 8(M_T\,\|G\|\,\|\hat{F}_2\|)^2\mathbb{E}\,|g_1(T)-g_2(T)|^2) \end{aligned} \tag{3.49}$$

Substitute (3.49) into (3.47)

$$\begin{aligned}\mathbb{E}\,|d(t)|^2 \leq & \frac{2M_T^2}{1-\alpha(T)}\Big[2T\mathbb{E}\int_t^T \left|\Pi(r)BB^*d(r)+\psi(r)\right|^2 dr + \|G\|^2\,\|\hat{F}_2\|^2\,\mathbb{E}\,|g_1(T)-g_2(T)|^2\Big] \\ \leq & \frac{2M_T^2}{1-\alpha(T)}\Big[4T\mathbb{E}\int_t^T (\left|\Pi(r)BB^*d(r)\right|^2 + |\psi(r)|^2)dr + \|G\|^2\,\|\hat{F}_2\|^2\,\mathbb{E}\,|g_1(T)-g_2(T)|^2\Big] \\ \leq & \frac{2M_T^2}{1-\alpha(T)}\Big[4T\mathscr{C}_\Pi^2(T)\,\|B\|^4\int_t^T \mathbb{E}\,|d(r)|^2\,dr + \Big(16T^2((\|M\|\,\|\hat{F}_1\|)^2 + (\mathscr{C}_\Pi(T)|\,\|F_1\|)^2 \\ & + \mathscr{C}_\Pi^2(T)((\|D\|\,\|F_2\|)^2 + (\|D_0\|\,\|F_0\|)^2) + (\|G\|\,\|\hat{F}_2\|)^2)\Big)\,|g_1-g_2|_\infty\Big] \end{aligned} \tag{3.50}$$

By Gronwall inequality, we obtain the result. □

Theorem 25 (Contraction Condition for the Case with Common Noise)**.** *The mapping*

$$\Upsilon : g \in C_{\mathscr{F}_0}(\mathscr{T};L^2(\Omega;H)) \longrightarrow y_g \in C_{\mathscr{F}_0}(\mathscr{T};L^2(\Omega;H))$$

admits a unique fixed point if

$$\mathscr{C}_2(T)e^{T\mathscr{C}_3(T)} < 1, \tag{3.51}$$

where

$$\begin{aligned}\mathscr{C}_2(T) &= 5M_T^2 T\left[T\left(\|B\|^4\mathscr{C}_1(T)+\|F_1\|^2\right)+\|F_0\|^2\right],\\ \mathscr{C}_3(T) &= 5M_T^2\left[\|D_0\|^2+T\|B\|^4\mathscr{C}_\Pi^2(T)\right].\end{aligned}\tag{3.52}$$

Proof. By (3.39), we have

$$\begin{aligned}y_{g_1}(t)-y_{g_2}(t) = &-\int_0^t S(t-r)BB^*\Pi(r)\left(y_{g_1}(r)-y_{g_2}(r)\right)dr\\ &+\int_0^t S(t-r)D_0\left(y_{g_1}(r)-y_{g_2}(r)\right)dW_0(r)\\ &+\int_0^t S(t-r)F_1\left(g_1(r)-g_2(r)\right)dr\\ &+\int_0^t S(t-r)F_0\left(g_1(r)-g_2(r)\right)dW_0(r)\\ &+\int_0^t S(t-r)BB^*\left(q_1(r)-q_2(r)\right)dr\\ = &\ \mathscr{I}_1+\mathscr{I}_2+\mathscr{I}_3+\mathscr{I}_4+\mathscr{I}_5.\end{aligned}\tag{3.53}$$

Calculating the square norm and taking expectation, we obtain

$$\mathbb{E}\left|y_{g_1}(t)-y_{g_2}(t)\right|^2 \le 5\left(\mathbb{E}\,|\mathscr{I}_1|^2+\mathbb{E}\,|\mathscr{I}_2|^2+\mathbb{E}\,|\mathscr{I}_3|^2+\mathbb{E}\,|\mathscr{I}_4|^2+\mathbb{E}\,|\mathscr{I}_5|^2\right).\tag{3.54}$$

For $\mathbb{E}\,|\mathscr{I}_5|^2$, using Lemma 24, we have

$$\begin{aligned}\mathbb{E}\,|\mathscr{I}_5|^2 &= \mathbb{E}\left|\int_0^t S(t-r)BB^*(q_1(r)-q_2(r))\,dr\right|^2\\ &\le T\,\mathbb{E}\int_0^t \left|S(t-r)BB^*(q_1(r)-q_2(r))\right|^2 dr\\ &\le M_T^2T^2\|B\|^4\mathbb{E}\,|q_1(t)-q_2(t)|^2\\ &\le M_T^2T^2\|B\|^4\mathscr{C}_1(T)\,|g_1-g_2|_\infty^2.\end{aligned}\tag{3.55}$$

Using a similar treatment for the remaining terms and collecting all contributions, we obtain

$$\mathbb{E}\left|y_{g_1}(t)-y_{g_2}(t)\right|^2 \le \mathscr{C}_2(T)\,|g_1-g_2|_\infty^2+\mathscr{C}_3(T)\int_0^t \mathbb{E}\left|y_{g_1}(r)-y_{g_2}(r)\right|^2 dr.\tag{3.56}$$

The result then follows from Grönwall's inequality. □

Remark 11. *It is straightforward to verify the following convergence properties as $T\to 0$:*

- $M_T \to 1$,

- $\alpha(T) \to 0$,

- $\mathscr{C}_\Pi(T) \to 1$,

- $\mathscr{C}_1(T) \to \theta > 0$,

- $\mathscr{C}_2(T) \to 0$,

- $T\mathscr{C}_3(T) \to 0$.

All of these functions are continuous in T. Therefore, it can be shown that the contraction condition is satisfied for a sufficiently small time horizon $T > 0$. Moreover, for such a small time horizon A3.3.2 is satisfied as we have $\alpha(T) \to 0$ as $T \to 0$. Thus, A3.3.2 does not impose any further restriction on the model parameters.

Suppose that (3.51) holds, we denote the unique fixed point of the mapping Υ associated with the mean field consistency condition (3.9) by $\bar{x} \in C_{\mathscr{F}_0}(\mathscr{T}; L^2(\Omega; H))$ and refer to it as the *mean field*.

Nash Equilibrium

Theorem 26 (Nash Equilibrium)**.** *Consider the Hilbert space-valued limiting system, described by* (3.7) *and* (3.8) *for $i \in \mathbb{N}$, and suppose the condition* (3.51) *holds. Then, the set of control laws $\{u_i^\circ\}_{i\in\mathbb{N}}$, where u_i° is given by*

$$u_i^\circ = -B^* \left(\Pi(t) x_i(t) - q(t)\right), \tag{3.57}$$

forms a unique Nash equilibrium for the limiting system where the mean field $\bar{x}(t) \in H$, the operator $\Pi(t) \in \mathscr{L}(H)$ and the pair of offset terms $(q(t) \in H, \tilde{q}(t) \in \mathscr{L}_2(V_{Q_0}, H))$, are characterized by the *unique fixed point of the following set of consistency equations*

$$\frac{d}{dt}\langle\Pi(t)x,x\rangle = -2\langle\Pi(t)x,Ax\rangle + \langle\Pi(t)BB^*\Pi(t)x,x\rangle - \langle\Delta(\Pi(t))x,x\rangle - \langle\Delta_0(\Pi(t))x,x\rangle - \langle Mx,x\rangle,$$

$$\Pi(T) = G, \quad x \in \mathscr{D}(A), \tag{3.58}$$

$$q(t) = S^*(T-t)G\hat{F}_2\bar{x}(T) - \int_t^T S^*(r-t)\Big(\Pi(r)BB^*q(r) - M\hat{F}_1\bar{x}(r) + \Gamma_1(\Pi(r)(F_2\bar{x}(r)+\sigma))$$
$$+ \Gamma_0(\Pi(r)(F_0\bar{x}(r)+\sigma_0)) - \tilde{q}(r)) + \Pi(r)F_1\bar{x}(r)\Big)dr - \int_t^T S^*(r-t)\tilde{q}(r)dW_0(r) \tag{3.59}$$

$$\bar{x}(t) = S(t)\bar{\xi} - \int_0^t S(t-r)((BB^*\Pi(r) - F_1)\bar{x}(r) - BB^*q(r))dr$$
$$+ \int_0^t S(t-r)((D_0+F_0)\bar{x}(r) + \sigma_0)\,dW_0(r). \tag{3.60}$$

Proof. The proof proceeds in a similar manner as that of Theorem 16. The equilibrium state is now given by

$$x_i^0(t) = S(t)\xi_i - \int_0^t S(t-r)(BB^*\Pi(r)x_i^0(r) - BB^*q(r) - F_1\bar{x}(r))dr$$
$$+ \int_0^t S(t-r)(Dx_i^0(r) + F_2\bar{x}(r) + \sigma)\,dW_i(r) + \int_0^t S(t-r)(D_0x_i^0(r) + F_0\bar{x}(r) + \sigma_0)\,dW_0(r), \tag{3.61}$$

Moreover, the consistency condition (3.9) is guaranteed by Proposition 23 with respect to the fixed-point $\bar{x}$, which is the mild solution of the SDE given by

$$d\bar{x}(t) = (A\bar{x}(t) - ((BB^*\Pi(t) - F_1)\bar{x}(t) - BB^*q(t))dr + ((D_0+F_0)\bar{x}(t) + \sigma_0)dW_0(t), \tag{3.62}$$

with $\bar{x}(0) = \bar{\xi}$ □

3.3.3 ε-Nash Property

In this section, we establish the ε-Nash property of the set of control laws $\{u_i^\circ\}_{i\in\mathscr{N}}$ given by (3.57) for the N-player game described by (3.3)-(3.4). Due to the symmetric properties (exchangeability) of agents, we study the case where agent $i=1$ deviates from the Nash equilibrium strategies. Specifically, we suppose that any agent i, $i\in\mathscr{N}$ and $i\neq 1$, employs the feedback strategy u_i° given by (3.57) and agent $i=1$ is allowed to choose an arbitrary control $u_1 \in \mathscr{U}^{[N],0}$. Therefore, we have a system of stochastic evolution equations given by

$$x_1^{[N]}(t) = S(t)\xi_1 + \int_0^t S(t-r)\left(Bu_1^{[N]}(r) + F_1x^{(N)}(r)\right)dr + \int_0^t S(t-r)p_1^{[N]}(r)dW_i(r)$$

$$+\int_0^t S(t-r)p_{1,0}^{[N]}(r)dW_0(r), \tag{3.63}$$

$$x_i^{[N]}(t) = S(t)\xi_i + \int_0^t S(t-r)\left(Bu_i^{[N],\circ}(r) + F_1 x^{(N)}(r)\right)dr + \int_0^t S(t-r)p_i^{[N]}(r)dW_i(r)$$
$$+\int_0^t S(t-r)p_{i,0}^{[N]}(r)dW_0(r), \quad i \in \mathcal{N}, \text{ and } i \neq 1, \tag{3.64}$$

where $p_i^{[N]}(t) := Dx_i^{[N]}(t) + Fx^{(N)}(t) + \sigma, p_{i,0}^{[N]}(t) := D_0 x_i^{[N]}(t) + F_0 x^{(N)}(t) + \sigma_0$. Furthermore, we recall that the cost functional of agent $i = 1$ in the N-player game is given by

$$J^{[N]}(u_1^{[N]}, u_{-1}) := \mathbb{E}\int_0^T \left|M^{\frac{1}{2}}\left(x_1^{[N]}(t) - x^{(N)}(t)\right)\right|^2 + \left|u_1^{[N]}(t)\right|^2 dt + \mathbb{E}\left|G^{\frac{1}{2}}\left(x_1^{[N]}(T) - x^{(N)}(T)\right)\right|^2. \tag{3.65}$$

The ε-Nash property indicates that

$$J_1^{[N]}(u_1^{[N],\circ}, u_{-1}^{[N],\circ}) \leq \inf_{\left\{u_1^{[N]} \in \mathscr{U}^{[N],0}\right\}} J_1^{[N]}(u_1^{[N]}, u_{-1}^{[N],\circ}) + \varepsilon_N, \tag{3.66}$$

where the sequence $\{\varepsilon_N\}_{N\in\mathbb{N}}$ converges to zero. To establish this property. The results that follow can be derived using a method similar to that employed in the previous chapter.

Lemma 27. *Consider the N coupled systems described by* (3.63)-(3.64)*. Then, the property*

$$\mathbb{E}\left[\sum_{i\in\mathcal{N}} \left|x_i^{[N]}(t)\right|_H^2\right] \leq C(u_1^{[N]})N, \tag{3.67}$$

holds uniformly for all $t \in \mathfrak{T}$. Here, the constant $C(u_1^{[N]})$ depends on the model parameters and $u_1^{[N]}$.

Proof. From (3.64), for agent i, $i \in \mathcal{N}$ and $i \neq 1$, we have $Bu_i^\circ(t) = -B^*\left(\Pi(t)x_i^{[N]}(t) - q(t)\right)$. Therefore,

$$\mathbb{E}\left|x_i^{[N]}(t)\right|^2 \leq C\mathbb{E}\left[|\xi_i|^2 + \int_0^t \left(\left|Bu_i^\circ(r) + F_1 x^{(N)}(r)\right|^2 + \left\|p_i^{[N]}\right\|_{\mathscr{L}_2}^2 + \left\|p_{i,0}^{[N]}\right\|_{\mathscr{L}_2}^2\right)dr\right]$$
$$\leq C\Big(\int_0^t \mathbb{E}\left|x_i^{[N]}(r)\right|^2 dr + \int_0^t \mathbb{E}\left|x^{(N)}(r)\right|^2 dr + 1\Big)$$
$$\leq C\Big(\int_0^t \mathbb{E}\left|x_i^{[N]}(r)\right|^2 dr + \frac{1}{N}\int_0^t \mathbb{E}\Big[\sum_{j\in\mathcal{N}}\left|x_j^{[N]}(r)\right|^2\Big]dr + 1\Big). \tag{3.68}$$

From (3.63), for agent $i = 1$, we still have $\mathbb{E}\int_0^T \left|u_1^{[N]}(r)\right|^2 dt < \infty$, and hence

$$\mathbb{E}\left|x_1^{[N]}(t)\right|^2 \leq C\mathbb{E}\left[|\xi_1|^2 + \int_0^t \left(\left|Bu_1^{[N]}(r)\right|^2 + \left|Fx^{(N)}(r)\right|^2 + \left\|p_1^{[N]}\right\|_{\mathscr{L}_2}^2 + \left\|p_{1,0}^{[N]}\right\|_{\mathscr{L}_2}^2\right)dr\right]$$
$$\leq C\Big(\int_0^t \mathbb{E}\left|x_1^{[N]}(r)\right|^2 dr + \frac{1}{N}\int_0^t \mathbb{E}\Big[\sum_{j\in\mathcal{N}}\left|x_j^{[N]}(r)\right|^2\Big]dr + 1\Big). \tag{3.69}$$

From (3.68) and (3.69), we obtain

$$\mathbb{E}\left[\sum_{i\in\mathcal{N}}\left|x_i^{[N]}(t)\right|^2\right]\leq C\left(N+\int_0^t\mathbb{E}\left[\sum_{i\in\mathcal{N}}\left|x_i^{[N]}(r)\right|^2\right]dr\right). \tag{3.70}$$

Applying Grönwall's inequality to the above equation results in (3.67). □

Similarly, we have the following results, which are directly from the lemma above

$$\mathbb{E}\left|x^{(N)}(t)\right|^2\leq C(u_1^{[N]}),\mathbb{E}\left|x_1^{[N]}(t)\right|^2\leq C(u_1^{[N]}), \tag{3.71}$$

At the equilibrium, we have

$$\mathbb{E}\left|x^{(N),\circ}(t)\right|^2\leq C^\circ,\quad \mathbb{E}\left|x_1^{[N],\circ}(t)\right|^2\leq C^\circ,\quad \forall t\in\mathfrak{T}. \tag{3.72}$$

where the constant C° does not depend on N.

Theorem 28 (Average State Error Bound for the Case with Common Noise)**.** *Suppose the state of any agent i, $i\in\mathcal{N}$ and $i\neq 1$, satisfies* (3.64)*, where the agent employs the strategy u_i° given by* (3.57)*. For any control law $u_1\in\mathscr{U}^{[N]}$ that agent $i=1$ chooses, we have*

$$\sup_{t\in\mathfrak{T}}\mathbb{E}\left|\bar{x}(t)-x^{(N)}(t)\right|_H^2\leq\frac{C(u_1^{[N]})}{N}. \tag{3.73}$$

Proof. The average state $x^{(N)}(t)$, by direct computation, is given by

$$\begin{aligned}x^{(N)}(t)=&S(t)x^{(N)}(0)-\int_0^t S(t-r)(BB^*\Pi(t)-F_1)x^{(N)}(r)\,dr+\int_0^t S(t-r)BB^*q(r)\,dr\\&+\frac{1}{N}\left[\sum_{i\in\mathcal{N}}\int_0^t S(t-r)p_i^{[N]}(r)\,dW_i(r)\right]+\int_0^t S(t-r)\left((D_0+F_0)x^{(N)}(r)+\sigma_0\right)dW_0(r)\\&+\frac{1}{N}\int_0^t S(t-r)B\left(u_1^{[N]}(r)+B^*x_1^{[N]}(r)-B^*q(r)\right)dr.\end{aligned} \tag{3.74}$$

Now, define $y^N(t):=\bar{x}(t)-x^{(N)}(t)$. Then, we have

$$\begin{aligned}y^N(t)=&S(t)y^N(0)-\int_0^t S(t-r)(BB^*\Pi(t)-F_1)y^{(N)}(r)\,dr+\int_0^t S(t-r)(D_0+F_0)y^{(N)}(r)\,dW_0(r)\\&-\frac{1}{N}\left[\sum_{i\in\mathcal{N}}\int_0^t S(t-r)p_i^{[N]}(r)\,dW_i(r)\right]-\frac{1}{N}\int_0^t S(t-r)B\left(u_1^{[N]}(r)+B^*x_1^{[N]}(r)-B^*q(r)\right)dr.\end{aligned} \tag{3.75}$$

Then, we obtain the following estimate:

$$\mathbb{E}\left|y^N(t)\right|^2 \leq C\int_0^t \mathbb{E}\left|y^N(r)\right|^2 dr + \frac{C}{N^2}\left[\mathbb{E}\left|\sum_{i\in\mathcal{N}}\int_0^t S(t-r)p_i^{[N]}(r)\,dW_i(r)\right|\right]^2 + \frac{C}{N^2}\left(\int_0^t \mathbb{E}\left|B\left(u_1^{[N]}(r)+B^*x_1^{[N]}(r)-B^*q(r)\right)\right|^2 dr\right). \tag{3.76}$$

Further, we estimate the stochastic fluctuation term:

$$\begin{aligned}\mathbb{E}\left|\sum_{i\in\mathcal{N}}\int_0^t S(t-r)p_i^{[N]}(r)\,dW_i(r)\right|^2 &= \sum_{i\in\mathcal{N}}\mathbb{E}\left|\int_0^t S(t-r)\left(Dx_i^{[N]}(r)+F_2x^{(N)}(r)+\sigma\right)dW_i(r)\right|^2\\ &\leq C\left(\int_0^t\sum_{i\in\mathcal{N}}\mathbb{E}\left|x_i^{[N]}(r)\right|^2 dr + N\int_0^t \mathbb{E}\left|x^{(N)}(r)\right|^2 dr + N\right)\\ &\leq CN. \end{aligned}\tag{3.77}$$

Also,

$$\int_0^t \mathbb{E}\left|B(u_1^{[N]}(r)+B^*x_1^{[N]}(r)-B^*q(r))\right|^2 dr \leq C\int_0^t (\mathbb{E}\left|x_1^{[N]}(r)\right|^2+1)dr \leq CN. \tag{3.78}$$

By (3.71), we derive

$$\mathbb{E}\left|y^N(t)\right|^2 \leq \frac{C}{N} + C\int_0^t \mathbb{E}\left|y^N(r)\right|^2 dr. \tag{3.79}$$

Applying grownwall inequality, 2.28 follows. □

Proposition 29 (Error Bounds for Agent $i=1$ for the Case with Common Noise))**.** *Let $x_1(t)$ and $x_1^{[N]}(t)$, respectively, denote the state of agent $i=1$ in the limiting game and the N-player game satisfying* (3.5) *and* (3.63)*. Moreover, let $J^\infty(u_1^\circ)$ and $J^{[N]}(u_1^\circ, u_{-1}^\circ)$, respectively, denote the cost functional of agent $i=1$ in the limiting game and the N-player game given by* (3.6) *and* (3.65)*.*

(i) If agent $i=1$ employs the control law u_1° given by (3.57)*, we have*

$$\sup_{t\in\mathfrak{T}}\mathbb{E}\left|x_1^\circ(t)-x_1^{[N],\circ}(t)\right|_H^2 \leq \frac{C^\circ}{N}, \tag{3.80}$$

$$\left|J_1^\infty(u_1^\circ)-J_1^{[N]}(u_1^{[N],\circ},u_{-1}^{[N],\circ})\right| \leq \frac{C^\circ}{\sqrt{N}}. \tag{3.81}$$

(ii) If agent $i=1$ employs any $u_1^{[N]}\in\mathscr{U}^{[N],0}$, we have

$$\sup_{t\in\mathfrak{T}}\mathbb{E}\left|x_1(t)-x_1^{[N]}(t)\right|_H^2 \leq \frac{C(u_1^{[N]})}{N}, \tag{3.82}$$

$$\left|J_1^\infty(u_1^{[N]})-J_1^{[N]}(u_1^{[N]},u_{-1}^{[N],\circ})\right| \leq \frac{C(u_1^{[N]})}{\sqrt{N}}. \tag{3.83}$$

Proof. For the case where agent $i = 1$ employs the control law u_1° given by (3.57), by direct computation, we have

$$
\begin{aligned}
x_1^\circ(t) - x_1^{[N],\circ}(t) = &-\int_0^t S(t-r)BB^*\Pi(r)(x_1^\circ(r) - x_1^{[N],\circ}(r))dr + \int_0^t S(t-r)F\left(\bar{x}(r) - x^{(N),\circ}(r)\right)dr \\
&+ \int_0^t S(t-r)\Big(D(x_1^\circ(r) - x_1^{[N],\circ}(r)) + E(\bar{x}(r) - x^{(N),\circ}(r))\Big)dW_1(r) \\
&+ \int_0^t S(t-r)\Big(D_0(x_1^\circ(r) - x_1^{[N],\circ}(r)) + E_0(\bar{x}(r) - x^{(N),\circ}(r))\Big)dW_0(r)
\end{aligned} \tag{3.84}
$$

Moreover, for the case where agent $i = 1$ employs an arbitrary control $u_1 \in \mathscr{U}^{[N,0]}$, we have

$$
\begin{aligned}
x_1(t) - x^{[N]}(t) = &\int_0^t S(t-r)F\left(\bar{x}(r) - x^{(N)}(r)\right)dr \\
&+ \int_0^t S(t-r)\Big(D(x_1(r) - x_1^{[N]}(r)) + E(\bar{x}(r) - x^{(N)}(r))\Big)dW_1(r) \\
&+ \int_0^t S(t-r)\Big(D_0(x_1(r) - x_1^{[N]}(r)) + E_0(\bar{x}(r) - x^{(N)}(r))\Big)dW_0(r).
\end{aligned} \tag{3.85}
$$

The remainder of the proof for (3.80) and (3.82) follows the method used in Proposition 19: taking square norms, expectations, applying Grönwall's inequality (see (3.76)), and leveraging the results from Theorem 28. Subsequently, the proof of (3.81) and (3.83) proceeds in the same way as in Proposition 19. □

Theorem 30. *(ε-Nash Equilibrium for the Case with Common Noise) Suppose that condition* (3.51) *holds. Then, the set of control laws $\{u_i^\circ\}_{i\in\mathcal{N}}$, where u_i° is given by* (3.57)*, forms an ε-Nash equilibrium for the N-player system described by* (3.3)-(3.4)*. That is, for any alternative control action $u_1 \in \mathscr{U}^{[N],0}$ that the representative agent $i = 1$ employs, there is a sequence of nonnegative numbers $\{\varepsilon_N\}_{N\in\mathbb{N}}$ converging to zero, such that*

$$
J_1^{[N]}(u_1^{[N],\circ}, u_{-1}^{[N],\circ}) \leq \inf_{\left\{u_1^{[N]} \in \mathscr{U}^{[N],0}\right\}} J_1^{[N]}(u_1^{[N]}, u_{-1}^{[N],\circ}) + \varepsilon_N, \tag{3.86}
$$

where $\varepsilon_N = O(\frac{1}{\sqrt{N}})$.

Proof. The proof follows in the same manner as in Theorem 20, based on Lemma 27–Proposition 29. □

3.4 Hilbert Space-Valued LQ Mean Field Games with Common Noise and Random Operators

In this section, we discuss the case where the model involves random operators. We begin by briefly recalling the concept of measurability for operator-valued functions.

Let $(\mathcal{S}, \mathcal{A})$ be a measurable space, and let E_1 and E_2 be two separable Banach spaces. We denote by $\sigma_{so}(E_1, E_2)$ the sigma-algebra generated by the strong operator topology on $\mathscr{L}(E_1; E_2)$. A function $f : \mathcal{S} \to \mathscr{L}(E_1; E_2)$ is said to be strongly measurable if it is $\sigma_{so}(E_1, E_2)$-measurable, or equivalently, if $f(s)e$ is an E_2-valued measurable function for every $e \in E_1$. Note that, since E_1 and E_2 are separable, the real-valued function $\|f(s)\|_{\mathscr{L}(E_1;E_2)}$ is measurable, although $f(s)$ is not necessarily measurable with respect to the Borel sigma-algebra generated by the operator norm topology. We refer the reader to Blasco and van Neerven (2010) and Figiel (1990) for more details on measurability, integrability, and the associated function spaces.

3.4.1 N-Player Game

In this section, we introduce LQ mean field games in Hilbert spaces with common noise and random operators. The N-player game is still given in the same form as that of Section 3.3.1. However, the coefficient operators are now stochastic processes taking values in appropriate operator spaces. Specifically, in the N-player game, the dynamics for agent i, $i \in \mathcal{N}$, are given by

$$\begin{aligned} x_i(t) = & S(t)\xi + \int_0^t S(t-r)(B(r)u_i(r) + F_1(r)x^{(N)}(r))\, dr + \\ & \int_0^t S(t-r)(D(r)x_i(t) + F_2(r)x^{(N)}(r) + \sigma)\, dW_i(r) \\ & + \int_0^t S(t-r)(D_0(r)x_i(t) + F_0(r)x^{(N)}(r) + \sigma_0)\, dW_0(r), \end{aligned} \tag{3.87}$$

and the cost functional is given by

$$J_i^{[N]}(u_i, u_{-i}) = \mathbb{E}\int_0^T \left(\left| M^{\frac{1}{2}}(r) \left(x_i(t) - \widehat{F}_1(r)x^{(N)}(t) \right) \right|^2 + |u_i(t)|^2 \right) dt + \mathbb{E} \left| G^{\frac{1}{2}} \left(x_i(T) - \widehat{F}_2 x^{(N)}(T) \right) \right|^2, \tag{3.88}$$

where the C_0-semigroup is defined as in (2.6) and all other operators are random as detailed below. We note that, in this case, due to the presence of random coefficients, the well-posedness of the N-player game does not directly fall under the framework of Theorem 21. However, the given framework in this theorem can be modified to accommodate more general settings of random coefficients. In this chapter, for simplicity, we restrict our attention to only coupled linear stochastic evolution equations given by (3.87). We now impose the following assumptions for the N-players game described above.

A 3.4.1. *$B : \Omega \times \mathfrak{T} \to \mathscr{L}(U;H)$ is $\mathscr{B}([0,t]) \otimes \mathscr{F}^0$ progressively strongly measurable with $\|B\|_{L(U;H)} \leq C_B$, $\lambda \times \mathbb{P}$-a.e.*

A3.4.2. *$M, F_1, \hat{F}_1 : \Omega \times \mathfrak{T} \to \mathscr{L}(H)$ are $\mathscr{B}([0,t]) \otimes \mathscr{F}^0$ progressively strongly measurable with $\|F_1\|_{L(H)} \leq C_{F_1}, \left\|\hat{F}_1\right\|_{L(H)} \leq C_{\hat{F}_1}, \|M\|_{L(H)} \leq C_M$, $\lambda \times \mathbb{P}$-a.e. In addition, M takes value as positive operators $\lambda \times \mathbb{P}$-a.e.*

A3.4.3. *$G, \hat{F}_2 \in \Omega \to \mathscr{L}(H)$, are strongly measurable with $\|G\|_{L(H)} \leq C_G, \left\|\hat{F}_2\right\|_{L(H)} \leq C_{\hat{F}_2}$, $\mathbb{P}$-a.s. In addition, G takes value as positive operators $\mathbb{P}$-a.s.*

A3.4.4. *$D, F_2 : \Omega \times \mathfrak{T} \to \mathscr{L}(H; \mathscr{L}_2(V_Q;H)))$ are $\mathscr{B}([0,t]) \otimes \mathscr{F}^0$ progressively strongly measurable with $\|D\| \leq C_D$, $\|F_2\| \leq C_{F_2}$, $\lambda \times \mathbb{P}$-a.e.*

A3.4.5. *$D_0, F_0 : \Omega \times \mathfrak{T} \to \mathscr{L}(H; \mathscr{L}_2((V_{Q_0};H)))$ are $\mathscr{B}([0,t]) \otimes \mathscr{F}^0$ progressively strongly measurable with $\|D_0\| \leq C_{D_0}, \|F_0\| \leq C_{F_0}$, $\lambda \times \mathbb{P}$-a.e.*

Theorem 31. *Under A3.4.1-A3.4.5, the N-player game given by* (3.87) *admits a unique mild solution in the space $\mathscr{H}^2(\mathfrak{T}; H^N)$.*

Proof. This result can be easily verified under A3.4.1–A3.4.5 by following the approach outlined in the proofs of Theorem 9 and Theorem 21. □

3.4.2 Limiting Game

Similar to Section 3.3.2, the limiting game is given by the following state equation

$$
\begin{aligned}
x_i(t) =& S(t)\xi_i + \int_0^t S(t-r)(B(r)u_i(r) + F_1(r)\bar{x}(r))dr + \int_0^t S(t-r)(D(r)x_i(r) + F_2(r)\bar{x}(r) + \sigma)\,dW_i(r) \\
&+ \int_0^t S(t-r)(D_0(r)x_i(r) + F_0(r)\bar{x}(r) + \sigma_0)\,dW_0(r),
\end{aligned} \tag{3.89}
$$

and the cost functional

$$
J_i^\infty(u_i) = \mathbb{E}\int_0^T \left(\left| M^{\frac{1}{2}}(r) \left(x_i(t) - \widehat{F}_1(r)\bar{x}(t) \right) \right|^2 + |u_i(t)|^2 \right) dt + \mathbb{E}\left| G^{\frac{1}{2}} \left(x_i(T) - \widehat{F}_2 \bar{x}(T) \right) \right|^2. \tag{3.90}
$$

Optimal Control

We now turn to the analysis of the optimal control problem characterized by (3.89) and (3.90) within the current framework. First, we note that under A3.4.1–A3.4.5, the H-valued process $\Gamma_1(\mathscr{R}(t))$ (and similarly $\Gamma_0(\mathscr{R}(t))$), where $\mathscr{R}(t)$ is now viewed as a process in $\mathscr{M}^2_{\mathscr{F}_0}(\mathfrak{T};\mathscr{L}_2(V_Q,H))$, is progressively measurable. Here, the processes $\Gamma_1(\mathscr{R}(t))$ and $\Gamma_0(\mathscr{R}(t))$ are defined by

$$\begin{aligned}\operatorname{tr}\left((D(t)x)\,Q^{1/2}\left(\mathscr{R}(t)Q^{1/2}\right)^*\right) &= \langle\Gamma_1(\mathscr{R}(t)),x\rangle, \quad \forall x\in H.\\ \operatorname{tr}\left((D_0(t)x)\,Q_0^{1/2}\left(\mathscr{R}(t)Q_0^{1/2}\right)^*\right) &= \langle\Gamma_0(\mathscr{R}(t)),x\rangle, \quad \forall x\in H.\end{aligned} \tag{3.91}$$

The progressive measurability of this process follows from the Pettis measurability theorem, which establishes the equivalence between measurability and weak measurability when H is separable. Similarly, given that $\mathscr{R}(t)$ is strongly progressively measurable in $\mathscr{L}(H)$, the $\mathscr{L}(H)$- valued process $\Delta_2(\mathscr{R}(t))$ defined by

$$\begin{aligned}\operatorname{Tr}\left(\mathscr{R}(t)\left((D(t)x)Q^{1/2}\right)\left((D(t)y)Q^{1/2}\right)^*\right) &= \langle\Delta_2(\mathscr{R}(t))x,y\rangle, \quad \forall x,y\in H,\\ \operatorname{Tr}\left(\mathscr{R}(t)\left((D_0(t)x)Q_0^{1/2}\right)\left((D_0(t)y)Q_0^{1/2}\right)^*\right) &= \langle\Delta_2^0(\mathscr{R}(t))x,y\rangle, \quad \forall x,y\in H,\end{aligned} \tag{3.92}$$

is also strongly progressively measurable in $\mathscr{L}(H)$.

Now, we apply the reformulation described in Remark 10 to write the state equation as

$$x(t) = S(t)\xi + \int_0^t S(t-r)(B(r)u(r)+F_1(r)g(r))\,dr + \int_0^t S(t-r)(\bar{D}(r)x(r)+\bar{p}(r))\,d\bar{W}(r). \tag{3.93}$$

The cost functional retains the same form. For convenience, it is given by

$$J(u) = \mathbb{E}\int_0^T \left(\left|M^{\frac{1}{2}}(r)\left(x(t)-\widehat{F}_1(r)g(t)\right)\right|^2 + |u_i(t)|^2\right)dt + \mathbb{E}\left|G^{\frac{1}{2}}\left(x(T)-\widehat{F}_2 g(T)\right)\right|^2. \tag{3.94}$$

The optimal control problem given by (3.93) and (3.94) is addressed in (Guatteri and Tessitore, 2005, Theorem 8.1). In summary, the optimal control is given by

$$u^\circ(t) = -B^*\left(\Pi(t)x(t)-q(t)\right), \tag{3.95}$$

where Π satisfies a stochastic Riccati equation given by

$$\begin{aligned}-d\Pi(t) = \Big[&A^*\Pi(t)+\Pi(t)A+\Delta_2(\Pi(t))+\Delta_2^0(\Pi(t))+\operatorname{Tr}(\bar{D}^*(t)\tilde{\Pi}(t)+\tilde{\Pi}(t)\bar{D}(t))\\ &-\Pi(t)B(t)B^*\Pi(t)-M(t)\Big]dt+\tilde{\Pi}(t)d\bar{W}(t).\end{aligned} \tag{3.96}$$

We refer to Guatteri and Tessitore (2005) for details such as the existence and uniqueness of the solution. Moreover, it is shown that

$$\|\Pi(t)\|_{\mathscr{L}(H)} \leq \mathscr{C}_\Pi(T), \quad \lambda \times \mathbb{P} - a.e. \tag{3.97}$$

Moreover, the offset process q under the current framework satisfies

$$\begin{aligned} q(t) =& S^*(T-t)G\hat{F}_2 g(T) - \int_t^T S^*(r-t)\Big(\Pi(r)B(r)B^*(r)q(r) - M\hat{F}_1(r)g(r) + \Gamma_1(\Pi(r)p(r)) \\ &+ \Gamma_0(\Pi(r)p_0(r) - \tilde{q}(r)) + \Pi(r)F_1(r)g(r)\Big)dr - \int_t^T S^*(r-t)\tilde{q}(r)dW_0(r), \end{aligned} \tag{3.98}$$

which is similar to (3.27). Under A3.4.1-A3.4.5, (3.98) still falls within the framework of Hu and Peng (1991) and Guatteri and Tessitore (2005). Therefore, the existence of a solution is still guaranteed in the space $\mathscr{M}^2_{\mathscr{F}_0}(\mathfrak{T};H) \times \mathscr{M}^2_{\mathscr{F}_0}(\mathfrak{T};\mathscr{L}_2(V_{Q_0},H))$.

Fixed Point Problem

The optimal state for agent i under the limiting game is given by

$$\begin{aligned} x_i^\circ(t) =& S(t)\xi_i - \int_0^t S(t-r)(B(r)B^*(r)\Pi(r)x_i^\circ(r) - B(r)B^*(r)q(r) - F_1(r)g(r))dr \\ &+ \int_0^t S(t-r)(D(r)x_i^\circ(r) + F_2(r)g(r) + \sigma)\, dW_i(r) \\ &+ \int_0^t S(t-r)(D_0(r)x_i^\circ(r) + F_0(r)g(r) + \sigma_0)\, dW_0(r). \end{aligned} \tag{3.99}$$

Due to A3.4.1-A3.4.5, the fixed point problem and associated equilibrium are very similar to the case with deterministic operators. We use the following proposition, which is paralleled to Proposition 23, to demonstrate the main difference.

Proposition 32. *For the optimal state of agent i satisfying* (3.99)*, we have*

$$\mathbb{E}\,|x_i^\circ(t)|^2 \leq C, \quad \forall t \in \mathfrak{T}, \tag{3.100}$$

where the constant C only depends on the parameters. Moreover, for a fixed $g \in \mathscr{M}^2_{\mathscr{F}_0}(\mathfrak{T};H)$, we have

$$\sup_{t\in\mathfrak{T}} \mathbb{E}\left|y_g(t) - x^{(N),\circ}(t)\right|_H^2 \to 0. \tag{3.101}$$

where

$$y_g(t) = S(t)\xi - \int_0^t S(t-r)\left(B(r)B^*(r)\Pi(r)y_g(r) - B(r)B^*(r)q(r) - F_1(r)g(r)\right)dr$$
$$+ \int_0^t S(t-r)\left(D_0(r)y_g(r) + F_0(r)g(r) + \sigma_0\right)dW_0(r). \tag{3.102}$$

Proof.

$$\begin{aligned}
\mathbb{E}|x_i^\circ(t)|^2 \le\, & C\Big(\mathbb{E}|\xi_i|^2 + \mathbb{E}\int_0^t |B(r)B^*(r)\Pi(r)x_i^\circ(r) - B(r)B^*(r)q(r) - F_1(r)g(r)|^2\, dr \\
& + \mathbb{E}\int_0^t |D(r)x_i^\circ(r) + F_2(r)g(r) + \sigma|^2\, dr + \mathbb{E}\int_0^t |D_0(r)x_i^\circ(r) + F_0(r)g(r) + \sigma_0|^2\, dr\Big) \\
\le\, & C\Big(\mathbb{E}|\xi_i|^2 + \mathbb{E}\int_0^t \left(\|B(r)\|^2\|\Pi(r)\|^2 + \|D(r)\|^2 + \|D_0(r)\|^2\right)|x_i^\circ(r)|^2 dr \\
& + \mathbb{E}\int_0^t \left(\|F_0(r)\|^2 + \|F_1(r)\|^2 + \|F_2(r)\|^2\right)|g(r)|^2 dr \\
& + \mathbb{E}\int_0^t \left(\|B(r)\|^2|q(r)|^2 + \|\sigma\|^2 + \|\sigma_0\|^2\right)dr\Big) \\
\le\, & C\Big(\mathbb{E}|\xi_i|^2 + (C_B^2C_\Pi^2(T) + C_D^2 + C_{D_0}^2)\mathbb{E}\int_0^t |x_i^\circ(r)|^2 dr \\
& + (C_{F_0}^2 + C_{F_1}^2 + C_{F_2}^2)\mathbb{E}\int_0^t |g(r)|^2 dr + C_B^2\mathbb{E}\int_0^t |q(r)|^2 dr \\
& + \mathbb{E}\int_0^t \left(\|\sigma\|^2 + \|\sigma_0\|^2\right)dr\Big) \\
\le\, & C(1 + \mathbb{E}\int_0^t |x_i(r)|^2\, dr),
\end{aligned} \tag{3.103}$$

where the second-to-last inequality follows from A3.4.1–A3.4.5 and (3.97). The proof of (3.101) uses the same argument as used in that of Proposition 23. □

Remark 12. *We note that the only difference between* (3.40) *and* (3.103) *is that the operator norms in* (3.40) *are now replaced by the essential bounds of the operator-valued processes specified in A3.4.1–A3.4.5. The rest of the analysis follows exactly as in Section 3.3.2-Section 3.3.3, with similar adjustments to those made for the transition from* (3.40) *to* (3.103)*, corresponding to the shift from deterministic to stochastic operator settings. We only highlight the main result here. The subsequent analysis proceeds.*

Theorem 33 (Contraction Condition for the Case with Common Noise and Random Operators)**.** *The mapping associated with the mean field consistency condition, i.e.*

$$\Upsilon : g \in C_{\mathscr{F}_0}(\mathscr{T}; L^2(\Omega; H)) \longrightarrow y_g \in C_{\mathscr{F}_0}(\mathscr{T}; L^2(\Omega; H))$$

admits a unique fixed point if

$$\mathscr{C}_2(T)e^{T\mathscr{C}_3(T)} < 1, \tag{3.104}$$

where

$$\begin{aligned}
&\alpha(T) = 16M_T^2 T C_{D_0}^2 \\
&\mathscr{C}_1(T) = \frac{2M_T^2}{1-\alpha}\Big(16T^2((C_M C_{\hat{F}_1})^2 + (\mathscr{C}_\Pi(t)C_{F_1})^2 + \mathscr{C}_\Pi^2(T)((C_D C_{F_2})^2 + (C_{D_0}C_{F_0})^2) + (C_G C_{\hat{F}_2})^2)\Big) \\
&\qquad\qquad \times \exp(\frac{8M_T^2}{1-\alpha}\mathscr{C}_\Pi^2(T)C_B^4) \\
&\mathscr{C}_2(T) = 5M_T^2 T\Big[T(C_B^4\mathscr{C}_1(T) + C_{F_1})^2) + C_{F_0}^2\Big] \\
&\mathscr{C}_3(T) = 5M_T^2\Big[C_{D_0}^2 + TC_B^4\mathscr{C}_\Pi^2(T)\Big],
\end{aligned} \tag{3.105}$$

with $\mathscr{C}_\Pi(T)$ *introduced in* (3.97) *and the other parameters defined in A3.4.1-A3.4.4.*

In addition, we note that the constant $\mathscr{C}_\Pi(T)$ converges to zero as T converges to zero (see (Guatteri and Tessitore, 2005, Proposition 5.12 and Theorem 3.2)). Obviously, the convergence properties of $\mathscr{C}_i(T), i = 1,2,3$ as $T \to 0$ are the same as those indicated in Remark 11. Therefore, the argument for the existence and uniqueness of the mean field remains valid in this case. That is, the unique mean field $\bar{x} \in C_{\mathscr{F}_0}(\mathscr{T}; L^2(\Omega; H))$ exists for some $T > 0$.

3.4.3 Nash and ε-Nash Equilibria

Theorem 34 (Equilibria for the case with Common Noise and Random Operators)**.** *Consider the Hilbert space-valued limiting system, described by* (3.89) *and* (3.90) *for* $i \in \mathbb{N}$*, and suppose that condition* (3.104) *holds. Then, the set of control laws* $\{u_i^\circ\}_{i\in\mathbb{N}}$*, where* u_i° *is given by*

$$u_i^\circ = -B^*\left(\Pi(t)x_i(t) - q(t)\right), \tag{3.106}$$

forms a unique Nash equilibrium for the limiting system where the mean field $\bar{x}(t) \in H$*, the operator* $\Pi(t) \in \mathscr{L}(H)$ *and the pair of offset terms* $(q(t) \in H, \tilde{q}(t) \in \mathscr{L}_2(V_{Q_0}, H))$*, are characterized by the* *unique fixed point of the following set of consistency equations*

$$
\begin{aligned}
d\Pi(t) = \Big[&A^*\Pi(t) + \Pi(t)A + \Delta_2(\Pi(t)) + \Delta_2^0(\Pi(t)) + \mathrm{Tr}(\bar{D}^*(t)\tilde{\Pi}(t) + \tilde{\Pi}(t)\bar{D}(t)) \\
&- \Pi(t)B(t)B^*\Pi(t) - M(t)\Big]dt + \tilde{\Pi}(t)d\bar{W}(t), \qquad (3.107)\\
q(t) = S^*(T-t)G\hat{F}_2\bar{x}(T) - &\int_t^T S^*(r-t)\Big(\Pi(r)B(r)B^*(r)q(r) - M(r)\hat{F}_1(r)\bar{x}(r) \\
&+ \Gamma_1(\Pi(r)(F_2(r)\bar{x}(r)+\sigma)) + \Gamma_0(\Pi(r)(F_0(r)\bar{x}(r)+\sigma_0)) \\
&- \tilde{q}(r)) + \Pi(r)F_1(r)\bar{x}(r)\Big)dr - \int_t^T S^*(r-t)\tilde{q}(r)dW_0(r), \qquad (3.108)\\
\bar{x}(t) = S(t)\xi - &\int_0^t S(t-r)((B(r)B^*(r)\Pi(r) - F_1(r))\bar{x}(r) - B(r)B^*(r)q(r))dr \\
&+ \int_0^t S(t-r)((D_0(r)+F_0(r))\bar{x}(r)+\sigma_0)\,dW_0(r), \qquad (3.109)
\end{aligned}
$$

Moreover, the ε-Nash property for the N-player game, described by (3.87)-(3.88), is established in the same manner and yields the same conclusion as detailed in Section 3.3.3.

3.5 Concluding Remarks

We extended the infinite-dimensional linear-quadratic mean field game framework developed in Chapter 2 by incorporating a common noise component. The presence of common noise introduces significant analytical challenges, as it makes both the offset equation for individual agents and the evolution of the mean field itself stochastic. Therefore, the consistency conditions naturally take the form of a fully coupled system of forward-backward stochastic evolution equations in Hilbert spaces. We began our analysis with the case in which the parameter operators are deterministic, and then proceeded to the more general setting where these operators are stochastic processes taking values in operator spaces and adapted to the filtration generated by the common noise. Under standard assumptions, we found that extending the model to include stochastic (operator-valued) coefficients maintains the fundamental nature of the results.

Conclusion

This thesis extends linear-quadratic mean field games in two main directions: from risk sensitivity with a major agent (Chapter 1) to infinite-dimensional settings (Chapter 2), while Chapter 3 partially combines the developments of the first two chapters. These extensions substantially enrich the theoretical foundation of mean field game theory and broaden its applicability to more realistic and complex systems—such as those involving risk-sensitive dominant participants or distributed dynamics over functional spaces—which may commonly be encountered in financial markets, energy systems, and large-scale engineering applications.

Due to the tractability of the linear-quadratic structure, it is usually the first choice when one seeks to model novel/new scenarios. Therefore, a natural future direction of this thesis is to develop more general classes of mean field games under the combined framework of risk sensitivity and infinite-dimensional settings.

Other future directions for Chapter 1 include applying the risk-sensitive major-minor framework to more realistic financial and economic models. Examples include large investor effects in markets—settings where one agent has outsized influence and all agents exhibit risk aversion. The framework could also be extended to a Stackelberg game structure, where the major agent plays the role of a leader and the minor agents act as followers.

Other future directions based on Chapter 2 and Chapter 3 include modeling strategic interactions in energy markets and limit order books. Both types of financial systems have previously been studied in infinite-dimensional settings Benth and Krühner (2023); Cont and Müller (2021). On the theoretical side, many frameworks from classical linear-quadratic mean field game theory—such as extended mean field games—may be further developed within this infinite-dimensional setting. This framework may also be connected to graphon mean field games due to similarities in their infinite-dimensional structure. Another important aspect is the presence of a double limit, where both the system dimension and the number of agents tend to infinity. Investigating the interplay between these two limits may lead to deeper theoretical insights.

Bibliography

Ahuja, S. (2016). Wellposedness of mean field games with common noise under a weak monotonicity condition. *SIAM Journal on Control and Optimization*, 54(1):30–48.

Başar, T. (2021). Robust designs through risk sensitivity: An overview. *Journal of Systems Science and Complexity*, 34:1634–1665.

Başar, T. and Olsder, G. J. (1998). *Dynamic noncooperative game theory*. SIAM.

Behr, M., Benner, P., and Heiland, J. (2019). Solution formulas for differential sylvester and lyapunov equations. *Calcolo*, 56(4):51.

Bensoussan, A., Chau, M. H. M., Lai, Y., and Yam, S. C. P. (2017). Linear-quadratic mean field Stackelberg games with state and control delays. *SIAM Journal on Control and Optimization*, 55(4):2748–2781.

Bensoussan, A., Feng, X., Huang, J., et al. (2021). Linear-quadratic-gaussian mean-field-game with partial observation and common noise. *Mathematical control and related fields*, 11(1):23–46.

Bensoussan, A., Frehse, J., Yam, P., et al. (2013). *Mean field games and mean field type control theory*, volume 101. Springer.

Bensoussan, A., Sung, K., Yam, S. C. P., and Yung, S.-P. (2016). Linear-quadratic mean field games. *Journal of Optimization Theory and Applications*, 169:496–529.

Benth, F. E. and Krühner, P. (2023). *Stochastic Models for Prices Dynamics in Energy and Commodity Markets: An Infinite-Dimensional Perspective*. Springer Nature.

Bielecki, T. R. and Pliska, S. R. (2003). Economic properties of the risk sensitive criterion for portfolio management. *Review of Accounting and Finance*.

Bielecki, T. R., Pliska, S. R., and Sherris, M. (2000). Risk sensitive asset allocation. *Journal of Economic Dynamics and Control*, 24(8):1145–1177.

Blasco, O. and van Neerven, J. (2010). Spaces of operator-valued functions measurable with respect to the strong operator topology. In *Vector measures, integration and related topics*, pages 65–78. Springer.

Bo, L. and Capponi, A. (2015). Systemic risk in interbanking networks. *SIAM Journal on Financial Mathematics*, 6(1):386–424.

Brzeźniak, Z., van Neerven, J. M., Veraar, M. C., and Weis, L. (2008). Itô's formula in umd banach spaces and regularity of solutions of the zakai equation. *Journal of Differential Equations*, 245(1):30–58.

Caines, P. E. and Kizilkale, A. C. (2017). ε-Nash equilibria for partially observed LQG mean field games with major player. *IEEE Transaction on Automatic Control*, 62(7):3225–3234.

Cardaliaguet, P., Delarue, F., Lasry, J.-M., and Lions, P.-L. (2019). *The master equation and the convergence problem in mean field games:(AMS-201)*. Princeton University Press.

Cardaliaguet, P. and Lehalle, C.-A. (2018). Mean field game of controls and an application to trade crowding. *Mathematics and Financial Economics*, 12(3):335–363.

Carmona, R., Delarue, F., et al. (2018a). *Probabilistic theory of mean field games with applications I-II*. Springer.

Carmona, R., Delarue, F., and Lacker, D. (2016). Mean field games with common noise. *The Annals of Probability*, 44(6):3740–3803.

Carmona, R., Fouque, J.-P., Mousavi, and Sun, L.-H. (2015). Mean field games and systemic risk. *Communications in Mathematical Sciences*, 13(4):911–933.

Carmona, R., Fouque, J.-P., Mousavi, S. M., and Sun, L.-H. (2018b). Systemic risk and stochastic games with delay. *Journal of Optimization Theory and Applications*, 179:366–399.

Carmona, R. and Lacker, D. (2015). A probabilistic weak formulation of mean field games and applications. *The Annals of Applied Probability*, 25(3):1189–1231.

Carmona, R. and Wang, P. (2017). An alternative approach to mean field game with major and minor players, and applications to herders impacts. *Applied Mathematics & Optimization*, 76(1):5–27.

Carmona, R. and Zhu, X. (2016). A probabilistic approach to mean field games with major and minor players. *The Annals of Applied Probability*, 26(3):1535–1580.

Casgrain, P. and Jaimungal, S. (2020). Mean-field games with differing beliefs for algorithmic trading. *Mathematical Finance*, 30(3):995–1034.

Chang, Y., Firoozi, D., and Benatia, D. (2025). Large banks and systemic risk: Insights from a mean-field game model. *Journal of Systems Science and Complexity*, 38(1):460–494.

Chen, Yan, Li, Tao, and Xin, Zhixian (2023). Risk-sensitive mean field games with major and minor players. *ESAIM: COCV*, 29.

Ciarlet, P. G. (2013). *Linear and nonlinear functional analysis with applications*, volume 130. SIAM.

Cohn, D. L. (2013). *Measure theory*, volume 5. Springer.

Cont, R. and Müller, M. S. (2021). A stochastic partial differential equation model for limit order book dynamics. *SIAM Journal on Financial Mathematics*, 12(2):744–787.

Cosso, A., Gozzi, F., Kharroubi, I., Pham, H., and Rosestolato, M. (2023). Optimal control of path-dependent McKean–Vlasov SDEs in infinite-dimension. *The Annals of Applied Probability*, 33(4):2863–2918.

Curtain, R. F. and Falb, P. L. (1970). Itô's lemma in infinite dimensions. *Journal of mathematical analysis and applications*, 31(2):434–448.

Da Prato, G. (1984). Direct solution of a Riccati equation arising in stochastic control theory. *Applied Mathematics and Optimization*, 11(1):191–208.

Da Prato, G. and Zabczyk, J. (2014). *Stochastic equations in infinite dimensions*. Cambridge university press.

Diagana, T. (2018). *Semilinear evolution equations and their applications*. Springer.

Duncan, T. E. (2013). Linear-exponential-quadratic gaussian control. *IEEE Transactions on Automatic Control*, 58(11):2910–2911.

Duncan, T. E. (2015). Linear exponential quadratic stochastic differential games. *IEEE Transactions on Automatic Control*, 61(9):2550–2552.

Dunyak, A. and Caines, P. E. (2022). Linear stochastic graphon systems with Q-space noise. In *2022 IEEE 61st Conference on Decision and Control (CDC)*, pages 3926–3932. IEEE.

Dunyak, A. and Caines, P. E. (2024). Quadratic optimal control of graphon q-noise linear systems. *arXiv preprint arXiv:2407.00212*.

Fabbri, G., Gozzi, F., and Swiech, A. (2017). Stochastic optimal control in infinite dimension. *Probability and Stochastic Modelling. Springer*.

Federico, S., Gozzi, F., et al. (2024a). On mean field games in infinite dimension. *arXiv preprint arXiv:2411.14604*.

Federico, S., Gozzi, F., and Ghilli, D. (2024b). Linear-quadratic mean field games in Hilbert spaces. *arXiv preprint arXiv:2402.14935*.

Figiel, T. (1990). Singular integral operators: a martingale approach. *Geometry of Banach spaces (Strobl, 1989)*, 158:95–110.

Firoozi, D. (2022). LQG mean field games with a major agent: Nash certainty equivalence versus probabilistic approach. *Automatica*, 146:110559.

Firoozi, D. and Caines, P. E. (2017). The execution problem in finance with major and minor traders: A mean field game formulation. In *Annals of the International Society of Dynamic Games (ISDG): Advances in Dynamic and Mean Field Games*, volume 15, pages 107–130. Birkhäuser Basel.

Firoozi, D. and Caines, P. E. (2020). ε-Nash equilibria for major–minor LQG mean field games with partial observations of all agents. *IEEE Transactions on Automatic Control*, 66(6):2778–2786.

Firoozi, D. and Jaimungal, S. (2022). Exploratory LQG mean field games with entropy regularization. *Automatica*, 139:110177.

Firoozi, D., Jaimungal, S., and Caines, P. E. (2020). Convex analysis for LQG systems with applications to major–minor LQG mean–field game systems. *Systems & Control Letters*, 142:104734.

Firoozi, D., Pakniyat, A., and Caines, P. E. (2022). A class of hybrid LQG mean field games with state-invariant switching and stopping strategies. *Automatica*, 141:110244.

Fleming, W. H. and Sheu, S. (2000). Risk-sensitive control and an optimal investment model. *Mathematical Finance*, 10(2):197–213.

Fleming, W. H. and Soner, H. M. (2006). *Controlled Markov processes and viscosity solutions*, volume 25. Springer Science & Business Media.

Fouque, J.-P. and Zhang, Z. (2018). Mean field game with delay: a toy model. *Risks*, 6(3):90.

Fujii, M. and Takahashi, A. (2022). A mean field game approach to equilibrium pricing with market clearing condition. *SIAM Journal on Control and Optimization*, 60(1):259–279.

Garnier, J., Papanicolaou, G., and Yang, T.-W. (2013). Large deviations for a mean field model of systemic risk. *SIAM Journal on Financial Mathematics*, 4(1):151–184.

Gawarecki, L. and Mandrekar, V. (2010). *Stochastic differential equations in infinite dimensions: with applications to stochastic partial differential equations*. Springer Science & Business Media.

Goldstein, J. A. (2017). *Semigroups of linear operators and applications*. Courier Dover Publications.

Gomes, D. and Nurbekyan, L. (2015). On the minimizers of calculus of variations problems in hilbert spaces. *Calculus of Variations and Partial Differential Equations*, 52(1):65–93.

Gomes, D. and Saúde, J. (2021). A mean-field game approach to price formation in electricity markets. *Dynamic Games and Applications*, 11(1):29–53.

Graber, P. J. (2016). Linear quadratic mean field type control and mean field games with common noise, with application to production of an exhaustible resource. *Applied Mathematics & Optimization*, 74(3):459–486.

Guatteri, G. and Tessitore, G. (2005). On the backward stochastic Riccati equation in infinite dimensions. *SIAM journal on control and optimization*, 44(1):159–194.

Hu, Y. and Peng, S. (1991). Adapted solution of a backward semilinear stochastic evolution equation. *Stochastic Analysis and Applications*, 9(4):445–459.

Hu, Y. and Tang, S. (2022). Stochastic LQ control and associated Riccati equation of PDEs driven by state-and control-dependent white noise. *SIAM Journal on Control and Optimization*, 60(1):435–457.

Huang, J. and Huang, M. (2017). Robust mean field linear-quadratic-gaussian games with unknown L^2-disturbance. *SIAM Journal on Control and Optimization*, 55(5):2811–2840.

Huang, M. (2010). Large-population LQG games involving a major player: the Nash certainty equivalence principle. *SIAM Journal on Control and Optimization*, 48(5):3318–3353.

Huang, M. (2021). Linear-quadratic mean field games with a major player: Nash certainty equivalence versus master equations. *Communications in Information and Systems, vol. 21, no. 3*.

Huang, M., Caines, P. E., and Malhamé, R. P. (2007). Large-population cost-coupled LQG problems with nonuniform agents: individual-mass behavior and decentralized ε-Nash equilibria. *IEEE transactions on automatic control*, 52(9):1560–1571.

Huang, M., Malhamé, R. P., and Caines, P. E. (2006). Large population stochastic dynamic games: Closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle. *Communications in Information & Systems*, 6(3):221–252.

Huang, M. and Yang, X. (2019). Linear quadratic mean field social optimization: asymptotic solvability. In *2019 IEEE 58th Conference on Decision and Control (CDC)*, pages 8210–8215. IEEE.

Huang, X. and Jaimungal, S. (2017). Robust stochastic games and systemic risk. *SSRN preprint: SSRN 3024021*.

Huang, X., Jaimungal, S., and Nourian, M. (2019). Mean-field game strategies for optimal execution. *Applied Mathematical Finance*, 26(2):153–185.

Hytönen, T., Van Neerven, J., Veraar, M., and Weis, L. (2016). *Analysis in Banach spaces*, volume 12. Springer.

Ichikawa, A. (1979). Dynamic programming approach to stochastic evolution equations. *SIAM Journal on Control and Optimization*, 17(1):152–174.

Ichikawa, A. (1982). Stability of semilinear stochastic evolution equations. *Journal of Mathematical Analysis and Applications*, 90(1):12–44.

Jacobson, D. (1973). Optimal stochastic linear systems with exponential performance criteria and their relation to deterministic differential games. *IEEE Transactions on Automatic control*, 18(2):124–131.

Karatzas, I. and Shreve, S. E. (1991). *Brownian motion and stochastic calculus*, volume 113. Springer Science & Business Media.

Kolokoltsov, V. N. and Troeva, M. (2019). On mean field games with common noise and mckean-vlasov spdes. *Stochastic Analysis and Applications*, 37(4):522–549.

Kumar, P. and Van Schuppen, J. (1981). On the optimal control of stochastic systems with an exponential-of-integral performance index. *Journal of mathematical analysis and applications*, 80(2):312–332.

Lacker, D. (2016). A general characterization of the mean field limit for stochastic differential games. *Probability Theory and Related Fields*, 165:581–648.

Lacker, D. and Le Flem, L. (2023). Closed-loop convergence for mean field games with common noise. *The Annals of Applied Probability*, 33(4):2681–2733.

Lacker, D. and Webster, K. (2015). Translation invariant mean field games with common noise. *Electronic Communications in Probability*, 20(56):1–13.

Lasry, J.-M. and Lions, P.-L. (2007). Mean field games. *Japanese journal of mathematics*, 2(1):229–260.

Lasry, J.-M. and Lions, P.-L. (2018). Mean-field games with a major player. *Comptes Rendus Mathematique*, 356(8):886–890.

Lehalle, C.-A. and Mouzouni, C. (2019). A mean field game of portfolio trading and its consequences on perceived correlations. *arXiv preprint arXiv:1902.09606*.

Li, M., Mou, C., Wu, Z., and Zhou, C. (2023a). Linear-quadratic mean field games of controls with non-monotone data. *Transactions of the American Mathematical Society*, 376(06):4105–4143.

Li, Z., Reppen, A. M., and Sircar, R. (2023b). A mean field games model for cryptocurrency mining. *Management Science*.

Lim, A. E. and Zhou, X. Y. (2005). A new risk-sensitive maximum principle. *IEEE transactions on automatic control*, 50(7):958–966.

Liu, H. and Firoozi, D. (2025). Hilbert space-valued LQ mean field games: An infinite-dimensional analysis. *(to appear) SIAM Journal on Control and Optimization*.

Liu, H., Firoozi, D., and Breton, M. (2025). LQG risk-sensitive single-agent and major-minor mean-field game systems: A variational framework. *SIAM Journal on Control and Optimization*, 63(4):2251–2281.

Moon, J. and Başar, T. (2016). Linear quadratic risk-sensitive and robust mean field games. *IEEE Transactions on Automatic Control*, 62(3):1062–1077.

Moon, J. and Başar, T. (2018). Linear quadratic mean field stackelberg differential games. *Automatica*, 97:200–213.

Moon, J. and Başar, T. (2019). Risk-sensitive mean field games via the stochastic maximum principle. *Dynamic Games and Applications*, 9(4):1100–1125.

Moon, J., Duncan, T. E., and Başar, T. (2018). Risk-sensitive zero-sum differential games. *IEEE Transactions on Automatic Control*, 64(4):1503–1518.

Nagai, H. (1996). Bellman equations of risk-sensitive control. *SIAM journal on control and optimization*, 34(1):74–101.

Nourian, M. and Caines, P. E. (2013). ε-Nash mean field game theory for nonlinear stochastic dynamical systems with major and minor agents. *SIAM Journal on Control and Optimization*, 51(4):3302–3331.

Nurbekyan, L. (2012). Calculus of variations in hilbert spaces. *Journal of Contemporary Mathematical Analysis*, 47:148–160.

Pan, Z. and Başar, T. (1996). Model simplification and optimal control of stochastic singularly perturbed systems under exponentiated quadratic cost. *SIAM Journal on Control and Optimization*, 34(5):1734–1766.

Peszat, S. and Zabczyk, J. (2007). *Stochastic partial differential equations with Lévy noise: An evolution equation approach*, volume 113. Cambridge University Press.

Ren, X. Y. and Firoozi, D. (2024). Risk-sensitive mean field games with common noise: a theoretical study with applications to interbank markets. *arXiv preprint arXiv:2403.03915*.

Saldi, N., Basar, T., and Raginsky, M. (2018). Discrete-time risk-sensitive mean-field games. *arXiv preprint arXiv:1808.03929*.

Saldi, N., Başar, T., and Raginsky, M. (2022). Partially observed discrete-time risk-sensitive mean field games. *Dynamic Games and Applications*, pages 1–32.

Shrivats, A., Firoozi, D., and Jaimungal, S. (2021). Principal agent mean field games in REC markets. *arXiv preprint arXiv:2112.11963*.

Shrivats, A. V., Firoozi, D., and Jaimungal, S. (2022). A mean-field game approach to equilibrium pricing in solar renewable energy certificate markets. *Mathematical Finance*, 32(3):779–824.

Tchuendom, R. F. (2018). Uniqueness for linear-quadratic mean field games with common noise. *Dynamic Games and Applications*, 8:199–210.

Tchuendom, R. F., Firoozi, D., and Breton, M. (2025). Ranking quantilized mean-field games with an application to early-stage venture investments. *arXiv preprint arXiv:2507.00853*.

Tchuendom, R. F., Malhamé, R., and Caines, P. (2019). A quantilized mean field game approach to energy pricing with application to fleets of plug-in electric vehicles. *2019 IEEE 58th conference on decision and control (CDC)*, pages 299–304.

Tembine, H., Zhu, Q., and Başar, T. (2013). Risk-sensitive mean-field games. *IEEE Transactions on Automatic Control*, 59(4):835–850.

Tessitore, G. (1992). Some remarks on the Riccati equation arising in an optimal control problem with state-and control-dependent noise. *SIAM Journal on Control and Optimization*, 30(3):717–744.

Toumi, N., Malhamé, R., and Le Ny, J. (2024). A mean field game approach for a class of linear quadratic discrete choice problems with congestion avoidance. *Automatica*, 160:111420.

Whittle, P. (1981). Risk-sensitive linear/quadratic/gaussian control. *Advances in Applied Probability*, 13(4):764–777.

Yong, J. and Zhou, X. Y. (1999). *Stochastic controls: Hamiltonian systems and HJB equations*, volume 43. Springer Science & Business Media.