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Newsvendor Problem with Inventory Costs for Data-Driven Supply Chain Optimization

par
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Résumé

Cette thèse est constituée de trois articles. Dans le premier article, un problème de vendeur de journaux est examiné où les coûts de stockage dans quatre phases de la chaîne d'approvisionnement (y compris les phases de production, transport, le saison de vente régulière et le saison de vente à rabais) sont pris en compte dans le modèle, tandis que nous supposons que la production et la demande totales sont accumulées linéairement dans le temps. Nous considérons une demande probabiliste pour la période de vente régulière, tout en considérant une distribution linéaire générale par morceaux (GPLD), ainsi qu'une distribution discrète de demande. Le but de ce problème est d'obtenir la quantité de production optimale qui maximise le profit total attendu dans la chaîne d'approvisionnement. Dans le deuxième article, le même problème est considéré tout en incorporant une fonction non-linéaire générale pour les fonctions de production cumulée et de demande cumulée au cours du temps, et une distribution discrète pour la demande incertaine. L'applicabilité du cas non-linéaire général est illustrée davantage en considérant les modèles bien connus de Bass (1969) et Wright (1936) pour les fonctions de demande et de production, respectivement, ce qui nous a aussi permis de réaliser des expériences numériques. Dans le troisième article, les mêmes modèles des premier et deuxième articles sont considérés, tout en supposant des scénarios de demande discrète et en prenant en compte le problème de la maximisation du profit dans le pire des cas. Malgré le fait que les fonctions de coût

de stockage soient non-linéaires, dans les trois articles, nous développons des processus de résolution optimaux et efficaces pour le problème. De plus, nous utilisons des données de demande réelles pour effectuer des études numériques, ce qui nous amène à utiliser des méthodes basées sur les données dans le problème. Plus précisément, dans le premier article, l'utilisation de la distribution GPLD nous permet de considérer l'histogramme de densité des données de demande comme la distribution de probabilité de la demande pour le problème du premier article. De plus, dans le deuxième article, nous ajustons un modèle de demande de Bass (Bass 1969) sur des données réelles, pour modéliser la fonction de la demande cumulative dans le temps.

Nous effectuons des expériences numériques approfondies, en utilisant des ensembles de données de demande synthétiques et réelles. Dans le premier article, les résultats indiquent que le nouveau modèle est généralement plus rentable que les approximations développées à l'aide du modèle standard de vendeur de journaux. Dans le deuxième article, nos résultats indiquent que la prise en compte de fonctions non-linéaires pour la production et la demande cumulées dans le problème, généralement conduit à des solutions plus profitables que le cas de la production et de la demande linéaires et que le problème standard du vendeur de journaux. Dans le troisième article, les résultats des études numériques montrent que la solution du problème dans le premier et le deuxième article n'est pas robuste, alors que la solution robuste est conservatrice.

Les modèles et solutions développés dans cette these, peuvent être utilisés pour aider les responsables de la fabrication et des achats à prendre des décisions sur la quantité à produire ou à commander, tout en maximisant le profit de la chaîne d'approvisionnement, et en profitant de leurs données de demande au fil du temps, en particulier lorsque la durée de la chaîne d'approvisionnement est longue et que les coûts de stockage sont élevés.

Mots-clés

Chaîne d'approvisionnement, Problème de vendeur de journaux, Coût de stockage, Demande incertaine, Distribution discrète, Distribution linéaire par morceaux.

Méthodes de recherche

Recherche opérationnelle, Analytics des données, Optimisation stochastique, Optimisation robuste, Solutions analytiques.

Abstract

This dissertation consists of three articles. In the first article, a newsvendor problem is studied where holding costs of four stages in the supply chain (including production, transportation, regular selling and discount seasons) are incorporated in the model, while assuming that the total production and demand are accumulated linearly over time. We consider an uncertain demand under a general piecewise linear distribution (GPLD), as well as discrete demand distribution, for the regular selling period. The goal of the problem is to obtain the optimal production amount which maximizes the total expected profit of the supply chain. In the second paper, we consider the same problem while considering a general nonlinear function for both cumulative production and cumulative demand functions over the course of time, and discrete distribution for the uncertain demand. The applicability of the general nonlinear case, is further illustrated by considering well-known models of Bass (1969) and Wright (1936) for demand and production functions of the problem, respectively, which also enabled us to conduct numerical experiments. In the third paper, we consider the same models of the first and second papers, under the assumption of discrete demand scenarios, while taking into account the problem of maximizing the profit under the worst case scenario. Despite the nonlinearity of the holding cost functions, in all the three papers, we develop optimal and efficient solution approaches for the problem. In addition, we take advantage of real demand data to conduct numerical studies which leads

us to utilize data-driven methods in the problem. More specifically, in the first paper, the use of GPLD distribution enables us to consider the density histogram of the demand data as the probability distribution of the demand for the problem in the first paper. Moreover, in the second paper, we fit a Bass demand model (Bass 1969) over real data, to model the cumulative demand function over time.

We conduct extensive numerical experiments, using both synthetic and real demand data. In the first paper, the results indicate that the new model is overall more profitable than the approximations developed using the standard newsvendor model. In the second paper, our results illustrate that considering nonlinear functions for cumulative production and demand in the problem, generally leads to more profitable solutions than the linear production and demand case, and the standard newsvendor problem. In the third paper, the results of numerical experiments demonstrate that the solution of the problem in the first and second papers, is not a robust one, while the robust solution is conservative.

The models and solutions developed in this dissertation, can be used to help manufacturing and purchasing managers decide on how much to produce or order, while maximizing the profit of the supply chain, and taking advantage of their demand data over time, especially when the supply chain has a long duration and high holding costs.

Keywords

Supply chain, Newsvendor problem, Inventory holding costs, Uncertain demand, Discrete distribution, Piecewise linear distribution.

Research Methods

Operations research, Data analytics, Stochastic optimization, Robust optimization, Analytical solutions.

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General Introduction

The newsvendor problem is one of the most widely studied topics in the inventory control literature and it has applications in a variety of industries including fashion and aviation (Qin *et al.* 2011). It involves a purchaser (or a producer) with one ordering (or production) opportunity for an item to be sold by the purchaser/producer, the amount of which is the single decision to be made. The objective is to determine the optimal order or production quantity so as to maximize the expected profit, based on a probability distribution of the (uncertain) demand for the product. After the order (or production) of the amount Q, the customer demand with an amount x becomes known. In case the order amount of Q is not sufficient to meet the demand x, an underage cost $p-v$ (where p is the selling price and v is the cost of the product) is incurred which is the profit loss. On the other hand, if the order is more than the demand, it is assumed that these leftovers can be sold according to their salvage value g (which is typically less than the unit cost v), which leads to an overage cost of $v-g$. Given $F(Q)$, which is the demand cumulative distribution function for a continuous demand distribution, indicating the probability that demand is less than or equal to Q, the optimal order (or production) amount Q^*, can be derived very efficiently and in closed-form using the equation $Q^* = F^{-1}\left(\frac{p-v}{p-g}\right)$. The notations and formulations for the standard newsvendor problem, which we use throughout this dissertation, are also utilized by Silver *et al.* (1998), and Qin *et al.* (2011).

The newsvendor model is typically applied to products which have short life cycle, such as fashion items, seasonal products (such as Christmas items), or perishable products. A weakness in the standard newsvendor problem is the assumption that the product demand occurs at a single point in time, although in practice, it is spread over a season during which the item is sold at a regular price. Besides, the sale of the leftovers can also happen over time, during another season, at a discount price (i.e., the salvage value). Since during these seasons, the produced (or purchased) products are in stock, inventory holding costs are also incurred. In addition, before starting the selling season, the products must be produced and then transported, both of which happen during an extended period of time which results in added inventory holding costs. The standard newsvendor model, does not consider these holding costs. Therefore, in this dissertation we study an extension of the standard newsvedor model, where we take into account the holding cost of the inventory over four stages of a supply chain consisting of production, transportation, and regular and discount selling periods. Qin *et al.* (2011) provide a review of the newsvendor literature.

In the first paper of this dissertation, we present an overview of newsvendor literature with holding cost, with a focus on papers which have holding costs and are relevant to the case considered in this thesis. More specifically, we review the papers Matsuo (1990), Chen and Chuang (2000), Tang *et al.* (2018), and Schlapp *et al.* (2022), which also take into account the holding cost of phases where inventory is stored over an extended period of time, and under some conditions, they are equivalent to the holding cost functions in the transportation or regular selling periods studied in this dissertation. Nevertheless, none of those papers consider the holding cost of all four periods of our supply chain, altogether in the same problem.

This dissertation consists of three papers. In the first paper, we study a newsvendor problem in the context of a supply chain with four periods including production, shipping, a regular selling phase, and a discount selling phase. More specifically, we consider the holding cost of the stock, which may vary over time during the production and demand periods and can be described as a linear function of time (i.e., cumulative production and demand are both linear functions of time). We then propose optimal solution procedures for the problem under a continuous general piecewise linear distribution and also discrete distribution. Furthermore, we propose approximations based on the standard newsvendor model. We perform numerical experiments using synthetic and real datasets. For each of the real data cases, a piecewise linear distribution is formed using the histogram of real demand data. The results demonstrate that the proposed model in all instances can generate a higher or equal expected profit compared with the approximation methods based on the standard newsvendor model, while the proposed approximations can still provide solutions that are on average close to optimal.

Given that, in practice, the production and demand can be nonlinear functions of time, in the second paper, we extend the study of the first paper to the case of nonlinear production and demand functions over time. We first derive the holding cost functions under general nonlinear demand and production functions, and then obtain the holding cost functions under a specific production and demand based on the well-known Wright (1936) and Bass (1969) models, respectively. These models are flexible and can represent a variety of functions via parameter adjustment of the model. In our experiments, the Bass demand functions in the regular selling season, are estimated using real data. The numerical results demonstrate that the new model always provides more profitable or equal solutions when compared to the approximations based on the linear demand and production case, and also the standard newsvendor model.

In the third paper, the robust version of the problems in the second and first papers under the discrete scenarios, is studied and an efficient solution process is proposed. Then, several numerical experiments are performed to analyze the robust solution compared with the solution of the stochastic problem studied in the first and second papers.

Chapter 1

A Newsvendor Problem with Holding Costs in a Multi-Stage Supply Chain

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Abstract

We study a generalized modeling framework for the newsvendor problem which involves multiple stages comprising production, shipping, selling and discount periods. In this context, the aspect of time is explicitly captured in the newsvendor model. Production and demand are gradually and linearly accumulated over time as opposed to occurring at a single point as in the classical newsvendor problem. Consequently, quantity-and-time-dependent holding costs must be explicitly taken into account, which is also motivated by the importance of holding costs in supply chains. We present efficient solution procedures to solve the problem under a general piecewise linear distribution (GPLD), and a discrete distri-

bution. Approximation methods based on adaptations of the standard newsvendor model are also proposed. We conduct numerical studies under both synthetic and real data, the results of which overall demonstrate the higher profitability of our modeling framework compared to the standard newsvendor model, in the presence of holding cost. The approximation methods also provide profits which are close to the optimal one. The managerial implications are as follows. The mathematical formulation and the solution approaches used in this paper can be utilized by decision makers to decide on the production or order quantity of an item which minimizes the total cost of the corresponding supply chain where the quantity-and-time-dependent holding costs are present, and this is especially relevant for cases with high holding cost, long production times and a long selling season.

1.1 Introduction

Inventory management is an essential part of manufacturing and retail operations. The main objective of this process is to determine the optimal amount of a product to be produced or ordered to maximize the corresponding profit of selling that item. The newsvendor problem is one of the most widely adopted and fundamental models in inventory management (Porteus 2002), especially in retail businesses. Nevertheless, the traditional newsvendor model relies on strict assumptions and many researchers have examined various extensions in order to enhance the model to deal with more realistic assumptions. Khouja (1999) and Qin *et al.* (2011) provide a literature review of such efforts. In this paper, we focus on an important shortcoming of the newsvendor model which has not been addressed in the literature before. In the classical newsvendor problem, the element of time is not taken into account, since the production and demand are assumed to occur at one single point in time. However, in reality, production, distribution and sales happen over an extended period of time. As a result, the inventory level varies over time and this has to be properly captured in order to correctly account for the inventory holding costs. Moreover, our model, may achieve solutions which are not obtainable by the standard newsvendor problem. For instance, the optimal production quantity in our case can be a value below the minimum demand. In addition, the holding costs of the production and selling phases in our problem have nonlinear functions which cannot be incorporated into the standard newsvendor model. Holding costs represent the product ownership cost, such as the costs of insurance, storage space and human resources needed for storing products which may depend on the amount of product and the duration of time for storing goods. The holding cost also includes the opportunity cost of the money invested in inventory (Porteus 1990). In the traditional newsvendor models, such a holding cost is typically only partially taken into account as part of the overage cost which is charged for the excess inventory at the end of the regular selling season (Eppen 1979, Kouvelis and Gutierrez 1997, Maggioni *et al.* 2019). However, in supply chain management, we generally encounter a holding cost which is incurred based not only on the quantity on-hand, but also on the duration of time each unit of item is stored.

Besides, as time progresses, the (perceived) quality of a product may decline (Ferguson and Koenigsberg 2007) which can cause the value and price of the product to decrease continuously over time, especially for short-life-cycle products and high technology items such as components of personal computers (Khouja and Park 2003, Sriram *et al.* 2010). Callioni *et al.* (2005) take into account the continuous devaluation of products as a hidden cost of inventory, especially for electronic hardware such as components of PCs, due to constant advances in technology. The price decline over time also occurs due to quality degradation; for example, for perishable and fresh products such as fruits (Blackburn and Scudder 2009) or meat (Rajan *et al.* 1992). The term "marginal value of time" is defined by Blackburn and Scudder (2009) to represent the value change of a unit item per unit time in a supply chain. This decrease in value cannot be taken into account by adjusting the selling price of the standard newsvendor model, since the selling price is constant for the entire regular or discount selling seasons. However, such kind of value loss or continuous selling price decline can be considered as a part of the time-and-quantity-dependent holding cost in our case.

It is clear that production, distribution and sales take place over a period time. For example, the Sport Obermeyer case (Hammond and Raman 2006) illustrates that production takes time due to limited capacity, and also that it requires some time to transport the products. Furthermore, retailers are very familiar with selling seasons (e.g., Halloween, Christmas, Valentine) which consist of a main selling period and a discount selling period. To this end, we extend the traditional newsvendor model to take into account four different phases: a production and a shipping phase prior to the two selling seasons. Contrary to standard newsvendor models which assume that the production and demand of an item occur at a single point in time, our model takes into account a more realistic perspective in which production and demand happen gradually over time and hence the holding cost in these phases is considered to be dependent on both the amount of the item and the duration of time each unit item is in stock. The total demand is the only uncertain parameter, and we assume that this total demand is spread uniformly over the first selling season. The objective is to find the optimal production quantity in order to maximize the total expected profit which is the difference between the total revenues from selling the product in the regular and discount seasons, and the total cost which includes production and holding costs.

The mathematical functions involved in the standard newsvendor model are all linear in terms of the decision variable which is the ordering or production quantity (Eppen 1979, Matsuo 1990, Mieghem and Rudi 2002, Levi *et al.* 2007, Maggioni *et al.* 2019). However, considering the quantity-and-time-dependent holding cost during each of the production and demand phases on its own, makes the model a nonlinear quadratic problem. Such a problem is in general more complex to solve than a linear model, especially when dealing with its stochastic version. In this paper, we first introduce and mathematically model the problem where we assume that the demand is known and that the production and demand functions over time are linear during the production and the selling seasons, respectively. The constant production and demand rates are frequently assumed in the literature of the economic production quantity (Cárdenas-Barrón 2001) and the economic order quantity (Moily 2015, Perera *et al.* 2017, Muriel *et al.* 2021), respectively. Moreover, we consider the case of demand uncertainty where the uncertainty set can be characterized by discrete scenarios or by a general piecewise linear distribution (GPLD). For these two cases, we provide efficient solution procedures to solve the problem. Furthermore, we present heuristics that rely on adapting certain parameters in the standard newsvendor problem. The advantage is that these adaptations allow to use the closed-form solution of the standard newsvendor problem. The disadvantage is that in some cases these approximate solutions do not provide good quality solutions. In some cases, the optimal solution (considering holding costs) can even not be obtained using the standard newsvendor model with adapted parameters. This is for example the case with a uniform demand distribution and high holding cost. In such a case, it may be that the optimal order quantity, when considering holding costs, is lower than the minimum demand level. Such a solution where the optimal production quantity is lower than the demand, can never be obtained by the standard newsvendor problem. In a relevant paper, Cachon and Kök (2007) examine a newsvendor problem with a nonlinear function for salvage value, which cannot be incorporated in the traditional newsvendor problem, and they propose estimations of the salvage value which result in using the traditional newsvendor problem as an approximation of the nonlinear problem.

Overall, the contributions of this paper are summarized as follows. We mathematically model the newsvendor problem with inventory build-up and depletion, and quantity-and-time-dependent holding costs, consisting of four phases in the supply chain including production, transportation, regular selling season and discount season. The proper consideration of the holding costs in the production and discount season is, to the best of our knowledge, novel, while Tang *et al.* (2018) and Schlapp *et al.* (2022) do consider such costs in the regular selling season. The holding cost related to the shipping phase, can directly be incorporated in the standard newsvendor model (Matsuo 1990, Chen and Chuang 2000, and Schlapp *et al.* 2022). We provide efficient solution procedures under GPLD and discrete distributions. We propose approximation methods which can be incorporated in the standard newsvendor problem (another such approximation based on the standard newsvendor model, is proposed by Tang *et al.* 2018 for the regular selling season, which we also examine in our paper). We conduct numerical experiments on both synthetic and real data, which demonstrate the higher average profitability (which can be more significant for higher holding costs) of our method over the standard newsvendor model and the proposed approximations, while the approximations provide relatively close results to the optimal profit.

The rest of this paper is organized as follows. In the next section a review of the relevant literature is presented. In Section 3, the problem is modeled and described in more detail. Section 4 first studies the problem under continuous distributions and provides an efficient solution method for the stochastic problem under piecewise linear distributions, and then an efficient solution procedure is presented for the stochastic optimization problem under discrete probability distributions. The heuristic methods based on adapting parameters in the standard newsvendor problem are presented in Section 5. Section 6 presents the numerical results. We provide the concluding remarks in Section 7.

1.2 Literature review

A number of newsvendor studies take the holding cost into account only in a limited way as part of an overage cost which depends only on the overstocked quantity at the end of the regular selling season. The discount given for the excess inventory, can also be incorporated in that framework as a part of the overage cost. In this context, Eppen (1979) studies a multilocation newsvendor problem where the holding cost is charged for the excess inventory and a penalty cost is occurred if there is a shortage, at each location. Kouvelis and Gutierrez (1997) also consider shortage costs if the demand exceeds the available inventory and overage cost if the demand is less than the inventory. In general, the overage cost in the newsvendor model is calculated as the acquisition cost minus the salvage value (Silver *et al.* 1998). Mieghem and Rudi (2002) study newsvendor networks where an effective unit holding cost is charged which is equal to the actual unit holding cost minus the unit salvage value. They also study the problem in a dynamic setting with multiple periods where the holding cost is charged for the stock carried over to the next period. The assumption of a linear demand over time is also incorporated by Urban (2002) which consider such a demand until the shelf is being replenished, and once the replenishment stops due to a lack of inventory, demand decreases at an accelerating rate. They consider holding cost for the amount of stock that is left when the selling phase is over. Levi *et al.* (2007) study the newsvendor problem under both single and multiple period settings, where the true demand distribution is not known and only independent samples of the true distribution are known. They consider a holding cost for each unit of item remaining at the end of a period and a penalty cost for each unit of unsatisfied demand. Chen *et al.* (2014) examine the newsvendor model with discrete demand and with single and multiple periods, where for each period a unit holding cost is considered for each excess item unit at the end of the period and also a unit backlog cost if a shortage occurs.

Maggioni *et al.* (2019) study a cost-based newsvendor model under common continuous probability distributions and consider a unit holding cost for each overstocked unit item after the regular demand is realized. Wu and Honhon (2023) study a newsvendor problem with two selling periods where they consider a holding cost for the unsold inventory at the end of the first period, which is carried to the second period. They also assume a salvage value for the leftover inventory at the end of the second period. We will next review papers in chronological order that take into account holding cost in the newsvendor model in a different (and usually more complex) way compared with the aforementioned stream of literature.

Bitran *et al.* (1986) study the production and selling of families of products in a newsvendor context. The production and selling of each product family are assumed to be single points in time and a holding cost is charged per unit product per period from the point of production to the point of selling. Therefore, a fixed number of discrete time periods is assumed between production and selling points, where holding cost is incurred for each of those periods and for each produced item. No holding cost is considered after the first selling point for selling the overstocked items with discount. Later, Matsuo (1990) studies a more realistic version of the problem in Bitran *et al.* (1986), where time is considered as continuous (as in our case) and a holding cost for each item in the product family is charged per unit product per unit time from the start of production of the next family of products in the sequence (this start time is a decision variable in their problem), to the point of selling, while in our case the holding cost from the start time (which is variable) of production of the same item to its selling is included in the problem. The holding cost function of the shipping phase in our case has a linear structure and can be considered equivalent to the holding cost function in Matsuo (1990) for a single item.

Chen and Chuang (2000) study a newsvendor problem where the seller provides a discount for a purchase made early. The holding cost is per unit item per unit time and the total inventory holding cost is calculated from the point of purchase (equivalent to a single production point) to the single point of demand realization (in their problem, the duration between these two points, is variable), which is equivalent to the total holding cost of our shipping phase.

Pal *et al.* (2015) add holding cost to a distribution free newsvendor model with customer balking which is studied in earlier works of Gallego and Moon (1993), Moon and Choi (1995), and Liao *et al.* (2011). They take into account a unit holding cost in the model which is a specific nonlinear function of the order quantity. They do not consider the production or selling phases as gradual periods and hence the corresponding holding cost occurred during the production or selling phases are not taken into account. Although in their notations the unit holding cost is defined as per unit item per unit time, since in the profit function they do not consider the duration of time for which the unit holding cost applies, their unit holding cost is equivalent to the quantity-dependent holding cost.

Similar to our approach, Tang *et al.* (2018) study the newsvendor problem with holding cost and a gradual linear demand in the regular selling season. However, they do not propose a solution approach for the problem under discrete distribution or GPLD, and do not take into account the gradual evolution of inventory in the production phase and discount selling season. They also suggest an approximation for the holding cost of the regular selling season, in a way to be able to use the closed-form solution of the standard newsvendor model. They specifically motivate the problem based on items with high unit holding costs, and show that an increase in unit holding cost results in a decrease in the optimal order quantity, and the gap increases between the optimal order quantity of (and the expected objective value of) the exact model and of the approximation based on the standard newsvendor model. These results are also in line with our numerical experiments.

Wang *et al.* (2019) study the newsvendor problem with a price dependent demand where the selling price is a decision variable along with the order quantity. They also take into account a unit holding cost charged for each unit of the order quantity which makes the total holding cost a linear function of the order quantity. Their holding cost is not defined as a function of time and is not representative of the holding cost in our production and selling seasons.

Schlapp *et al.* (2022) introduce a newsvendor problem where the timing of inventory arrival is a decision variable (which may be before or after the start of the selling season, unlike our case where production occurs before the distribution and selling seasons). In this work, the starting point of the selling season is unknown (as opposed to a known starting point in our case), and the demand may follow a nonlinear pattern. The same as our paper, they also consider a regular selling season in which demand occurs gradually, and where quantity-and-time-dependent holding costs occur since the time the inventory becomes available. They also do not take into account the holding cost of gradual production and discount selling periods. Their holding cost can be equivalent to the holding cost in our paper for the shipping phase and the regular selling season (assuming a constant demand rate and a fixed period of time for the selling season). Nevertheless, unlike our paper, their solution process is a heuristic one. They also study the case in which a regular selling season is deterministic (which is the assumption of our model as well), and the same aforementioned equivalencies also hold for this case.

To our knowledge, the holding cost of linear gradual production, and linear gradual demand in the discount selling season, are not incorporated in the newsvendor models in the literature. We also take into account the holding cost of a linear gradual demand for the regular selling season which is first studied by Tang *et al.* (2018) (and also considered later by Schlapp *et al.* 2022) in the newsvendor model. The holding cost of each of these three periods, is a nonlinear function in terms of the decision variable which is the production quantity (each of the holding cost functions of the two selling seasons is also nonlinear in terms of the uncertain demand) which cannot be incorporated in the well known closed-form solution of the standard newsvendor model, unlike the holding cost of the transportation phase which is a linear function of the production quantity and, under conditions mentioned previously, is equivalent to the holding cost function used in Matsuo (1990) and Chen and Chuang (2000). We incorporate the holding cost of all these four phases of supply chain in the model, in order to enhance the applicability of the model in practice.

1.3 Problem description

The assumptions of our model for the four phases of the supply chain in this paper, are described as follows.

1. Production phase: In the first phase, the production of a single product is done according to the quantity decision. Due to a limited production capacity available at any point in time, the length of the production phase essentially varies depending on the total production quantity decision. The initial inventory is zero and it accumulates over time according to the production rate. The starting time of production is flexible.

2. Shipping phase: The second phase involves the shipment of the produced quantity from the production facility to the seller based on a fixed lead time. The regular selling season may start immediately after the shipment arrives, or alternatively, one can set the duration of the shipping phase equal to the time for the shipping plus any additional time until the regular selling season begins. In this period, the inventory level does not change.

3. Regular selling season, i.e., the first selling season: This corresponds to the main selling season of the product. The starting time and duration of this season is

fixed, and the inventory level declines gradually, since we assume that the demand is spread over this whole first season.

4. Discount season, i.e., the second selling season: If the demand in the first selling season is less than the stocked amount, i.e., the production quantity, then the unsold items will be sold at a lower price in the second selling season. The start of the discount season coincides with the end of the regular season. The duration of this discount season is variable, and it lasts until the whole stock is sold.

We assume that there is no overlap between these phases. Moreover, for each of these phases, we take into account the corresponding holding costs, in addition to the revenue and production cost functions of the standard newsvendor model. The objective of the problem is to find the optimal production quantity which maximizes the total profit of the supply chain. Except for the holding cost of the shipping phase, the holding cost of the other phases cannot be incorporated in the standard newsvendor model.

The following notation is considered to model the problem, some of which are the same as the ones considered by Silver *et al.* (1998) for the standard newsvendor model. First, we let Q be the production (order) quantity, which is our main decision variable, and define the deterministic parameters as follows. We define p as selling price during the regular selling season, g as selling price during the discount season, where $g < p$. We also let v be the unit production cost, r the production rate (amount of item produced per unit time), u the demand rate during the discount season (amount of item sold per unit time), h_1, h_2, h_3, h_4 the holding cost per unit item per unit time during its corresponding periods 1 to 4, t_2 the duration of the shipment period, which is independent of Q, and t_3 the duration of the regular selling season, which is also independent of Q. We have one uncertain parameter which is x, the uncertain total demand in the regular selling season. We define t_1 as the duration of the production period, which depends on Q and r, and t_4 as the duration of the discount selling season, which depends on $(Q-x)$ and u. Note that in our definitions, t_2 and t_3 are constants, and t_1 and t_4 are dependent variables. The demand accumulates gradually and linearly over time and the demand rate during the first selling season is $\tau = \frac{x}{t_3}$. We hence assume that the uncertain demand will be equally spread over the fixed regular selling season.

We must decide on the production quantity before knowing the realization of the uncertain demand. If the production quantity Q is less than the total regular demand, i.e., if $Q \leq x$, then we will have sold all the inventory before or by the end of the first selling season at the regular price. As a consequence, since we do not satisfy the complete demand, we have lost revenues in the regular selling season, and there will be no products to be sold at the reduced price, and no holding cost will occur during the discount season. On the other hand, if we produce more than the regular demand, i.e., $Q > x$, then there will be no lost revenue and we will have excess inventory left at the end of the regular selling season, which will be sold during the discount season and will result in additional holding costs. In the model, we assume that the demand rate for the discount season is fixed and known, and hence the duration of the discount season becomes dependent on the quantity ordered. Therefore, depending on the realized demand relative to the production quantity, we will have a different total revenue and total cost, and consequently a different total profit function for the two selling seasons. As a result, we build the profit function of the model for two cases of $Q \leq x$ and $Q > x$ as follows.

Case 1: $Q \leq x$

In production phase, the inventory is gradually built up at the production rate. The length of this production period is $t_1 = \frac{Q}{r}$, and the average inventory on-hand is $\frac{Q}{2}$. Hence, since the unit holding cost h_1 is charged per unit item per unit time, the total holding cost in period 1 is equal to $h_1 \frac{Q}{r} \frac{Q}{2} = \frac{h_1}{2r} Q^2$. We also have a production cost of vQ (a transportation cost can be considered in the model as a part of the production cost). In the second phase, the produced Q item units are transported which takes t_2 time units. Hence, in period 2, a total holding cost of $h_2 t_2 Q$ is incurred. In the third phase, the inventory is sold at the regular price p and since $Q \leq x$, all the stock amount will have been sold by the end of the first selling season and the revenue will be pQ. Since the demand rate τ during the first selling season is x/t_3 and $Q \leq x$, the inventory will become zero after $\frac{Q}{\tau} = t_3 \frac{Q}{x}$ time units after the start of the selling season and during this time the average inventory level is $\frac{Q}{2}$. Hence, the corresponding total inventory holding cost during this season is $h_3 \frac{Q}{2} t_3 \frac{Q}{x} = \frac{h_3 t_3 Q^2}{2x}$. Figure 1.1 illustrates the evolution of the inventory level over time during these four phases of the supply chain for the Case 1: $Q \leq x$. Overall, we will have the following total profit function $P(Q,x)$ for this case:

$$pQ - vQ - \frac{h_1}{2r} Q^2 - h_2 t_2 Q - \frac{h_3 t_3 Q^2}{2x}$$

Case 2: $Q > x$

For this case the production and holding cost functions related to the first and second phases are the same as in the first case, since those functions are independent of the demand realization. For the third phase which is the regular selling season, since $Q > x$, only x unit items are sold at the regular price and the excess inventory of $Q - x$ units is sold at the discount price during the fourth period. Hence, in this case the revenue is $px + g[Q - x]$. The average inventory during the regular selling season is $\frac{Q+Q-x}{2} = Q - \frac{x}{2}$. Therefore, the corresponding total holding cost during period 3 is $h_3 t_3 \left[Q - \frac{x}{2}\right]$. Moreover, the average inventory during the discount season is $\left[\frac{Q-x}{2}\right]$, and the length of the discount season is $t_4 = \left[\frac{Q-x}{u}\right]$. This results in a holding cost of $h_4 \left[\frac{Q-x}{u}\right]\left[\frac{Q-x}{2}\right] = \frac{h_4}{2u}[Q-x]^2$. Figure 1.1 illustrates the evolution of the inventory level over time during these four phases of the supply chain for the Case 2: $Q > x$. Overall, we will have the following profit function $P(Q,x)$ for this case:

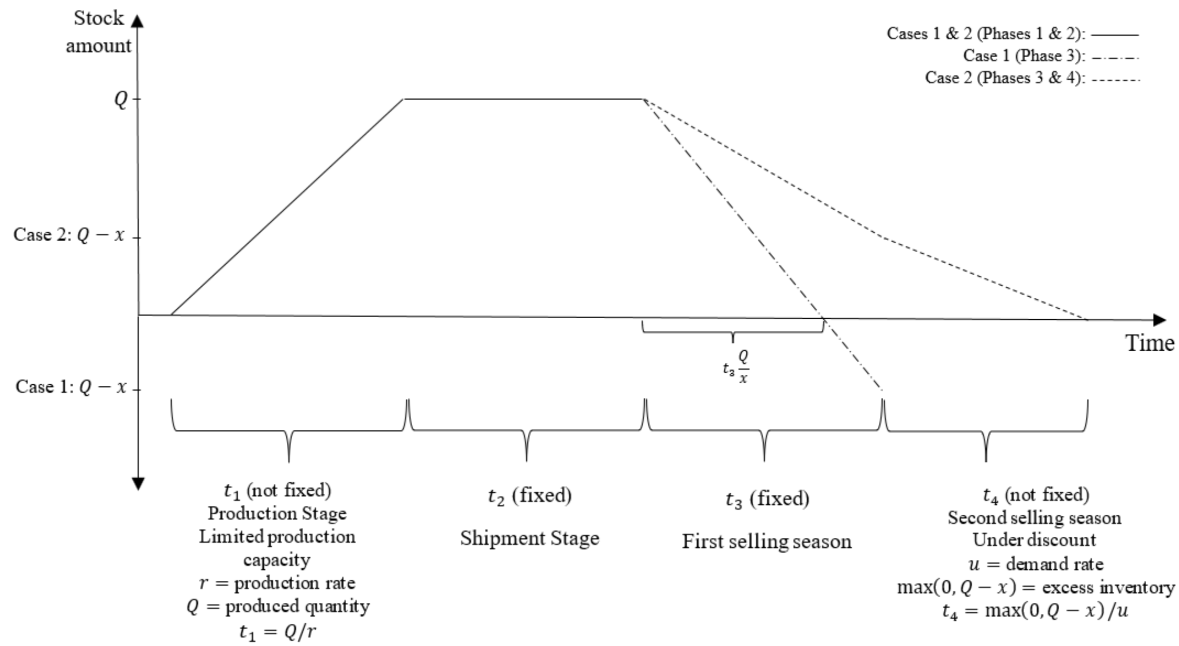

Figure 1.1: An illustration of the stock level over time for Case 1: $Q \leq x$, and Case 2: $Q > x$.

$$px + g\left[Q - x\right] - vQ - \frac{h_1}{2r}Q^2 - h_2 t_2 Q - h_3 t_3 \left[Q - \frac{x}{2}\right] - \frac{h_4}{2u}\left[Q - x\right]^2$$

The holding cost of the regular selling season, is equivalent to that of Tang *et al.* (2018) if $h = h_3 t_3$ where Tang *et al.* (2018) define h as the unit holding cost. More details are given in the Appendix 1.9.1.

Since we borrowed some of the notations from Silver *et al.* (1998), the formulations for the total cost of production, and the revenues for the first and second selling seasons in the above two cases (which are also in the standard newsvendor model) are the same as the ones formulated by Silver *et al.* (1998).

In practice, it is rarely the case to know the demand beforehand and instead we have a probability distribution of the demand which may be continuous or discrete, which are examined in the next sections.

In this part, we also demonstrate how our model can take into account the case of a price which decreases over time (as motivated in the introduction), where p and g indicate the price at the start of the first and second selling seasons, respectively. Let μ_a and μ_b be the price decrease per unit item per unit time during the regular and discount seasons, respectively. This leads to a lower total revenue. In order to be able to incorporate the price decrease in the model, we only need to update some parameters of the model as follows: $h'_3 = h_3 + \mu_a$, $h'_4 = h_4 + \mu_b$, $g' = g + (t_3\mu_a)$. It is worth noting that the parameter g needs to be updated in this way, since the additional μ_a in the h'_3 must only be applied to the amount $\left[Q - \frac{x}{2}\right] - [Q - x] = \frac{x}{2}$ when $Q > x$. This is due to the fact that when we use h'_3, the corresponding holding cost when $Q > x$, will be $h'_3 t_3 \left[Q - \frac{x}{2}\right]$, and $t_3\mu_a$ is excessively applied to the leftover amount of $Q - x$. Hence, in order to compensate for this redundant additional cost, we add $t_3\mu_a$ to the original g, which will add an additional revenue of $t_3\mu_a(Q - x)$, and hence balancing the excessive cost of $t_3\mu_a(Q - x)$. Such adjustment of revenue is not needed when $Q \leq x$, since there is no leftover in that case.

1.4 Solution procedures

1.4.1 Stochastic optimization under a continuous distribution

In this part, we study the problem under a continuous demand distribution. First, we present the model under a general continuous distribution. However, since this model does not yield a closed form solution, we propose a model under a general piecewise linear distribution which can be solved by an efficient solution procedure. The uniform, triangular and trapezoidal distributions and also the density histogram are special cases of the GPLD.

Newsvendor with time-dependent holding cost under general distribution

Proposition 1. *Let $f(x)$ be the probability density function of the uncertain parameter x, $F(x)$ be the corresponding cumulative distribution function, and $E[P(Q,x)]$ be the expected profit. Then, we have:*

$$E[P(Q,x)] = -vQ - \frac{h_1 Q^2}{2r} - h_2 Q t_2 + \left[p - g + \frac{h_3 t_3}{2}\right]\left[QF(Q) - \int_0^Q F(x)\,dx\right] + [g - h_3 t_3]\,QF(Q)$$
$$- \frac{h_4}{u}\int_0^Q \left[\int_0^x F(x)\,dx\right]dx + pQ[1 - F(Q)] - \frac{h_3 t_3}{2}Q^2 \int_Q^\infty \frac{1}{x} f(x)\,dx$$

while,

$$\frac{dE[P(Q,x)]}{dQ} = p - v - h_2 t_2 - \frac{h_1 Q}{r} + [g - h_3 t_3 - p]\,F(Q) - \frac{h_4}{u}\int_0^Q F(x)\,dx - h_3 t_3 Q \int_Q^\infty \frac{1}{x} f(x)\,dx.$$

Proof. See the Appendix 1.9.2.

The holding cost function and its derivative for the regular selling season given in Proposition 1, are equivalent to those proposed by Tang *et al.* (2018), if $h = h_3 t_3$ where Tang *et al.* (2018) define h as the unit holding cost. More details are presented in the the Appendix 1.9.1.

Since the given profit function has a single decision variable Q, we can find its optimal solution using the first order condition as long as we can obtain the explicit values of the terms $\int_0^Q F(x)\,dx$ and $\int_Q^\infty \frac{1}{x} f(x)\,dx$ in terms of only one variable which is Q, for a given distribution function. This is not possible for a general distribution. However, in the next sections, we present some probability density functions under which we can analytically solve the problem using the above formulation for each interval of the given distribution.

The concavity results of the problem under the general distribution case are also presented in the Appendix 1.9.2.

Newsvendor with time-dependent holding cost under GPLD

In order to introduce an approach to solve the newsvendor model under a general piecewise linear distribution (GPLD), we first define the GPLD. This type of distributions can be used to represent any distribution comprising linear components such as uniform, triangular, and trapezoidal distributions. We formulate the density function $f(x)$ of the GPLD as follows.

$$f(x) = \begin{cases} \left[\frac{f^-(L_n)-f^+(L_{n-1})}{L_n-L_{n-1}}\right][x-L_{n-1}]+f^+(L_{n-1}) & for\, x\in[L_{n-1},L_n]\,, for\, n=1,...,N \\ 0 & for\, x>L_N\ or\ x<L_0 \end{cases}$$

where for all $n=1,...,N$, $f^-(L_n)=\lim\limits_{x\to L_n^-} f(x)$ is the left-hand limit of function $f(x)$ at point L_n, and for all $n=0,...,N$, $f^+(L_n)=\lim\limits_{x\to L_n^+} f(x)$ is the right-hand limit of function $f(x)$ at point L_n, which are all known constants, and

$$F(x) = \begin{cases} 0 & x\le L_0 \\ \left[\frac{f^-(x)+f^+(L_0)}{2}\right]x & for\, x\in(L_0,L_1] \\ \left[\sum\limits_{k=1}^{n-1}\left[\frac{f^-(L_k)+f^+(L_{k-1})}{2}\right][L_k-L_{k-1}]\right]+\left[\frac{f^-(x)+f^+(L_{n-1})}{2}\right][x-L_{n-1}] & for\, x\in[L_{n-1},L_n]\,, for\, n=2,...,N \\ 1 & for\, x\ge L_N \end{cases}$$

In our newsvendor problem, we let $L_0=0$, since in our case, the demand cannot be negative.

Note that the equation $F(L_N)=\sum\limits_{k=1}^{N}\left[\frac{f^-(L_k)+f^+(L_{k-1})}{2}\right][L_k-L_{k-1}]=1$ always holds true, since $F(x)$ is a cumulative distribution function. Figure 1.2 illustrates the probability density function $f(x)$ of the GPLD. The GPLD is reduced to the uniform distribution when we have $N=2$, $f^+(L_0)=f^-(L_1)=0$, and $f^+(L_1)=f^-(L_2)=\frac{1}{L_2-L_1}$. The triangular distribution is derived when $N=3$, $f^+(L_0)=f^-(L_1)=f^+(L_1)=f^-(L_3)=0$, and $f^-(L_2)=f^+(L_2)=\frac{2}{L_3-L_1}$. We can have the trapezoidal distribution when $N=4$, $f^+(L_0)=f^-(L_1)=f^+(L_1)=f^-(L_4)=0$, $f^-(L_2)=f^+(L_2)=f^-(L_3)=f^+(L_3)=$

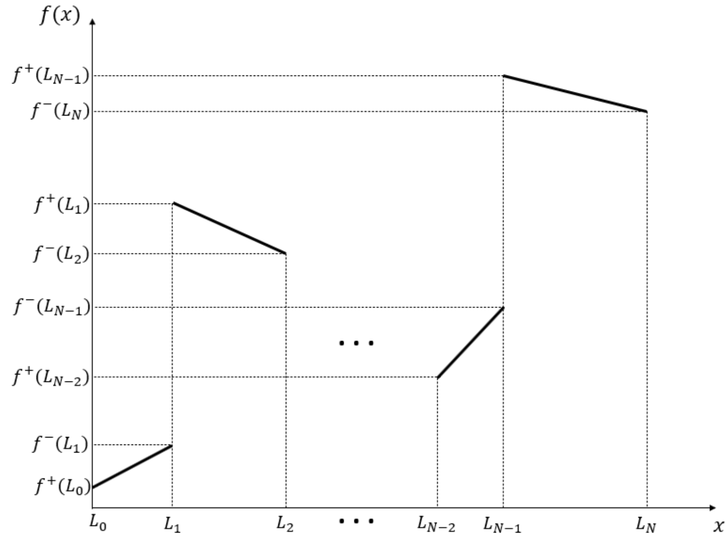

Figure 1.2: An illustration of the probability density function $f(x)$ of the general piecewise linear distribution (GPLD). The sum of the areas below the bold lines must be equal to 1.

$\frac{2}{L_4+L_3-L_1-L_2}$. The density histogram can be derived when N is the number of intervals of the histogram and $f^+(L_0) = f^-(L_1)$, $f^+(L_{N-1}) = f^-(L_N)$, and $f^+(L_{i-1}) = f^-(L_i)$ for all $i = 2, ..., N-1$.

Proposition 2. *Let x be a random variable with the general piecewise linear distribution (GPLD). Then, for the optimal order quantity Q^*_{GPLD} the following equation holds,*

$$Q^*_{GPLD} = \underset{Q \in \left\{0, L_1, L_2, ..., L_N, Q_1, Q_2, ..., Q_{N-1}, Q_N, Q_{N+1}, \overline{Q}\right\}}{\arg\max} E\left[P(Q,x)\right]$$

where, $\{L_1, L_2, ..., L_N\}$ are the parameters of the GPLD, which represent the endpoints of the intervals of the distribution ($L_0 = 0$), $\overline{Q}$ is the maximum production or purchase quantity, $E\left[P_j(Q,x)\right]$ is the expected profit function of the jth interval,

$Q_1 = \left\{Q \mid 0 \leq Q \leq \overline{Q}, Q \leq L_1, \frac{dE[P_1(Q,x)]}{dQ} = 0\right\}$,

for all $k = 2, ..., N-1$, $Q_k = \left\{Q \mid 0 \leq Q \leq \overline{Q}, L_{k-1} \leq Q \leq L_k, \frac{dE[P_k(Q,x)]}{dQ} = 0\right\}$,

$$Q_N = \left\{ Q \mid 0 \leq Q \leq \overline{Q}, L_{N-1} \leq Q \leq L_N, \frac{dE[P_N(Q,x)]}{dQ} = 0 \right\}, \textit{ and}$$

$$Q_{N+1} = \Bigg\{ Q \mid 0 \leq Q \leq \overline{Q}, L_N \leq Q, Q = \frac{1}{\frac{h_1}{r} + \frac{h_4}{u}} \left[-v - h_2 t_2 + g - h_3 t_3 - \frac{h_4}{u} \left[\frac{f^-(L_1) - f^+(0)}{6} L_1^2 + \frac{f^+(0)}{2} L_1^2 \right] \right.$$
$$\left. - \frac{h_4}{u} \left[\sum_{j=1}^{N-1} \left[\left[[L_{j+1} - L_j] \sum_{m=1}^{j} \left[\frac{f^-(L_m) + f^+(L_{m-1})}{2} \right] [L_m - L_{m-1}] \right] + \left[\frac{f^-(L_{j+1}) - f^+(L_j)}{6 [L_{j+1} - L_j]} \right] [L_{j+1} - L_j]^3 + \frac{f^+(L_j)}{2} [L_{j+1} - L_j]^2 \right] \right] + \frac{h_4}{u} L_N \right] \Bigg\}.$$

Proof. See the Appendix 1.9.3.

The proposition given above, does not only allow to calculate the optimal order quantity but also the corresponding optimal expected profit. We will show in the numerical results for the GPLD case that our model which incorporates nonlinear holding costs provides higher average profit gains compared with linear approximations based on the standard newsvendor problem. The explicit formulations of $E\left[P_j(Q,x)\right]$, for all $j = 1, 2, \ldots, N, N+1$, and $\frac{dE\left[P_j(Q,x)\right]}{dQ}$, for all $j = 1, 2, \ldots, N$, are derived in the Appendix 1.9.3.

1.4.2 Stochastic optimization under a discrete distribution

In this part, we propose an efficient solution procedure for the stochastic model of the problem under a discrete distribution. In order to represent the uncertain parameter with multiple demand scenarios, we first define x_j, the demand under the jth scenario, k_j as the probability of the the jth scenario, and N as the total number of scenarios. When $N = 1$, the problem reduces to the deterministic case.

An efficient solution procedure

We present an efficient solution which requires a maximization over only $2N + 3$ number of values. This solution is presented and proved in the form of the following proposition.

Proposition 3. *The optimal solution, under the demand with discrete distribution, is the one that maximizes the following problem.*

$$\max\left\{z_1(0), z_1\left(\frac{p-v-h_2t_2}{\frac{h_1}{r}+h_3t_3\sum_{j=1}^{N}\left[\frac{k_j}{x_j}\right]}\right),\right.$$
$$z_n(x_{n-1}): \forall n=2..N, z_n\left(\frac{-v-h_2t_2+[g-h_3t_3]\sum_{j=1}^{n-1}k_j+\frac{h_4}{u}\sum_{j=1}^{n-1}k_jx_j+p\sum_{j=n}^{N}k_j}{\frac{h_1}{r}+\frac{h_4}{u}\sum_{j=1}^{n-1}k_j+h_3t_3\sum_{j=n}^{N}\frac{k_j}{x_j}}\right): \forall n=2..N,$$
$$\left. z_{N+1}(x_N), z_{N+1}(\overline{Q}), z_{N+1}\left(\frac{-v-h_2t_2+[g-h_3t_3]\sum_{j=1}^{N}k_j+\frac{h_4}{u}\sum_{j=1}^{N}k_jx_j}{\frac{h_1}{r}+\frac{h_4}{u}\sum_{j=1}^{N}k_j}\right)\right\}$$

where all demand scenarios are sorted in increasing order so that $0 \leq x_1 \leq x_2 \leq \ldots \leq x_N \leq \overline{Q}$*. Moreover, for all* $n = 2, \ldots, N$*,* $z_n(Q)$ *is the total expected profit of producing* Q *items when* $x_{n-1} \leq Q < x_n$*.* $z_1(Q)$ *and* $z_{N+1}(Q)$ *are the total expected profit of producing* Q *items when* $0 \leq Q < x_1$ *and* $x_N \leq Q \leq \overline{Q}$*, respectively.*[1]

Proof. See the Appendix 1.9.4.

As in the case of GPLD, our numerical results under discrete distribution, will demonstrate that our problem leads to solutions with a higher average expected profit in contrast with the approximations developed based on the basic newsvendor model.

1.5 Approximation methods based on the standard newsvendor model

In this part, we introduce new approximations to the first, third and fourth phases of the problem, where we have nonlinear holding cost functions. We do this by linearizing the corresponding holding cost functions in different ways. The motivation behind this part is that when we have such linear costs, they can be incorporated into the standard newsvendor model to approximately solve our problem. Hence, companies may use their existing standard newsvendor solution (with simple adjustment of parameters) in order to ap-

[1]The explicit formulations of $z_i(Q)$, for all $i = 1, 2, \ldots, N, N+1$, are derived in the Appendix 1.9.4.

proximately solve the newsvendor problem with gradual inventory, and quantity-and-time-dependent holding cost. Moreover, this enables us to further examine the performance of the closed-form solution of the standard newsvendor model compared to the optimal solution of our model, when holding cost is present. Tang *et al.* (2018) has also previously proposed approximating the holding cost function of the regular selling season, so that the closed form solution of the newsvendor model can be used to solve the problem.

Table 1.1 demonstrates an overview of the cost functions used to approximate the problem, according to the approximate holding cost functions for phases 1, 3, and 4.

In phase 1, our holding cost function is $\frac{h_1}{2r}Q^2$. We use $\frac{h_1\bar{x}}{2r}Q$ as an approximation, where $\bar{x}$ is the average demand. This can be interpreted as approximating either the average inventory level as $\frac{\bar{x}}{2}$, or the duration of the phase 1 as $t_1 = \frac{\bar{x}}{r}$. Adding the holding cost $\frac{h_1\bar{x}}{2r}Q$ to the standard newsvendor model, will be equivalent to increasing the unit cost v by $\frac{h_1\bar{x}}{2r}$. We combine this approximation of phase 1 with the exact holding cost functions of the other periods, and denote the resulting problem as $ANVH_1^1$.

As a second approximation for phase 1, we use $\frac{h_1 Q_{NV}^*}{2r}Q$ to approximate the total holding cost of phase 1, where Q_{NV}^* is the optimal solution of the standard newsvendor model which does not take into account any holding cost for any of the phases 1 to 4. This can be interpreted as approximating either the average inventory level as $\frac{Q_{NV}^*}{2}$, or the duration of the phase 1 as $t_1 = \frac{Q_{NV}^*}{r}$. Adding the holding cost $\frac{h_1 Q_{NV}^*}{2r}Q$ to the standard newsvendor model, will be equivalent to increasing the unit production cost v by $\frac{h_1 Q_{NV}^*}{2r}$. When this approximation for phase 1 is combined with the exact holding cost functions of the other periods, the resulting problem is denoted by $ANVH_1^2$.

In phase 3, we approximate the corresponding total holding cost function as $h_3 t_3 Q$, which is linear function of Q and would be equivalent for the total holding cost of keeping the produced quantity Q (which is the stock amount at the beginning of phase 3) for a total time of t_3. Adding this holding cost to the standard newsvendor model, will be equivalent to increasing the purchasing cost v by h_3t_3. We combine this approximation of phase 3 with the exact holding cost functions of the other periods, and denote the resulting problem as $ANVH_3^1$.

We follow the following analytical logic for obtaining the second approximation of the total holding cost of phase 3. First, we remind that when $Q \leq x$, the total holding cost of phase 3 is $\frac{h_3t_3Q^2}{2x}$. Then, we note that in this case the inequality $Q \leq x$ results in the inequalities $\frac{Q}{x} \leq 1$ and $\frac{Q^2}{x} \leq Q$. Consequently, the inequality $\frac{h_3t_3Q^2}{2x} \leq \frac{h_3t_3}{2}Q$ always holds as well, when $Q \leq x$.

Moreover, when $Q > x$, the total holding cost of phase 3 becomes $h_3t_3\left[Q - \frac{x}{2}\right]$. In this case, the inequality $Q > x$ results in the inequalities $\frac{Q}{2} - \frac{x}{2} > 0$ and $Q - \frac{x}{2} > \frac{Q}{2}$. Consequently, the inequality $h_3t_3\left[Q - \frac{x}{2}\right] > \frac{h_3t_3}{2}Q$ always hold when $Q > x$.

Therefore, we note that the function $\frac{h_3t_3}{2}Q$ overestimates the total holding cost of phase 3 when $Q \leq x$, and underestimate it when $Q > x$. Hence, we suggest using the function $\frac{h_3t_3}{2}Q$ to approximate the total holding cost of phase 3, since those overestimation and underestimation taken together, may neutralize their deviation from the exact function of the total holding in phase 3. We combine this approximation for phase 3 with the exact holding cost functions of the other phases and denote the corresponding problem as $ANVH_3^2$.

As a third approximation for phase 3, we use the approximation proposed by Tang *et al.* (2018) which is half of the multiplication of the unit holding cost (denoted by h in Tang *et al.* 2018) and the expected demand (i.e., the average demand, $\bar{x}$), which is $\frac{h}{2}\bar{x}$. In a problem with an objective of loss minimization, this corresponds to adding the constant cost of $\frac{h}{2}\bar{x}$ to the objective function. As explained in the problem description section, the holding cost of phase 3 will be equivalent to that of the regular selling season in Tang *et al.* (2018) if $h = h_3t_3$. Hence, for the third approximation of phase 3, we use $\frac{h_3t_3}{2}\bar{x}$. We then combine this approximation of phase 3 with the exact holding cost functions of the other periods and denote the resulting problem as $ANVH_3^3$. It is worth noting that replacing $\bar{x}$ with Q in this approximation of phase 3, will result in our second approximation of this phase (i.e., $\frac{h_3 t_3}{2} Q$).

In order to approximate the holding cost function of the phase 4, we propose three different approximations as follows, which are proposed in a way that it is possible to incorporate them in the standard newsvendor model by updating the parameter g. The models of the problem created by using these approximations are denoted as $ANVH_4^1$, $ANVH_4^2$ and $ANVH_4^3$, respectively.

1. We first approximate $\frac{h_4}{2u}[Q-x]^2 = \frac{h_4}{2u}\left[Q^2+x^2-2xQ\right]$ as $\frac{h_4}{2u}[\bar{x}Q+\bar{x}x-2\bar{x}Q] = \frac{h_4}{2u}[\bar{x}x-\bar{x}Q] = -\frac{h_4}{2u}\bar{x}[Q-x]$, where $\bar{x}$ is the average demand. Here, for Q^2, we approximated one Q term as $\bar{x}$; for x^2, we approximated one x term as $\bar{x}$; and the term $-2xQ$ is approximated as $-2\bar{x}Q$. The resulting approximation is $-\frac{h_4}{2u}\bar{x}[Q-x]$ which would be equivalent to updating g as $g+\frac{h_4}{2u}\bar{x}$ in the standard newsvendor model.

2. As a second approximation, we approximate $\frac{h_4}{2u}[Q-x]^2 = \frac{h_4}{2u}\left[Q^2+x^2-2xQ\right]$ as $\frac{h_4}{2u}[\bar{x}Q+\bar{x}x-2\bar{x}x] = \frac{h_4}{2u}[\bar{x}Q-\bar{x}x] = \frac{h_4}{2u}\bar{x}[Q-x]$, where $\bar{x}$ is the average demand. The terms Q^2 and x^2 are approximated similar to the first approximation above. However, the term $-2xQ$ is approximated as $-2\bar{x}x$. This results in the approximation function $\frac{h_4}{2u}\bar{x}[Q-x]$, which would be equivalent to updating g as $g-\frac{h_4}{2u}\bar{x}$ in the standard newsvendor model.

3. As a third approximation, we approximate $\frac{h_4}{2u}[Q-x]^2$ as $\frac{h_4}{2u}[Q-x]$. So, in the standard newsvendor model, the parameter g can be updated as $g-\frac{h_4}{2u}$.

Table 1.1: An overview of the developed approximations and their exact counterparts for phases 1, 3 and 4.

	Phase 1		Phase 3			Phase 4		
Approx.	$ANVH_1^1$	$ANVH_1^2$	$ANVH_3^1$	$ANVH_3^2$	$ANVH_3^3$	$ANVH_4^1$	$ANVH_4^2$	$ANVH_4^3$
	$\frac{h_1 \bar{x}}{2r} Q$	$\frac{h_1 Q_{NV}^*}{2r} Q$	$h_3 t_3 Q$	$\frac{h_3 t_3}{2} Q$	$\frac{h_3 t_3}{2} \bar{x}$	$-\frac{h_4}{2u} \bar{x} [Q-x]$	$\frac{h_4}{2u} \bar{x} [Q-x]$	$\frac{h_4}{2u} [Q-x]$
Exact	$\frac{h_1}{2r} Q^2$		$\frac{h_3 t_3 Q^2}{2x}$ if $Q \leq x$, $h_3 t_3 \left[Q - \frac{x}{2}\right]$ if $Q > x$			$\frac{h_4}{2u} [Q-x]^2$		

1.6 Numerical results

In this section, the numerical results of the proposed approaches are presented under both synthetic and real demand data, using GPLD and discrete distributions, and compared with the standard newsvendor model and the approximation methods.

1.6.1 Experiments with synthetic data

GPLD

We first compare the solution of the standard newsvendor model with that of our new model under the GPLD. Irrespective of the values of the unit holding costs, the other parameters are kept the same and are set as follows: $p = 20$, $g = 9$, $v = 10$, $r = 0.03$ units per day (e.g., if $Q = 2.5$, then $t_1 = \frac{Q}{r} = \frac{2.5}{0.03} \approx 83.3$ days), $u = 0.02$ units per day (e.g., if $Q - x = 0.5$, then $t_4 = \frac{Q-x}{u} = \frac{0.5}{0.02} = 25$ days), $t_2 = 56$ days (i.e., the duration of the shipping is assumed to be 8 weeks), $t_3 = 42$ days (i.e., the length of the regular selling season is 6 weeks), and $\overline{Q} = 4$ where $Q \leq \overline{Q}$ (i.e., $\overline{Q}$ is the maximum production quantity). Furthermore, the unit holding cost rates (per unit per day) for the four phases are all assumed to be equal. Different values for this unit holding cost are considered: 0.00275, 0.004125, 0.0055, 0.006875, and 0.00825. These values correspond respectively to an annual holding cost of approximately 10%, 15%, 20%, 25% and 30% of the unit cost $v = 10$ (Schlapp *et al.* 2022 also use a similar method for the values of unit holding cost in their numerical experiments, with percentage levels of 12% to 36%). Moreover, for $n = 0,...,N$, L_n is derived by dividing the interval $[0,3]$ into $N = 11$ sub-intervals, using $N+1 = 12$ points, where $L_0 = 0$, $L_1 = 1$, $L_{11} = 3$, and for $i = 2,...,10$, L_i is derived by dividing the interval $[1,3]$ into 10 equidistant sub-intervals. The values of $f(L_n)$ are as follows (rounded to 3 decimal points). $f^+(L_0) = f^-(L_1) = 0$, $f^+(L_1) = f^-(L_{11}) = 0.137$, $f^-(L_2) = f^+(L_2) = f^-(L_{10}) = f^+(L_{10}) = 0.548$, $f^-(L_3) = f^+(L_3) = f^-(L_9) = f^+(L_9) = 0.411$, $f^-(L_4) = f^+(L_4) = f^-(L_8) = f^+(L_8) = 0.479$, $f^-(L_5) = f^+(L_5) = f^-(L_7) = f^+(L_7) = 0.685$, and $f^-(L_6) = f^+(L_6) = 0.616$. We use the efficient solution for the GPLD case (Proposition 2) to solve the generated instances (in order to be able to examine our models further, we will also consider different sets of parameters using real data in subsequent parts of the numerical studies).

The results are shown in the Table 1.2, where Q^*_{NV} and Q^*_{NVH} are the solutions of the standard newsvendor model and our model, respectively, and $NVH(Q^*_{NV})$ and $NVH(Q^*_{NVH})$ are their corresponding profit when they are used in our model as the production (order) quantity. In all of our numerical results (in this and next parts), the standard newsvendor solution is derived by letting $h_1 = h_2 = h_3 = h_4 = 10^{-100} \approx 0$ (for the unit holding cost, the value 10^{-100} is used instead of 0, since in some of our formulations the unit holding cost is the denominator) in our model. The profit gain of our model compared with the standard newsvendor model is calculated as $\frac{NVH(Q^*_{NVH}) - NVH(Q^*_{NV})}{NVH(Q^*_{NV})}$ and is given as a percentage. The results indicate that when holding cost is present, the average profit gain is 0.94%, and the profit corresponding to the standard newsvendor solution ($NVH(Q^*_{NV})$) is always less than that of our new model ($NVH(Q^*_{NVH})$). Moreover, Q^*_{NVH} is always less than Q^*_{NV}, in the presence of holding cost, and as the unit holding cost increases, the gap between the $NVH(Q^*_{NVH})$ and $NVH(Q^*_{NV})$ grows as well.

Table 1.2: Comparison of the new model with the standard newsvendor model, under the GPLD.

h_1, h_2, h_3, h_4	Q^*_{NVH}	$NVH(Q^*_{NVH})$	Q^*_{NV}	$NVH(Q^*_{NV})$	Profit gain
$10^{-100} \approx 0$	2.758	19.145	2.758	19.145	0%
0.00275	2.644	18.145	2.758	18.110	0.19%
0.004125	2.583	17.672	2.758	17.593	0.45%
0.0055	2.526	17.216	2.758	17.076	0.82%
0.006875	2.472	16.777	2.758	16.559	1.32%
0.00825	2.423	16.352	2.758	16.041	1.94%

We now compare the solution of the approximations with the one of our new model under the GPLD with the same parameters used for comparison of our model and the standard newsvendor. The results are shown in the tables 1.3, 1.4 and 1.5 for phases 1, 3 and 4, respectively (Note that in each row, the values which are very close to each other, are represented with more fractional digits to better indicate the difference. Moreover, the results in these tables are aimed at finding the best approximation for each phase, and hence the profit gains are not reported for these specific tables). Note again that when we approximate a specific phase separately in Tables 1.3, 1.4, and 1.5, we use the exact holding cost function for the other phases. We can observe that for phase 1, $ANVH_1^2$ provides closer results to the optimal profit compared to $ANVH_1^1$. For phase 3, $ANVH_3^1$ outperforms $ANVH_3^2$ and $ANVH_3^3$ (Tang *et al.* 2018). For phase 4, $ANVH_4^3$ provides higher profits compared with $ANVH_4^1$. $ANVH_4^2$ outperforms $ANVH_4^3$ only when unit holding cost is 0.00275, while in other instances and also on average, $ANVH_4^3$ results in higher profits than $ANVH_4^2$. Hence, we can conclude that $ANVH_4^3$ provides the best results for phase 4.

Based on the above results, $ANVH_1^2$, $ANVH_3^1$ and $ANVH_4^3$ are selected to form $ANVH$, which is the approximate model with all phases having a holding cost function which can be incorporated into the standard newsvendor problem. The corresponding numerical results of the $ANVH$ are demonstrated in the Table 1.6, where the profit gain of our model compared with $ANVH$ is calculated as $\frac{NVH(Q^*_{NVH})-NVH(Q^*_{ANVH})}{NVH(Q^*_{ANVH})}$ and is presented as a percentage. The results indicate that as the unit holding cost increases, the profits $NVH(Q^*_{NVH})$ and $NVH(Q^*_{ANVH})$ decrease, while the profit gain of our model increase. Furthermore, $ANVH$ results in solutions close to the optimal one, and the corresponding average profit gain of our model is 0.04%. We conclude that this approximation performs extremely well for this case.

Table 1.3: Comparison of the new model with its approximations for phase 1, under the GPLD.

h_1, h_2, h_3, h_4	Q^*_{NVH}	$NVH(Q^*_{NVH})$	$Q^*_{ANVH^1_1}$	$NVH\left(Q^*_{ANVH^1_1}\right)$	$Q^*_{ANVH^2_1}$	$NVH\left(Q^*_{ANVH^2_1}\right)$
0.00275	2.644	18.1452	2.674	18.1429	2.667	**18.1439**
0.004125	2.583	17.6721	2.629	17.6670	2.618	**17.6691**
0.0055	2.526	17.2165	2.582	17.2082	2.568	**17.2119**
0.006875	2.472	16.7769	2.538	16.7654	2.521	**16.7706**
0.00825	2.423	16.3519	2.495	16.3369	2.476	**16.3439**

Table 1.4: Comparison of the new model with its approximations for phase 3, under the GPLD.

h_1, h_2, h_3, h_4	Q^*_{NVH}	$NVH(Q^*_{NVH})$	$Q^*_{ANVH^1_3}$	$NVH\left(Q^*_{ANVH^1_3}\right)$	$Q^*_{ANVH^2_3}$	$NVH\left(Q^*_{ANVH^2_3}\right)$	$Q^*_{ANVH^3_3}$	$NVH\left(Q^*_{ANVH^3_3}\right)$
0.00275	2.64389	18.14519947	2.64371	**18.14519939**	2.655	18.1449	2.666	18.1439
0.004125	2.58327	17.67205442	2.5829	**17.67205403**	2.601	17.6713	2.618	17.6691
0.0055	2.52564	17.21649977	2.52496	**17.21649866**	2.547	17.2153	2.570	17.2114
0.006875	2.47242	16.77693330	2.4714	**16.77693049**	2.498	16.7752	2.525	16.7694
0.00825	2.42289	16.35190312	2.4215	**16.35189736**	2.452	16.3495	2.483	16.3417

Table 1.5: Comparison of the new model with its approximations for phase 4, under the GPLD.

h_1, h_2, h_3, h_4	Q^*_{NVH}	$NVH\,(Q^*_{NVH})$	$Q^*_{ANVH^1_4}$	$NVH\left(Q^*_{ANVH^1_4}\right)$	$Q^*_{ANVH^2_4}$	$NVH\left(Q^*_{ANVH^2_4}\right)$	$Q^*_{ANVH^3_4}$	$NVH\left(Q^*_{ANVH^3_4}\right)$
0.00275	2.644	18.145199	2.685	18.1408	2.639	**18.145132**	2.650	18.145090
0.004125	2.583	17.672054	2.646	17.6623	2.574	17.671851	2.592	**17.671869**
0.0055	2.526	17.216500	2.605	17.2002	2.513	17.216084	2.535	**17.216286**
0.006875	2.472	16.776933	2.564	16.7545	2.456	16.776218	2.481	**16.776720**
0.00825	2.423	16.351903	2.523	16.3232	2.403	16.350805	2.431	**16.351714**

Table 1.6: Comparison of the new model with its approximation ($ANVH$), under the GPLD.

h_1, h_2, h_3, h_4	Q^*_{NVH}	$NVH\,(Q^*_{NVH})$	Q^*_{ANVH}	$NVH\,(Q^*_{ANVH})$	Profit gain
0.00275	2.644	18.1452	2.674	18.1429	0.01%
0.004125	2.583	17.672	2.627	17.667	0.03%
0.0055	2.526	17.216	2.579	17.209	0.04%
0.006875	2.472	16.777	2.531	16.768	0.06%
0.00825	2.423	16.352	2.486	16.341	0.07%

We next provide a sensitivity analysis on the following three parameters: the duration of the first selling season (t_3), the production rate (r), and the demand rate in the discount season (u). We analyze these parameters since they affect the duration of their respective phase and they cannot be incorporated in the basic newsvendor model. The base case for the sensitivity analysis is the instance used in this subsection, with the holding cost for all phases fixed at 0.0055. The results are presented in the Tables 1.7, 1.8, and 1.9 for different values of t_3, r, and u, respectively. It is worth noting that a zero value for the parameters r and u cannot be implemented in our solution procedure, since those parameters exist in some denominators of the formulations. Therefore, instead of zero, we consider the number 10^{-100} for such cases. As shown in the Table 1.7, as the duration of the third period (i.e., t_3) increases, the profit of Q^*_{NVH}, Q^*_{NV} and Q^*_{ANVH} decreases, while the average profit gain in the presence of a nonzero t_3, is 0.83% and 0.042%, over Q^*_{NV} and Q^*_{ANVH}, respectively. On the other hand, as demonstrated in the Tables 1.8, and 1.9, an increase in the production rate and the demand rate in the discount period, results in an increase in profit of Q^*_{NVH}, Q^*_{NV} and Q^*_{ANVH}. The average profit gain (excluding the case of $r = 10^{-100} \approx 0$) in the Table 1.8 is 0.97% and 0.056% over Q^*_{NV} and Q^*_{ANVH}, respectively, while in the Table 1.9, it is 0.87% and 0.044% (excluding the instance of $u = 10^{-100} \approx 0$) over Q^*_{NV} and Q^*_{ANVH}, respectively.

Table 1.7: Comparison of the new model with its approximation and the standard newsvendor model, under the GPLD, for different values of t_3.

t_3	Q^*_{NVH}	NVH (Q^*_{NVH})	Q^*_{NV}	NVH (Q^*_{NV})	Profit gain	Q^*_{ANVH}	NVH (Q^*_{ANVH})	Profit gain
0	2.570	17.574	2.758	17.482	0.53%	2.628	17.566	0.048%
21	2.548	17.394	2.758	17.279	0.67%	2.604	17.386	0.046%
31.5	2.537	17.305	2.758	17.177	0.74%	2.591	17.297	0.044%
42	2.526	17.216	2.758	17.076	0.82%	2.579	17.209	0.042%
52.5	2.515	17.129	2.758	16.974	0.91%	2.566	17.122	0.040%
63	2.504	17.041	2.758	16.873	1.00%	2.554	17.035	0.039%

Table 1.8: Comparison of the new model with its approximation and the standard newsvendor model, under the GPLD, for different values of r.

r	Q^*_{NVH}	NVH (Q^*_{NVH})	Q^*_{NV}	NVH (Q^*_{NV})	Profit gain	Q^*_{ANVH}	NVH (Q^*_{ANVH})	Profit gain
$10^{-100} \approx 0$	0	0	2.758	$-2.092 * 10^{98}$	100.00%	0	0	0%
0.015	2.442	16.651	2.758	16.379	1.66%	2.526	16.632	0.117%
0.0225	2.497	17.024	2.758	16.843	1.07%	2.561	17.013	0.064%
0.03	2.526	17.216	2.758	17.076	0.82%	2.579	17.209	0.042%
0.0375	2.544	17.334	2.758	17.215	0.69%	2.590	17.329	0.031%
0.045	2.556	17.414	2.758	17.308	0.61%	2.597	17.410	0.024%

Table 1.9: Comparison of the new model with its approximation and the standard newsvendor model, under the GPLD, for different values of u.

u	Q^*_{NVH}	NVH (Q^*_{NVH})	Q^*_{NV}	NVH (Q^*_{NV})	Profit gain	Q^*_{ANVH}	NVH (Q^*_{ANVH})	Profit gain
$10^{-100} \approx 0$	1	9.538	2.758	$-2.32 * 10^{97}$	100.00%	1	9.538	0%
0.01	2.497	17.145	2.758	16.960	1.09%	2.555	17.136	0.053%
0.015	2.516	17.192	2.758	17.037	0.91%	2.571	17.184	0.046%
0.02	2.526	17.216	2.758	17.076	0.82%	2.579	17.209	0.042%
0.025	2.532	17.231	2.758	17.099	0.77%	2.584	17.224	0.040%
0.03	2.536	17.241	2.758	17.115	0.87%	2.587	17.235	0.038%

Discrete distribution

In this part, we present the numerical results under the discrete distribution with $N = 100$ scenarios where the x_j values for $j = 1,...,N$ are drawn randomly and uniformly from the interval $(1,3)$. For k_j values for $j = 1,...,N$, the weight of each scenario j is drawn randomly and uniformly from the interval $(0,10)$ and then the weights are normalized (via dividing each weight by the sum of all weights) to derive the scenario probabilities k_j for $j = 1,...,N$. The other parameters of the problem are the same as the ones used in section 1.6.1. The results are shown in Table 1.10. We observe that, as in the GPLD case, as the unit holding cost increases, the values of Q^*_{NVH} and Q^*_{ANVH}, and also the profits $NVH(Q^*_{NV})$, $NVH\left(Q^*_{ANVH}\right)$ and $NVH(Q^*_{NVH})$, decrease. When holding cost is present, the average profit gain of our model compared with the standard newsvendor model, is 0.64%, while the average profit gain of our model compared with $ANVH$, is 0.04%. $ANVH$ provides profit close to the optimal one in all instances, with the exception of the case where the unit holding cost is the highest, where the profit gain of our model compared with $ANVH$ is 0.172%.

Table 1.10: Comparison of the new model with its approximation and the standard newsvendor model, under the discrete distribution.

h_1, h_2, h_3, h_4	Q^*_{NVH}	NVH (Q^*_{NVH})	Q^*_{NV}	NVH (Q^*_{NV})	Profit gain	Q^*_{ANVH}	NVH (Q^*_{ANVH})	Profit gain
$10^{-100} \approx 0$	2.757	18.850	2.757	18.850	0%	2.757	18.850	0%
0.00275	2.683	17.8229	2.757	17.807	0.09%	2.722	17.8208	0.012%
0.004125	2.6341	17.3361	2.757	17.286	0.29%	2.6388	17.3356	0.003%
0.0055	2.596	16.8565	2.757	16.765	0.55%	2.634	16.8530	0.021%
0.006875	2.538	16.3859	2.757	16.243	0.88%	2.597	16.3845	0.009%
0.00825	2.488	15.941	2.757	15.722	1.39%	2.594	15.914	0.172%

1.6.2 Experiments with real data

SKU bb5419c49b

GPLD In this part, we perform experiments on real demand data provided by Shen *et al.* (2020). We consider the daily order quantities made by customers for the SKU bb5419c49b over a period of one month (March 2018), in order to form a density histogram, which represents the demand distribution of the regular selling season, and is a special case of the GPLD. For this density histogram, we use 5 bins (since we have only 31 days of daily demand data, the number of bins should be small) which result in the following values for parameters of the GPLD: $L_0 = 0, L_1 = 0.8, L_2 = 1.6, L_3 = 2.4, L_4 = 3.2$, and $L_5 = 4$, while $f(L_n)$ values are (rounded to 3 decimal points) $f^+(L_0) = f^-(L_1) = 0.202$, $f^+(L_1) = f^-(L_2) = 0.323$, $f^+(L_2) = f^-(L_3) = 0.444$, $f^+(L_3) = f^-(L_4) = 0.242$, and $f^+(L_4) = f^-(L_5) = 0.040$. We also consider the average selling price of this SKU during the month as the selling price of the daily regular selling period ($p = 83.935$). The other parameters are assumed as follows: selling price during the discount period $g = 50$, production rate $r = 0.04$ units per hour, the demand rate during the discount period $u = 0.02$ units per hour, $t_2 = 8$ hours, $t_3 = 24$ hours, the unit cost $v = 60$, and $\overline{Q} = 10$. The unit holding cost (per unit per hour) for the four phases are all assumed to be equal. Different values for this unit holding cost are considered: 0.000685, 0.0010275, 0.00137, 0.0017125, and 0.002055. These values correspond respectively to an annual holding cost of approximately 10%, 15%, 20%, 25% and 30% of the unit cost $v = 60$.

The results are presented in the Table 1.11 which demonstrate that as the unit holding cost increases, both Q^*_{NVH} and Q^*_{ANVH}, and also the profits $NVH\left(Q^*_{NV}\right)$, $NVH\left(Q^*_{ANVH}\right)$ and $NVH\left(Q^*_{NVH}\right)$ decrease, while the profit gains of the new model increases. In the presence of holding cost, the average profit gain of the optimal solution compared to the standard newsvendor model and $ANVH$, are 0.0032% and 0.0004%, respectively. Although in this case both the standard newsvendor and $ANVH$ models give profits which are very close to the optimal profit $NVH\left(Q^*_{NVH}\right)$, $ANVH$ still provides closer results to the optimal one, than the standard newsvendor model.

Table 1.11: Comparison of the new model with its approximation ($ANVH$), under the GPLD (density histogram) derived from real data (SKU bb5419c49b - $t_3 = 1$ day).

h_1, h_2, h_3, h_4	Q^*_{NVH}	NVH $\left(Q^*_{NVH}\right)$	Q^*_{NV}	NVH $\left(Q^*_{NV}\right)$	Profit gain	Q^*_{ANVH}	NVH $\left(Q^*_{ANVH}\right)$	Profit gain
$10^{-100} \approx 0$	2.2447	31.353208	2.2447	31.353208	0%	2.2447	31.353208	0%
0.000685	2.2393	31.259783	2.2447	31.259560	0.0007%	2.2412	31.259755	0.0001%
0.0010275	2.2366	31.213237	2.2447	31.212736	0.0016%	2.2394	31.213176	0.0002%
0.00137	2.2339	31.166802	2.2447	31.165912	0.0029%	2.2377	31.166694	0.0003%
0.0017125	2.2312	31.120477	2.2447	31.119088	0.0045%	2.2359	31.120309	0.0005%
0.002055	2.2285	31.074261	2.2447	31.072264	0.0064%	2.2341	31.074021	0.0008%

We next examine the same data over a longer regular selling season and distribution phase. Similar to synthetic data, we assume a regular selling season of 42 days and a distribution phase of 56 days. Hence, $t_2 = 24 * 56 = 1344$ hours, and $t_3 = 24 * 42 = 1008$ hours. We also multiply the daily demand data by 42, in order to simulate the demand of this longer regular selling season (due to having higher demand values as a result of a longer regular selling season, the production and discount selling seasons are likely to become longer as well). By doing so, we have 31 observations of demand data for a regular selling season with a duration of 42 days. We also let $\overline{Q} = 300$. Creating a density histogram with 5 bins result in the values for parameters of the GPLD as follows: $L_0 = 0$, $L_1 = 33.6$, $L_2 = 67.2$, $L_3 = 100.8$, $L_4 = 134.4$, $L_5 = 168$, and $f(L_n)$ values (rounded to 5 decimal points) are $f^+(L_0) = f^-(L_1) = 0.00480$, $f^+(L_1) = f^-(L_2) = 0.00768$, $f^+(L_2) = f^-(L_3) = 0.01056$, $f^+(L_3) = f^-(L_4) = 0.00576$, and $f^+(L_4) = f^-(L_5) = 0.00096$.

The results are demonstrated in Table 1.12, which indicate a smaller profit and a larger profit gain of our model, as the holding cost increases. The average profit gain of our model in the presence of holding cost (i.e., excluding the case of a unit holding cost of $10^{-100} \approx 0$), is 11.41% and 0.63% , over the standard newsvendor model and $ANVH$, respectively. The higher profit gains in this case ($t_3 = 42$ days) compared with the case of $t_3 = 1$ day, can be due to having a higher duration for periods in the supply chain which results in higher holding costs.

Table 1.12: Comparison of the new model with its approximation ($ANVH$), under the GPLD (density histogram) derived from real data (SKU bb5419c49b - $t_3 = 42$ days).

h_1, h_2, h_3, h_4	Q^*_{NVH}	NVH (Q^*_{NVH})	Q^*_{NV}	NVH (Q^*_{NV})	Profit gain	Q^*_{ANVH}	NVH (Q^*_{ANVH})	Profit gain
$10^{-100} \approx 0$	94.278	1316.835	94.278	1316.835	0%	94.278	1316.835	0%
0.000685	83.932	1107.926	94.278	1086.542	1.97%	87.500	1105.387	0.23%
0.0010275	79.493	1017.173	94.278	971.396	4.71%	84.113	1012.719	0.44%
0.00137	75.435	933.923	94.278	856.250	9.07%	80.728	927.829	0.66%
0.0017125	71.699	857.252	94.278	741.104	15.67%	77.345	850.050	0.85%
0.002055	68.238	786.404	94.278	625.958	25.63%	73.963	778.737	0.98%

Discrete distribution We use the created density histogram of data to form the discrete distribution where the demand scenarios are considered as the middle point of the bins, and the weight of each scenario is the value of the probability density function of that scenario in the density histogram. The scenario weights are then normalized (by dividing each weight by the sum of the weights) to obtain the scenario probabilities. This results in the following parameters for the discrete distribution: $x_1 = 0.4$, $x_2 = 1.2$, $x_3 = 2.0$, $x_4 = 2.8$, and $x_5 = 3.6$, while the scenario probabilities are (rounded to 3 decimal points) 0.161, 0.258, 0.355, 0.194, 0.032. The other problem parameters are considered the same as the ones used in section 1.6.2.

The results are given in Table 1.13, which show that the three examined models give the same solution ($Q^*_{NVH} = Q^*_{NV} = Q^*_{ANVH} = 2$) for the examined values of the unit holding cost, under the given discrete distribution. It is worth noting that according to our experiments, increasing the unit holding cost to 0.09, would still give the same result of $Q^*_{NVH} = Q^*_{NV} = Q^*_{ANVH} = 2$; however, for a unit holding cost of 0.1, $Q^*_{NVH} = 1.875$ and $NVH\left(Q^*_{NVH}\right) = 21.312$, while $Q^*_{NV} = Q^*_{ANVH} = 2$ and $NVH\left(Q^*_{NV}\right) = NVH\left(Q^*_{ANVH}\right) = 21.271$ (rounded to 3 decimal points), which result in a profit gain of 0.19% for the new model. Moreover, using a unit holding cost of 0.15, results in $NVH\left(Q^*_{NV}\right) = 15.853$ (for $Q^*_{NV} = 2$), $Q^*_{ANVH} = 1.2$ and $NVH\left(Q^*_{ANVH}\right) = 18.016$, while $Q^*_{NVH} = 1.254$ and $NVH\left(Q^*_{NVH}\right) = 18.027$ with a profit gain of 13.71% and 0.06% over the standard newsvendor and $ANVH$, respectively. Therefore, on average, our new model is more profitable than the standard newsvendor, although in this case, it provides the same solution for the instances where the unit holding cost is up to around 0.09.

Table 1.13: Comparison of the new model with its approximation and the standard newsvendor model, under the discrete distribution derived from real data (SKU bb5419c49b - $t_3 = 1$ day).

h_1, h_2, h_3, h_4	Q^*_{NVH}, Q^*_{NV}, Q^*_{ANVH}	Profit	Profit gain
$10^{-100} \approx 0$	2.0	32.107	0%
0.000685	2.0	32.032	0%
0.0010275	2.0	31.995	0%
0.00137	2.0	31.958	0%
0.0017125	2.0	31.921	0%
0.002055	2.0	31.884	0%

We also examine the data over the longer regular selling season (which would likely result in a longer production and discount season as well) and distribution phase, for the discrete distribution case. Using a process similar to the one described for the case of $t_3 = 1$ day, we generate the discrete distribution which result in 5 demand scenarios as $x_1 = 16.8$, $x_2 = 50.4$, $x_3 = 84$, $x_4 = 117.6$, and $x_5 = 151.2$, and the scenario probabilities (rounded to 3 decimal points) of 0.161, 0.258, 0.355, 0.194, 0.032, respectively. The results are in Table 1.14. The profit gain of our model is zero compared to the two other models, except for when the unit holding cost is 0.002055 (the highest value in the table) where the profit gain is 1.02% over the other two models. In this case, the optimal solution $Q^*_{NVH} = 71.811$ is between the two demand scenarios $x_2 = 50.4$ and $x_3 = 84$, and cannot be obtained by the standard newsvendor model. We also examined the case of a unit holding cost of 0.003 which results in $Q^*_{NVH} = Q^*_{ANVH} = 50.4$ and $NVH\left(Q^*_{NVH}\right) = NVH\left(Q^*_{ANVH}\right) = 646.777$, and a profit gain of 24.13% over the standard newsvendor model ($NVH\left(Q^*_{NV}\right) = 521.051$).

Table 1.14: Comparison of the new model with its approximation and the standard newsvendor model, under the discrete distribution derived from real data (SKU bb5419c49b - $t_3 = 42$ days).

h_1, h_2, h_3, h_4	Q^*_{NVH}	NVH (Q^*_{NVH})	Q^*_{NV}	NVH (Q^*_{NV})	Profit gain	Q^*_{ANVH}	NVH (Q^*_{ANVH})	Profit gain
$10^{-100} \approx 0$	84	1348.479	84	1348.479	0%	84	1348.479	0%
0.000685	84	1159.550	84	1159.550	0%	84	1159.550	0%
0.0010275	84	1065.085	84	1065.085	0%	84	1065.085	0%
0.00137	84	970.620	84	970.620	0%	84	970.620	0%
0.0017125	84	876.156	84	876.156	0%	84	876.156	0%
0.002055	71.811	789.644	84	781.691	1.02%	84	781.691	1.02%

SKU d17d9135b0

We next perform experiments on SKU d17d9135b0 which shows that for even under a regular selling season of 1 day, the new model can provide considerable profit gains over the standard newsvendor model and $ANVH$.

GPLD For the SKU d17d9135b0, the parameters of the problem are assumed as follows: $p = 15.886$ (the average selling price during the month), $g = 8.886$, $r = 0.2$ units per hour, $u = 0.04$ units per hour, $t2 = 8$ hours, $t3 = 24$ hours, the unit cost $v = 9.5$, and $\overline{Q} = 250$. The unit holding cost (per unit per hour) for all four phases are considered to have the same value. Different values for the unit holding cost are taken into account: 0.0001085, 0.00016275, 0.000217, 0.00027125, and 0.0003255, which respectively represent an annual unit holding cost of roughly 10%, 15%, 20%, 25%, and 30% of the unit cost $v = 9.5$. We generate a density histogram of the daily demand data, with 5 bins, which result in the following parameters for the GPLD: $L_0 = 0$, $L_1 = 11.4$, $L_2 = 22.8$, $L_3 = 34.2$, $L_4 = 45.6$, $L_5 = 57$, and $f(L_n)$ values (rounded to 4 decimal points) as $f^+(L_0) = f^-(L_1) = 0.0679$, $f^+(L_1) = f^-(L_2) = 0.0113$, $f^+(L_2) = f^-(L_3) = 0.0028$, $f^+(L_3) = f^-(L_4) = 0.0028$, and $f^+(L_4) = f^-(L_5) = 0.0028$. The numerical results are presented in Table 1.15. We observe an average profit gain of 0.63% and 0.31% over the standard newsvendor model and $ANVH$, respectively, in the presence of holding cost (i.e., excluding the case of a unit holding cost of $10^{-100} \approx 0$). Moreover, the profit gain increases as the unit holding cost becomes larger.

Table 1.15: Comparison of the new model with its approximation ($ANVH$), under the GPLD (density histogram) derived from real data (SKU d17d9135b0).

h_1, h_2, h_3, h_4	Q^*_{NVH}	NVH (Q^*_{NVH})	Q^*_{NV}	NVH (Q^*_{NV})	Profit gain	Q^*_{ANVH}	NVH (Q^*_{ANVH})	Profit gain
$10^{-100} \approx 0$	26.002	47.883	26.002	47.883	0%	26.002	47.883	0%
0.0001085	23.228	47.250	26.002	47.162	0.19%	25.408	47.196	0.11%
0.00016275	22.596	46.983	26.002	46.802	0.39%	25.111	46.877	0.23%
0.000217	22.285	46.726	26.002	46.442	0.61%	24.815	46.574	0.33%
0.00027125	21.985	46.476	26.002	46.081	0.86%	24.518	46.286	0.41%
0.0003255	21.694	46.235	26.002	45.721	1.12%	24.221	46.013	0.48%

Discrete distribution The discrete distribution is generated from the density histogram using a process similar to the one described for the SKU bb5419c49b, which results in demand scenarios of $x_1 = 5.7$, $x_2 = 17.1$, $x_3 = 28.5$, $x_4 = 39.9$, and $x_5 = 51.3$, with scenario probabilities (rounded to 3 decimal points) of 0.774, 0.129, 0.032, 0.032, and 0.032, respectively. The results are given in Table 1.16 which indicate an average profit gain of 1.15% for our model over the other two models, when holding cost is present (i.e., excluding the case where unit holding cost is $10^{-100} \approx 0$). As the unit holding cost increases, the profit gain becomes larger as well.

Table 1.16: Comparison of the new model with its approximation and the standard newsvendor model, under the discrete distribution derived from real data (SKU d17d9135b0).

h_1, h_2, h_3, h_4	Q^*_{NVH}	NVH (Q^*_{NVH})	Q^*_{NV}	NVH (Q^*_{NV})	Profit gain	Q^*_{ANVH}	NVH (Q^*_{ANVH})	Profit gain
$10^{-100} \approx 0$	28.5	48.143	28.5	48.143	0%	28.5	48.143	0%
0.0001085	26.058	47.277	28.5	47.268	0.02%	28.5	47.268	0.02%
0.00016275	19.010	47.033	28.5	46.831	0.43%	28.5	46.831	0.43%
0.000217	17.1	46.894	28.5	46.393	1.08%	28.5	46.393	1.08%
0.00027125	17.1	46.762	28.5	45.956	1.75%	28.5	45.956	1.75%
0.0003255	17.1	46.630	28.5	45.519	2.44%	28.5	45.519	2.44%

1.7 Conclusion

In this paper, we introduced a new variant of the well-known newsvendor problem, where we take into account the element of time over four phases of the supply chain and the corresponding quantity-and-time-dependent holding cost which is neglected in the standard newsvendor model. We then mathematically modeled the problem, which resulted in a nonlinear formulation, and then developed exact solution approaches for the stochastic version of the model under general piecewise linear distribution and discrete distribution cases. We also provided numerical studies using both synthetic and real data, and showed that in the presence of holding cost, our new model outperforms the standard newsvendor model, by resulting in more profitable solutions on average. We also proposed and examined new linear approximations of the model, which can be solved using the well-known closed-form solution of the standard newsvendor model, and provide profits close to the optimal ones.

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1.9 Appendices

1.9.1 Explanation on the holding cost of the regular selling season, proposed by Tang *et al.* (2018)

Tang *et al.* (2018) formulated the holding cost of a linear gradual demand for the regular selling season, in the case of $Q \leq x$, as $\frac{hQ^2}{2x}$ (given the notation of Q and x as defined in our paper) where they define h as the unit holding cost. Hence, in this case, the holding cost of $\frac{h_3t_3Q^2}{2x}$ would be equivalent to $\frac{hQ^2}{2x}$ proposed by Tang *et al.* (2018), if $h = h_3t_3$. Moreover, for the case of $Q > x$, Tang *et al.* (2018) formulated the holding cost function of the regular selling season with linear gradual demand as $h\left[Q - \frac{x}{2}\right]$ (given the notation of Q and x as defined in our paper). Therefore, the holding cost of $h\left[Q - \frac{x}{2}\right]$ proposed by Tang *et al.* (2018), would be equivalent to the holding cost of $h_3t_3\left[Q - \frac{x}{2}\right]$ in our paper in this case, if $h = h_3t_3$.

Furthermore, Tang *et al.* (2018) derived the holding cost function of the regular selling season as $-h\int_0^Q (Q-x) f(x)\,dx - \frac{h}{2}\int_0^Q xf(x)\,dx - \frac{h}{2}Q^2\int_Q^\infty \frac{1}{x}f(x)\,dx$ (given x, Q, $f(x)$, and an objective of maximization of profit, as defined in our paper) which can be simplified as $\frac{h}{2}\int_0^Q xf(x)\,dx - hQ\int_0^Q f(x)\,dx - \frac{h}{2}Q^2\int_Q^\infty \frac{1}{x}f(x)\,dx$. As shown in the Appendix 1.9.2, we also first derive the holding cost function of the regular selling season as $\frac{h_3t_3}{2}\int_0^Q xf(x)\,dx - h_3t_3Q\int_0^Q f(x)\,dx - \frac{h_3t_3}{2}Q^2\int_Q^\infty \frac{1}{x}f(x)\,dx$, which would be equivalent to that of Tang *et al.* (2018), if $h = h_3t_3$. This holding cost function is simplified further in our paper in Proposition 1 and its proof, as $\frac{h_3t_3}{2}\left[QF(Q) - \int_0^Q F(x)\,dx\right] - h_3t_3QF(Q) -$

$\frac{h_3t_3}{2}Q^2\int_Q^\infty \frac{1}{x}f(x)\,dx$. Tang *et al.* (2018), also obtained the derivative of the holding cost function of the regular selling season as $-h\int_0^Q f(x) - hQ\int_Q^\infty \frac{1}{x}f(x)\,dx$ (assuming x, Q, $f(x)$, and a profit maximization objective, as defined in our paper), which would be equivalent to $-h_3t_3F(Q) - h_3t_3Q\int_Q^\infty \frac{1}{x}f(x)\,dx$, as given in the formula of $\frac{dE[P(Q,x)]}{dQ}$ in our paper in Proposition 1 and its proof, if $h = h_3t_3$.

1.9.2 Proof of Proposition 1 and concavity results for the general distribution case

Proof.

$$E[P(Q,x)] = \underbrace{-vQ - \frac{h_1Q^2}{2r}}_{Phase\,1}\underbrace{-h_2Qt_2}_{Phase\,2}$$
$$+\underbrace{\int_0^Q \left[\underbrace{px}_{(Phase\,3)} + \underbrace{g[Q-x]}_{(Phase\,4)}\underbrace{-h_3\left[\frac{2Q-x}{2}\right]t_3}_{(Phase\,3)}\underbrace{-\frac{h_4}{2u}[Q-x]^2}_{(Phase\,4)}\right]f(x)\,dx}_{Phase\,3\,and\,4,\,Q>x} + \underbrace{\int_Q^\infty \left[pQ - \frac{h_3Q^2}{2x}t_3\right]f(x)\,dx}_{Phase\,3,\,Q<x}$$

$$= -vQ - \frac{h_1Q^2}{2r} - h_2Qt_2 + \left[p - g + \frac{h_3t_3}{2} + \frac{h_4}{u}Q\right]\int_0^Q xf(x)\,dx + \left[g - h_3t_3 - \frac{h_4}{2u}Q\right]Q\int_0^Q f(x)\,dx$$
$$-\frac{h_4}{2u}\left[\int_0^Q x^2 f(x)\,dx\right] + pQ\int_Q^\infty f(x)\,dx - \frac{h_3t_3}{2}Q^2\int_Q^\infty \frac{1}{x}f(x)\,dx$$

$$= -vQ - \frac{h_1Q^2}{2r} - h_2Qt_2 + \left[p - g + \frac{h_3t_3}{2} + \frac{h_4}{u}Q\right]\left[QF(Q) - \int_0^Q F(x)\,dx\right] + \left[g - h_3t_3 - \frac{h_4}{2u}Q\right]QF(Q)$$
$$-\frac{h_4}{2u}\left[Q^2F(Q) - 2\left[\int_0^Q xF(x)\,dx\right]\right] + pQ[1-F(Q)] - \frac{h_3t_3}{2}Q^2\int_Q^\infty \frac{1}{x}f(x)\,dx$$

$$= -vQ - \frac{h_1Q^2}{2r} - h_2Qt_2 + \left[p - g + \frac{h_3t_3}{2} + \frac{h_4}{u}Q\right]\left[QF(Q) - \int_0^Q F(x)\,dx\right] + \left[g - h_3t_3 - \frac{h_4}{2u}Q\right]QF(Q)$$
$$-\frac{h_4}{2u}\left[Q^2F(Q) - 2\left[Q\int_0^Q F(x)\,dx - \int_0^Q\left[\int_0^x F(x)\,dx\right]dx\right]\right] + pQ[1-F(Q)] - \frac{h_3t_3}{2}Q^2\int_Q^\infty \frac{1}{x}f(x)\,dx$$

$$= -vQ - \frac{h_1Q^2}{2r} - h_2Qt_2 + \left[p - g + \frac{h_3t_3}{2} + \frac{h_4}{u}Q\right]\left[QF(Q) - \int_0^Q F(x)\,dx\right] + \left[g - h_3t_3 - \frac{h_4}{2u}Q\right]QF(Q)$$
$$-\frac{h_4}{2u}\left[2Q^2F(Q) - 2Q\int_0^Q F(x)\,dx - Q^2F(Q) + 2\int_0^Q\left[\int_0^x F(x)\,dx\right]dx\right] + pQ[1-F(Q)] - \frac{h_3t_3}{2}Q^2\int_Q^\infty \frac{1}{x}f(x)\,dx$$

$$= -vQ - \frac{h_1Q^2}{2r} - h_2Qt_2 + \left[p - g + \frac{h_3t_3}{2} + \frac{h_4}{u}Q - \frac{h_4}{u}Q\right]\left[QF(Q) - \int_0^Q F(x)\,dx\right]$$
$$+\left[g - h_3t_3 - \frac{h_4}{2u}Q + \frac{h_4}{2u}Q\right]QF(Q) - \frac{h_4}{u}\int_0^Q\left[\int_0^x F(x)\,dx\right]dx + pQ[1-F(Q)] - \frac{h_3t_3}{2}Q^2\int_Q^\infty \frac{1}{x}f(x)\,dx$$

$$= -vQ - \frac{h_1Q^2}{2r} - h_2Qt_2 + \left[p - g + \frac{h_3t_3}{2}\right]\left[QF(Q) - \int_0^Q F(x)\,dx\right] + [g - h_3t_3]\,QF(Q)$$
$$-\frac{h_4}{u}\int_0^Q\left[\int_0^x F(x)\,dx\right]dx + pQ[1-F(Q)] - \frac{h_3t_3}{2}Q^2\int_Q^\infty \frac{1}{x}f(x)\,dx.$$

The derivative of the above expected profit function in terms of Q is,

$$\frac{dE[P(Q,x)]}{dQ} = -v - \frac{h_1Q}{r} - h_2t_2 + \left[p - g + \frac{h_3t_3}{2}\right][Qf(Q)] + [g - h_3t_3][Qf(Q) + F(Q)]$$
$$-\frac{h_4}{u}\left[\int_0^Q F(x)\,dx\right] + p[1 - F(Q) - Qf(Q)] - \frac{h_3t_3}{2}\left[Q^2\int_Q^\infty \frac{1}{x}f(x)\,dx\right]'$$

$$= p - v - h_2t_2 - \frac{h_1Q}{r} - \frac{h_3t_3}{2}Qf(Q) + [g - h_3t_3 - p]\,F(Q) - \frac{h_4}{u}\left[\int_0^Q F(x)\,dx\right] - \frac{h_3t_3}{2}\left[Q^2\int_Q^\infty \frac{1}{x}f(x)\,dx\right]'$$

where,

$$\left[Q^2\int_Q^\infty \frac{1}{x}f(x)\,dx\right]' = 2Q\int_Q^\infty \frac{1}{x}f(x)\,dx + Q^2\left[\frac{1}{\infty}f(\infty) - \frac{1}{Q}f(Q)\right] = 2Q\int_Q^\infty \frac{1}{x}f(x)\,dx - Qf(Q)$$

hence,

$$\frac{dE[P(Q,x)]}{dQ} = p - v - h_2t_2 - \frac{h_1Q}{r} - \frac{h_3t_3}{2}Qf(Q) + [g - h_3t_3 - p]\,F(Q) - \frac{h_4}{u}\left[\int_0^Q F(x)\,dx\right] - \frac{h_3t_3}{2}\left[2Q\int_Q^\infty \frac{1}{x}f(x)\,dx - Qf(Q)\right]$$

$$= p - v - h_2 t_2 - \frac{h_1 Q}{r} + [g - h_3 t_3 - p] F(Q) - \frac{h_4}{u} \int_0^Q F(x)\, dx - h_3 t_3 Q \int_Q^{\infty} \frac{1}{x} f(x)\, dx. \blacksquare$$

To obtain the concavity results of the problem under the general distribution case, we now derive the second derivative of $E[P(Q,x)]$ for the general distribution case, as below.

$$\begin{aligned}
\frac{d^2 E[P(Q,x)]}{dQ^2} &= -\frac{h_1}{r} + [g - h_3 t_3 - p] f(Q) - \frac{h_4}{u} F(Q) - h_3 t_3 \left[\int_Q^{\infty} \frac{1}{x} f(x)\, dx + Q \left[\frac{1}{\infty} f(\infty) - \frac{1}{Q} f(Q) \right] \right] \\
&= -\frac{h_1}{r} + [g - h_3 t_3 - p] f(Q) - \frac{h_4}{u} F(Q) - h_3 t_3 \left[\int_Q^{\infty} \frac{1}{x} f(x)\, dx - f(Q) \right] \\
&= -\frac{h_1}{r} + [g - p] f(Q) - \frac{h_4}{u} F(Q) - h_3 t_3 \int_Q^{\infty} \frac{1}{x} f(x)\, dx
\end{aligned}$$

The term $-\frac{h_1}{r}$ has a negative value. Moreover, since $p > g$ and for all $Q \geq 0$ we have $f(Q) \geq 0$, the inequality $[g - p] f(Q) \leq 0$ holds for all $Q \geq 0$. Additionally, for all $Q \geq 0$ we have $F(Q) \geq 0$, while $-\frac{h_4}{u}$ has a negative value. This implies that the inequality $-\frac{h_4}{u} F(Q) \leq 0$ holds for any $Q \geq 0$. Also, the inequality $\int_Q^{\infty} \frac{1}{x} f(x)\, dx \geq 0$ holds for any $Q > 0$, since the term $\int_Q^{\infty} \frac{1}{x} f(x)\, dx$ is an integration of a non-negative function when we have $Q > 0$. Given these implications, we can conclude that for any $Q > 0$, the inequality $\frac{d^2 E[P(Q,x)]}{dQ^2} \leq 0$ holds, which implies the concavity of the profit function $E[P(Q,x)]$ for any $Q > 0$. When $h_1 > 0$, $\frac{d^2 E[P(Q,x)]}{dQ^2} < 0$ holds for any $Q > 0$, which implies that given $h_1 > 0$ and $Q > 0$, the profit function $E[P(Q,x)]$ is concave downward.

1.9.3 Proof of Proposition 2

Proof. Since the expected profit function $E[P(Q,x)]$ and its derivative depend on $F(Q)$, and the value of $F(Q)$ differs based on the interval of Q, we decompose the problem into the $(N+1)$ intervals in the formulation of $F(Q)$, and reach a closed form optimal solution for each interval (i.e., $Q_1, Q_2, ..., Q_N, Q_{N+1}$) using the first order condition. Then, the optimal order quantity is the one with the highest expected profit among the $(N+1)$

solutions. Moreover, the points $\{0, L_1, L_2, ..., L_N\}$, as the end-points of these intervals, must also be considered as the potential optimal solutions.

First, let $0 \leq Q < L_1$. In this case,

$$\int_Q^{\infty} \frac{1}{x} f(x)\,dx = \int_Q^{L_1} \frac{1}{x} f(x)\,dx + \sum_{n=2}^{N} \int_{L_{n-1}}^{L_n} \frac{1}{x} f(x)\,dx =$$

$$= \int_Q^{L_1} \frac{1}{x}\left[\left[\frac{f^-(L_1) - f^+(0)}{L_1}\right][x] + f^+(0)\right]dx + \sum_{n=2}^{N} \int_{L_{n-1}}^{L_n} \frac{1}{x}\left[\left[\frac{f^-(L_n) - f^+(L_{n-1})}{L_n - L_{n-1}}\right][x - L_{n-1}] + f^+(L_{n-1})\right]dx$$

$$= f^-(L_1) - f^+(0) + f^+(0)\ln L_1 - \left[\frac{f^-(L_1) - f^+(0)}{L_1}\right]Q + f^+(0)\ln Q$$

$$+ \sum_{n=2}^{N} \left[\left[\left[\frac{f^-(L_n) - f^+(L_{n-1})}{L_n - L_{n-1}}\right][L_n - L_{n-1}\ln L_n] + f^+(L_{n-1})\ln L_n\right] - \left[\left[\frac{f^-(L_n) - f^+(L_{n-1})}{L_n - L_{n-1}}\right][L_{n-1} - L_{n-1}\ln L_{n-1}] + f^+(L_{n-1})\ln L_{n-1}\right]\right]$$

$$= f^-(L_1)\left[1 - \frac{Q}{L_1}\right] + f^+(0)\left[\frac{Q}{L_1} - 1 + \ln L_1 + \ln Q\right]$$

$$+ \sum_{n=2}^{N} \left[\left[\frac{f^-(L_n) - f^+(L_{n-1})}{L_n - L_{n-1}}\right]\left[L_n - L_{n-1} + L_{n-1}\left[\ln\frac{L_{n-1}}{L_n}\right]\right] + f^+(L_{n-1})\left[\ln\frac{L_n}{L_{n-1}}\right]\right]$$

In this interval, $F(0) = 0$ and for $Q \in (0, L_1)$ we have

$$F(Q) = \left[\frac{f^-(x) + f^+(0)}{2}\right]Q = \left[\frac{\left[\left[\frac{f^-(L_1) - f^+(0)}{L_1}\right]Q + f^+(0)\right] + f^+(0)}{2}\right]Q = \left[\frac{f^-(L_1) - f^+(0)}{2L_1}\right]Q^2 + Qf^+(0)$$

and

$$\int_0^Q F(x)\,dx = \int_0^Q \left[\left[\frac{f^-(L_1) - f^+(0)}{2L_1}\right]x^2 + xf^+(0)\right]dx = \left[\frac{f^-(L_1) - f^+(0)}{6L_1}\right]Q^3 + \frac{f^+(0)}{2}Q^2$$

and for $\frac{dE[P_1(Q,x)]}{dQ}$ we have:

$$\begin{aligned}
\frac{dE\left[P_1\left(Q,x\right)\right]}{dQ} =& p-v-h_2t_2-\frac{h_1Q}{r}+\left[g-h_3t_3-p\right]F\left(Q\right)-\frac{h_4}{u}\int_0^Q F\left(x\right)dx-h_3t_3Q\int_Q^\infty \frac{1}{x}f\left(x\right)dx \\
=& p-v-h_2t_2-\frac{h_1Q}{r}+\left[g-h_3t_3-p\right]\left[\left[\frac{f^-\left(L_1\right)-f^+\left(0\right)}{2L_1}\right]Q^2+Qf^+\left(0\right)\right] \\
&-\frac{h_4}{u}\left[\left[\frac{f^-\left(L_1\right)-f^+\left(0\right)}{6L_1}\right]Q^3+\frac{f^+\left(0\right)}{2}Q^2\right] \\
&-h_3t_3Q\left[f^-\left(L_1\right)\left[1-\frac{Q}{L_1}\right]+f^+\left(0\right)\left[\frac{Q}{L_1}-1+\ln L_1+\ln Q\right]\right. \\
&\left.+\sum_{n=2}^{N}\left[\left[\frac{f^-\left(L_n\right)-f^+\left(L_{n-1}\right)}{L_n-L_{n-1}}\right]\left[L_n-L_{n-1}+L_{n-1}\left[\ln\frac{L_{n-1}}{L_n}\right]\right]+f^+\left(L_{n-1}\right)\left[\ln\frac{L_n}{L_{n-1}}\right]\right]\right]
\end{aligned}$$

By using FOC we have,

$$Q_1=\left\{Q \mid \frac{dE\left[P_1\left(Q,x\right)\right]}{dQ}=0\right\}$$

Moreover, in this interval

$$\begin{aligned}
\int_0^Q\left[\int_0^x F\left(x\right)dx\right]dx &= \int_0^Q\left[\left[\frac{f^-\left(L_1\right)-f^+\left(0\right)}{6L_1}\right]x^3+\frac{f^+\left(0\right)}{2}x^2\right]dx \\
&= \left[\frac{f^-\left(L_1\right)-f^+\left(0\right)}{24L_1}\right]Q^4+\frac{f^+\left(0\right)}{6}Q^3
\end{aligned}$$

and for $E\left[P_1\left(Q,x\right)\right]$ we have:

$$
\begin{aligned}
E\left[P_1(Q,x)\right] = & -vQ - \frac{h_1 Q^2}{2r} - h_2 Q t_2 + \left[p - g + \frac{h_3 t_3}{2}\right]\left[QF(Q) - \int_0^Q F(x)\,dx\right] + \left[g - h_3 t_3\right] QF(Q) \\
& - \frac{h_4}{u}\int_0^Q \left[\int_0^x F(x)\,dx\right] dx + pQ\left[1 - F(Q)\right] - \frac{h_3 t_3}{2} Q^2 \int_Q^\infty \frac{1}{x} f(x)\,dx \\
= & -vQ - \frac{h_1 Q^2}{2r} - h_2 Q t_2 + \left[p - g + \frac{h_3 t_3}{2}\right]\left[\left[\frac{f^-(L_1) - f^+(0)}{3L_1}\right] Q^3 + \frac{f^+(0)}{2} Q^2\right] \\
& + \left[g - h_3 t_3\right]\left[\left[\frac{f^-(L_1) - f^+(0)}{2L_1}\right] Q^3 + Q^2 f^+(0)\right] \\
& - \frac{h_4}{u}\left[\left[\frac{f^-(L_1) - f^+(0)}{24L_1}\right] Q^4 + \frac{f^+(0)}{6} Q^3\right] \\
& + pQ\left[1 - \left[\frac{f^-(L_1) - f^+(0)}{2L_1}\right] Q^2 - Q f^+(0)\right] \\
& - \frac{h_3 t_3}{2} Q^2 \left[f^-(L_1)\left[1 - \frac{Q}{L_1}\right] + f^+(0)\left[\frac{Q}{L_1} - 1 + \ln L_1 + \ln Q\right]\right. \\
& \left. + \sum_{n=2}^{N}\left[\left[\frac{f^-(L_n) - f^+(L_{n-1})}{L_n - L_{n-1}}\right]\left[L_n - L_{n-1} + L_{n-1}\left[\ln\frac{L_{n-1}}{L_n}\right]\right] + f^+(L_{n-1})\left[\ln\frac{L_n}{L_{n-1}}\right]\right]\right]
\end{aligned}
$$

Now, for all $k = 2, \ldots, N-1$, let $L_{k-1} \leq Q \leq L_k$. In this case,

$$
\int_Q^\infty \frac{1}{x} f(x)\,dx = \int_Q^{L_k} \frac{1}{x} f(x)\,dx + \sum_{n=k+1}^{N} \int_{L_{n-1}}^{L_n} \frac{1}{x} f(x)\,dx
$$

$$
= \int_Q^{L_k} \frac{1}{x}\left[\left[\frac{f^-(L_k) - f^+(L_{k-1})}{L_k - L_{k-1}}\right]\left[x - L_{k-1}\right] + f^+(L_{k-1})\right] dx
$$

$$
+ \sum_{n=k+1}^{N} \int_{L_{n-1}}^{L_n} \frac{1}{x}\left[\left[\frac{f^-(L_n) - f^+(L_{n-1})}{L_n - L_{n-1}}\right]\left[x - L_{n-1}\right] + f^+(L_{n-1})\right] dx
$$

$$
= \left[\frac{f^-(L_k) - f^+(L_{k-1})}{L_k - L_{k-1}}\right]\left[L_k - L_{k-1}\ln L_k\right] + f^+(L_{k-1})\ln L_k - \left[\frac{f^-(L_k) - f^+(L_{k-1})}{L_k - L_{k-1}}\right]\left[Q - L_{k-1}\ln Q\right] - f^+(L_{k-1})\ln Q
$$

$$
\begin{aligned}
+ \sum_{n=k+1}^{N} & \left[\left[\left[\frac{f^-(L_n) - f^+(L_{n-1})}{L_n - L_{n-1}}\right]\left[L_n - L_{n-1}\ln L_n\right] + f^+(L_{n-1})\ln L_n\right]\right. \\
& \left. - \left[\left[\frac{f^-(L_n) - f^+(L_{n-1})}{L_n - L_{n-1}}\right]\left[L_{n-1} - L_{n-1}\ln L_{n-1}\right] + f^+(L_{n-1})\ln L_{n-1}\right]\right]
\end{aligned}
$$

$$= \left[\frac{f^{-}(L_k) - f^{+}(L_{k-1})}{L_k - L_{k-1}}\right]\left[L_k - Q + L_{k-1}\ln\frac{Q}{L_k}\right] + f^{+}(L_{k-1})\left[\ln\frac{L_k}{Q}\right]$$

$$+ \sum_{n=k+1}^{N}\left[\left[\frac{f^{-}(L_n) - f^{+}(L_{n-1})}{L_n - L_{n-1}}\right]\left[L_n - L_{n-1} + L_{n-1}\left[\ln\frac{L_{n-1}}{L_n}\right]\right] + f^{+}(L_{n-1})\left[\ln\frac{L_n}{L_{n-1}}\right]\right]$$

In this interval, where for all $k = 2, ..., N-1$, $L_{k-1} \leq Q \leq L_k$, we have

$$\begin{aligned}
F(Q) &= \left[\sum_{j=1}^{k-1}\left[\frac{f^{-}(L_j) + f^{+}(L_{j-1})}{2}\right]\left[L_j - L_{j-1}\right]\right] + \left[\frac{f^{-}(Q) + f^{+}(L_{k-1})}{2}\right][Q - L_{k-1}] \\
&= \left[\sum_{j=1}^{k-1}\left[\frac{f^{-}(L_j) + f^{+}(L_{j-1})}{2}\right]\left[L_j - L_{j-1}\right]\right] + \left[\frac{\left[\left[\frac{f^{-}(L_k) - f^{+}(L_{k-1})}{L_k - L_{k-1}}\right][Q - L_{k-1}] + f^{+}(L_{k-1})\right] + f^{+}(L_{k-1})}{2}\right][Q - L_{k-1}] \\
&= \left[\sum_{j=1}^{k-1}\left[\frac{f^{-}(L_j) + f^{+}(L_{j-1})}{2}\right]\left[L_j - L_{j-1}\right]\right] + \left[\frac{f^{-}(L_k) - f^{+}(L_{k-1})}{2[L_k - L_{k-1}]}\right][Q - L_{k-1}]^2 + f^{+}(L_{k-1})[Q - L_{k-1}]
\end{aligned}$$

and

$$
\begin{aligned}
\int_0^Q F(x)\,dx &= \int_0^{L_1}[F(x)]\,dx + \left[\sum_{j=1}^{k-2}\left[\int_{L_j}^{L_{j+1}}[F(x)]\,dx\right]\right] + \int_{L_{k-1}}^{Q} F(x)\,dx \\
&= \int_0^{L_1}\left[\left[\frac{f^-(x)+f^+(0)}{2}\right]x\right]dx + \left[\sum_{j=1}^{k-2}\left[\int_{L_j}^{L_{j+1}}\left[\left[\sum_{m=1}^{j}\left[\frac{f^-(L_m)+f^+(L_{m-1})}{2}\right][L_m-L_{m-1}]\right]+\left[\frac{f^-(x)+f^+(L_j)}{2}\right][x-L_j]\right]dx\right]\right] \\
&\quad + \int_{L_{k-1}}^{Q}\left[\left[\sum_{j=1}^{k-1}\left[\frac{f^-(L_j)+f^+(L_{j-1})}{2}\right][L_j-L_{j-1}]\right]+\left[\frac{f^-(L_k)-f^+(L_{k-1})}{2[L_k-L_{k-1}]}\right][x-L_{k-1}]^2+f^+(L_{k-1})[x-L_{k-1}]\right]dx \\
&= \int_0^{L_1}\left[\left[\frac{\left[\left[\frac{f^-(L_1)-f^+(0)}{L_1}\right]x+f^+(0)\right]+f^+(0)}{2}\right]x\right]dx \\
&\quad + \left[\sum_{j=1}^{k-2}\left[\int_{L_j}^{L_{j+1}}\left[\left[\sum_{m=1}^{j}\left[\frac{f^-(L_m)+f^+(L_{m-1})}{2}\right][L_m-L_{m-1}]\right]+\left[\frac{\left[\left[\frac{f^-(L_{j+1})-f^+(L_j)}{L_{j+1}-L_j}\right][x-L_j]+f^+(L_j)\right]+f^+(L_j)}{2}\right][x-L_j]\right]dx\right]\right] \\
&\quad + \left[\sum_{j=1}^{k-1}\left[\frac{f^-(L_j)+f^+(L_{j-1})}{2}\right][L_j-L_{j-1}]\right][Q-L_{k-1}]+\left[\frac{f^-(L_k)-f^+(L_{k-1})}{6[L_k-L_{k-1}]}\right][Q-L_{k-1}]^3+\frac{f^+(L_{k-1})}{2}[Q-L_{k-1}]^2 \\
&= \int_0^{L_1}\left[\frac{f^-(L_1)-f^+(0)}{2L_1}x^2+f^+(0)x\right]dx \\
&\quad + \left[\sum_{j=1}^{k-2}\left[\int_{L_j}^{L_{j+1}}\left[\left[\sum_{m=1}^{j}\left[\frac{f^-(L_m)+f^+(L_{m-1})}{2}\right][L_m-L_{m-1}]\right]+\left[\frac{f^-(L_{j+1})-f^+(L_j)}{2[L_{j+1}-L_j]}\right][x-L_j]^2+f^+(L_j)[x-L_j]\right]dx\right]\right] \\
&\quad + \left[\sum_{j=1}^{k-1}\left[\frac{f^-(L_j)+f^+(L_{j-1})}{2}\right][L_j-L_{j-1}]\right][Q-L_{k-1}]+\left[\frac{f^-(L_k)-f^+(L_{k-1})}{6[L_k-L_{k-1}]}\right][Q-L_{k-1}]^3+\frac{f^+(L_{k-1})}{2}[Q-L_{k-1}]^2 \\
&= \frac{f^-(L_1)-f^+(0)}{6}L_1^2+\frac{f^+(0)}{2}L_1^2 \\
&\quad + \left[\sum_{j=1}^{k-2}\left[\left[x\sum_{m=1}^{j}\left[\frac{f^-(L_m)+f^+(L_{m-1})}{2}\right][L_m-L_{m-1}]\right]+\left[\frac{f^-(L_{j+1})-f^+(L_j)}{6[L_{j+1}-L_j]}\right][x-L_j]^3+\frac{f^+(L_j)}{2}[x-L_j]^2\right]_{L_j}^{L_{j+1}}\right] \\
&\quad + \left[\sum_{j=1}^{k-1}\left[\frac{f^-(L_j)+f^+(L_{j-1})}{2}\right][L_j-L_{j-1}]\right][Q-L_{k-1}]+\left[\frac{f^-(L_k)-f^+(L_{k-1})}{6[L_k-L_{k-1}]}\right][Q-L_{k-1}]^3+\frac{f^+(L_{k-1})}{2}[Q-L_{k-1}]^2 \\
&= \frac{f^-(L_1)-f^+(0)}{6}L_1^2+\frac{f^+(0)}{2}L_1^2 \\
&\quad + \left[\sum_{j=1}^{k-2}\left[\left[[L_{j+1}-L_j]\sum_{m=1}^{j}\left[\frac{f^-(L_m)+f^+(L_{m-1})}{2}\right][L_m-L_{m-1}]\right]+\left[\frac{f^-(L_{j+1})-f^+(L_j)}{6[L_{j+1}-L_j]}\right][L_{j+1}-L_j]^3+\frac{f^+(L_j)}{2}[L_{j+1}-L_j]^2\right]\right] \\
&\quad + \left[\sum_{j=1}^{k-1}\left[\frac{f^-(L_j)+f^+(L_{j-1})}{2}\right][L_j-L_{j-1}]\right][Q-L_{k-1}]+\left[\frac{f^-(L_k)-f^+(L_{k-1})}{6[L_k-L_{k-1}]}\right][Q-L_{k-1}]^3+\frac{f^+(L_{k-1})}{2}[Q-L_{k-1}]^2
\end{aligned}
$$

and for $\frac{dE[P_k(Q,x)]}{dQ}$ we have:

$$
\begin{aligned}
\frac{dE\left[P_k(Q,x)\right]}{dQ} =& p-v-h_2t_2-\frac{h_1Q}{r}+\left[g-h_3t_3-p\right]F(Q)-\frac{h_4}{u}\int_0^Q F(x)\,dx-h_3t_3Q\int_Q^{\infty}\frac{1}{x}f(x)\,dx \\
=& p-v-h_2t_2-\frac{h_1Q}{r} \\
&+\left[g-h_3t_3-p\right]\Bigg[\Bigg[\sum_{j=1}^{k-1}\left[\frac{f^-\left(L_j\right)+f^+\left(L_{j-1}\right)}{2}\right]\left[L_j-L_{j-1}\right]\Bigg]+\left[\frac{f^-\left(L_k\right)-f^+\left(L_{k-1}\right)}{2\left[L_k-L_{k-1}\right]}\right]\left[Q-L_{k-1}\right]^2+f^+\left(L_{k-1}\right)\left[Q-L_{k-1}\right]\Bigg] \\
&-\frac{h_4}{u}\Bigg[\frac{f^-\left(L_1\right)-f^+(0)}{6}L_1^2+\frac{f^+(0)}{2}L_1^2 \\
&+\Bigg[\sum_{j=1}^{k-2}\Bigg[\Bigg[\left[L_{j+1}-L_j\right]\sum_{m=1}^{j}\left[\frac{f^-\left(L_m\right)+f^+\left(L_{m-1}\right)}{2}\right]\left[L_m-L_{m-1}\right]\Bigg]+\left[\frac{f^-\left(L_{j+1}\right)-f^+\left(L_j\right)}{6\left[L_{j+1}-L_j\right]}\right]\left[L_{j+1}-L_j\right]^3+\frac{f^+\left(L_j\right)}{2}\left[L_{j+1}-L_j\right]^2\Bigg]\Bigg] \\
&+\Bigg[\sum_{j=1}^{k-1}\left[\frac{f^-\left(L_j\right)+f^+\left(L_{j-1}\right)}{2}\right]\left[L_j-L_{j-1}\right]\Bigg]\left[Q-L_{k-1}\right]+\left[\frac{f^-\left(L_k\right)-f^+\left(L_{k-1}\right)}{6\left[L_k-L_{k-1}\right]}\right]\left[Q-L_{k-1}\right]^3+\frac{f^+\left(L_{k-1}\right)}{2}\left[Q-L_{k-1}\right]^2\Bigg] \\
&-h_3t_3Q\Bigg[\left[\frac{f^-\left(L_k\right)-f^+\left(L_{k-1}\right)}{L_k-L_{k-1}}\right]\left[L_k-Q+L_{k-1}\ln\frac{Q}{L_k}\right]+f^+\left(L_{k-1}\right)\left[\ln\frac{L_k}{Q}\right] \\
&+\sum_{n=k+1}^{N}\Bigg[\left[\frac{f^-\left(L_n\right)-f^+\left(L_{n-1}\right)}{L_n-L_{n-1}}\right]\left[L_n-L_{n-1}+L_{n-1}\left[\ln\frac{L_{n-1}}{L_n}\right]\right]+f^+\left(L_{n-1}\right)\left[\ln\frac{L_n}{L_{n-1}}\right]\Bigg]\Bigg]
\end{aligned}
$$

By using FOC, for all $k=2,\ldots,N-1$, we have,

$$
Q_k=\left\{Q \mid \frac{dE\left[P_k(Q,x)\right]}{dQ}=0\right\}
$$

Moreover, in this interval

$$\int_0^Q \left[\int_0^x F(x)\,dx\right] dx=\int_0^{L_1}\left[\int_0^x F(x)\,dx\right] dx+\sum_{i=1}^{k-2}\left[\int_{L_i}^{L_{i+1}}\left[\int_0^x F(x)\,dx\right] dx\right]+\int_{L_{k-1}}^{Q}\left[\int_0^x F(x)\,dx\right] dx$$

$$=\left[\frac{f^-(L_1)-f^+(0)}{24L_1}\right]L_1^4+\frac{f^+(0)}{6}L_1^3$$

$$+\sum_{i=1}^{k-2}\Bigg[\int_{L_i}^{L_{i+1}}\Bigg[\frac{f^-(L_1)-f^+(0)}{6}L_1^2+\frac{f^+(0)}{2}L_1^2$$

$$+\Bigg[\sum_{j=1}^{(i+1)-2}\Bigg[\Bigg[[L_{j+1}-L_j]\sum_{m=1}^{j}\left[\frac{f^-(L_m)+f^+(L_{m-1})}{2}\right][L_m-L_{m-1}]\Bigg]+\left[\frac{f^-(L_{j+1})-f^+(L_j)}{6[L_{j+1}-L_j]}\right][L_{j+1}-L_j]^3+\frac{f^+(L_j)}{2}[L_{j+1}-L_j]^2\Bigg]\Bigg]$$

$$+\Bigg[\sum_{j=1}^{(i+1)-1}\left[\frac{f^-(L_j)+f^+(L_{j-1})}{2}\right][L_j-L_{j-1}]\Bigg][x-L_{(i+1)-1}]$$

$$+\left[\frac{f^-(L_{(i+1)})-f^+(L_{(i+1)-1})}{6[L_{(i+1)}-L_{(i+1)-1}]}\right][x-L_{(i+1)-1}]^3+\frac{f^+(L_{(i+1)-1})}{2}[x-L_{(i+1)-1}]^2\Bigg]dx\Bigg]$$

$$+\int_{L_{k-1}}^{Q}\Bigg[\frac{f^-(L_1)-f^+(0)}{6}L_1^2+\frac{f^+(0)}{2}L_1^2$$

$$+\Bigg[\sum_{j=1}^{k-2}\Bigg[\Bigg[[L_{j+1}-L_j]\sum_{m=1}^{j}\left[\frac{f^-(L_m)+f^+(L_{m-1})}{2}\right][L_m-L_{m-1}]\Bigg]+\left[\frac{f^-(L_{j+1})-f^+(L_j)}{6[L_{j+1}-L_j]}\right][L_{j+1}-L_j]^3+\frac{f^+(L_j)}{2}[L_{j+1}-L_j]^2\Bigg]\Bigg]$$

$$+\Bigg[\sum_{j=1}^{k-1}\left[\frac{f^-(L_j)+f^+(L_{j-1})}{2}\right][L_j-L_{j-1}]\Bigg][x-L_{k-1}]+\left[\frac{f^-(L_k)-f^+(L_{k-1})}{6[L_k-L_{k-1}]}\right][x-L_{k-1}]^3+\frac{f^+(L_{k-1})}{2}[x-L_{k-1}]^2\Bigg]dx$$

$$=\left[\frac{f^-(L_1)-f^+(0)}{24}\right]L_1^3+\frac{f^+(0)}{6}L_1^3$$

$$+\sum_{i=1}^{k-2}\Bigg[\Bigg[\left[\frac{f^-(L_1)-f^+(0)}{6}L_1^2+\frac{f^+(0)}{2}L_1^2\right]x$$

$$+\Bigg[\sum_{j=1}^{i-1}\Bigg[\Bigg[[L_{j+1}-L_j]\sum_{m=1}^{j}\left[\frac{f^-(L_m)+f^+(L_{m-1})}{2}\right][L_m-L_{m-1}]\Bigg]+\left[\frac{f^-(L_{j+1})-f^+(L_j)}{6[L_{j+1}-L_j]}\right][L_{j+1}-L_j]^3+\frac{f^+(L_j)}{2}[L_{j+1}-L_j]^2\Bigg]\Bigg]x$$

$$+\Bigg[\sum_{j=1}^{i}\left[\frac{f^-(L_j)+f^+(L_{j-1})}{4}\right][L_j-L_{j-1}]\Bigg][x-L_i]^2+\left[\frac{f^-(L_{i+1})-f^+(L_i)}{24[L_{i+1}-L_i]}\right][x-L_i]^4+\frac{f^+(L_i)}{6}[x-L_i]^3\Bigg]_{L_i}^{L_{i+1}}\Bigg]$$

$$+\Bigg[\left[\frac{f^-(L_1)-f^+(0)}{6}L_1^2+\frac{f^+(0)}{2}L_1^2\right]x$$

$$+\Bigg[\sum_{j=1}^{k-2}\Bigg[\Bigg[[L_{j+1}-L_j]\sum_{m=1}^{j}\left[\frac{f^-(L_m)+f^+(L_{m-1})}{2}\right][L_m-L_{m-1}]\Bigg]+\left[\frac{f^-(L_{j+1})-f^+(L_j)}{6[L_{j+1}-L_j]}\right][L_{j+1}-L_j]^3+\frac{f^+(L_j)}{2}[L_{j+1}-L_j]^2\Bigg]\Bigg]x$$

$$+\Bigg[\sum_{j=1}^{k-1}\left[\frac{f^-(L_j)+f^+(L_{j-1})}{4}\right][L_j-L_{j-1}]\Bigg][x-L_{k-1}]^2+\left[\frac{f^-(L_k)-f^+(L_{k-1})}{24[L_k-L_{k-1}]}\right][x-L_{k-1}]^4+\frac{f^+(L_{k-1})}{6}[x-L_{k-1}]^3\Bigg]_{L_{k-1}}^{Q}$$

$$=\left[\frac{f^-(L_1)-f^+(0)}{24}\right]L_1^3+\frac{f^+(0)}{6}L_1^3$$

$$+\sum_{i=1}^{k-2}\Bigg[\left[\frac{f^-(L_1)-f^+(0)}{6}L_1^2+\frac{f^+(0)}{2}L_1^2\right][L_{i+1}-L_i]$$

$$+\Bigg[\sum_{j=1}^{i-1}\Bigg[\Bigg[[L_{j+1}-L_j]\sum_{m=1}^{j}\left[\frac{f^-(L_m)+f^+(L_{m-1})}{2}\right][L_m-L_{m-1}]\Bigg]+\left[\frac{f^-(L_{j+1})-f^+(L_j)}{6[L_{j+1}-L_j]}\right][L_{j+1}-L_j]^3+\frac{f^+(L_j)}{2}[L_{j+1}-L_j]^2\Bigg]\Bigg][L_{i+1}-L_i]$$

$$+\Bigg[\sum_{j=1}^{i}\left[\frac{f^-(L_j)+f^+(L_{j-1})}{4}\right][L_j-L_{j-1}]\Bigg][L_{i+1}-L_i]^2+\left[\frac{f^-(L_{i+1})-f^+(L_i)}{24[L_{i+1}-L_i]}\right][L_{i+1}-L_i]^4+\frac{f^+(L_i)}{6}[L_{i+1}-L_i]^3\Bigg]$$

$$+\left[\frac{f^-(L_1)-f^+(0)}{6}L_1^2+\frac{f^+(0)}{2}L_1^2\right][Q-L_{k-1}]$$

$$+\Bigg[\sum_{j=1}^{k-2}\Bigg[\Bigg[[L_{j+1}-L_j]\sum_{m=1}^{j}\left[\frac{f^-(L_m)+f^+(L_{m-1})}{2}\right][L_m-L_{m-1}]\Bigg]+\left[\frac{f^-(L_{j+1})-f^+(L_j)}{6[L_{j+1}-L_j]}\right][L_{j+1}-L_j]^3+\frac{f^+(L_j)}{2}[L_{j+1}-L_j]^2\Bigg]\Bigg][Q-L_{k-1}]$$

$$+\Bigg[\sum_{j=1}^{k-1}\left[\frac{f^-(L_j)+f^+(L_{j-1})}{4}\right][L_j-L_{j-1}]\Bigg][Q-L_{k-1}]^2+\left[\frac{f^-(L_k)-f^+(L_{k-1})}{24[L_k-L_{k-1}]}\right][Q-L_{k-1}]^4+\frac{f^+(L_{k-1})}{6}[Q-L_{k-1}]^3$$

and for $E\left[P_k\left(Q,x\right)\right]$ we have:

$$
\begin{aligned}
E\left[P_k\left(Q,x\right)\right] =& - vQ - \frac{h_1Q^2}{2r} - h_2Qt_2 + \left[p-g+\frac{h_3t_3}{2}\right]\left[QF\left(Q\right)-\int_0^Q F\left(x\right)dx\right] + \left[g-h_3t_3\right]QF\left(Q\right) \\
& - \frac{h_4}{u}\int_0^Q\left[\int_0^x F\left(x\right)dx\right]dx + pQ\left[1-F\left(Q\right)\right] - \frac{h_3t_3}{2}Q^2\int_Q^\infty \frac{1}{x}f\left(x\right)dx \\
=& - vQ - \frac{h_1Q^2}{2r} - h_2Qt_2 \\
& + \left[p-g+\frac{h_3t_3}{2}\right]\left[Q\left[\left[\sum_{j=1}^{k-1}\left[\frac{f^-\left(L_j\right)+f^+\left(L_{j-1}\right)}{2}\right]\left[L_j-L_{j-1}\right]\right]+\left[\frac{f^-\left(L_k\right)-f^+\left(L_{k-1}\right)}{2\left[L_k-L_{k-1}\right]}\right]\left[Q-L_{k-1}\right]^2+f^+\left(L_{k-1}\right)\left[Q-L_{k-1}\right]\right]\right. \\
& - \frac{f^-\left(L_1\right)-f^+\left(0\right)}{6}L_1^2 - \frac{f^+\left(0\right)}{2}L_1^2 \\
& - \left[\sum_{j=1}^{L_{k-2}}\left[\left[\left[L_{j+1}-L_j\right]\sum_{m=1}^{j}\left[\frac{f^-\left(L_m\right)+f^+\left(L_{m-1}\right)}{2}\right]\left[L_m-L_{m-1}\right]\right]+\left[\frac{f^-\left(L_{j+1}\right)-f^+\left(L_j\right)}{6\left[L_{j+1}-L_j\right]}\right]\left[L_{j+1}-L_j\right]^3+\frac{f^+\left(L_j\right)}{2}\left[L_{j+1}-L_j\right]^2\right]\right] \\
& - \left[\sum_{j=1}^{k-1}\left[\frac{f^-\left(L_j\right)+f^+\left(L_{j-1}\right)}{2}\right]\left[L_j-L_{j-1}\right]\right]\left[Q-L_{k-1}\right] - \left[\frac{f^-\left(L_k\right)-f^+\left(L_{k-1}\right)}{6\left[L_k-L_{k-1}\right]}\right]\left[Q-L_{k-1}\right]^3 - \frac{f^+\left(L_{k-1}\right)}{2}\left[Q-L_{k-1}\right]^2\Bigg] \\
& + \left[g-h_3t_3\right]Q\left[\left[\sum_{j=1}^{k-1}\left[\frac{f^-\left(L_j\right)+f^+\left(L_{j-1}\right)}{2}\right]\left[L_j-L_{j-1}\right]\right]+\left[\frac{f^-\left(L_k\right)-f^+\left(L_{k-1}\right)}{2\left[L_k-L_{k-1}\right]}\right]\left[Q-L_{k-1}\right]^2+f^+\left(L_{k-1}\right)\left[Q-L_{k-1}\right]\right] \\
& + pQ\left[1-\left[\left[\sum_{j=1}^{k-1}\left[\frac{f^-\left(L_j\right)+f^+\left(L_{j-1}\right)}{2}\right]\left[L_j-L_{j-1}\right]\right]+\left[\frac{f^-\left(L_k\right)-f^+\left(L_{k-1}\right)}{2\left[L_k-L_{k-1}\right]}\right]\left[Q-L_{k-1}\right]^2+f^+\left(L_{k-1}\right)\left[Q-L_{k-1}\right]\right]\right] \\
& - \frac{h_4}{u}\left[\left[\frac{f^-\left(L_1\right)-f^+\left(0\right)}{24}\right]L_1^3+\frac{f^+\left(0\right)}{6}L_1^3\right. \\
& + \sum_{i=1}^{k-2}\left[\left[\frac{f^-\left(L_1\right)-f^+\left(0\right)}{6}L_1^2+\frac{f^+\left(0\right)}{2}L_1^2\right]\left[L_{i+1}-L_i\right]\right. \\
& + \left[\sum_{j=1}^{i-1}\left[\left[\left[L_{j+1}-L_j\right]\sum_{m=1}^{j}\left[\frac{f^-\left(L_m\right)+f^+\left(L_{m-1}\right)}{2}\right]\left[L_m-L_{m-1}\right]\right]+\left[\frac{f^-\left(L_{j+1}\right)-f^+\left(L_j\right)}{6\left[L_{j+1}-L_j\right]}\right]\left[L_{j+1}-L_j\right]^3+\frac{f^+\left(L_j\right)}{2}\left[L_{j+1}-L_j\right]^2\right]\right]\left[L_{i+1}-L_i\right] \\
& + \left.\left[\sum_{j=1}^{i}\left[\frac{f^-\left(L_j\right)+f^+\left(L_{j-1}\right)}{4}\right]\left[L_j-L_{j-1}\right]\right]\left[L_{i+1}-L_i\right]^2+\left[\frac{f^-\left(L_{i+1}\right)-f^+\left(L_i\right)}{24\left[L_{i+1}-L_i\right]}\right]\left[L_{i+1}-L_i\right]^4+\frac{f^+\left(L_i\right)}{6}\left[L_{i+1}-L_i\right]^3\right] \\
& + \left[\frac{f^-\left(L_1\right)-f^+\left(0\right)}{6}L_1^2+\frac{f^+\left(0\right)}{2}L_1^2\right]\left[Q-L_{k-1}\right] \\
& + \left[\sum_{j=1}^{k-2}\left[\left[\left[L_{j+1}-L_j\right]\sum_{m=1}^{j}\left[\frac{f^-\left(L_m\right)+f^+\left(L_{m-1}\right)}{2}\right]\left[L_m-L_{m-1}\right]\right]+\left[\frac{f^-\left(L_{j+1}\right)-f^+\left(L_j\right)}{6\left[L_{j+1}-L_j\right]}\right]\left[L_{j+1}-L_j\right]^3+\frac{f^+\left(L_j\right)}{2}\left[L_{j+1}-L_j\right]^2\right]\right]\left[Q-L_{k-1}\right] \\
& + \left.\left[\sum_{j=1}^{k-1}\left[\frac{f^-\left(L_j\right)+f^+\left(L_{j-1}\right)}{4}\right]\left[L_j-L_{j-1}\right]\right]\left[Q-L_{k-1}\right]^2+\left[\frac{f^-\left(L_k\right)-f^+\left(L_{k-1}\right)}{24\left[L_k-L_{k-1}\right]}\right]\left[Q-L_{k-1}\right]^4+\frac{f^+\left(L_{k-1}\right)}{6}\left[Q-L_{k-1}\right]^3\right] \\
& - \frac{h_3t_3}{2}Q^2\left[\left[\frac{f^-\left(L_k\right)-f^+\left(L_{k-1}\right)}{L_k-L_{k-1}}\right]\left[L_k-Q+L_{k-1}\ln\frac{Q}{L_k}\right]+f^+\left(L_{k-1}\right)\left[\ln\frac{L_k}{Q}\right]\right. \\
& + \left.\sum_{n=k+1}^{N}\left[\left[\frac{f^-\left(L_n\right)-f^+\left(L_{n-1}\right)}{L_n-L_{n-1}}\right]\left[L_n-L_{n-1}+L_{n-1}\left[\ln\frac{L_{n-1}}{L_n}\right]\right]+f^+\left(L_{n-1}\right)\left[\ln\frac{L_n}{L_{n-1}}\right]\right]\right]
\end{aligned}
$$

Now, let $L_{N-1}\leq Q\leq L_N$. In this case,

$$\int_Q^{\infty}\frac{1}{x}f(x)\,dx=\int_Q^{L_N}\frac{1}{x}f(x)\,dx$$

$$=\int_Q^{L_N}\frac{1}{x}\left[\left[\frac{f^-(L_N)-f^+(L_{N-1})}{L_N-L_{N-1}}\right][x-L_{N-1}]+f^+(L_{N-1})\right]dx$$

$$=\left[\frac{f^-(L_N)-f^+(L_{N-1})}{L_N-L_{N-1}}\right][L_N-L_{N-1}\ln L_N]+f^+(L_{N-1})\ln L_N-\left[\frac{f^-(L_N)-f^+(L_{N-1})}{L_N-L_{N-1}}\right][Q-L_{N-1}\ln Q]-f^+(L_{N-1})\ln Q$$

$$=\left[\frac{f^-(L_N)-f^+(L_{N-1})}{L_N-L_{N-1}}\right]\left[L_N-Q+L_{N-1}\ln\frac{Q}{L_N}\right]+f^+(L_{N-1})\left[\ln\frac{L_N}{Q}\right]$$

In this interval $(L_{N-1}\le Q\le L_N)$, we can also follow the same procedure we used for the kth interval, to derive $F(Q)$ and $\int_0^Q F(x)\,dx$ as follows.

$$\begin{aligned}F(Q)&=\left[\sum_{j=1}^{N-1}\left[\frac{f^-(L_j)+f^+(L_{j-1})}{2}\right][L_j-L_{j-1}]\right]+\left[\frac{f^-(Q)+f^+(L_{N-1})}{2}\right][Q-L_{N-1}]\\&=\left[\sum_{j=1}^{N-1}\left[\frac{f^-(L_j)+f^+(L_{j-1})}{2}\right][L_j-L_{j-1}]\right]+\left[\frac{f^-(L_N)-f^+(L_{N-1})}{2[L_N-L_{N-1}]}\right][Q-L_{N-1}]^2+f^+(L_{N-1})[Q-L_{N-1}]\end{aligned}$$

and

$$\begin{aligned}\int_0^Q F(x)\,dx&=\int_0^{L_1}[F(x)]\,dx+\left[\sum_{j=1}^{N-2}\left[\int_{L_j}^{L_{j+1}}[F(x)]\,dx\right]\right]+\int_{L_{N-1}}^Q F(x)\,dx\\&=\frac{f^-(L_1)-f^+(0)}{6}L_1^2+\frac{f^+(0)}{2}L_1^2\\&+\left[\sum_{j=1}^{N-2}\left[\left[[L_{j+1}-L_j]\sum_{m=1}^{j}\left[\frac{f^-(L_m)+f^+(L_{m-1})}{2}\right][L_m-L_{m-1}]\right]+\left[\frac{f^-(L_{j+1})-f^+(L_j)}{6[L_{j+1}-L_j]}\right][L_{j+1}-L_j]^3+\frac{f^+(L_j)}{2}[L_{j+1}-L_j]^2\right]\right]\\&+\left[\sum_{j=1}^{N-1}\left[\frac{f^-(L_j)+f^+(L_{j-1})}{2}\right][L_j-L_{j-1}]\right][Q-L_{N-1}]+\left[\frac{f^-(L_N)-f^+(L_{N-1})}{6[L_N-L_{N-1}]}\right][Q-L_{N-1}]^3+\frac{f^+(L_{N-1})}{2}[Q-L_{N-1}]^2\end{aligned}$$

and for $\frac{dE[P_N(Q,x)]}{dQ}$ we have:

$$
\begin{aligned}
\frac{dE[P_N(Q,x)]}{dQ} =& p - v - h_2t_2 - \frac{h_1Q}{r} + [g-h_3t_3-p]F(Q) - \frac{h_4}{u}\int_0^Q F(x)\,dx - h_3t_3Q\int_Q^\infty \frac{1}{x}f(x)\,dx \\
=& p - v - h_2t_2 - \frac{h_1 Q}{r} \\
&+[g-h_3t_3-p]\left[\left[\sum_{j=1}^{N-1}\left[\frac{f^-(L_j)+f^+(L_{j-1})}{2}\right][L_j-L_{j-1}]\right] + \left[\frac{f^-(L_N)-f^+(L_{N-1})}{2[L_N-L_{N-1}]}\right][Q-L_{N-1}]^2 + f^+(L_{N-1})[Q-L_{N-1}]\right] \\
&-\frac{h_4}{u}\left[\frac{f^-(L_1)-f^+(0)}{6}L_1^2 + \frac{f^+(0)}{2}L_1^2\right. \\
&+\left[\sum_{j=1}^{N-2}\left[\left[[L_{j+1}-L_j]\sum_{m=1}^{j}\left[\frac{f^-(L_m)+f^+(L_{m-1})}{2}\right][L_m-L_{m-1}]\right]+\left[\frac{f^-(L_{j+1})-f^+(L_j)}{6[L_{j+1}-L_j]}\right][L_{j+1}-L_j]^3+\frac{f^+(L_j)}{2}[L_{j+1}-L_j]^2\right]\right] \\
&+\left.\left[\sum_{j=1}^{N-1}\left[\frac{f^-(L_j)+f^+(L_{j-1})}{2}\right][L_j-L_{j-1}]\right][Q-L_{N-1}] + \left[\frac{f^-(L_N)-f^+(L_{N-1})}{6[L_N-L_{N-1}]}\right][Q-L_{N-1}]^3+\frac{f^+(L_{N-1})}{2}[Q-L_{N-1}]^2\right] \\
&-h_3t_3Q\left[\left[\frac{f^-(L_N)-f^+(L_{N-1})}{L_N-L_{N-1}}\right]\left[L_N-Q+L_{N-1}\ln\frac{Q}{L_N}\right]+f^+(L_{N-1})\left[\ln\frac{L_N}{Q}\right]\right]
\end{aligned}
$$

By using FOC, we have,

$$Q_N = \left\{Q \mid \frac{dE[P_N(Q,x)]}{dQ} = 0\right\}$$

Moreover, in this interval $(L_{N-1} \leq Q \leq L_N)$, we can again follow the same procedure we used for the kth interval, to derive $\int_0^Q [\int_0^x F(x)\,dx]\,dx$ as below.

$$
\begin{aligned}
\int_0^Q\left[\int_0^x F(x)\,dx\right]dx =& \int_0^{L_1}\left[\int_0^x F(x)\,dx\right]dx + \sum_{i=1}^{N-2}\left[\int_{L_i}^{L_{i+1}}\left[\int_0^x F(x)\,dx\right]dx\right] + \int_{L_{N-1}}^{Q}\left[\int_0^x F(x)\,dx\right]dx \\
=& \left[\frac{f^-(L_1)-f^+(0)}{24}\right]L_1^3 + \frac{f^+(0)}{6}L_1^3 \\
&+\sum_{i=1}^{N-2}\left[\left[\frac{f^-(L_1)-f^+(0)}{6}L_1^2+\frac{f^+(0)}{2}L_1^2\right][L_{i+1}-L_i]\right. \\
&+\left[\sum_{j=1}^{i-1}\left[\left[[L_{j+1}-L_j]\sum_{m=1}^{j}\left[\frac{f^-(L_m)+f^+(L_{m-1})}{2}\right][L_m-L_{m-1}]\right]+\left[\frac{f^-(L_{j+1})-f^+(L_j)}{6[L_{j+1}-L_j]}\right][L_{j+1}-L_j]^3+\frac{f^+(L_j)}{2}[L_{j+1}-L_j]^2\right]\right][L_{i+1}-L_i] \\
&+\left.\left[\sum_{j=1}^{i}\left[\frac{f^-(L_j)+f^+(L_{j-1})}{4}\right][L_j-L_{j-1}]\right][L_{i+1}-L_i]^2+\left[\frac{f^-(L_{i+1})-f^+(L_i)}{24[L_{i+1}-L_i]}\right][L_{i+1}-L_i]^4+\frac{f^+(L_i)}{6}[L_{i+1}-L_i]^3\right] \\
&+\left[\frac{f^-(L_1)-f^+(0)}{6}L_1^2+\frac{f^+(0)}{2}L_1^2\right][Q-L_{N-1}] \\
&+\left[\sum_{j=1}^{N-2}\left[\left[[L_{j+1}-L_j]\sum_{m=1}^{j}\left[\frac{f^-(L_m)+f^+(L_{m-1})}{2}\right][L_m-L_{m-1}]\right]+\left[\frac{f^-(L_{j+1})-f^+(L_j)}{6[L_{j+1}-L_j]}\right][L_{j+1}-L_j]^3+\frac{f^+(L_j)}{2}[L_{j+1}-L_j]^2\right]\right][Q-L_{N-1}] \\
&+\left[\sum_{j=1}^{N-1}\left[\frac{f^-(L_j)+f^+(L_{j-1})}{4}\right][L_j-L_{j-1}]\right][Q-L_{N-1}]^2+\left[\frac{f^-(L_N)-f^+(L_{N-1})}{24[L_N-L_{N-1}]}\right][Q-L_{N-1}]^4+\frac{f^+(L_{N-1})}{6}[Q-L_{N-1}]^3
\end{aligned}
$$

and for $E\left[P_N\left(Q,x\right)\right]$ we have:

$$
\begin{aligned}
E\left[P_N\left(Q,x\right)\right] =& -vQ-\frac{h_1Q^2}{2r}-h_2Qt_2+\left[p-g+\frac{h_3t_3}{2}\right]\left[QF\left(Q\right)-\int_0^Q F\left(x\right)dx\right]+\left[g-h_3t_3\right]QF\left(Q\right)\\
& -\frac{h_4}{u}\int_0^Q\left[\int_0^x F\left(x\right)dx\right]dx+pQ\left[1-F\left(Q\right)\right]-\frac{h_3t_3}{2}Q^2\int_Q^\infty\frac{1}{x}f\left(x\right)dx\\
=& -vQ-\frac{h_1Q^2}{2r}-h_2Qt_2\\
& +\left[p-g+\frac{h_3t_3}{2}\right]\left[Q\left[\left[\sum_{j=1}^{N-1}\left[\frac{f^-\left(L_j\right)+f^+\left(L_{j-1}\right)}{2}\right]\left[L_j-L_{j-1}\right]\right]+\left[\frac{f^-\left(L_N\right)-f^+\left(L_{N-1}\right)}{2\left[L_N-L_{N-1}\right]}\right]\left[Q-L_{N-1}\right]^2+f^+\left(L_{N-1}\right)\left[Q-L_{N-1}\right]\right]\right.\\
& -\frac{f^-\left(L_1\right)-f^+\left(0\right)}{6}L_1^2-\frac{f^+\left(0\right)}{2}L_1^2\\
& -\left[\sum_{j=1}^{N-2}\left[\left[\left[L_{j+1}-L_j\right]\sum_{m=1}^{j}\left[\frac{f^-\left(L_m\right)+f^+\left(L_{m-1}\right)}{2}\right]\left[L_m-L_{m-1}\right]\right]+\left[\frac{f^-\left(L_{j+1}\right)-f^+\left(L_j\right)}{6\left[L_{j+1}-L_j\right]}\right]\left[L_{j+1}-L_j\right]^3+\frac{f^+\left(L_j\right)}{2}\left[L_{j+1}-L_j\right]^2\right]\right]\\
& -\left[\sum_{j=1}^{N-1}\left[\frac{f^-\left(L_j\right)+f^+\left(L_{j-1}\right)}{2}\right]\left[L_j-L_{j-1}\right]\right]\left[Q-L_{N-1}\right]-\left[\frac{f^-\left(L_N\right)-f^+\left(L_{N-1}\right)}{6\left[L_N-L_{N-1}\right]}\right]\left[Q-L_{N-1}\right]^3-\frac{f^+\left(L_{N-1}\right)}{2}\left[Q-L_{N-1}\right]^2\Bigg]\\
& +\left[g-h_3t_3\right]Q\left[\left[\sum_{j=1}^{N-1}\left[\frac{f^-\left(L_j\right)+f^+\left(L_{j-1}\right)}{2}\right]\left[L_j-L_{j-1}\right]\right]+\left[\frac{f^-\left(L_N\right)-f^+\left(L_{N-1}\right)}{2\left[L_N-L_{N-1}\right]}\right]\left[Q-L_{N-1}\right]^2+f^+\left(L_{N-1}\right)\left[Q-L_{N-1}\right]\right]\\
& -\frac{h_4}{u}\left[\left[\frac{f^-\left(L_1\right)-f^+\left(0\right)}{24}\right]L_1^3+\frac{f^+\left(0\right)}{6}L_1^3\right.\\
& +\sum_{i=1}^{N-2}\left[\left[\frac{f^-\left(L_1\right)-f^+\left(0\right)}{6}L_1^2+\frac{f^+\left(0\right)}{2}L_1^2\right]\left[L_{i+1}-L_i\right]\right.\\
& +\left[\sum_{j=1}^{i-1}\left[\left[\left[L_{j+1}-L_j\right]\sum_{m=1}^{j}\left[\frac{f^-\left(L_m\right)+f^+\left(L_{m-1}\right)}{2}\right]\left[L_m-L_{m-1}\right]\right]+\left[\frac{f^-\left(L_{j+1}\right)-f^+\left(L_j\right)}{6\left[L_{j+1}-L_j\right]}\right]\left[L_{j+1}-L_j\right]^3+\frac{f^+\left(L_j\right)}{2}\left[L_{j+1}-L_j\right]^2\right]\right]\left[L_{i+1}-L_i\right]\\
& +\left[\sum_{j=1}^{i}\left[\frac{f^-\left(L_j\right)+f^+\left(L_{j-1}\right)}{4}\right]\left[L_j-L_{j-1}\right]\right]\left[L_{i+1}-L_i\right]^2+\left[\frac{f^-\left(L_{i+1}\right)-f^+\left(L_i\right)}{24\left[L_{i+1}-L_i\right]}\right]\left[L_{i+1}-L_i\right]^4+\frac{f^+\left(L_i\right)}{6}\left[L_{i+1}-L_i\right]^3\Bigg]\\
& +\left[\frac{f^-\left(L_1\right)-f^+\left(0\right)}{6}L_1^2+\frac{f^+\left(0\right)}{2}L_1^2\right]\left[Q-L_{N-1}\right]\\
& +\left[\sum_{j=1}^{N-2}\left[\left[\left[L_{j+1}-L_j\right]\sum_{m=1}^{j}\left[\frac{f^-\left(L_m\right)+f^+\left(L_{m-1}\right)}{2}\right]\left[L_m-L_{m-1}\right]\right]+\left[\frac{f^-\left(L_{j+1}\right)-f^+\left(L_j\right)}{6\left[L_{j+1}-L_j\right]}\right]\left[L_{j+1}-L_j\right]^3+\frac{f^+\left(L_j\right)}{2}\left[L_{j+1}-L_j\right]^2\right]\right]\left[Q-L_{N-1}\right]\\
& +\left[\sum_{j=1}^{N-1}\left[\frac{f^-\left(L_j\right)+f^+\left(L_{j-1}\right)}{4}\right]\left[L_j-L_{j-1}\right]\right]\left[Q-L_{N-1}\right]^2+\left[\frac{f^-\left(L_N\right)-f^+\left(L_{N-1}\right)}{24\left[L_N-L_{N-1}\right]}\right]\left[Q-L_{N-1}\right]^4+\frac{f^+\left(L_{N-1}\right)}{6}\left[Q-L_{N-1}\right]^3\Bigg]\\
& +pQ\left[1-\left[\left[\sum_{j=1}^{N-1}\left[\frac{f^-\left(L_j\right)+f^+\left(L_{j-1}\right)}{2}\right]\left[L_j-L_{j-1}\right]\right]+\left[\frac{f^-\left(L_N\right)-f^+\left(L_{N-1}\right)}{2\left[L_N-L_{N-1}\right]}\right]\left[Q-L_{N-1}\right]^2+f^+\left(L_{N-1}\right)\left[Q-L_{N-1}\right]\right]\right]\\
& -\frac{h_3t_3}{2}Q^2\left[\left[\frac{f^-\left(L_N\right)-f^+\left(L_{N-1}\right)}{L_N-L_{N-1}}\right]\left[L_N-Q+L_{N-1}\ln\frac{Q}{L_N}\right]+f^+\left(L_{N-1}\right)\left[\ln\frac{L_N}{Q}\right]\right]
\end{aligned}
$$

Also, let $L_N \leq Q$. In this case,

$$\int_Q^\infty \frac{1}{x}f\left(x\right)dx = 0$$

In this interval $F(Q)=1$, and

$$
\begin{aligned}
\int_0^Q F(x)\,dx =& \int_0^{L_1}[F(x)]\,dx+\left[\sum_{j=1}^{N-1}\left[\int_{L_j}^{L_{j+1}}[F(x)]\,dx\right]\right]+\int_{L_N}^{Q}F(x)\,dx\\
=& \int_0^{L_1}\left[\left[\frac{f^-(x)+f^+(0)}{2}\right]x\right]dx+\left[\sum_{j=1}^{N-1}\left[\int_{L_j}^{L_{j+1}}\left[\left[\sum_{m=1}^{j}\left[\frac{f^-(L_m)+f^+(L_{m-1})}{2}\right][L_m-L_{m-1}]\right]+\left[\frac{f^-(x)+f^+(L_j)}{2}\right][x-L_j]\right]dx\right]\right]+Q-L_N\\
=& \int_0^{L_1}\left[\left[\frac{\left[\left[\frac{f^-(L_1)-f^+(0)}{L_1}\right]x+f^+(0)\right]+f^+(0)}{2}\right]x\right]dx\\
&+\left[\sum_{j=1}^{N-1}\left[\int_{L_j}^{L_{j+1}}\left[\left[\sum_{m=1}^{j}\left[\frac{f^-(L_m)+f^+(L_{m-1})}{2}\right][L_m-L_{m-1}]\right]+\left[\frac{\left[\left[\frac{f^-(L_{j+1})-f^+(L_j)}{L_{j+1}-L_j}\right][x-L_j]+f^+(L_j)\right]+f^+(L_j)}{2}\right][x-L_j]\right]dx\right]\right]+Q-L_N\\
=& \int_0^{L_1}\left[\frac{f^-(L_1)-f^+(0)}{2L_1}x^2+f^+(0)x\right]dx\\
&+\left[\sum_{j=1}^{N-1}\left[\int_{L_j}^{L_{j+1}}\left[\left[\sum_{m=1}^{j}\left[\frac{f^-(L_m)+f^+(L_{m-1})}{2}\right][L_m-L_{m-1}]\right]+\left[\frac{f^-(L_{j+1})-f^+(L_j)}{2[L_{j+1}-L_j]}\right][x-L_j]^2+f^+(L_j)[x-L_j]\right]dx\right]\right]+Q-L_N\\
=& \frac{f^-(L_1)-f^+(0)}{6}L_1^2+\frac{f^+(0)}{2}L_1^2\\
&+\left[\sum_{j=1}^{N-1}\left[\left[x\sum_{m=1}^{j}\left[\frac{f^-(L_m)+f^+(L_{m-1})}{2}\right][L_m-L_{m-1}]\right]+\left[\frac{f^-(L_{j+1})-f^+(L_j)}{6[L_{j+1}-L_j]}\right][x-L_j]^3+\frac{f^+(L_j)}{2}[x-L_j]^2\right]_{L_j}^{L_{j+1}}\right]+Q-L_N\\
=& \frac{f^-(L_1)-f^+(0)}{6}L_1^2+\frac{f^+(0)}{2}L_1^2\\
&+\left[\sum_{j=1}^{N-1}\left[\left[[L_{j+1}-L_j]\sum_{m=1}^{j}\left[\frac{f^-(L_m)+f^+(L_{m-1})}{2}\right][L_m-L_{m-1}]\right]+\left[\frac{f^-(L_{j+1})-f^+(L_j)}{6[L_{j+1}-L_j]}\right][L_{j+1}-L_j]^3+\frac{f^+(L_j)}{2}[L_{j+1}-L_j]^2\right]\right]+Q-L_N
\end{aligned}
$$

and for $\frac{dE[P_{N+1}(Q,x)]}{dQ}$ we have:

$$
\begin{aligned}
\frac{dE[P_{N+1}(Q,x)]}{dQ} =& p-v-h_2t_2-\frac{h_1Q}{r}+[g-h_3t_3-p]F(Q)-\frac{h_4}{u}\int_0^Q F(x)\,dx-h_3t_3Q\int_Q^{\infty}\frac{1}{x}f(x)\,dx\\
=& p-v-h_2t_2-\frac{h_1Q}{r}+[g-h_3t_3-p]-\frac{h_4}{u}\left[\frac{f^-(L_1)-f^+(0)}{6}L_1^2+\frac{f^+(0)}{2}L_1^2\right]\\
&-\frac{h_4}{u}\left[\sum_{j=1}^{N-1}\left[\left[[L_{j+1}-L_j]\sum_{m=1}^{j}\left[\frac{f^-(L_m)+f^+(L_{m-1})}{2}\right][L_m-L_{m-1}]\right]+\left[\frac{f^-(L_{j+1})-f^+(L_j)}{6[L_{j+1}-L_j]}\right][L_{j+1}-L_j]^3+\frac{f^+(L_j)}{2}[L_{j+1}-L_j]^2\right]\right]\\
&-\frac{h_4}{u}Q+\frac{h_4}{u}L_N\\
=& -v-h_2t_2-\left[\frac{h_1}{r}+\frac{h_4}{u}\right]Q+g-h_3t_3-\frac{h_4}{u}\left[\frac{f^-(L_1)-f^+(0)}{6}L_1^2+\frac{f^+(0)}{2}L_1^2\right]\\
&-\frac{h_4}{u}\left[\sum_{j=1}^{N-1}\left[\left[[L_{j+1}-L_j]\sum_{m=1}^{j}\left[\frac{f^-(L_m)+f^+(L_{m-1})}{2}\right][L_m-L_{m-1}]\right]+\left[\frac{f^-(L_{j+1})-f^+(L_j)}{6[L_{j+1}-L_j]}\right][L_{j+1}-L_j]^3+\frac{f^+(L_j)}{2}[L_{j+1}-L_j]^2\right]\right]+\frac{h_4}{u}L_N
\end{aligned}
$$

By using FOC we have,

$$Q_{N+1} = \frac{1}{\frac{h_1}{r} + \frac{h_4}{u}} \left[-v - h_2 t_2 + g - h_3 t_3 - \frac{h_4}{u} \left[\frac{f^-(L_1) - f^+(0)}{6} L_1^2 + \frac{f^+(0)}{2} L_1^2 \right] \right.$$
$$\left. - \frac{h_4}{u} \left[\sum_{j=1}^{N-1} \left[\left[[L_{j+1} - L_j] \sum_{m=1}^{j} \left[\frac{f^-(L_m) + f^+(L_{m-1})}{2} \right] [L_m - L_{m-1}] \right] + \left[\frac{f^-(L_{j+1}) - f^+(L_j)}{6 [L_{j+1} - L_j]} \right] [L_{j+1} - L_j]^3 + \frac{f^+(L_j)}{2} [L_{j+1} - L_j]^2 \right] \right] + \frac{h_4}{u} L_N \right]$$

Moreover, in this interval

$$
\begin{aligned}
\int_0^Q \left[\int_0^x F(x)\,dx\right]dx &= \int_0^{L_1}\left[\int_0^x F(x)\,dx\right]dx + \left[\sum_{i=1}^{N-1}\left[\int_{L_i}^{L_{i+1}}\left[\int_0^x F(x)\,dx\right]dx\right]\right] + \int_{L_N}^{Q}\left[\int_0^x F(x)\,dx\right]dx \\
&= \left[\frac{f^-(L_1)-f^+(0)}{24}\right]L_1^3 + \frac{f^+(0)}{6}L_1^3 \\
&\quad + \sum_{i=1}^{N-2}\Bigg[\left[\frac{f^-(L_1)-f^+(0)}{6}L_1^2+\frac{f^+(0)}{2}L_1^2\right][L_{i+1}-L_i] \\
&\quad + \Bigg[\sum_{j=1}^{i-1}\Bigg[\Bigg[[L_{j+1}-L_j]\sum_{m=1}^{j}\left[\frac{f^-(L_m)+f^+(L_{m-1})}{2}\right][L_m-L_{m-1}]\Bigg]+\left[\frac{f^-(L_{j+1})-f^+(L_j)}{6[L_{j+1}-L_j]}\right][L_{j+1}-L_j]^3+\frac{f^+(L_j)}{2}[L_{j+1}-L_j]^2\Bigg]\Bigg][L_{i+1}-L_i] \\
&\quad + \Bigg[\sum_{j=1}^{i}\left[\frac{f^-(L_j)+f^+(L_{j-1})}{4}\right][L_j-L_{j-1}]\Bigg][L_{i+1}-L_i]^2+\left[\frac{f^-(L_{i+1})-f^+(L_i)}{24[L_{i+1}-L_i]}\right][L_{i+1}-L_i]^4+\frac{f^+(L_i)}{6}[L_{i+1}-L_i]^3\Bigg] \\
&\quad + \left[\frac{f^-(L_1)-f^+(0)}{6}L_1^2+\frac{f^+(0)}{2}L_1^2\right][L_N-L_{N-1}] \\
&+ \Bigg[\sum_{j=1}^{N-2}\Bigg[\Bigg[[L_{j+1}-L_j]\sum_{m=1}^{j}\left[\frac{f^-(L_m)+f^+(L_{m-1})}{2}\right][L_m-L_{m-1}]\Bigg]+\left[\frac{f^-(L_{j+1})-f^+(L_j)}{6[L_{j+1}-L_j]}\right][L_{j+1}-L_j]^3+\frac{f^+(L_j)}{2}[L_{j+1}-L_j]^2\Bigg]\Bigg][L_N-L_{N-1}] \\
&\quad + \Bigg[\sum_{j=1}^{N-1}\left[\frac{f^-(L_j)+f^+(L_{j-1})}{4}\right][L_j-L_{j-1}]\Bigg][L_N-L_{N-1}]^2+\left[\frac{f^-(L_N)-f^+(L_{N-1})}{24[L_N-L_{N-1}]}\right][L_N-L_{N-1}]^4+\frac{f^+(L_{N-1})}{6}[L_N-L_{N-1}]^3 \\
&\quad + \int_{L_N}^{Q}\Bigg[\frac{f^-(L_1)-f^+(0)}{6}L_1^2+\frac{f^+(0)}{2}L_1^2 \\
&+ \Bigg[\sum_{j=1}^{N-1}\Bigg[\Bigg[[L_{j+1}-L_j]\sum_{m=1}^{j}\left[\frac{f^-(L_m)+f^+(L_{m-1})}{2}\right][L_m-L_{m-1}]\Bigg]+\left[\frac{f^-(L_{j+1})-f^+(L_j)}{6[L_{j+1}-L_j]}\right][L_{j+1}-L_j]^3+\frac{f^+(L_j)}{2}[L_{j+1}-L_j]^2\Bigg]\Bigg]+x-L_N\Bigg]dx \\
&= \left[\frac{f^-(L_1)-f^+(0)}{24}\right]L_1^3 + \frac{f^+(0)}{6}L_1^3 \\
&\quad + \sum_{i=1}^{N-1}\Bigg[\left[\frac{f^-(L_1)-f^+(0)}{6}L_1^2+\frac{f^+(0)}{2}L_1^2\right][L_{i+1}-L_i] \\
&\quad + \Bigg[\sum_{j=1}^{i-1}\Bigg[\Bigg[[L_{j+1}-L_j]\sum_{m=1}^{j}\left[\frac{f^-(L_m)+f^+(L_{m-1})}{2}\right][L_m-L_{m-1}]\Bigg]+\left[\frac{f^-(L_{j+1})-f^+(L_j)}{6[L_{j+1}-L_j]}\right][L_{j+1}-L_j]^3+\frac{f^+(L_j)}{2}[L_{j+1}-L_j]^2\Bigg]\Bigg][L_{i+1}-L_i] \\
&\quad + \Bigg[\sum_{j=1}^{i}\left[\frac{f^-(L_j)+f^+(L_{j-1})}{4}\right][L_j-L_{j-1}]\Bigg][L_{i+1}-L_i]^2+\left[\frac{f^-(L_{i+1})-f^+(L_i)}{24[L_{i+1}-L_i]}\right][L_{i+1}-L_i]^4+\frac{f^+(L_i)}{6}[L_{i+1}-L_i]^3\Bigg] \\
&\quad + \Bigg[\left[\frac{f^-(L_1)-f^+(0)}{6}L_1^2+\frac{f^+(0)}{2}L_1^2\right]x \\
&\quad + \Bigg[\sum_{j=1}^{N-1}\Bigg[\Bigg[[L_{j+1}-L_j]\sum_{m=1}^{j}\left[\frac{f^-(L_m)+f^+(L_{m-1})}{2}\right][L_m-L_{m-1}]\Bigg]+\left[\frac{f^-(L_{j+1})-f^+(L_j)}{6[L_{j+1}-L_j]}\right][L_{j+1}-L_j]^3+\frac{f^+(L_j)}{2}[L_{j+1}-L_j]^2\Bigg]\Bigg]x \\
&\quad + \frac{[x-L_N]^2}{2}\Bigg]_{L_N}^{Q} \\
&= \left[\frac{f^-(L_1)-f^+(0)}{24}\right]L_1^3 + \frac{f^+(0)}{6}L_1^3 \\
&\quad + \sum_{i=1}^{N-1}\Bigg[\left[\frac{f^-(L_1)-f^+(0)}{6}L_1^2+\frac{f^+(0)}{2}L_1^2\right][L_{i+1}-L_i] \\
&\quad + \Bigg[\sum_{j=1}^{i-1}\Bigg[\Bigg[[L_{j+1}-L_j]\sum_{m=1}^{j}\left[\frac{f^-(L_m)+f^+(L_{m-1})}{2}\right][L_m-L_{m-1}]\Bigg]+\left[\frac{f^-(L_{j+1})-f^+(L_j)}{6[L_{j+1}-L_j]}\right][L_{j+1}-L_j]^3+\frac{f^+(L_j)}{2}[L_{j+1}-L_j]^2\Bigg]\Bigg][L_{i+1}-L_i] \\
&\quad + \Bigg[\sum_{j=1}^{i}\left[\frac{f^-(L_j)+f^+(L_{j-1})}{4}\right][L_j-L_{j-1}]\Bigg][L_{i+1}-L_i]^2+\left[\frac{f^-(L_{i+1})-f^+(L_i)}{24[L_{i+1}-L_i]}\right][L_{i+1}-L_i]^4+\frac{f^+(L_i)}{6}[L_{i+1}-L_i]^3\Bigg] \\
&\quad + \left[\frac{f^-(L_1)-f^+(0)}{6}L_1^2+\frac{f^+(0)}{2}L_1^2\right][Q-L_N] \\
&\quad + \Bigg[\sum_{j=1}^{N-1}\Bigg[\Bigg[[L_{j+1}-L_j]\sum_{m=1}^{j}\left[\frac{f^-(L_m)+f^+(L_{m-1})}{2}\right][L_m-L_{m-1}]\Bigg]+\left[\frac{f^-(L_{j+1})-f^+(L_j)}{6[L_{j+1}-L_j]}\right][L_{j+1}-L_j]^3+\frac{f^+(L_j)}{2}[L_{j+1}-L_j]^2\Bigg]\Bigg][Q-L_N] \\
&\quad + \frac{[Q-L_N]^2}{2}
\end{aligned}
$$

and for $E\left[P_{N+1}(Q,x)\right]$ we have:

$$
\begin{aligned}
E\left[P_{N+1}(Q,x)\right] = & -vQ-\frac{h_1Q^2}{2r}-h_2Qt_2+\left[p-g+\frac{h_3t_3}{2}\right]\left[QF(Q)-\int_0^Q F(x)\,dx\right]+\left[g-h_3t_3\right]QF(Q) \\
& -\frac{h_4}{u}\int_0^Q\left[\int_0^x F(x)\,dx\right]dx+pQ\left[1-F(Q)\right]-\frac{h_3t_3}{2}Q^2\int_Q^\infty\frac{1}{x}f(x)\,dx \\
= & -vQ-\frac{h_1Q^2}{2r}-h_2Qt_2+\left[p-g+\frac{h_3t_3}{2}\right]\left[Q-\frac{f^-(L_1)-f^+(0)}{6}L_1^2-\frac{f^+(0)}{2}L_1^2\right. \\
& -\left[\sum_{j=1}^{N-1}\left[\left[\left[L_{j+1}-L_j\right]\sum_{m=1}^{j}\left[\frac{f^-(L_m)+f^+(L_{m-1})}{2}\right]\left[L_m-L_{m-1}\right]\right]+\left[\frac{f^-\left(L_{j+1}\right)-f^+\left(L_j\right)}{6\left[L_{j+1}-L_j\right]}\right]\left[L_{j+1}-L_j\right]^3+\frac{f^+\left(L_j\right)}{2}\left[L_{j+1}-L_j\right]^2\right]\right]-Q+L_N\Bigg] \\
& +\left[g-h_3t_3\right]Q-\frac{h_4}{u}\left[\left[\frac{f^-(L_1)-f^+(0)}{24}\right]L_1^3+\frac{f^+(0)}{6}L_1^3+\sum_{i=1}^{N-1}\left[\left[\frac{f^-(L_1)-f^+(0)}{6}L_1^2+\frac{f^+(0)}{2}L_1^2\right]\left[L_{i+1}-L_i\right]\right.\right. \\
& +\left[\sum_{j=1}^{i-1}\left[\left[\left[L_{j+1}-L_j\right]\sum_{m=1}^{j}\left[\frac{f^-(L_m)+f^+(L_{m-1})}{2}\right]\left[L_m-L_{m-1}\right]\right]+\left[\frac{f^-\left(L_{j+1}\right)-f^+\left(L_j\right)}{6\left[L_{j+1}-L_j\right]}\right]\left[L_{j+1}-L_j\right]^3+\frac{f^+\left(L_j\right)}{2}\left[L_{j+1}-L_j\right]^2\right]\right]\left[L_{i+1}-L_i\right] \\
& +\left[\sum_{j=1}^{i}\left[\frac{f^-\left(L_j\right)+f^+\left(L_{j-1}\right)}{4}\right]\left[L_j-L_{j-1}\right]\right]\left[L_{i+1}-L_i\right]^2+\left[\frac{f^-(L_{i+1})-f^+(L_i)}{24\left[L_{i+1}-L_i\right]}\right]\left[L_{i+1}-L_i\right]^4+\frac{f^+(L_i)}{6}\left[L_{i+1}-L_i\right]^3\Bigg] \\
& +\left[\frac{f^-(L_1)-f^+(0)}{6}L_1^2+\frac{f^+(0)}{2}L_1^2\right]\left[Q-L_N\right] \\
& +\left[\sum_{j=1}^{N-1}\left[\left[\left[L_{j+1}-L_j\right]\sum_{m=1}^{j}\left[\frac{f^-(L_m)+f^+(L_{m-1})}{2}\right]\left[L_m-L_{m-1}\right]\right]+\left[\frac{f^-\left(L_{j+1}\right)-f^+\left(L_j\right)}{6\left[L_{j+1}-L_j\right]}\right]\left[L_{j+1}-L_j\right]^3+\frac{f^+\left(L_j\right)}{2}\left[L_{j+1}-L_j\right]^2\right]\right]\left[Q-L_N\right] \\
& \left.+\frac{\left[Q-L_N\right]^2}{2}\right]
\end{aligned}
$$

Finally, since $\{0, L_1, ..., L_N\}$ are the endpoints of the given intervals, they may also be the optimal solution. We also assume that $\overline{Q}$ is a big number such that $\overline{Q} > Q_{N+1}$. $\overline{Q}$ is introduced here since $Q = \infty$ is not possible to be achieved, although it can theoretically be a potential optimal solution of the problem.■

1.9.4 Proof of Proposition 3

First let us remind you that given a realization of scenario j, one of the following cases will determine the total profit related to scenario j depending on whether or not x_j is greater than Q.

Case 1: $Q \leq x_j$

$$pQ - vQ - \frac{h_1}{2r}Q^2 - h_2 t_2 Q - \frac{h_3 t_3 Q^2}{2x_j}$$

Case 2: $Q > x_j$

$$px_j + g\left[Q - x_j\right] - vQ - \frac{h_1}{2r}Q^2 - h_2 t_2 Q - h_3 t_3 \left[Q - \frac{x_j}{2}\right] - \frac{h_4}{2u}\left[Q - x_j\right]^2$$

If $0 \leq Q < x_1$, then all demand scenarios will be greater than the production quantity Q, hence, the second stage profit of scenario j is equal to $pQ - \frac{h_3 t_3 Q^2}{2x}$ for all $j = 1,..,N$. Therefore, $z_1(Q) = -vQ - \frac{h_1}{2r}Q^2 - h_2 t_2 Q + \sum_{j=1}^{N} k_j \left[pQ - \frac{h_3 t_3 Q^2}{2x_j}\right]$; and the optimal production quantity is either 0 or the solution of $\frac{dz_1}{dQ} = 0$. So, we have:

$$\frac{dz_1}{dQ} = -v - \frac{h_1}{r}Q - h_2 t_2 + p - h_3 t_3 Q \sum_{j=1}^{N} \left[\frac{k_j}{x_j}\right] = -v - h_2 t_2 + p - Q\left[\frac{h_1}{r} + h_3 t_3 \sum_{j=1}^{N} \left[\frac{k_j}{x_j}\right]\right] = 0$$

which results in:

$$Q = \frac{p - v - h_2 t_2}{\frac{h_1}{r} + h_3 t_3 \sum_{j=1}^{N} \left[\frac{k_j}{x_j}\right]}$$

Therefore, if $0 \leq Q < x_1$, then the optimal total profit in this interval is either $z_1(0)$ or $z_1\left(\frac{p - v - h_2 t_2}{\frac{h_1}{r} + h_3 t_3 \sum_{j=1}^{N} \left[\frac{k_j}{x_j}\right]}\right)$. Please note that since $z_1(x_1) = z_2(x_1)$, we only consider x_1 in calculating $z_2(Q)$ which will come next.

$\forall n = 2, \ldots, N$: if $x_{n-1} \leq Q < x_n$, then for all $j = 1,..,n-1$ all demand scenarios will be less than or equal to the production quantity Q, hence, the second stage profit of scenario j is equal to $px_j + g\left[Q - x_j\right] - h_3 t_3 \left[Q - \frac{x_j}{2}\right] - \frac{h_4}{2u}\left[Q - x_j\right]^2$. Also, for all $j = n,..,N$, the demand scenarios will be greater than the production quantity Q, hence, the second stage profit of scenario j is equal to $pQ - \frac{h_3 t_3 Q^2}{2x_j}$. Therefore,

$$z_n(Q) = -vQ - \frac{h_1}{2r}Q^2 - h_2 t_2 Q + \sum_{j=1}^{n-1} k_j \left[px_j + g\left[Q - x_j\right] - h_3 t_3 \left[Q - \frac{x_j}{2}\right] - \frac{h_4}{2u}\left[Q - x_j\right]^2\right] + \sum_{j=n}^{N} k_j \left[pQ - \frac{h_3 t_3 Q^2}{2x_j}\right]$$

and the optimal production quantity is either x_{n-1} or the solution of $\frac{dz_n}{dQ}=0$. So, we have:

$$\begin{aligned}\frac{dz_n}{dQ} &= -v-\frac{h_1}{r}Q-h_2t_2+\sum_{j=1}^{n-1}k_j\left[g-h_3t_3-\frac{h_4}{u}\left[Q-x_j\right]\right]+\sum_{j=n}^{N}k_j\left[p-\frac{h_3t_3}{x_j}Q\right]\\ &= -v-h_2t_2+\sum_{j=1}^{n-1}k_j\left[g-h_3t_3+\frac{h_4}{u}x_j\right]+\sum_{j=n}^{N}k_jp-\left[\frac{h_1}{r}+\frac{h_4}{u}\sum_{j=1}^{n-1}k_j+\sum_{j=n}^{N}k_j\left[\frac{h_3t_3}{x_j}\right]\right]Q\\ &=0\end{aligned}$$

which results in:

$$Q=\frac{-v-h_2t_2+[g-h_3t_3]\sum_{j=1}^{n-1}k_j+\frac{h_4}{u}\sum_{j=1}^{n-1}k_jx_j+p\sum_{j=n}^{N}k_j}{\frac{h_1}{r}+\frac{h_4}{u}\sum_{j=1}^{n-1}k_j+h_3t_3\sum_{j=n}^{N}\frac{k_j}{x_j}}$$

Therefore, for all $n=2,\dots,N$, if $x_{n-1}\le Q<x_n$, then the optimal total profit in this interval is either $z_n\left(x_{n-1}\right)$ or $z_n\left(\frac{-v-h_2t_2+[g-h_3t_3]\sum_{j=1}^{n-1}k_j+\frac{h_4}{u}\sum_{j=1}^{n-1}k_jx_j+p\sum_{j=n}^{N}k_j}{\frac{h_1}{r}+\frac{h_4}{u}\sum_{j=1}^{n-1}k_j+h_3t_3\sum_{j=n}^{N}\frac{k_j}{x_j}}\right)$. It is worth noting that $z_n\left(x_n\right)=z_{n+1}\left(x_n\right)$, hence, we only consider x_n when examining $z_{n+1}\left(Q\right)$.

If $x_N\le Q\le\overline{Q}$, then for all $j=1,\dots,N$, the demand scenarios will be less than or equal to the production quantity Q, hence, the second stage profit of scenario j is equal to $px_j+g\left[Q-x_j\right]-h_3t_3\left[Q-\frac{x_j}{2}\right]-\frac{h_4}{2u}\left[Q-x_j\right]^2$. Therefore,

$$z_{N+1}\left(Q\right)=-vQ-\frac{h_1}{2r}Q^2-h_2t_2Q+\sum_{j=1}^{N}k_j\left[px_j+g\left[Q-x_j\right]-h_3t_3\left[Q-\frac{x_j}{2}\right]-\frac{h_4}{2u}\left[Q-x_j\right]^2\right]$$

and in this interval the optimal production quantity can be x_N, $\overline{Q}$ or the solution of $\frac{dz_{N+1}}{dQ}=0$. So, we have:

$$
\begin{aligned}
\frac{dz_{N+1}}{dQ} =& -v - \frac{h_1}{r}Q - h_2t_2 + \sum_{j=1}^{N} k_j \left[g - h_3t_3 - \frac{h_4}{u}\left[Q - x_j\right]\right] \\
=& -v - \left[\frac{h_1}{r} + \frac{h_4}{u}\sum_{j=1}^{N} k_j\right] Q - h_2t_2 + \left[g - h_3t_3\right]\sum_{j=1}^{N} k_j + \frac{h_4}{u}\sum_{j=1}^{N} k_jx_j \\
=& 0
\end{aligned}
$$

which results in:

$$
Q = \frac{-v - h_2t_2 + \left[g - h_3t_3\right]\sum_{j=1}^{N} k_j + \frac{h_4}{u}\sum_{j=1}^{N} k_jx_j}{\frac{h_1}{r} + \frac{h_4}{u}\sum_{j=1}^{N} k_j}
$$

Therefore, the optimal total profit in this interval can be $z_{N+1}\left(x_N\right)$, $z_{N+1}\left(\overline{Q}\right)$, or $z_{N+1}\left(\frac{-v-h_2t_2+\left[g-h_3t_3\right]\sum_{j=1}^{N} k_j+\frac{h_4}{u}\sum_{j=1}^{N} k_jx_j}{\frac{h_1}{r}+\frac{h_4}{u}\sum_{j=1}^{N} k_j}\right)$.

Since the union of the sets $0 \leq Q < x_1$, $x_{n-1} \leq Q < x_n : \forall n = 2,\ldots,N$, and $x_N \leq Q \leq \overline{Q}$, is the set $0 \leq Q \leq \overline{Q}$, we can conclude that the optimal solution to the original problem is the one that maximizes the following problem:

$$
\begin{aligned}
\max \Bigg\{ & z_1\left(0\right), z_1\left(\frac{p - v - h_2t_2}{\frac{h_1}{r} + h_3t_3\sum_{j=1}^{N}\left[\frac{k_j}{x_j}\right]}\right), \\
& z_n\left(x_{n-1}\right) : \forall n = 2..N, z_n\left(\frac{-v - h_2t_2 + \left[g - h_3t_3\right]\sum_{j=1}^{n-1} k_j + \frac{h_4}{u}\sum_{j=1}^{n-1} k_jx_j + p\sum_{j=n}^{N} k_j}{\frac{h_1}{r} + \frac{h_4}{u}\sum_{j=1}^{n-1} k_j + h_3t_3\sum_{j=n}^{N}\frac{k_j}{x_j}}\right) : \forall n = 2..N, \\
& z_{N+1}\left(x_N\right), z_{N+1}\left(\overline{Q}\right), z_{N+1}\left(\frac{-v - h_2t_2 + \left[g - h_3t_3\right]\sum_{j=1}^{N} k_j + \frac{h_4}{u}\sum_{j=1}^{N} k_jx_j}{\frac{h_1}{r} + \frac{h_4}{u}\sum_{j=1}^{N} k_j}\right) \Bigg\} \blacksquare
\end{aligned}
$$

The above problem obtains a maximum value over $2N + 3$ values. Hence, it is quite efficient even for very large number of scenarios.

Chapter 2

Newsvendor Problem under Nonlinear Production and Demand with Holding Cost

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Abstract

This paper studies the newsvendor problem in the context of a supply chain where holding costs are incurred over time. The supply chain consists of four phases including production, transportation, as well as the two selling phases as in the standard newsvendor model, i.e., a regular and a discount season. The cumulative production and demand functions are assumed to have a general nonlinear form over the time of their corresponding seasons and their holding cost, which depends on the amount of time and quantity that an item

is stored during each season, are taken into consideration in the newsvendor model. We first mathematically derive the total holding cost of the production and demand phases and then provide the solution process of the stochastic problem in its general form, under discrete distribution, by partially leveraging the solution process of the problem under linear production and demand functions in a former study (Ghaniabadi et al. 2023). To shed light on the applicability of the general model and its solution, we use the general model and solution to derive the holding costs and solution procedure under discrete distribution, given Wright's (Wright 1936) model for the production and Bass (Bass 1969) model for the demand phase, which are well-known and able to represent a variety of functions. The corresponding numerical experiments are then performed over synthetic and real datasets, where the demand function of the first selling season is estimated from real data.

2.1 Introduction

The newsvendor model is one of the most well-studied problems in the literature of inventory control. Holding costs play a major role in a supply chain and consequently it is important to have a comprehensive consideration of such costs in an inventory management problem. Nevertheless, the standard newsvendor model does not take into consideration the holding costs which occur over time during various phases of a supply chain. In this paper, we study a supply chain of four phases which include production, shipping, regular and discount selling seasons, where the holding cost of the inventory over these phases are taken into account (Ghaniabadi *et al.* 2023). While the inventory during the transportation period remains constant (Ghaniabadi *et al.* 2023), for the production and the two selling seasons, we consider nonlinear cumulative functions for the amount of inventory over time. The goal is to obtain an optimal quantity for production, so that the expected profit of the supply chain is maximized. Ghaniabadi *et al.* (2023) study the same problem while only considering the linear production and demand functions over time. In practical situations, however, this is not always the case and we can encounter a nonlinear demand function over time. For instance, when a new product is introduced to the market, at the beginning (the product introduction phase) and at the end (after the product saturation) of the first selling season, we may have lower demand than in the middle of the first selling season. Moreover, the production function can also be nonlinear, for instance, due to the learning effects in production. The nonlinearities in production and demand functions impact the amount of incurred holding cost. Hence, in this article, we model the newsvendor problem under general production and demand functions which are nonlinear over time and formulate their corresponding holding costs. We then provide a solution procedure for the problem under the discrete distribution by extending the solution framework of Ghaniabadi *et al.* (2023) which was developed under the same distribution for the case of linear production and demand functions. In order to illustrate the applicability of our model and solution process, we derive the holding costs of the production and demand phases under Wright's (Wright 1936) and Bass (Bass 1969) models, respectively, using the developed general model and provide its solution via the proposed general solution procedure. Both the models of Wright (1936) and Bass (1969) are well-known in the literature and are flexible which enables them to model a variety of functions by adjusting their parameters. We also conduct numerical experiments, with synthetic and real datasets, for the problem under the Wright (1936) and Bass (1969) models (for production and demand, respectively) and compare the profitability of the optimal solution with solutions for the basic newsvendor model without considering holding costs, the approximations based on the basic newsvendor problem with consideration of holding costs, and the problem under the linear case taken from Ghaniabadi *et al.* (2023). The results of the experiments show that the novel model under nonlinear production and demand, brings forth solutions that, on average, have higher expected profit than the solutions of simpler models examined by Ghaniabadi *et al.* (2023), evaluated under the nonlinear case. Moreover, in contrast to the results of Ghaniabadi *et al.* (2023), there are instances where the approximations based on the basic newsvendor model provide solutions with higher expected profits than solutions of the linear production and demand case proposed by Ghaniabadi *et al.* (2023). Nevertheless, considering all our experiments as a whole, the results indicate that the linear case still generally provides better solutions than the approximate solutions based on the basic newsvendor model, when the nonlinear cumulative production function (in our experiments according to the Wright's model) is approximated as a linear function with a specific constant production rate, while the nonlinear cumulative demand function for the discount period (in our experiments based on the Bass model) is approximated as a linear function with a specific constant demand rate.

Given the fact that holding costs are generally considered in the newsvendor literature in a limited way (Ghaniabadi *et al.* 2023), besides our paper, some other studies aimed at taking holding costs into consideration using functions that represent the real-world conditions more accurately than the standard newsvendor problem, especially in regards to considering the holding costs of inventory over time. Matsuo (1990), Chen and Chuang (2000), Tang *et al.* (2018), Schlapp *et al.* (2022), and Ghaniabadi *et al.* (2023) consider such holding costs in the newsvendor problem, which we outline in the next section.

We organize this paper as follows. We review the relevant literature in Section 2. In Section 3, the new problem is formulated and the holding costs functions under general nonlinear cumulative production and demand functions are derived for the production and the two demand seasons. The solution procedure for the stochastic model of the general problem under the discrete distribution is presented in Section 4. Section 5 studies the problem under the Wright's model (Wright 1936) for the cumulative production function over time and Bass model (Bass 1969) for the cumulative demand functions over time for the regular and discount selling periods, and derives the corresponding holding cost functions and the solution procedures based on the findings of the general case. The results of the numerical experiments, and conclusions are given in Sections 6 and 7, respectively.

2.2 Literature review

In the traditional newsvendor literature, holding costs are only taken into account in a very limited way, by considering that the holding costs for the overstocked items are part of the overage costs. Ghaniabadi *et al.* (2023) review some of the papers in this area, e.g., Eppen (1979), Urban (2002), Levi *et al.* (2007), and Maggioni *et al.* (2019). In particular, Urban (2002) study a newsvendor model where the demand is a function of the stock amount displayed on the shelf, and demand rate is constant (as is the assumption of Ghaniabadi *et al.* 2023) up to the point the shelf is refilled, and the cumulative demand function becomes nonlinear (although they do not consider a general nonlinear function, unlike our paper) with a decreasing demand rate up to the end of the period. The holding cost is incurred only for the excess inventory at the end of the period. Ghaniabadi *et al.* (2023) also review some other papers (Pal *et al.* 2015 and Wang *et al.* 2019) which consider the holding cost in a different way, although they still do not examine the holding cost of the stock amount over a phase. Pal *et al.* (2015) consider a holding cost which is a nonlinear function of the order quantity, for the newsvendor problem with customer balking, where backlogging is partially allowed and only the mean and variance of the demand distribution are known. Moreover, Wang *et al.* (2019) study a multi-product newsvendor model, where the total holding cost of the order quantities of all products, must not exceed a certain threshold.

Only a small number of papers in the newsvendor literature (including this paper) consider the holding cost as a result of holding inventory over a period of time, which we outline as follows.

Matsuo (1990) studies a multi-item newsvendor problem motivated by production and selling of fashion products where the goal is to find the optimal sequence and amount for production for families of products. As explained by Ghaniabadi *et al.* (2023), the holding cost studied by Matsuo (1990), is a linear function of the production amount and is equivalent to the holding cost of the transportation period considered by Ghaniabadi *et al.* (2023). We also take into account the same holding cost used by Ghaniabadi *et al.* (2023) for the transportation phase. The problem of Matsuo (1990) is the continuous-time version of the problem studied by Bitran *et al.* (1986) which considers discrete time periods and their holding cost.

Chen and Chuang (2000) examine a problem more complex than the standard newsvendor model where, in addition to the optimal order quantity, the optimal time of purchasing needs to be determined, while a discount is applied for an early procurement, and there is a holding cost from the order time until the order is satisfied. As noted by Ghaniabadi *et al.* (2023), the holding cost in the transportation period of the article by Ghaniabadi *et al.* (2023), which is also considered in our paper, is equivalent to the holding cost of Chen and Chuang (2000), if the time between order and demand satisfaction is fixed and equal to the length of the transportation period.

Tang *et al.* (2018) assume a constant demand rate for the regular selling phase and consider its holding cost, while also proposing an approximation based on the standard newsvendor model and examining its solution in the new model according to the objective value. Their holding cost function is equivalent to the holding cost of our model in the first selling phase assuming a linear cumulative demand (in the same way explained by Ghaniabadi *et al.* 2023).

Schlapp *et al.* (2022) examine a newsvendor problem where timing and quantity of inventory are decision variables, the start and duration of the selling season are uncertain, the availability of inventory may be before or after the start of the selling phase, and there is holding cost from the start time of inventory availability, till the end of the selling period.

The holding cost of the third period in our paper would be equivalent to the one proposed by Schlapp *et al.* (2022) if, in their paper, the inventory timing is at the start of a regular selling phase which has a known start and end point in time (i.e., a deterministic regular selling period, which they also examine as a special case). Furthermore, given a fixed inventory timing which coincides with the start time of our transportation period (before the selling starts), and a known duration for the regular selling period, the holding cost of both our transportation and first-selling periods become equivalent to the holding cost in the paper of Schlapp *et al.* (2022). The same equivalencies with Schlapp *et al.* (2022) were also explained by Ghaniabadi *et al.* (2023) for the linear case. Schlapp *et al.* (2022) provide some characteristics of the optimal solution, however, for numerical studies, they use an approximate solution approach, for three specific demand rates, two of which would result in a nonlinear cumulative demand function.

Ghaniabadi *et al.* (2023) proposed an optimal approach for the linear case of our problem, and in this paper, we provide an optimal solution approach for the general nonlinear case. Ghaniabadi *et al.* (2023) take into consideration the holding cost of a supply chain of four gradual phases consisting of production, transportation, a regular and a discount selling phase, while assuming a linear function for cumulative production and demand. Our model is an extension of their problem to general nonlinear cumulative production and demand functions. Given constant production and demand rates (i.e., linear cumulative production and demand functions) our model becomes the same as the one studied by Ghaniabadi *et al.* (2023). The holding cost of our transportation phase is the same as the one used by Ghaniabadi *et al.* (2023), and given linear cumulative production and demand functions, the holding cost in other periods would also be the same as the ones incorporated by Ghaniabadi *et al.* (2023). We incorporate the same assumptions made by Ghaniabadi *et al.* (2023) where the duration of the second and third phases (i.e., transportation and regular selling periods) are known, and the length of the first and fourth phases (i.e., the production and discount periods) are variable and a higher production quantity, and also a higher stock amount at the beginning of the fourth phase, result in a larger duration for these two periods, respectively, based on how the stock level changes over time. Ghaniabadi *et al.* (2023) also propose new approximations via transforming the basic newsvendor problem for the first, third and fourth phases and selecting the best one (while considering an approximation by Tang et al. 2018 for the regular selling period) for each period. They then examine the expected profits of solutions of their model, the approximations and the basic newsvendor model, according to their problem. We use the same numerical experiments of Ghaniabadi *et al.* (2023) under the discrete distribution, to examine our new model with the solutions obtained by Ghaniabadi *et al.* (2023). Ghaniabadi *et al.* (2023) also propose optimal solution procedures for their problem given a general piece-wise linear distribution and discrete distribution for the demand. The solution approach of our paper is also an extension of the solution procedure of Ghaniabadi *et al.* (2023) for discrete distribution, to the case of general nonlinear cumulative production and demand functions. Only under linear cumulative production and demand functions, our solution approach becomes the same as the solution approach of Ghaniabadi *et al.* (2023) for the discrete distribution case.

2.3 Problem formulation

In this section, we formulate the newsvendor problem with quantity-and-time-dependent holding cost and nonlinear production and demand functions. Throughout this paper, for the standard newsvendor model, we use the same notation as Silver *et al.* (1998) and Ghaniabadi *et al.* (2023). Moreover, our notation for the unit holding cost in each period, the demand scenarios and their probabilities, and also the maximum production amount, are the same as the ones used by Ghaniabadi *et al.* (2023).

We first formulate the total holding cost function for the production phase. We use $S_1(t)$ to denote the total production amount up to time t during the production phase. For the special case of the linear production (Ghaniabadi *et al.* 2023), we have $S_1(t) = rt$ where r is a constant production rate, $t \in [0, t_1]$, and $t_1 = \frac{Q}{r}$.

Proposition 4. *The total holding cost over the production phase under the nonlinear production function is equal to*

$$h_1 \left[\int_0^{\left\{t | t \geq 0, t = S_1^{-1}(Q)\right\}} S_1(t)\, dt \right].$$

Proof. The average inventory during the production period is $\frac{\int_0^{t_1} S_1(t)dt}{t_1}$ where $t_1 = \{t | t \geq 0, Q = S_1(t)\} = \left\{t | t \geq 0, t = S_1^{-1}(Q)\right\}$. Hence, the average inventory is $\frac{\int_0^{\left\{t | t \geq 0, t = S_1^{-1}(Q)\right\}} S_1(t)dt}{t_1}$ and we will have the following total holding cost over the production phase.

$$h_1 \frac{\int_0^{\left\{t | t \geq 0, t = S_1^{-1}(Q)\right\}} S_1(t)\, dt}{t_1} t_1$$

which is the multiplication of h_1, the average inventory, and t_1, and can be further simplified as follows.

$$h_1 \left[\int_0^{\left\{t | t \geq 0, t = S_1^{-1}(Q)\right\}} S_1(t)\, dt \right]. \blacksquare$$

We next formulate the total holding cost function when demand is greater than the production quantity. A summary of the notations for this case (i.e., $x \geq Q$), is presented as below.

- $D_3(t)$: The total demand up to time t of the first selling season when the demand $x = 1$, where $D_3(0) = 0$ and $D_3(t_3) = 1$. As an example, for the special case of the linear demand case (Tang *et al.* 2018, Ghaniabadi *et al.* 2023), $D_3(t) = \frac{t}{t_3}$ where

$t \in [0, t_3]$. For any given cumulative demand function $\Gamma_3(t)$ (with the assumption of $\Gamma_3(0) = 0$), we can derive $D_3(t)$ as follows: $D_3(t) = \frac{\Gamma_3(t)}{\Gamma_3(t_3)}$, which results in $D_3(0) = 0$, and $D_3(t_3) = 1$.

- $X_3(t) = xD_3(t)$: Total demand up to time t of the first selling season, where $X_3(0) = 0$ and $X_3(t_3) = x$. Given this definition of $X_3(t)$, $D_3(t)$ is the standardized $X_3(t)$.
- $S_3(t) = Q - xD_3(t)$: Net stock amount at time t of the first selling season, where $S_3(0) = Q$, $S_3\left(D_3^{-1}\left(\frac{Q}{x}\right)\right) = 0$.
- $S_4(t) = 0$: Stock amount at time t of the second selling season

Proposition 5. *The total holding cost over the first selling season for a given x and under the nonlinear demand function when $x \geq Q$, is equal to*

$$h_3\left[Q\left\{t|t \geq 0, t = D_3^{-1}\left(\frac{Q}{x}\right)\right\} - x\int_0^{\left\{t|t \geq 0, t = D_3^{-1}\left(\frac{Q}{x}\right)\right\}} D_3(t)\,dt\right].$$

Proof. Let $S_3(t)$ be the stock amount at time t of the first selling season and let Q be the produced quantity. Given $D_3(t)$ as defined above, and x as the predicted demand for the future second selling season, we can derive $S_3(t)$ as follows.

$$S_3(t) = Q - xD_3(t)$$

Then, we note that when the stock amount during the first selling season becomes zero, we have $S_3(t) = Q - xD_3(t) = 0$. Therefore, at $t = \left\{t|t \geq 0, t = D_3^{-1}\left(\frac{Q}{x}\right)\right\}$ we reach the zero stock level during the first selling season. We can illustrate the stock level over four periods of the problem for the first case where the demand is greater than the produced quantity (i.e., $x \geq Q$) using Figure 2.1 (under the linear production and demand, this figure will be the same as the one presented by Ghaniabadi *et al.* 2023 for the case of $x \geq Q$).

We can now formulate the average inventory during this period as

$\frac{\int_0^{\left\{t|t\geq 0,t=D_3^{-1}\left(\frac{Q}{x}\right)\right\}}[Q-xD_3(t)]dt}{\left\{t|t\geq 0,t=D_3^{-1}\left(\frac{Q}{x}\right)\right\}}$ which is equal to $\frac{Q\left\{t|t\geq 0,t=D_3^{-1}\left(\frac{Q}{x}\right)\right\}-x\int_0^{\left\{t|t\geq 0,t=D_3^{-1}\left(\frac{Q}{x}\right)\right\}}D_3(t)dt}{\left\{t|t\geq 0,t=D_3^{-1}\left(\frac{Q}{x}\right)\right\}}$.
Hence, we will have the following total holding cost over the first selling season.

$$h_3\frac{Q\left\{t|t\geq 0,t=D_3^{-1}\left(\frac{Q}{x}\right)\right\}-x\int_0^{\left\{t|t\geq 0,t=D_3^{-1}\left(\frac{Q}{x}\right)\right\}}D_3\left(t\right)dt}{\left\{t|t\geq 0,t=D_3^{-1}\left(\frac{Q}{x}\right)\right\}}\left\{t|t\geq 0,t=D_3^{-1}\left(\frac{Q}{x}\right)\right\}$$

which is the multiplication of h_3, the average inventory, and the time during which we have on-hand inventory, and can be further simplified as follows.

$$h_3\left[Q\left\{t|t\geq 0,t=D_3^{-1}\left(\frac{Q}{x}\right)\right\}-x\int_0^{\left\{t|t\geq 0,t=D_3^{-1}\left(\frac{Q}{x}\right)\right\}}D_3\left(t\right)dt\right].\blacksquare$$

We now formulate the total holding cost function where the demand is less than or equal to the production quantity. A summary of the notations for this case (i.e., $x\leq Q$), is presented as follows.

- $D_3\left(t\right)$: as defined for the case of $x\geq Q$.
- $X_3\left(t\right)=xD_3\left(t\right)$: as defined for the case of $x\geq Q$.
- $S_3\left(t\right)=Q-xD_3\left(t\right)$: Net stock amount at time t of the first selling season, where $S_3\left(0\right)=Q$, and $S_3\left(t_3\right)=Q-x$.
- $S_4\left(t\right)=\left[Q-x\right]-X_4\left(t\right)$: Stock amount at time t of the second selling season, where $S_4\left(0\right)=Q-x$ and $S_4\left(t_4\right)=0$.
- $X_4\left(t\right)$: Total demand up to time t of the second selling season, where $X_4\left(0\right)=0$ and $X_4\left(t_4\right)=Q-x$. As an example, for the special case of the linear demand case (Ghaniabadi *et al.* 2023), $X_4\left(t\right)=ut$ where u is a constant, $t\in\left[0,t_4\right]$, and $t_4=\left[\frac{Q-x}{u}\right]$.

Proposition 6. *The total holding cost over the first and second selling seasons for a given x and under the nonlinear demand function when $x \leq Q$, is equal to*

$$h_3 \left[Qt_3 - x \int_0^{t_3} D_3(t)\, dt \right] + h_4 \left[[Q-x] \left\{ t | t \geq 0, t = X_4^{-1}(Q-x) \right\} - \int_0^{\left\{ t | t \geq 0, t = X_4^{-1}(Q-x) \right\}} X_4(t)\, dt \right].$$

Proof. Similar to the previous case, in this case we also have $S_3(t) = Q - xD_3(t)$. However, since the stock amount does not finish before the end of the first selling season, the average inventory during the first selling season is $\frac{\int_0^{t_3} [Q - xD_3(t)] dt}{t_3}$ which is equal to $\frac{Qt_3 - x \int_0^{t_3} D_3(t) dt}{t_3}$. Hence, we will have the following total holding cost over the first selling season.

$$h_3 \frac{Qt_3 - x \int_0^{t_3} D_3(t)\, dt}{t_3} t_3$$

which is the multiplication of h_3, the average inventory, and t_3, and can be further simplified as follows

$$h_3 \left[Qt_3 - x \int_0^{t_3} D_3(t)\, dt \right].$$

Since $x \leq Q$, at the beginning of the second selling season the stock amount is equal to $Q - x$. Therefore, the stock amount at time t of the second selling season is $S_4(t) = [Q-x] - X_4(t)$. The stock level over four periods of the problem for the second case where the demand is less than or equal to the produced quantity (i.e., $x \leq Q$), can be illustrated using Figure 2.2 (given a linear production and demand, this figure will be the same one illustrated by Ghaniabadi *et al.* 2023 for $x \leq Q$).

Now, we can calculate the average inventory during the second selling season as $\frac{\int_0^{t_4} [[Q-x] - X_4(t)] dt}{t_4}$ which is then equal to $\frac{[Q-x] t_4 - \int_0^{t_4} X_4(t) dt}{t_4}$ where

$$t_4 = \{ t | t \geq 0, Q - x = X_4(t) \} = \left\{ t | t \geq 0, t = X_4^{-1}(Q-x) \right\}.$$

Therefore, the average inventory is $\frac{[Q-x]\left\{t|t\geq 0,t=X_4^{-1}(Q-x)\right\}-\int_0^{\left\{t|t\geq 0,t=X_4^{-1}(Q-x)\right\}}X_4(t)dt}{t_4}$ and we will have the following total holding cost over the second selling season.

$$h_4\frac{[Q-x]\left\{t|t\geq 0,t=X_4^{-1}(Q-x)\right\}-\int_0^{\left\{t|t\geq 0,t=X_4^{-1}(Q-x)\right\}}X_4(t)\,dt}{t_4}t_4$$

which is the multiplication of h_4, the average inventory, and t_4, and can be further simplified as follows.

$$h_4\left[[Q-x]\left\{t|t\geq 0,t=X_4^{-1}(Q-x)\right\}-\int_0^{\left\{t|t\geq 0,t=X_4^{-1}(Q-x)\right\}}X_4(t)\,dt\right]$$

Therefore, when the demand is less than the production quantity, we will have the following holding costs over the first and second selling seasons, in total.

$$h_3\left[Qt_3-x\int_0^{t_3}D_3(t)\,dt\right]+h_4\left[[Q-x]\left\{t|t\geq 0,t=X_4^{-1}(Q-x)\right\}-\int_0^{\left\{t|t\geq 0,t=X_4^{-1}(Q-x)\right\}}X_4(t)\,dt\right].\blacksquare$$

The production cost and revenues of the regular and discount selling periods, have the same formulations as in the papers of Ghaniabadi *et al.* (2023) and Silver *et al.* (1998), which are in common with the basic newsvendor problem as well.

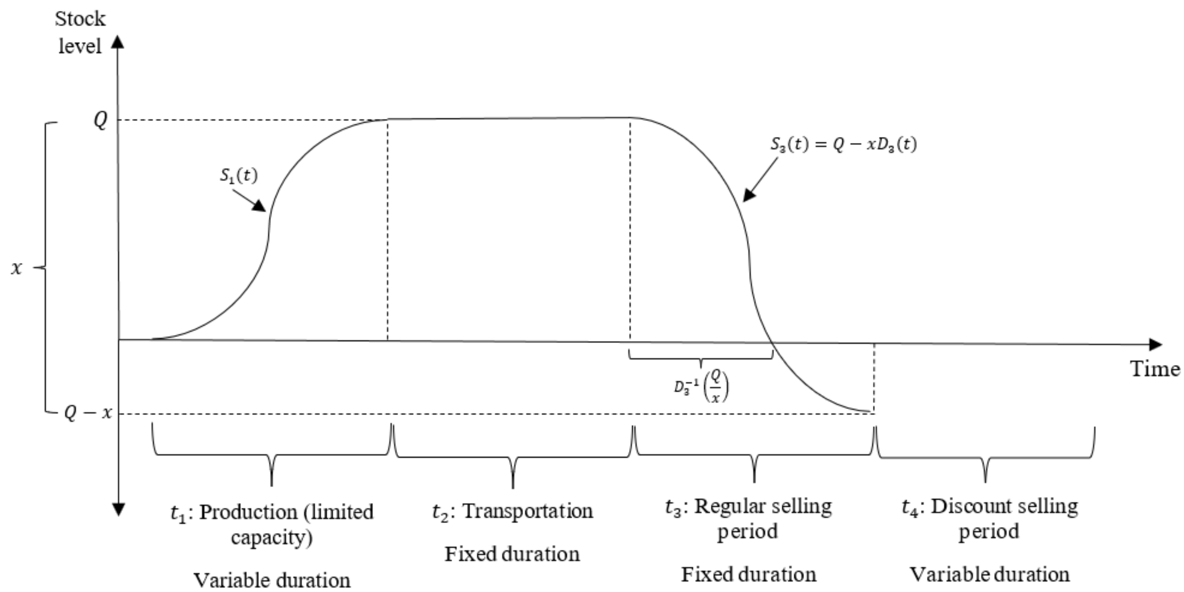

Figure 2.1: Depiction of the gradual stock amount over the supply chain periods under nonlinear production and demand functions for Case 1: $x \geq Q$.

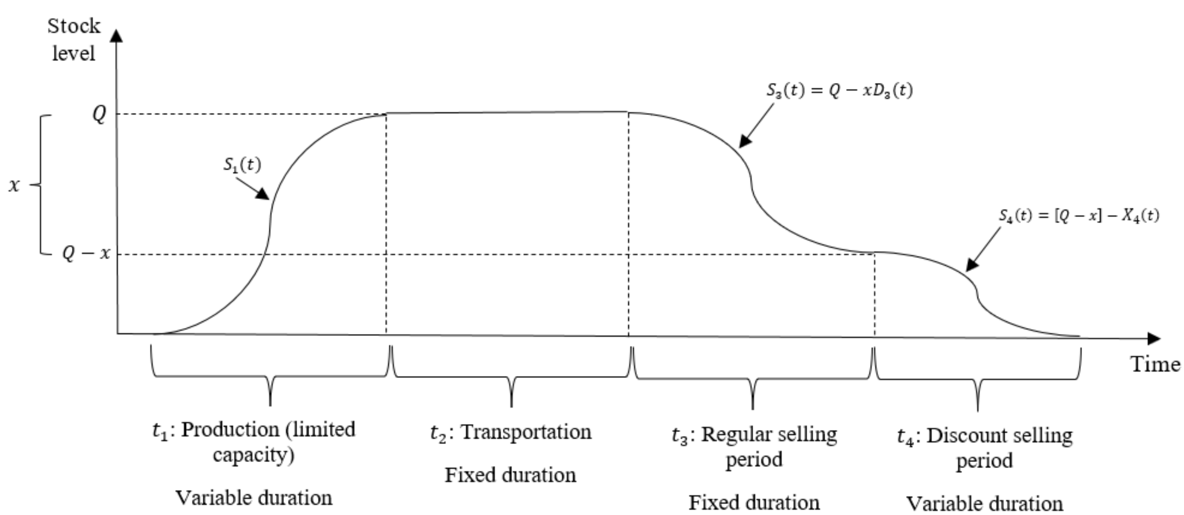

Figure 2.2: Representing the gradual stock level over four phases of supply chain with nonlinear production and demand functions for Case 2: $x \leq Q$.

2.4 The stochastic problem under discrete distributions

We let x_j be the demand in the jth scenario, while we define k_j as the probability of jth scenario (Ghaniabadi *et al.* 2023). We present the following proposition in order to solve the corresponding stochastic optimization problem under nonlinear production and demand functions. The problem under linear production and demand studied by Ghaniabadi *et al.* (2023), is a special case of this nonlinear case, and the following proposition is an extension of the solution process of the linear production and demand case under discrete distributions proposed by Ghaniabadi *et al.* (2023), to the nonlinear production and demand case.

Proposition 7. *The optimal profit and the corresponding optimal production (order) quantity can be found by solving the following maximization problem.*

$$\max\left\{z_1(0), z_1(Q_1^*), z_n(x_{n-1}) : \forall n = \{2,...,N\}, z_n(Q_n^*) : \forall n = \{2,...,N\},\right.$$
$$\left. z_{N+1}(x_N), z_{N+1}\left(Q_{N+1}^*\right), z_{N+1}\left(\overline{Q}\right)\right\}$$

where $0 \leq x_1 \leq x_2 \leq \ldots \leq x_N \leq \overline{Q}$ *and we define*
$H_1(Q) = h_1\left[\int_0^{\left\{t|t\geq 0, t=S_1^{-1}(Q)\right\}} S_1(t)\,dt\right]$,
$H_3^j(Q) = h_3\left[Q\left\{t|t\geq 0, t=D_3^{-1}\left(\frac{Q}{x_j}\right)\right\} - x_j\int_0^{\left\{t|t\geq 0, t=D_3^{-1}\left(\frac{Q}{x_j}\right)\right\}} D_3(t)\,dt\right]$, *and*
$H_4^j(Q) = h_4\left[\left[Q-x_j\right]\left\{t|t\geq 0, t=X_4^{-1}\left(Q-x_j\right)\right\} - \int_0^{\left\{t|t\geq 0, t=X_4^{-1}\left(Q-x_j\right)\right\}} X_4(t)\,dt\right]$.
Moreover,

$$z_1(Q) = -vQ - H_1(Q) - h_2t_2Q + \sum_{j=1}^{N} k_j\left[pQ - H_3^j(Q)\right]$$

where $z_1(Q)$ *is the expected profit corresponding to* Q *while* $0 \leq Q \leq x_1$*;*

$$z_n(Q) = -vQ - H_1(Q) - h_2t_2Q$$
$$+ \sum_{j=1}^{n-1} k_j \left[px_j + g[Q - x_j] - h_3\left[Qt_3 - x_j\int_0^{t_3} D_3(t)\,dt\right] - H_4^j(Q)\right] + \sum_{j=n}^{N} k_j \left[pQ - H_3^j(Q)\right].$$

where $z_n(Q)$ is the expected profit corresponding to Q while $x_{n-1} \leq Q \leq x_n$ for all $n = 2, ..., N$; and

$$z_{N+1}(Q) = -vQ - H_1(Q) - h_2t_2Q + \sum_{j=1}^{N} k_j \left[px_j + g[Q - x_j] - h_3\left[Qt_3 - x_j\int_0^{t_3} D_3(t)\,dt\right] - H_4^j(Q)\right]$$

where $z_{N+1}(Q)$ is the expected profit corresponding to Q while $x_N \leq Q \leq \overline{Q}$.

Furthermore,

$$Q_1^* = \left\{ Q \mid -v - \frac{dH_1(Q)}{dQ} - h_2t_2 + p - \sum_{j=1}^{N} k_j \frac{dH_3^j(Q)}{dQ} = 0, Q \in (0, x_1) \right\},$$

$$Q_n^* = \left\{ Q \mid -v - \frac{dH_1(Q)}{dQ} - h_2t_2 + \sum_{j=1}^{n-1} k_j \left[g - h_3t_3 - \frac{dH_4^j(Q)}{dQ}\right] + \sum_{j=n}^{N} k_j \left[p - \frac{dH_3^j(Q)}{dQ}\right] = 0, Q \in (x_{n-1}, x_n) \right\},$$

and

$$Q_{N+1}^* = \left\{ Q \mid -v - \frac{dH_1(Q)}{dQ} - h_2t_2 + \sum_{j=1}^{N} k_j \left[g - h_3t_3 - \frac{dH_4^j(Q)}{dQ}\right] = 0, Q \in \left(x_N, \overline{Q}\right) \right\}.$$

Proof. Similar to the linear production and demand case (Ghaniabadi *et al.* 2023), we let $0 \leq x_1 \leq x_2 \leq \ldots \leq x_N \leq \overline{Q}$ (i.e, sorting scenarios of demand in increasing order), and also we note that for a given x_j, the corresponding profit can be derived according to one of the following two cases.

Case 1: if $Q \leq x_j$

$$pQ - vQ - H_1(Q) - h_2 t_2 Q - H_3^j(Q),$$

where $H_1(Q) = h_1 \left[\int_0^{\{t|t\geq 0, t=S_1^{-1}(Q)\}} S_1(t)\, dt \right]$, and

$H_3^j(Q) = h_3 \left[Q \left\{ t | t \geq 0, t = D_3^{-1}\left(\frac{Q}{x_j}\right) \right\} - x_j \int_0^{\left\{t|t\geq 0, t=D_3^{-1}\left(\frac{Q}{x_j}\right)\right\}} D_3(t)\, dt \right]$.

Case 2: if $Q \geq x_j$

$$px_j + g\left[Q - x_j\right] - vQ - H_1(Q) - h_2 t_2 Q - h_3 \left[Qt_3 - x_j \int_0^{t_3} D_3(t)\, dt \right] - H_4^j(Q)$$

where $H_1(Q) = h_1 \left[\int_0^{\{t|t\geq 0, t=S_1^{-1}(Q)\}} S_1(t)\, dt \right]$, and

$H_4^j(Q) = h_4 \left[\left[Q - x_j\right] \left\{ t | t \geq 0, t = X_4^{-1}\left(Q - x_j\right) \right\} - \int_0^{\left\{t|t\geq 0, t=X_4^{-1}\left(Q - x_j\right)\right\}} X_4(t)\, dt \right]$.

When $0 \leq Q \leq x_1$, we have $Q \leq x_j$ for all $j = 1, ..., N$ (as was the case in the linear production and demand problem of Ghaniabadi *et al.* 2023), in which case the corresponding expected profit function is $z_1(Q) = -vQ - H_1(Q) - h_2 t_2 Q + \sum_{j=1}^{N} k_j \left[pQ - H_3^j(Q) \right]$. We then use first order condition as follows.

$$\frac{dz_1}{dQ} = -v - \frac{dH_1(Q)}{dQ} - h_2 t_2 + p - \sum_{j=1}^{N} k_j \frac{dH_3^j(Q)}{dQ} = 0$$

We consequently have

$$Q_1^* = \left\{ Q \mid -v - \frac{dH_1(Q)}{dQ} - h_2 t_2 + p - \sum_{j=1}^{N} k_j \frac{dH_3^j(Q)}{dQ} = 0, Q \in (0, x_1) \right\}.$$

As was also the case for the linear production and demand version of the problem by Ghaniabadi *et al.* (2023), when $x_{n-1} \leq Q \leq x_n$ for all $n = 2, ..., N$, we have $Q \leq x_j$ for all $j = n, ..., N$, and the revenues and costs corresponding to such scenario j is

$pQ - H_3^j(Q)$; on the other hand, we have $Q \geq x_j$ for all $j = 1, ..., n-1$, in which case the revenues and costs corresponding to the demand scenario j becomes $px_j + g\left[Q - x_j\right] - h_3\left[Qt_3 - x_j \int_0^{t_3} D_3(t)\,dt\right] - H_4^j(Q)$. Hence, the expected profit function $z_n(Q)$ can be formulated as

$$z_n(Q) = -vQ - H_1(Q) - h_2t_2Q + \sum_{j=1}^{n-1} k_j \left[px_j + g\left[Q - x_j\right] - h_3\left[Qt_3 - x_j\int_0^{t_3} D_3(t)\,dt\right] - H_4^j(Q)\right] + \sum_{j=n}^{N} k_j \left[pQ - H_3^j(Q)\right].$$

We can then utilize first-order condition as below.

$$\frac{dz_n}{dQ} = -v - \frac{dH_1(Q)}{dQ} - h_2t_2 + \sum_{j=1}^{n-1} k_j \left[g - h_3t_3 - \frac{dH_4^j(Q)}{dQ}\right] + \sum_{j=n}^{N} k_j \left[p - \frac{dH_3^j(Q)}{dQ}\right] = 0$$

As a result, we can then have

$$Q_n^* = \left\{ Q \mid -v - \frac{dH_1(Q)}{dQ} - h_2t_2 + \sum_{j=1}^{n-1} k_j \left[g - h_3t_3 - \frac{dH_4^j(Q)}{dQ}\right] + \sum_{j=n}^{N} k_j \left[p - \frac{dH_3^j(Q)}{dQ}\right] = 0, Q \in (x_{n-1}, x_n) \right\}.$$

When $x_N \leq Q \leq \overline{Q}$, we have $Q \geq x_j$ for all $j = 1, ..., N$ (as we had in the linear production and demand version of the problem by Ghaniabadi *et al.* 2023), in which case the corresponding expected profit function $z_{N+1}(Q)$ can be calculated as

$$z_{N+1}(Q) = -vQ - H_1(Q) - h_2t_2Q + \sum_{j=1}^{N} k_j \left[px_j + g\left[Q - x_j\right] - h_3\left[Qt_3 - x_j\int_0^{t_3} D_3(t)\,dt\right] - H_4^j(Q)\right]$$

Using first order condition, we can have

$$\frac{dz_{N+1}}{dQ} = -v - \frac{dH_1(Q)}{dQ} - h_2t_2 + \sum_{j=1}^{N} k_j \left[g - h_3t_3 - \frac{dH_4^j(Q)}{dQ} \right] = 0$$

in which case, we can reach

$$Q_{N+1}^* = \left\{ Q \mid -v - \frac{dH_1(Q)}{dQ} - h_2t_2 + \sum_{j=1}^{N} k_j \left[g - h_3t_3 - \frac{dH_4^j(Q)}{dQ} \right] = 0, Q \in \left(x_N, \overline{Q}\right) \right\}.$$

Aside from the set $\left\{Q_1^*, ..., Q_{N+1}^*\right\}$ which were derived for their corresponding intervals using first order condition (as in the paper of Ghaniabadi *et al.* 2023 for the linear case), the endpoints of intervals (i.e., 0, $\{x_1, x_2, ..., x_N\}$, and $\overline{Q}$) must also be considered in the set of possible solutions (which was also the case in the linear case studied by Ghaniabadi *et al.* 2023). Therefore, overall, we can achieve the optimal profit and the corresponding optimal decision variable by solving the following problem (which would be the same as the one proposed by Ghaniabadi *et al.* 2023 for discrete distribution, given linear cumulative production and demand functions of time).

$$\begin{aligned} \max \{ & z_1(0), z_1(Q_1^*), z_n(x_{n-1}) : \forall n = \{2, ..., N\}, z_n(Q_n^*) : \forall n = \{2, ..., N\}, \\ & z_{N+1}(x_N), z_{N+1}\left(Q_{N+1}^*\right), z_{N+1}\left(\overline{Q}\right) \}. \blacksquare \end{aligned}$$

2.5 The problem under Wright's model for production and Bass model for demand

In this part, we model the holding costs of production and demand phases, under models of Wright (1936) and Bass (1969), respectively, and derive the corresponding solution process, all based on the models and solutions we developed from the general case. We specifically chose the models of Wright (1936) and Bass (1969), since they are widely-adopted in the literature and also since they can model different nonlinear functions via adjustments in their parameters. Figures 2.3 and 2.4 demonstrate some examples of such functions for Wright (1936) and Bass (1969) models, respectively. Under Wright (1936) model, the cumulative production at time t, can be obtained using the formula $\left[\frac{t}{a}\right]^{\frac{1}{1-b}}$, while a and b are parameters of the model. Under the model of Bass (1969), the cumulative demand at time t can be calculated via the formula $m\frac{1-e^{-[p+q]t}}{1+\frac{q}{p}e^{-[p+q]t}}$ where p, q, and m are parameters of the Bass model.

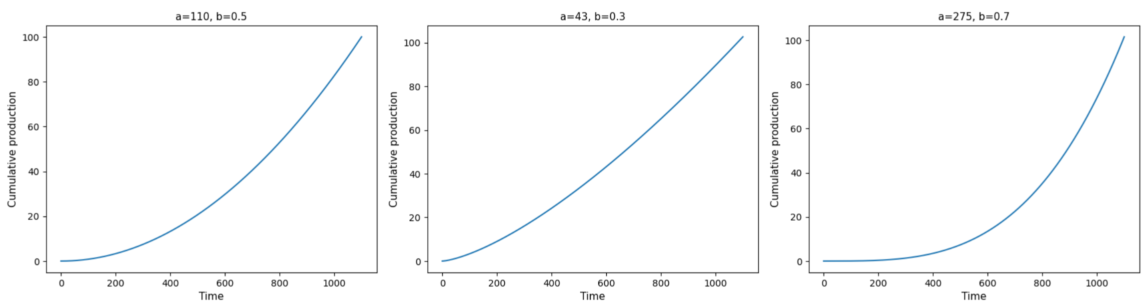

Figure 2.3: Examples of cumulative production functions according to the model of Wright (1936).

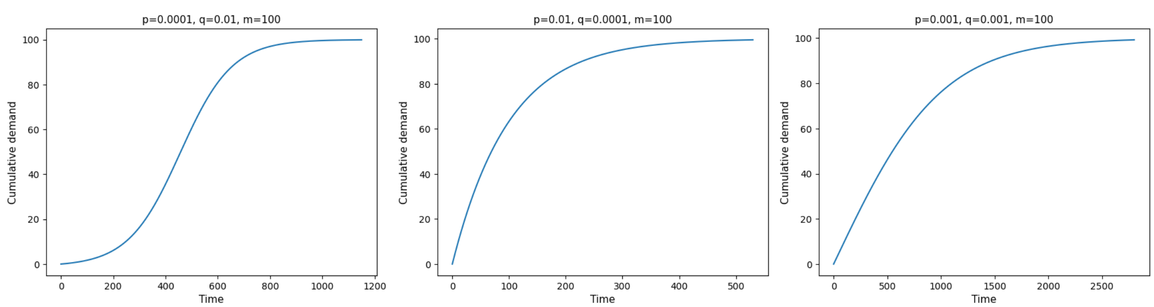

Figure 2.4: Examples of cumulative demand functions according to the model of Bass (1969).

Proposition 8. *The total holding cost over the production phase under the Wright's model (Wright 1936) is equal to*

$$H_1(Q) = h_1 a \left[\frac{1-b}{2-b}\right] Q^{2-b}$$

Proof. Under Wright's model (Wright 1936), the total time required for the production of $S_1(t)$ units can be derived as $t = a[S_1(t)]^{1-b}$ (Wright 1936, Teplitz 1991, Anzanello and Fogliatto 2011), where $a > 0$ and $0 < b < 1$ are parameters of the model. Hence, we can have $S_1(t) = \left[\frac{t}{a}\right]^{\frac{1}{1-b}}$ and $\left\{t | t \geq 0, t = S_1^{-1}(Q)\right\} = \{t | t \geq 0, Q = S_1(t)\} = \left\{t | t \geq 0, Q = \left[\frac{t}{a}\right]^{\frac{1}{1-b}}\right\} = aQ^{1-b}$.

We can now derive the holding cost function for the production phase under the Wright's model (Wright 1936), using Proposition 4, as follows.

$$\begin{aligned} H_1(Q) = h_1 \left[\int_0^{\left\{t | t \geq 0, t = S_1^{-1}(Q)\right\}} S_1(t)\, dt\right] &= h_1 \left[\int_0^{aQ^{1-b}} \left[\frac{t}{a}\right]^{\frac{1}{1-b}} dt\right] \\ &= h_1 a^{\frac{1}{b-1}} \left[\frac{1-b}{2-b} t^{\frac{2-b}{1-b}}\right]_0^{aQ^{1-b}} \\ &= h_1 a^{\frac{1}{b-1}} \left[\frac{1-b}{2-b}\right] \left[aQ^{1-b}\right]^{\frac{2-b}{1-b}} \\ &= h_1 a^{\frac{-1}{1-b} + \frac{2-b}{1-b}} \left[\frac{1-b}{2-b}\right] Q^{2-b} \\ &= h_1 a \left[\frac{1-b}{2-b}\right] Q^{2-b}. \blacksquare \end{aligned}$$

Proposition 9. *The total holding cost over the regular selling season under the Bass demand model (Bass 1969) when $x \geq Q$ and for a given x, is equal to*

$$H_3(Q, x) = h_3 \left[Q \left[\frac{\ln\left[QF_3(t_3)\frac{q_3}{p_3} + x\right] - \ln\left[x - QF_3(t_3)\right]}{p_3 + q_3}\right] - x\gamma\right]$$

where

$$\gamma = \frac{1}{F_3(t_3)}\left[\left[\frac{1+\frac{p_3}{q_3}}{p_3+q_3}\ln\left[\frac{QF_3(t_3)\frac{q_3}{p_3}+x}{x-QF_3(t_3)}+\frac{q_3}{p_3}\right]-\frac{p_3}{q_3}\left[\frac{\ln\left[QF_3(t_3)\frac{q_3}{p_3}+x\right]-\ln\left[x-QF_3(t_3)\right]}{p_3+q_3}\right]\right]-\frac{1+\frac{p_3}{q_3}}{p_3+q_3}\ln\left[1+\frac{q_3}{p_3}\right]\right]$$

and $F_3(t_3) = \frac{1-e^{-[p_3+q_3]t_3}}{1+\frac{q_3}{p_3}e^{-[p_3+q_3]t_3}}$.

Proof. According to the Bass demand model (Bass 1969), the cumulative demand function (for period 3) is $\Gamma_3(t) = m_3F_3(t) = m_3\frac{1-e^{-[p_3+q_3]t}}{1+\frac{q_3}{p_3}e^{-[p_3+q_3]t}}$. Therefore, we can have

$$\begin{aligned}
\left\{t|t\geq 0, t = D_3^{-1}\left(\frac{Q}{x}\right)\right\} &= \left\{t|t\geq 0, \frac{Q}{x} = D_3(t)\right\} \\
&= \left\{t|t\geq 0, \frac{Q}{x} = \frac{m_3F_3(t)}{m_3F_3(t_3)}\right\} \\
&= \left\{t|t\geq 0, \frac{Q}{x} = \frac{1-e^{-[p_3+q_3]t}}{F_3(t_3)\left[1+\frac{q_3}{p_3}e^{-[p_3+q_3]t}\right]}\right\} \\
&= \left\{t|t\geq 0, QF_3(t_3)+QF_3(t_3)\frac{q_3}{p_3}e^{-[p_3+q_3]t} = x - xe^{-[p_3+q_3]t}\right\} \\
&= \left\{t|t\geq 0, e^{-[p_3+q_3]t}\left[QF_3(t_3)\frac{q_3}{p_3}+x\right] = x-QF_3(t_3)\right\} \\
&= \left\{t|t\geq 0, e^{-[p_3+q_3]t} = \frac{x-QF_3(t_3)}{QF_3(t_3)\frac{q_3}{p_3}+x}\right\} \\
&= \left\{t|t\geq 0, e^{-[p_3+q_3]t} = \frac{x-QF_3(t_3)}{QF_3(t_3)\frac{q_3}{p_3}+x}\right\} \\
&= \left\{t|t\geq 0, -[p_3+q_3]t = \ln\left[x-QF_3(t_3)\right]-\ln\left[QF_3(t_3)\frac{q_3}{p_3}+x\right]\right\} \\
&= \left\{t|t\geq 0, t = \frac{\ln\left[x-QF_3(t_3)\right]-\ln\left[QF_3(t_3)\frac{q_3}{p_3}+x\right]}{-[p_3+q_3]}\right\} \\
&= \left\{t|t\geq 0, t = \frac{\ln\left[QF(t_3)\frac{q_3}{p_3}+x\right]-\ln\left[x-QF_3(t_3)\right]}{p_3+q_3}\right\}
\end{aligned}$$

Since under Bass model, $0 \leq F_3(t_3) \leq 1$ which (given that $Q \geq 0$) implies that $0 \leq QF_3(t_3) \leq Q$, and also since in this case we have $Q \leq x$, then we can have $0 \leq QF_3(t_3) \leq$

$Q \leq x$ which implies that $0 \leq x - QF_3(t_3) \leq x$. On the other hand, since $QF_3(t_3)\frac{q_3}{p_3} \geq 0$, $QF_3(t_3)\frac{q_3}{p_3} + x \geq x$. Therefore, we can conclude that $QF_3(t_3)\frac{q_3}{p_3} + x \geq x - QF_3(t_3) \geq 0$ which implies that $\ln[x - QF_3(t_3)] \leq \ln\left[QF_3(t_3)\frac{q_3}{p_3} + x\right]$. Moreover, $x \geq 1$ must always hold (which is not restrictive since one can simply scale the parameters of the problem accordingly in order for $x \geq 1$ to hold true), so as to make sure that $QF_3(t_3)\frac{q_3}{p_3} + x \geq 1$ and that $\ln\left(QF_3(t_3)\frac{q_3}{p_3} + x\right) \geq 0$. Hence, since we also have positive p_3 and q_3, we can then have

$$\left\{t \middle| t \geq 0, t = D_3^{-1}\left(\frac{Q}{x}\right)\right\} = \left\{t \middle| t \geq 0, t = \frac{\ln\left[QF_3(t_3)\frac{q_3}{p_3} + x\right] - \ln[x - QF_3(t_3)]}{p_3 + q_3}\right\}$$
$$= \frac{\ln\left[QF_3(t_3)\frac{q_3}{p_3} + x\right] - \ln[x - QF_3(t_3)]}{p_3 + q_3}$$

where $F_3(t_3) = \frac{1 - e^{-[p_3+q_3]t_3}}{1 + \frac{q_3}{p_3}e^{-[p_3+q_3]t_3}}$.

Moreover, $\int_E^L F_3(t)\,dt$ for any $E, L \in \mathbb{R}$ and $L > E$, can be derived as follows.

$$\begin{aligned}
\int_E^L F_3(t)\,dt &= \int_E^L \frac{1-e^{-[p_3+q_3]t}}{1+\frac{q_3}{p_3}e^{-[p_3+q_3]t}}dt \\
&= \int_E^L \left[\frac{1}{1+\frac{q_3}{p_3}e^{-[p_3+q_3]t}} - \frac{e^{-[p_3+q_3]t}}{1+\frac{q_3}{p_3}e^{-[p_3+q_3]t}}\right] dt \\
&= \int_E^L \left[\frac{e^{[p_3+q_3]t}}{e^{[p_3+q_3]t}+\frac{q_3}{p_3}} - \frac{1}{e^{[p_3+q_3]t}+\frac{q_3}{p_3}}\right] dt \\
&= \int_E^L \left[\frac{e^{[p_3+q_3]t}}{e^{[p_3+q_3]t}+\frac{q_3}{p_3}} - \frac{p_3}{q_3}\left[\frac{\frac{q_3}{p_3}}{e^{[p_3+q_3]t}+\frac{q_3}{p_3}}\right]\right] dt \\
&= \int_E^L \left[\frac{e^{[p_3+q_3]t}}{e^{[p_3+q_3]t}+\frac{q_3}{p_3}} - \frac{p_3}{q_3}\left[\frac{\frac{q_3}{p_3}+e^{[p_3+q_3]t}-e^{[p_3+q_3]t}}{e^{[p_3+q_3]t}+\frac{q_3}{p_3}}\right]\right] dt \\
&= \int_E^L \left[\frac{e^{[p_3+q_3]t}}{e^{[p_3+q_3]t}+\frac{q_3}{p_3}} - \frac{p_3}{q_3}\left[1-\frac{e^{[p_3+q_3]t}}{e^{[p_3+q_3]t}+\frac{q_3}{p_3}}\right]\right] dt \\
&= \int_E^L \left[\frac{e^{[p_3+q_3]t}}{e^{[p_3+q_3]t}+\frac{q_3}{p_3}} - \frac{p_3}{q_3} + \frac{p_3}{q_3}\left[\frac{e^{[p_3+q_3]t}}{e^{[p_3+q_3]t}+\frac{q_3}{p_3}}\right]\right] dt \\
&= \int_E^L \left[\left[1+\frac{p_3}{q_3}\right]\left[\frac{e^{[p_3+q_3]t}}{e^{[p_3+q_3]t}+\frac{q_3}{p_3}}\right] - \frac{p_3}{q_3}\right] dt \\
&= \left[\frac{1+\frac{p_3}{q_3}}{p_3+q_3}\ln\left[e^{[p_3+q_3]t}+\frac{q_3}{p_3}\right] - \frac{p_3}{q_3}t\right]_E^L
\end{aligned}$$

We then derive $\int_0^{\left\{t|t\geq 0, t=D_3^{-1}\left(\frac{Q}{x}\right)\right\}} D_3(t)\,dt$ as follows.

$$\begin{aligned}
&\int_0^{\left\{t|t\geq 0,t=D_3^{-1}\left(\frac{Q}{x}\right)\right\}} D_3(t)\,dt = \int_0^{\frac{\ln\left[QF_3(t_3)\frac{q_3}{p_3}+x\right]-\ln\left[x-QF_3(t_3)\right]}{p_3+q_3}} \frac{1-e^{-[p_3+q_3]t}}{F_3(t_3)\left[1+\frac{q_3}{p_3}e^{-[p_3+q_3]t}\right]}dt \\
&=\frac{1}{F_3(t_3)}\int_0^{\frac{\ln\left[QF_3(t_3)\frac{q_3}{p_3}+x\right]-\ln\left[x-QF_3(t_3)\right]}{p_3+q_3}} \frac{1-e^{-[p_3+q_3]t}}{1+\frac{q_3}{p_3}e^{-[p_3+q_3]t}}dt \\
&=\frac{1}{F_3(t_3)}\left[\left[\frac{1+\frac{p_3}{q_3}}{p_3+q_3}\ln\left[e^{[p_3+q_3]t}+\frac{q_3}{p_3}\right]-\frac{p_3}{q_3}t\right]_0^{\frac{\ln\left[QF_3(t_3)\frac{q_3}{p_3}+x\right]-\ln\left[x-QF_3(t_3)\right]}{p_3+q_3}}\right] \\
&=\frac{1}{F_3(t_3)}\left[\left[\frac{1+\frac{p_3}{q_3}}{p_3+q_3}\ln\left[e^{[p_3+q_3]\left[\frac{\ln\left[QF_3(t_3)\frac{q_3}{p_3}+x\right]-\ln\left[x-QF_3(t_3)\right]}{p_3+q_3}\right]}+\frac{q_3}{p_3}\right]-\frac{p_3}{q_3}\left[\frac{\ln\left[QF_3(t_3)\frac{q_3}{p_3}+x\right]-\ln\left[x-QF_3(t_3)\right]}{p_3+q_3}\right]\right]-\frac{1+\frac{p_3}{q_3}}{p_3+q_3}\ln\left[1+\frac{q_3}{p_3}\right]\right] \\
&=\frac{1}{F_3(t_3)}\left[\left[\frac{1+\frac{p_3}{q_3}}{p_3+q_3}\ln\left[e^{\left[\ln\left[QF_3(t_3)\frac{q_3}{p_3}+x\right]-\ln\left[x-QF_3(t_3)\right]\right]}+\frac{q_3}{p_3}\right]-\frac{p_3}{q_3}\left[\frac{\ln\left[QF_3(t_3)\frac{q_3}{p_3}+x\right]-\ln\left[x-QF_3(t_3)\right]}{p_3+q_3}\right]\right]-\frac{1+\frac{p_3}{q_3}}{p_3+q_3}\ln\left[1+\frac{q_3}{p_3}\right]\right] \\
&=\frac{1}{F_3(t_3)}\left[\left[\frac{1+\frac{p_3}{q_3}}{p_3+q_3}\ln\left[\frac{e^{\ln\left[QF_3(t_3)\frac{q_3}{p_3}+x\right]}}{e^{\ln\left[x-QF_3(t_3)\right]}}+\frac{q_3}{p_3}\right]-\frac{p_3}{q_3}\left[\frac{\ln\left[QF_3(t_3)\frac{q_3}{p_3}+x\right]-\ln\left[x-QF_3(t_3)\right]}{p_3+q_3}\right]\right]-\frac{1+\frac{p_3}{q_3}}{p_3+q_3}\ln\left[1+\frac{q_3}{p_3}\right]\right] \\
&=\frac{1}{F_3(t_3)}\left[\left[\frac{1+\frac{p_3}{q_3}}{p_3+q_3}\ln\left[\frac{QF_3(t_3)\frac{q_3}{p_3}+x}{x-QF_3(t_3)}+\frac{q_3}{p_3}\right]-\frac{p_3}{q_3}\left[\frac{\ln\left[QF_3(t_3)\frac{q_3}{p_3}+x\right]-\ln\left[x-QF_3(t_3)\right]}{p_3+q_3}\right]\right]-\frac{1+\frac{p_3}{q_3}}{p_3+q_3}\ln\left[1+\frac{q_3}{p_3}\right]\right]
\end{aligned}$$

Now, we can derive the holding cost function of the regular selling season when $x \geq Q$ and for a given x, and under the Bass demand model (Bass 1969), using Proposition 5 as follows.

$$\begin{aligned}
H_3(Q,x) =& h_3\left[Q\left\{t|t\geq 0, t=D_3^{-1}\left(\frac{Q}{x}\right)\right\}-x\int_0^{\left\{t|t\geq 0,t=D_3^{-1}\left(\frac{Q}{x}\right)\right\}} D_3(t)\,dt\right] \\
=& h_3\left[Q\left[\frac{\ln\left[QF_3(t_3)\frac{q_3}{p_3}+x\right]-\ln\left[x-QF_3(t_3)\right]}{p_3+q_3}\right]-x\gamma\right]
\end{aligned}$$

where

$$\gamma=\frac{1}{F_3(t_3)}\left[\left[\frac{1+\frac{p_3}{q_3}}{p_3+q_3}\ln\left[\frac{QF_3(t_3)\frac{q_3}{p_3}+x}{x-QF_3(t_3)}+\frac{q_3}{p_3}\right]-\frac{p_3}{q_3}\left[\frac{\ln\left[QF_3(t_3)\frac{q_3}{p_3}+x\right]-\ln\left[x-QF_3(t_3)\right]}{p_3+q_3}\right]\right]-\frac{1+\frac{p_3}{q_3}}{p_3+q_3}\ln\left[1+\frac{q_3}{p_3}\right]\right]$$

and $F_3(t_3)=\frac{1-e^{-[p_3+q_3]t_3}}{1+\frac{q_3}{p_3}e^{-[p_3+q_3]t_3}}$.■

Proposition 10. *The total holding cost over the regular and discount selling seasons under the Bass demand model (Bass 1969) for a given x when $x \leq Q$, is equal to*

$$h_3\left[Qt_3 - x\int_0^{t_3} D_3(t)\,dt\right] + H_4(Q,x) = h_3\left[Qt_3 - \frac{x}{F_3(t_3)}\left[\frac{1+\frac{p_3}{q_3}}{p_3+q_3}\ln\left[e^{[p_3+q_3]t_3} + \frac{q_3}{p_3}\right] - \frac{p_3}{q_3}t_3 - \frac{1+\frac{p_3}{q_3}}{p_3+q_3}\ln\left[1+\frac{q_3}{p_3}\right]\right]\right]$$
$$+ h_4\left[[Q-x]\left[\frac{\ln\left([Q-x]\frac{q_4}{p_4}+m_4\right) - \ln(m_4 - Q + x)}{p_4+q_4}\right] - \int_0^{\left\{t|t\geq 0, t=X_4^{-1}(Q-x)\right\}} X_4(t)\,dt\right]$$

where $F_3(t_3) = \frac{1-e^{-[p_3+q_3]t_3}}{1+\frac{q_3}{p_3}e^{-[p_3+q_3]t_3}}$ *and*

$$\int_0^{\left\{t|t\geq 0, t=X_4^{-1}(Q-x)\right\}} X_4(t)\,dt = m_4\left[\left[\frac{1+\frac{p_4}{q_4}}{p_4+q_4}\ln\left(\frac{[Q-x]\frac{q_4}{p_4}+m_4}{m_4-Q+x} + \frac{q_4}{p_4}\right) - \frac{p_4}{q_4}\left[\frac{\ln\left([Q-x]\frac{q_4}{p_4}+m_4\right) - \ln(m_4-Q+x)}{p_4+q_4}\right]\right] - \left[\frac{1+\frac{p_4}{q_4}}{p_4+q_4}\ln\left(1+\frac{q_4}{p_4}\right)\right]\right].$$

Proof. According to Proposition 6, and using the formulation for $\int_E^L F_3(t)\,dt$ for any $E, L \in \mathbb{R}$ and $L > E$, which we obtained in the proof of Proposition 9, we first derive $h_3\left[Qt_3 - x\int_0^{t_3} D_3(t)\,dt\right]$ under the Bass demand model (Bass 1969), as follows.

$$\begin{aligned}
h_3\left[Qt_3 - x\int_0^{t_3} D_3(t)\,dt\right] &= h_3\left[Qt_3 - x\int_0^{t_3}\frac{\Gamma_3(t)}{\Gamma_3(t_3)}dt\right] \\
&= h_3\left[Qt_3 - x\int_0^{t_3}\frac{m_3F_3(t)}{m_3F_3(t_3)}dt\right] \\
&= h_3\left[Qt_3 - \frac{x}{F_3(t_3)}\int_0^{t_3} F_3(t)\,dt\right] \\
&= h_3\left[Qt_3 - \frac{x}{F_3(t_3)}\left[\left[\frac{1+\frac{p_3}{q_3}}{p_3+q_3}\ln\left[e^{[p_3+q_3]t} + \frac{q_3}{p_3}\right] - \frac{p_3}{q_3}t\right]_0^{t_3}\right]\right] \\
&= h_3\left[Qt_3 - \frac{x}{F_3(t_3)}\left[\frac{1+\frac{p_3}{q_3}}{p_3+q_3}\ln\left[e^{[p_3+q_3]t_3} + \frac{q_3}{p_3}\right] - \frac{p_3}{q_3}t_3 - \frac{1+\frac{p_3}{q_3}}{p_3+q_3}\ln\left[1+\frac{q_3}{p_3}\right]\right]\right]
\end{aligned}$$

where $F_3(t_3) = \frac{1-e^{-[p_3+q_3]t_3}}{1+\frac{q_3}{p_3}e^{-[p_3+q_3]t_3}}$.

Moreover, according to Proposition 6, we derive $h_4\left[[Q-x]\left\{t|t\geq 0, t = X_4^{-1}(Q-x)\right\} - \int_0^{\left\{t|t\geq 0, t=X_4^{-1}(Q-x)\right\}} X_4(t)\,dt\right]$ under the Bass demand model (Bass 1969) as follows.

We first derive $\left\{t|t\geq 0, t = X_4^{-1}(Q-x)\right\}$ as below.

$$
\begin{aligned}
\left\{t|t \geq 0, t = X_4^{-1}(Q-x)\right\} &= \{t|t \geq 0, Q-x = X_4(t)\} \\
&= \{t|t \geq 0, Q-x = m_4 F_4(t)\} \\
&= \left\{t|t \geq 0, Q-x = m_4 \frac{1-e^{-[p_4+q_4]t}}{1+\frac{q_4}{p_4}e^{-[p_4+q_4]t}}\right\} \\
&= \left\{t|t \geq 0, Q-x+[Q-x]\frac{q_4}{p_4}e^{-[p_4+q_4]t} = m_4 - m_4 e^{-[p_4+q_4]t}\right\} \\
&= \left\{t|t \geq 0, e^{-[p_4+q_4]t}\left[[Q-x]\frac{q_4}{p_4}+m_4\right] = m_4 - Q + x\right\} \\
&= \left\{t|t \geq 0, e^{-[p_4+q_4]t} = \frac{m_4-Q+x}{[Q-x]\frac{q_4}{p_4}+m_4}\right\} \\
&= \left\{t|t \geq 0, -[p_4+q_4]t = \ln(m_4-Q+x) - \ln\left([Q-x]\frac{q_4}{p_4}+m_4\right)\right\} \\
&= \left\{t|t \geq 0, t = \frac{\ln(m_4-Q+x) - \ln\left([Q-x]\frac{q_4}{p_4}+m_4\right)}{-[p_4+q_4]}\right\} \\
&= \left\{t|t \geq 0, t = \frac{\ln\left([Q-x]\frac{q_4}{p_4}+m_4\right) - \ln(m_4-Q+x)}{p_4+q_4}\right\}
\end{aligned}
$$

Since in this case we have $Q \geq x$, hence $Q-x \geq 0$, and since $\frac{q_4}{p_4} \geq 0$ and $m_4 \geq 0$, we have $[Q-x]\frac{q_4}{p_4}+m_4 \geq m_4-Q+x \geq 0$. This implies that $\ln\left([Q-x]\frac{q_4}{p_4}+m_4\right) \geq \ln(m_4-Q+x)$. Moreover, $m_4 \geq 1$ must hold (this is not restrictive, since one can simply scale the problem parameters in such a way that $m_4 \geq 1$), so as to make sure that $[Q-x]\frac{q_4}{p_4}+m_4 \geq 1$ (since we have $Q-x \geq 0$, and $\frac{q_4}{p_4} \geq 0$), so that the inequality $0 \leq \ln\left([Q-x]\frac{q_4}{p_4}+m_4\right)$ would also then hold, which implies that $0 \leq \ln\left([Q-x]\frac{q_4}{p_4}+m_4\right) - \ln(m_4-Q+x)$, since $\ln\left([Q-x]\frac{q_4}{p_4}+m_4\right) \geq \ln(m_4-Q+x)$. Hence, also given that $p_4+q_4 > 0$, we can then have

$$\left\{t|t\geq 0,t=X_4^{-1}(Q-x)\right\}=\left\{t|t\geq 0,t=\frac{\ln\left([Q-x]\frac{q_4}{p_4}+m_4\right)-\ln(m_4-Q+x)}{p_4+q_4}\right\}$$
$$=\frac{\ln\left([Q-x]\frac{q_4}{p_4}+m_4\right)-\ln(m_4-Q+x)}{p_4+q_4}$$

We can then also derive $\int_0^{\left\{t|t\geq 0,t=X_4^{-1}(Q-x)\right\}} X_4(t)\,dt$ under the Bass demand model (Bass 1969) as follows.

$$\int_0^{\left\{t|t\geq 0,t=X_4^{-1}(Q-x)\right\}} X_4(t)\,dt=\int_0^{\frac{\ln\left([Q-x]\frac{q_4}{p_4}+m_4\right)-\ln(m_4-Q+x)}{p_4+q_4}} m_4F_4(t)\,dt$$
$$=m_4\int_0^{\frac{\ln\left([Q-x]\frac{q_4}{p_4}+m_4\right)-\ln(m_4-Q+x)}{p_4+q_4}}\frac{1-e^{-[p_4+q_4]t}}{1+\frac{q_4}{p_4}e^{-[p_4+q_4]t}}dt$$

Using the formulation for $\int_E^L F_3(t)\,dt$ for any $E,L\in\mathbb{R}$ and $L>E$, which we derived in the proof of Proposition 9, we can similarly obtain $\int_E^L F_4(t)\,dt$, and then have

$$\int_0^{\left\{t|t\geq 0,t=X_4^{-1}(Q-x)\right\}} X_4(t)\,dt=m_4\left[\left[\frac{1+\frac{p_4}{q_4}}{p_4+q_4}\ln\left(e^{[p_4+q_4]t}+\frac{q_4}{p_4}\right)-\frac{p_4}{q_4}t\right]_0^{\frac{\ln\left([Q-x]\frac{q_4}{p_4}+m_4\right)-\ln(m_4-Q+x)}{p_4+q_4}}\right]$$
$$=m_4\left[\left[\frac{1+\frac{p_4}{q_4}}{p_4+q_4}\ln\left(e^{[p_4+q_4]\frac{\ln\left([Q-x]\frac{q_4}{p_4}+m_4\right)-\ln(m_4-Q+x)}{p_4+q_4}}+\frac{q_4}{p_4}\right)-\frac{p_4}{q_4}\left[\frac{\ln\left([Q-x]\frac{q_4}{p_4}+m_4\right)-\ln(m_4-Q+x)}{p_4+q_4}\right]\right]-\left[\frac{1+\frac{p_4}{q_4}}{p_4+q_4}\ln\left(1+\frac{q_4}{p_4}\right)\right]\right]$$
$$=m_4\left[\left[\frac{1+\frac{p_4}{q_4}}{p_4+q_4}\ln\left(\frac{e^{\ln\left([Q-x]\frac{q_4}{p_4}+m_4\right)}}{e^{\ln(m_4-Q+x)}}+\frac{q_4}{p_4}\right)-\frac{p_4}{q_4}\left[\frac{\ln\left([Q-x]\frac{q_4}{p_4}+m_4\right)-\ln(m_4-Q+x)}{p_4+q_4}\right]\right]-\left[\frac{1+\frac{p_4}{q_4}}{p_4+q_4}\ln\left(1+\frac{q_4}{p_4}\right)\right]\right]$$
$$=m_4\left[\left[\frac{1+\frac{p_4}{q_4}}{p_4+q_4}\ln\left(\frac{[Q-x]\frac{q_4}{p_4}+m_4}{m_4-Q+x}+\frac{q_4}{p_4}\right)-\frac{p_4}{q_4}\left[\frac{\ln\left([Q-x]\frac{q_4}{p_4}+m_4\right)-\ln(m_4-Q+x)}{p_4+q_4}\right]\right]-\left[\frac{1+\frac{p_4}{q_4}}{p_4+q_4}\ln\left(1+\frac{q_4}{p_4}\right)\right]\right]$$

The holding cost function for the fourth period under the Bass demand model (Bass 1969) can then be derived as follows.

$$
\begin{aligned}
H_4(Q,x) =& h_4\left[[Q-x]\left\{t|t\geq 0, t = X_4^{-1}(Q-x)\right\} - \int_0^{\left\{t|t\geq 0, t=X_4^{-1}(Q-x)\right\}} X_4(t)\,dt\right] \\
=& h_4\left[[Q-x]\left[\frac{\ln\left([Q-x]\frac{q_4}{p_4}+m_4\right) - \ln(m_4-Q+x)}{p_4+q_4}\right] - \int_0^{\left\{t|t\geq 0, t=X_4^{-1}(Q-x)\right\}} X_4(t)\,dt\right]
\end{aligned}
$$

where

$$
\int_0^{\left\{t|t\geq 0, t=X_4^{-1}(Q-x)\right\}} X_4(t)\,dt = m_4\left[\left[\frac{1+\frac{p_4}{q_4}}{p_4+q_4}\ln\left(\frac{[Q-x]\frac{q_4}{p_4}+m_4}{m_4-Q+x}+\frac{q_4}{p_4}\right) - \frac{p_4}{q_4}\left[\frac{\ln\left([Q-x]\frac{q_4}{p_4}+m_4\right)-\ln(m_4-Q+x)}{p_4+q_4}\right]\right] - \left[\frac{1+\frac{p_4}{q_4}}{p_4+q_4}\ln\left(1+\frac{q_4}{p_4}\right)\right]\right]
$$

Now that we derived $h_3\left[Qt_3 - x\int_0^{t_3} D_3(t)\,dt\right]$ and $H_4(Q,x)$ for a given x and for the case of $x \leq Q$, and under the Bass demand model (Bass 1969), we can then use the Proposition 6 to reach the statement of Proposition 10.■

Proposition 11. *The formulations for* $\frac{dH_1(Q)}{dQ}$, $\frac{dH_3^j(Q)}{dQ}$, *and* $\frac{dH_4^j(Q)}{dQ}$ *(as defined in Proposition 7 to obtain* Q_1^*, Q_n^* *for* $n = 2, ..., N$, *and* Q_{N+1}^**) under Wright's model for production, and Bass model for demand in the regular and discount selling seasons, are as follows.*

$$
\frac{dH_1(Q)}{dQ} = h_1 a[1-b][2-b]Q^{1-b}
$$

$$
\begin{aligned}
\frac{dH_3^j(Q)}{dQ} =& \frac{h_3}{p_3+q_3}\left[QF_3(t_3)\left[\frac{q_3}{QF_3(t_3)q_3+p_3x_j}+\frac{1}{x_j-QF_3(t_3)}\right]+\ln\left[QF_3(t_3)\frac{q_3}{p_3}+x_j\right]-\ln\left[x_j-QF_3(t_3)\right]\right] \\
&-\frac{h_3x_jq_3\left[1+\frac{p_3}{q_3}\right]}{p_3[p_3+q_3]}\left[\frac{1}{\frac{QF_3(t_3)\frac{q_3}{p_3}+x_j}{x_j-QF_3(t_3)}+\frac{q_3}{p_3}}\left[\frac{1}{x_j-QF_3(t_3)}+\frac{QF_3(t_3)+x_j}{\left[x_j-QF_3(t_3)\right]^2}\right]\right] \\
&+\frac{h_3x_jp_3}{q_3[p_3+q_3]}\left[\frac{q_3}{QF_3(t_3)q_3+p_3x_j}+\frac{1}{x_j-QF_3(t_3)}\right]
\end{aligned}
$$

where $F_3(t_3) = \frac{1-e^{-[p+q]t_3}}{1+\frac{q}{p}e^{-[p+q]t_3}}$;

and

$$
\begin{aligned}
\frac{dH_4^j(Q)}{dQ} =& h_4 \left[\frac{[Q-x_j]}{p_4+q_4} \left[\frac{q_4}{p_4\left[[Q-x_j]\frac{q_4}{p_4}+m_4\right]} + \frac{1}{m_4-Q+x_j} \right] + \left[\frac{\ln\left([Q-x_j]\frac{q_4}{p_4}+m_4\right) - \ln\left(m_4-Q+x_j\right)}{p_4+q_4} \right] \right] \\
& - \frac{h_4 m_4}{p_4} \left[\frac{1}{\frac{[Q-x_j]\frac{q_4}{p_4}+m_4}{m_4-Q+x_j} + \frac{q_4}{p_4}} \right] \left[\frac{1}{m_4-Q+x_j} \right] \\
& + \frac{h_4 m_4}{p_4+q_4} \left[\frac{1}{[Q-x_j]\frac{q_4}{p_4}+m_4} + \frac{p_4}{q_4\left[m_4-Q+x_j\right]} \right].
\end{aligned}
$$

Proof. Given $H_1(Q)$ under the Wright's model, as derived in the Proposition 8, we can obtain the corresponding $\frac{dH_1(Q)}{dQ}$ as follows.

$$
\frac{dH_1(Q)}{dQ} = h_1 a \left[\frac{1-b}{2-b}\right][2-b]\,Q^{1-b} = h_1 a [1-b][2-b]\,Q^{1-b}
$$

Under the Bass model for demand, we obtained $H_3(Q,x)$ for the case of $Q \leq x$, in the Proposition 9. Given $H_3^j(Q) = H_3\left(Q,x_j\right)$, we can derive the corresponding $\frac{dH_3^j(Q)}{dQ}$ as follows.

$$
\begin{aligned}
\frac{dH_3^j(Q)}{dQ} =& h_3 \left[Q \left[\frac{1}{p_3+q_3} \left[\frac{F_3(t_3)\frac{q_3}{p_3}}{QF_3(t_3)\frac{q_3}{p_3}+x_j} + \frac{F_3(t_3)}{x_j-QF_3(t_3)} \right] \right] + \frac{\ln\left[QF_3(t_3)\frac{q_3}{p_3}+x_j\right] - \ln\left[x_j-QF_3(t_3)\right]}{p_3+q_3} \right] \\
& - \frac{h_3 x_j}{F_3(t_3)} \left[\frac{1+\frac{p_3}{q_3}}{p_3+q_3} \left[\frac{1}{\frac{QF_3(t_3)\frac{q_3}{p_3}+x_j}{x_j-QF_3(t_3)} + \frac{q_3}{p_3}} \right] \left[\frac{F_3(t_3)\frac{q_3}{p_3}}{x_j-QF_3(t_3)} + F_3(t_3)\frac{QF_3(t_3)\frac{q_3}{p_3}+x_j}{\left[x_j-QF_3(t_3)\right]^2} \right] \right] \\
& + \frac{h_3 x_j}{F_3(t_3)} \left[\frac{p_3}{q_3}\right] \left[\frac{1}{p_3+q_3} \left[\frac{F_3(t_3)\frac{q_3}{p_3}}{QF_3(t_3)\frac{q_3}{p_3}+x_j} + \frac{F_3(t_3)}{x_j-QF_3(t_3)} \right] \right] \\
=& \frac{h_3}{p_3+q_3} \left[QF_3(t_3) \left[\frac{q_3}{QF_3(t_3)q_3+p_3x_j} + \frac{1}{x_j-QF_3(t_3)} \right] + \ln\left[QF_3(t_3)\frac{q_3}{p_3}+x_j\right] - \ln\left[x_j-QF_3(t_3)\right] \right] \\
& - \frac{h_3 x_j q_3 \left[1+\frac{p_3}{q_3}\right]}{p_3\left[p_3+q_3\right]} \left[\frac{1}{\frac{QF_3(t_3)\frac{q_3}{p_3}+x_j}{x_j-QF_3(t_3)} + \frac{q_3}{p_3}} \left[\frac{1}{x_j-QF_3(t_3)} + \frac{QF_3(t_3)+x_j}{\left[x_j-QF_3(t_3)\right]^2} \right] \right] \\
& + \frac{h_3 x_j p_3}{q_3\left[p_3+q_3\right]} \left[\frac{q_3}{QF_3(t_3)q_3+p_3x_j} + \frac{1}{x_j-QF_3(t_3)} \right]
\end{aligned}
$$

where $F_3(t_3) = \frac{1-e^{-[p+q]t_3}}{1+\frac{q}{p}e^{-[p+q]t_3}}$.

In the Proposition 10, we derived $H_4(Q,x)$ under the Bass demand model. Considering $H_4^j(Q) = H_4\left(Q,x_j\right)$, we obtain the corresponding $\frac{dH_4^j(Q)}{dQ}$ as follows.

$$\frac{dH_4^j(Q)}{dQ} = h_4\left[\frac{[Q-x_j]}{p_4+q_4}\left[\frac{\frac{q_4}{p_4}}{[Q-x_j]\frac{q_4}{p_4}+m_4}+\frac{1}{m_4-Q+x_j}\right]+\left[\frac{\ln\left([Q-x_j]\frac{q_4}{p_4}+m_4\right)-\ln\left(m_4-Q+x_j\right)}{p_4+q_4}\right]\right]$$
$$-h_4m_4\left[\frac{1+\frac{p_4}{q_4}}{p_4+q_4}\right]\left[\frac{1}{\frac{[Q-x_j]\frac{q_4}{p_4}+m_4}{m_4-Q+x_j}+\frac{q_4}{p_4}}\left[\frac{\frac{q_4}{p_4}}{m_4-Q+x_j}\right]\right]$$
$$+h_4m_4\left[\frac{p_4}{q_4}\right]\left[\frac{1}{p_4+q_4}\right]\left[\frac{\frac{q_4}{p_4}}{[Q-x_j]\frac{q_4}{p_4}+m_4}+\frac{1}{m_4-Q+x_j}\right]$$
$$= h_4\left[\frac{[Q-x_j]}{p_4+q_4}\left[\frac{q_4}{p_4\left[[Q-x_j]\frac{q_4}{p_4}+m_4\right]}+\frac{1}{m_4-Q+x_j}\right]+\left[\frac{\ln\left([Q-x_j]\frac{q_4}{p_4}+m_4\right)-\ln\left(m_4-Q+x_j\right)}{p_4+q_4}\right]\right]$$
$$-\frac{h_4m_4}{p_4}\left[\frac{1}{\frac{[Q-x_j]\frac{q_4}{p_4}+m_4}{m_4-Q+x_j}+\frac{q_4}{p_4}}\right]\left[\frac{1}{m_4-Q+x_j}\right]$$
$$+\frac{h_4m_4}{p_4+q_4}\left[\frac{1}{[Q-x_j]\frac{q_4}{p_4}+m_4}+\frac{p_4}{q_4\left[m_4-Q+x_j\right]}\right].\blacksquare$$

We can use $H_1(Q)$, $H_3^j(Q)$, and $H_4^j(Q)$, obtained in Propositions 8, 9, and 10, respectively, and also $\frac{dH_1(Q)}{dQ}$, $\frac{dH_3^j(Q)}{dQ}$, and $\frac{dH_4^j(Q)}{dQ}$ obtained in Proposition 11, in the solution procedure given in Proposition 7, in order to obtain the solution process of the problem under Wright's model for production and Bass demand model.

2.6 Numerical experiments

In this section, we present the numerical experiments on synthetic and real data. We adopted the same parameters used by Ghaniabadi *et al.* (2023) in their numerical experiments, and incorporate new parameters for the nonlinear production and demand functions, under Wright's and Bass models, respectively. As in the paper of Ghaniabadi *et al.* (2023), we denote the solution of the standard newsvendor model as Q^*_{NV}, the solution of the newsvendor problem with holding cost and linear production and demand function as Q^*_{NVH}, and the solution of its approximations based on the standard newsvendor model as Q^*_{ANVH}. We also denote the optimal solution of the new model under the nonlinear production and demand functions (here under Wright's and Bass models, respectively) as Q^*_{NVHN}. We evaluate the aforementioned solutions of the four models based on their expected profit when used in the new model under the nonlinear production and demand functions (denoted as $NVHN\left(Q^*_{NV}\right)$, $NVHN\left(Q^*_{NVH}\right)$, $NVHN\left(Q^*_{ANVH}\right)$, and $NVHN\left(Q^*_{NVHN}\right)$, respectively) and the profit gain of Q^*_{NVHN} over the other solutions. The same scheme was also used by Ghaniabadi *et al.* (2023) to evaluate Q^*_{NV}, Q^*_{NVH}, and Q^*_{ANVH} based on their expected profit under linear production and demand functions, and the profit gain of Q^*_{NVH} over Q^*_{NV} and Q^*_{ANVH}. In this paper, we also utilize the same values of Q^*_{NV}, Q^*_{NVH}, and Q^*_{ANVH} achieved by Ghaniabadi *et al.* (2023), since we use the same data.

2.6.1 Synthetic dataset

We first provide the numerical results under synthetic data of Ghaniabadi *et al.* (2023) (i.e., $p = 20$, $v = 10$, $g = 9$, $r = 0.03$ units per day (for the linear production case in phase 1), $t_2 = 56$ days, $t_3 = 42$ days, $u = 0.02$ units per day (for the linear demand case in phase 4), $\overline{Q} = 4$, and the same values used by Ghaniabadi *et al.* (2023) for h_1, h_2, h_3, and h_4, and also for demand scenarios x_j and their probabilities k_j for $j = 1, ..., 100$).

In addition, we assume the parameters of the Wright's model (for the nonlinear production) in a way that at time $\frac{\overline{Q}}{r}$ (which is the maximum duration of the first period under the linear production case) a production level around (we use the terms "around" or "roughly", since we used the rounded numbers. Hence, although there might be small differences due to precision, in principle they are the same) $\overline{Q}$ is achieved according to the Wright's model. By doing so, both the linear and nonlinear production reach around the same production level (i.e., $\overline{Q}$) at time $\frac{\overline{Q}}{r}$. By letting $a = 71.867$, and $b = 0.556$ in the Wright's model, it takes roughly 133 days to reach a production amount of $\overline{Q} = 4$, while in the linear production case also at time $\frac{\overline{Q}}{r} = \frac{4}{0.03} \simeq 133$ days, a production amount of $\overline{Q} = 4$ is reached.

Conversely, given a Wright's production model with parameters $a = 71.867$ and $b = 0.556$, which takes roughly 133 days to reach a production of $\overline{Q} = 4$, one can approximate such nonlinear production function with a linear one with a production rate of $r = \frac{4}{133} \simeq 0.03$. Similarly, for any nonlinear production function, where it takes an amount of time equal to t_1 to achieve a production level of $\overline{Q}$, one can approximate that nonlinear production function with a linear one which has a production rate $r = \frac{\overline{Q}}{t_1}$.

For the parameters of Bass model for the regular selling season, we make use of the dataset of SKU aa71ad878e provided by Shen *et al.* (2020), which contains the order data over a period of 31 days. We first consider the daily demand data and then for each day, calculate the cumulative demand up to the end of that day. In order to be able to fit $F_3(t_3)$, we then normalize the cumulative demand values via diving them by the total cumulative demand at the end of the 31st day. Hence, we will have 31 observational data of normalized cumulative demand, to be able to fit $F_3(t_3)$, which can be fitted over the time data $[1, 2, ..., 31]$. However, since in our synthetic data we assumed $t_3 = 42$ days, we fit those 31 values of normalized cumulative demand over the time data $\left[\frac{42}{31}, 2 * \frac{42}{31}, 3 * \frac{42}{31}, ..., 30 * \frac{42}{31}, 42\right]$ which also contains 31 values. By doing so using the Python Scipy library, we reach the values $p_3 = 0.0178$ and $q_3 = 0.0865$ (here rounded to 4 decimal points).

For the discount selling season, we assume the parameters of the Bass model in a way that at time $\frac{\overline{Q} - x_1}{u}$ (which is the maximum duration scenario for the discount selling season under the linear demand case, where x_1 is the minimum demand scenario), both the linear and nonlinear demand cases reach a total demand level close to $\overline{Q} - x_1$ (which is the maximum leftover scenario at the end of the regular selling season). By letting $p_4 = 0.001$, $q_4 = 0.046$, and $m_4 = 3.11$, under the corresponding Bass model, it approximately takes 149 days to achieve a total demand of $\overline{Q} - x_1 \simeq 2.983$, which is the same demand level reached by the linear demand case with $u = 0.02$ units per day, at time $\frac{\overline{Q} - x_1}{u} \simeq \frac{2.983}{0.02} \simeq 149$

days.

On the other hand, a given Bass model with parameters $p_4 = 0.001$, $q_4 = 0.046$, and $m_4 = 3.11$, which takes roughly 149 days to reach a total demand of $\overline{Q} - x_1 \simeq 2.983$, can be approximated with a linear cumulative demand function with a demand rate of $u = \frac{2.983}{149} \simeq 0.02$. Likewise, any given nonlinear cumulative demand function, which takes an amount of time equal to t_4 to reach a demand level of $\overline{Q} - x_1$, can be approximated using a linear function with a demand rate of $u = \frac{\overline{Q}-x_1}{t_4}$.

The results of the numerical experiments are presented in the Table 2.1 (the values are rounded, except for the unit holding cost, and when profit gain is zero, which are exact values), where the profit gains of Q^*_{NVHN} over Q^*_{NVH}, Q^*_{ANVH}, and Q^*_{NV}, are calculated as $\frac{NVHN(Q^*_{NVHN})-NVHN(Q^*_{NVH})}{NVHN(Q^*_{NVH})}$, $\frac{NVHN(Q^*_{NVHN})-NVHN(Q^*_{ANVH})}{NVHN(Q^*_{ANVH})}$, and

$\frac{NVHN(Q^*_{NVHN})-NVHN(Q^*_{NV})}{NVHN(Q^*_{NV})}$, respectively. The results show that a higher holding cost results in a smaller optimal quantity Q^*_{NVHN} and also a lower optimal profit $NVHN\left(Q^*_{NVHN}\right)$, which is in line with the findings of Ghaniabadi *et al.* (2023) for the linear production and demand case. However, unlike the results of Ghaniabadi *et al.* (2023), Q^*_{NVH} does not always have a higher expected profit than Q^*_{ANVH}. For unit holding costs of 0.00275, 0.0055, and 0.006875, $NVHN\left(Q^*_{ANVH}\right)$ is closer to the optimal profit $NVHN\left(Q^*_{NVHN}\right)$ than $NVHN\left(Q^*_{NVH}\right)$, while for a unit holding cost of 0.00825, $NVHN\left(Q^*_{NVH}\right)$ is closer to the optimal profit, and for the unit holding cost 0.004125, $Q^*_{NVH} = Q^*_{NVHN}$. In the presence of holding cost (i.e., excluding the case of the unit holding cost $10^{-100} \simeq 0$), the standard newsvendor model provides the least expected profit, due to ignoring holding costs. When holding cost is present, Q^*_{NVHN} provides an average profit gain of 0.017%, 0.011%, and 0.48%, over Q^*_{NVH}, Q^*_{ANVH}, and Q^*_{NV}, respectively. This indicate that under this case of synthetic data, Q^*_{NVH} and Q^*_{ANVH} result in profits close to the optimal one.

Table 2.1: Comparison of the model under Wright's production and Bass demand with simpler models, for synthetic data.

$h_1, h_2,$ h_3, h_4	Q^*_{NVHN}	$NVHN$ (Q^*_{NVHN})	$NVHN$ (Q^*_{NVH})	Profit gain over NVH	$NVHN$ (Q^*_{ANVH})	Profit gain over $ANVH$	$NVHN$ (Q^*_{NV})	Profit gain over NV
$10^{-100} \simeq 0$	2.7574	18.849600	18.849600	0%	18.849600	0%	18.849600	0%
0.00275	2.7170	17.879761	17.878759	0.006%	17.879607	0.001%	17.868648	0.06%
0.004125	2.6341	17.414470	17.414470	0%	17.414452	0.0001%	17.378171	0.21%
0.0055	2.6329	16.957515	16.955536	0.012%	16.957437	0.0005%	16.887695	0.41%
0.006875	2.5956	16.508684	16.499985	0.053%	16.508488	0.001%	16.397219	0.68%
0.00825	2.5112	16.070512	16.068425	0.013%	16.062167	0.052%	15.906742	1.03%

2.6.2 Real-world datasets

SKU bb5419c49b

We now consider the same parameters used by Ghaniabadi *et al.* (2023) for the SKU bb5419c49b (from the dataset provided by Shen *et al.* 2020), which are as follows. $p = 83.935$, $v = 60$, $g = 50$, $r = 0.04$ units per hour (for the production phase of the linear case), $t_2 = 8$ hours, $t_3 = 24$ hours, $u = 0.02$ units per hour (for the discount selling season of the linear demand case), $\overline{Q} = 10$, and the same values for unit holding cost, demand scenarios and their probabilities which are utilized by Ghaniabadi *et al.* (2023).

Given that we have 31 days of data and the regular selling season is 24 hours, we aggregate the hourly demands of all 31 days, in a single day. Therefore, this new aggregated dataset, contains 24 values and each value is equal to the sum of the demand in that specific hour of the day over all 31 days. We then calculate the cumulative demands of the aggregated hourly dataset, which are then normalized by dividing each value by the total cumulative demand in the last hour. This normalized dataset is then used along with its corresponding time data $[1, 2, ..., 24]$ in order to fit $F_3(t_3)$ using Python Scipy library which result in the parameters $p_3 = 0.00313$ and $q_3 = 0.28360$ (here rounded to 5 decimal points) for the Bass model of the regular selling season.

For the parameters of the Wright's model for production and the Bass model for the discount selling season we incorporate the same method used for the synthetic data case. For the Wright's model, given $a = 66.788$, and $b = 0.427$, a production level of around $\overline{Q} = 10$ can be achieved in 250 hours, while in the linear production case with $r = 0.04$ units per hour, it also takes $\frac{\overline{Q}}{r} = \frac{10}{0.04} = 250$ hours to reach a production amount of $\overline{Q} = 10$. Moreover, with $p_4 = 0.0001$, $q_4 = 0.032$ and $m_4 = 9.601$ for the parameters of the Bass model for the discount selling season, it approximately takes 480 hours to achieve a total demand level of $\overline{Q} - x_1 = 9.6$, whereas in the linear demand case also, a demand of $\overline{Q} - x_1 = 9.6$ is reached in $\frac{\overline{Q}-x_1}{u} = \frac{9.6}{0.02} = 480$ hours.

Table 2.2 presents the result of the corresponding numerical experiments (the optimal profits are rounded to 3 decimal points), which are in line with Ghaniabadi *et al.* (2023), where a higher holding cost result in a lower expected profit, while $Q^*_{NVHN} = Q^*_{NVH} = Q^*_{ANVH} = Q^*_{NV} = 2$. According to our experiments, this result remained the same up to a unit holding cost of around 0.07. However, for a unit holding cost of 0.08, $Q^*_{NVHN} = 1.6$ and $NVHN(Q^*_{NVHN}) = 19.3146$ (rounded to 4 decimal points), whereas $Q^*_{NVH} = Q^*_{ANVH} = Q^*_{NV} = 2$ with an expected profit of 19.0999 (rounded to 4 decimal points), resulting in a profit gain of 1.124% (rounded to 3 decimal points) for Q^*_{NVHN}.

Table 2.2: Comparison of the model under Wright's production and Bass demand with simpler models, for SKU bb5419c49b ($t_3 = 1$ day).

h_1, h_2, h_3, h_4	Q^*_{NVHN}	$NVHN\left(Q^*_{NVHN}\right)$	Profit gain over NVH, $ANVH$ and NV
$10^{-100} \simeq 0$	2	32.107	0%
0.000685	2	31.995	0%
0.0010275	2	31.940	0%
0.00137	2	31.884	0%
0.0017125	2	31.828	0%
0.002055	2	31.773	0%

Ghaniabadi *et al.* (2023) also studied the same SKU bb5419c49b under the case of $t_3 = 42$ days, and $t_2 = 56$ days, while updating the demand scenarios and their probabilities according to this longer regular selling season, and also letting $\overline{Q} = 300$. The rest of the parameters were kept the same as the case of $t_3 = 24$ hours, and $t_2 = 8$ hours.

In order to have a Bass model similar to the case of $t_3 = 24$ hours, for this regular selling season of 42 days, we keep the same values of the normalized cumulative demand data, while transforming the corresponding time data from $[1, 2, ..., 24]$ to $[1 * 42, 2 * 42, ..., 24 * 42]$. Fitting the corresponding $F_3(t_3)$ using Python Scipy library, results in $p_3 = 0.0000746$ and $q_3 = 0.0067525$ (here rounded to seven decimal points).

For the parameters of the Wright's model for production phase and Bass model for the discount season, we again use the same method as in the synthetic data case. With $a = 264.937$, and $b = 0.414$ for the Wright's model, in production phase we reach an amount of $\overline{Q} = 300$ at roughly 7500 hours of production. In the linear production case, it also takes $\frac{\overline{Q}}{r} = \frac{300}{0.04} = 7500$ hours to achieve a production level of $\overline{Q} = 300$. For a Bass model of discount season, with parameters $p_4 = 2 * 10^{-6}$, $q_4 = 0.001$ and $m_4 = 283.3$, around 14160 hours is required for a demand level of $\overline{Q} - x_1 = 283.2$, while in the linear demand case, a demand level $\overline{Q} - x_1 = 283.2$ is achieved at $\frac{\overline{Q}-x_1}{u} = \frac{283.2}{0.02} = 14160$ hours.

Table 2.3 demonstrates the corresponding results (the profit values and the non-zero profit gains are rounded), where up to a unit holding cost of 0.00137, $Q^*_{NVHN} = Q^*_{NVH} = Q^*_{ANVH} = Q^*_{NV} = 84$. However, for the unit holding cost 0.0017125, $Q^*_{NVHN} = 67.2$, while $Q^*_{NVH} = Q^*_{ANVH} = Q^*_{NV} = 84$, resulting in a profit gain of 1.24% for the new model over the other ones. Under the unit holding cost 0.002055, the profit gain of Q^*_{NVHN} is even more significant, which is 4.24% over Q^*_{NVH}, and 9.87% over both Q^*_{ANVH} and Q^*_{NV}.

Table 2.3: Comparison of the model under Wright's production and Bass demand with simpler models, for SKU bb5419c49b ($t_3 = 42$ days).

h_1, h_2, h_3, h_4	Q^*_{NVHN}	$NVHN$ (Q^*_{NVHN})	$NVHN$ (Q^*_{NVH})	Profit gain over NVH	$NVHN\,(Q^*_{ANVH})$, $NVHN\,(Q^*_{NV})$	Profit gain over $ANVH$ and NV
$10^{-100} \simeq 0$	84	1348.479	1348.479	0%	1348.479	0%
0.000685	84	1107.487	1107.487	0%	1107.487	0%
0.0010275	84	986.991	986.991	0%	986.991	0%
0.00137	84	866.494	866.494	0%	866.494	0%
0.0017125	67.2	755.245	745.998	1.24%	745.998	1.24%
0.002055	50.4	687.229	659.250	4.24%	625.502	9.87%

SKU d17d9135b0

We also use the same parameters considered by Ghaniabadi *et al.* (2023) for the SKU d17d9135b0 (one of the SKU datasets provided by Shen *et al.* 2020), which are as follows. $p = 15.886$, $v = 9.5$, $g = 8.886$, $r = 0.2$ units per hour, $t_2 = 8$ hours, $t_3 = 24$ hours, $u = 0.04$ units per hour, $\overline{Q} = 250$, and the same values used by Ghaniabadi *et al.* (2023) for the unit holding cost, and demand scenarios and their probabilities.

In order to have the parameters of the Bass model for the regular selling season, we use the same method described for the numerical experiments of SKU bb5419c49b under the case of $t_3 = 24$ hours, which results in fitted parameters $p_3 = 0.00785$ and $q_3 = 0.22066$ (here rounded to five decimal points). For the parameters of Wright's model for production and Bass model for the discount selling season, we use the same method explained for the numerical experiments of synthetic data. Under the Wright's model, by letting $a = 66.251$, and $b = 0.468$, it approximately takes 1250 hours to reach a production level of $\overline{Q} = 250$. In the linear production, the time required for a production amount of $\overline{Q} = 250$, is also $\frac{\overline{Q}}{r} = \frac{250}{0.2} = 1250$ hours. For the Bass model of the discount selling season, with parameters $p_4 = 10^{-5}$, $q_4 = 0.002$ and $m_4 = 244.53$, a demand level of $\overline{Q} - x_1 = 244.3$ is achieved after roughly 6107.5 hours, while in the linear demand case, it also takes $\frac{\overline{Q}-x_1}{u} = \frac{244.3}{0.04} = 6107.5$ hours to reach a demand level of $\overline{Q} - x_1 = 244.3$ in the discount season.

The results of the corresponding numerical experiments are presented in the Table 2.4 (the profit values and non-zero profit gains are rounded to 3 decimal points). For the unit holding cost of 0.0001085 and 0.00016275, the profit gain over the linear production and demand case is 1.609% and 0.536%, respectively, while for a unit holding cost of 0.000217, 0.00027125 and 0.0003255, $Q^*_{NVHN} = Q^*_{NVH} = 17.1$. When holding cost exists, the average profit gain of Q^*_{NVHN} over Q^*_{NVH} is 0.429%, while the average profit gain of Q^*_{NVHN} over both Q^*_{ANVH} and Q^*_{NV} is 6.467%. This demonstrates that in this numerical experiment for the SKU d17d9135b0, Q^*_{NVH} on average provides expected profits considerably closer to the optimal one, compared with both Q^*_{ANVH} and Q^*_{NV}.

Table 2.4: Comparison of the model under Wright's production and Bass demand with simpler models, for SKU d17d9135b0.

h_1, h_2, h_3, h_4	Q^*_{NVHN}	$NVHN$ (Q^*_{NVHN})	$NVHN$ (Q^*_{NVH})	Profit gain over NVH	$NVHN\,(Q^*_{ANVH})$, $NVHN\,(Q^*_{NV})$	Profit gain over $ANVH$ and NV
$10^{-100} \simeq 0$	28.5	48.143	48.143	0%	48.143	0%
0.0001085	17.1	46.411	45.676	1.609%	45.422	2.178%
0.00016275	17.1	45.907	45.662	0.536%	44.061	4.188%
0.000217	17.1	45.402	45.402	0%	42.701	6.326%
0.00027125	17.1	44.898	44.898	0%	41.341	8.605%
0.0003255	17.1	44.394	44.394	0%	39.980	11.039%

2.7 Conclusion

In this article, we studied a generalization of the newsvendor model examined by Ghaniabadi *et al.* (2023), where we assume general nonlinear cumulative production and demand functions, as opposed to linear ones. The supply chain of the product comprises four periods as considered by Ghaniabadi *et al.* (2023), which includes production, transportation, a regular and a discount selling season. The holding cost of the general nonlinear production and demand functions are then derived, while keeping the holding cost of the transportation phase as in the paper of Ghaniabadi *et al.* (2023) due to the fact that the item quantity is assumed not to change during this phase. A solution process for the corresponding stochastic optimization problem under discrete distribution is proposed by extending the solution procedure of Ghaniabadi *et al.* (2023) which was given for the linear case of the problem. To numerically illustrate the application of the general problem and its solution, we model the holding costs of the problem and derive the corresponding solution procedure of the stochastic problem (by making use of the general model and solution), under the models of Wright (1936) for production and Bass (1969) for the two demand periods, which are flexible and notable models in the literature. We then conduct the numerical experiments of the problem under these specific production and demand function, over synthetic and real sets datasets. We fit the cumulative demand function of the first selling phase over real sets of demand data which we preprocessed. To do so, we take advantage of Scipy Python library which performs the least squares method (commonly used in data science and machine learning) to fit functions over a dataset. The results of experiments demonstrate that the new model which makes use of the more realistic nonlinear production and demand functions, provide solutions that are on average with higher expected profit than the linear case of the problem, the basic newsvendor model, and approximations based on the basic newsvendor problem. Contrary to the results of Ghaniabadi *et al.* (2023), there are instances where the standard newsvendor based approximations give solutions with higher expected profit than the optimal solution of the linear case. The numerical experiments are performed in a way that the Wright's model for the first period and the Bass model for the fourth period, take roughly the same amount of time as their linear counterparts take, given the same maximum production and maximum leftover (at the end of the third phase) values, respectively. This results in solutions which on average of our experiments altogether, are closer to the optimal solution than the approximations based on the basic newsvendor model proposed by Ghaniabadi *et al.* (2023). This indicates that linear case is a reasonably effective approximation when the nonlinear production and demand (for the discount period) functions are approximated by linear functions in a way that roughly the same amount of time is needed to reach the same level of maximum production and maximum leftover, respectively. This corresponds to a constant production rate of the linear case which is equal or close to $\overline{Q}$ divided by the time it takes for the nonlinear cumulative production function to have a total production amount of $\overline{Q}$, and a constant demand rate of the linear case in the fourth period, equal or close to $\overline{Q} - x_1$ divided by the time it takes for the nonlinear cumulative demand function to reach a total demand level of $\overline{Q} - x_1$ in the discount selling phase. A managerial insight of this study is that, accurately considering the form of production and demand functions over time (especially when nonlinearities are involved in that form) for calculating inventory costs and accordingly making decisions on production or ordering quantity, can result in having substantial more profits in the supply chain.

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Chapter 3

Robust Newsvendor Problem with Inventory Costs

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Abstract

In this article, we examine a robust newsvendor model while explicitly considering the holding costs of the corresponding supply chain which consists of a production and a shipment period, a sales season with regular pricing, and a subsequent discount sale season, with a discrete uncertainty set for the demand in the regular sales phase. The production and demand functions are assumed to follow a general nonlinear pattern over time (while the linear function is a special case of this) which affect the amount of total holding cost. We develop an optimal and efficient solution procedure for the problem and perform several experiments leveraging artificial and real data for demand, by comparing

the robust solution, and the solution of the stochastic problem under a discrete probability distribution. The results indicate the non-robustness of the stochastic solution, and the conservativeness of the robust solution.

3.1 Introduction

One of the main issues in supply chain management is the optimization of supply chain decisions based on its multiple phases and the corresponding involved costs and revenues, instead of optimizing according to only a single period. Consequently, in this paper we study a supply chain of four periods which incorporates production, shipment, and a regular sales period, and a second sales period for the discount season, and take into account revenues and costs of the supply chain according to the well-known newsvendor model, with a focus on holding costs (Ghaniabadi *et al.* 2023a and Ghaniabadi *et al.* 2023b). We assume that the demand in the first sales season is uncertain and only the discrete demand scenarios are known (without having the information on the probability distribution of the demand scenarios). We aim to identify the optimal quantity for production, based on a risk averse decision maker who wants to maximize the profit of this supply chain under the worst case that can happen in the first selling phase for the demand scenario with respect to its impact on the profit. Ghaniabadi *et al.* (2023a) and Ghaniabadi *et al.* (2023b) study the same problem under the assumption of knowing the probability distribution of the demand scenarios, while optimizing the production decision based on the expected profit, rather than the worst case. As opposed to the stochastic optimization models developed by Ghaniabadi *et al.* (2023a) and Ghaniabadi *et al.* (2023b), in robust optimization, a major assumption is that we do not have full information on the distribution of the uncertain parameter. Hence, in this article, we examine the robust optimization of the newsvendor problem where holding costs of production and two sales periods given general nonlinear production and demand functions (as in the paper of Ghaniabadi *et al.* 2023b) are taken into consideration, while the case of the linear production and demand functions (as in the paper of Ghaniabadi *et al.* 2023a) is a special case of the nonlinear one. We propose an efficient and optimal solution process for this robust optimization problem, and present the corresponding numerical findings for both linear and nonlinear cases of the problem, using artificial and real demand data as in the articles of Ghaniabadi *et al.* (2023a) and Ghaniabadi *et al.* (2023b), where for the nonlinear case we assume a cumulative production pattern according to Wright's model (Wright 1936) and a cumulative demand pattern according to Bass (1969) demand model (as in the paper of Ghaniabadi *et al.* 2023b), in order to be able to conduct numerical results for the nonlinear production and demand case. Based on our analysis of the experiments, the solution of the stochastic optimization model, which considers the expected profit to optimize production quantity, is not robust, in the sense that it results in low profits (and even a loss in some instances) in the worst case, compared with the solution of the robust optimization problem. On the other hand, the robust solution is conservative, which leads to low expected profits in contrast to the solution of the stochastic optimization problem (which were derived by Ghaniabadi *et al.* 2023a and Ghaniabadi *et al.* 2023b, for linear and nonlinear cases, respectively).

We organize the remaining of this article as follows. In Section 2, we review the related literature for the problem. In Section 3, the robust optimization problem under the discrete uncertainty set is modeled and the corresponding solution process is proposed and proved. In Section 4, we provide the numerical studies, under both linear and nonlinear cases, using artificial and real demand data. In Section 5, the conclusions of the paper are provided.

3.2 Literature review

We divide the related literature in two parts. First we review the papers which incorporate holding costs over the progression of time, into the newsvendor problem. Then we discuss some of the newsvendor literature which take into account the robust decision making.

Matsuo (1990), and Chen and Chuang (2000) studied newsvendor problems with the addition of a time-dependent holding cost as a linear function of the newsvendor decision that is the production or order quantity. We also incorporate such a holding cost in our shipment period, where the inventory level is constant and hence we have a linear function for the holding cost (Ghaniabadi *et al.* 2023a, Ghaniabadi *et al.* 2023b). Tang *et al.* (2018) examined a newsvendor model with additional holding costs for the sale period with a regular pricing, that is a nonlinear function of the decision variable (i.e., the order/production quantity). Given a linear demand model over time, the holding cost function in our third period (i.e., the regular sale period) will be equivalent to the holding cost function of Tang *et al.* (2018) for the same period (as explained by Ghaniabadi *et al.* 2023a). Schlapp *et al.* (2022) also consider a nonlinear model for the progression of cumulative demand over the course of time and the holding cost function that it entails. As explained by Ghaniabadi *et al.* (2023a) and Ghaniabadi *et al.* (2023b) (for linear and nonlinear cases, respectively), the holding cost functions in the paper of Schlapp *et al.* (2022), can be considered equivalent to those in our paper for the shipment and first sales periods (i.e., second and third phases of our supply chain). Ghaniabadi *et al.* (2023a) studied the same four-phase supply chain as in this paper, with newsvendor holding costs, where cumulative production and demand progress linearly over time, under uncertain demand for the first sales phase, for two cases of a general piecewise linear and discrete distributions, and provided the corresponding solution approaches. Under the linear production and demand, over our supply chain, we also take into consideration the same holding cost functions as Ghaniabadi *et al.* (2023a) did, for the discrete demand case. Ghaniabadi *et al.* (2023b) evaluated the same problem as the paper of Ghaniabadi *et al.* (2023a), while extending their model and solution process to the case of nonlinear models for the progression of the cumulative production and demand over time, under discrete demand distribution. For our general nonlinear case and also its special case of Wright's production (Wright 1936) and Bass demand (Bass 1969) models, we also consider the same holding cost functions proved by Ghaniabadi *et al.* (2023b) for these cases.

The problems studied by Ghaniabadi *et al.* (2023a) and Ghaniabadi *et al.* (2023b), assume that full information on the demand probability distribution is available, unlike in robust optimization, where it is assumed that we have only a partial knowledge of the distribution of the uncertain parameter. Scarf (1958) studied the robust newsvendor where only the mean and variance of the demand are assumed to be known (as opposed to our case where we assume we do not have such information and only the demand scenarios are known) and developed a closed form solution under the worst case distribution. Ahmed *et al.* (2007), study a risk averse newsvendor model with multiple selling periods where holding cost is charged for each period for the excess inventory. In addition to the first and second moments, Perakis and Roels (2008) also assume the availability of partial knowledge on the shape (including support, symmetry and mode) of the distribution and provide tractable robust solutions which are not conservative. Natarajan *et al.* (2018) study a distributionally robust newsvendor problem with multiple items, where they take into account the asymmetry in the probability distribution of the demand, and holding costs incur for the unsold inventory. Their numerical results demonstrate that using asymmetry information in the distributionally robust problem is more beneficial compared to the case where only the covariance information is utilized in the problem. Das *et al.* (2021) study the distributionally robust optimization of the newsvendor problem where the αth moment of the demand distribution is known, for any $\alpha \geq 1$. Gao *et al.* (2022) study a new regularization for distributionally robust optimization under Wasserstein metric, and apply their approach to a set of problems, including a multi-item newsvendor problem where holding costs are considered for the excess inventory of each item.

To our knowledge, no paper in the robust newsvendor literature takes into consideration the nonlinear holding cost functions over time for production and demand phases, as in our problem.

3.3 Robust optimization under discrete uncertainty set

In this part, we present a solution approach for the robust optimization of the problem studied by Ghaniabadi *et al.* (2023a) and Ghaniabadi *et al.* (2023b). We study the robust optimization problem where we assume only the support of the uncertain parameter is known under discrete uncertainty set (i.e., we assume that only the scenarios are known, without their probabilities). The goal of our robust problem is to maximize profit under the worst case scenario. We propose an efficient solution procedure to solve the problem.

We consider the same notations incorporated by Ghaniabadi *et al.* (2023b) which had borrowed some of them from Ghaniabadi *et al.* (2023a) and Silver *et al.* (1998). We let $z_j(Q)$ be the profit under scenario x_j and a production amount of Q. Moreover, let $z_j^{'}(Q)$ and $z_j^{"}(Q)$ be the profit of producing an amount of Q under the scenario x_j, where $Q \leq x_j$ and $Q \geq x_j$, respectively. Then we can have $z_j(Q) = \begin{cases} z_j^{'}(Q) & Q \leq x_j \\ z_j^{"}(Q) & Q \geq x_j \end{cases}, \forall j = 1, ..., N$ (where N is the total number of possible scenarios). As formulated in the paper Ghaniabadi *et al.* (2023b), for the newsvendor problem with the holding cost corresponding to general nonlinear functions for production and demand periods, $z_j^{'}(Q)$ is composed of the revenue pQ minus the costs $vQ + H_1(Q) + h_2 t_2 Q + H_3^j(Q)$, while $z_j^{"}(Q)$ is equal to the revenue $px_j + g\left[Q - x_j\right]$ subtracted by the costs $vQ + H_1(Q) + h_2 t_2 Q + h_3\left[Qt_3 - x_j \int_0^{t_3} D_3(t)\, dt\right] +$

$H_4^j(Q)$, where the holding costs $H_1(Q)$, $H_3^j(Q)$, and $H_4^j(Q)$, are as formulated by Ghaniabadi *et al.* (2023b), and represent the holding costs in the production phase, the regular sales season (when the demand is greater than the production quantity), and the discount season (when the demand is less than the production quantity), respectively. Moreover, according to Ghaniabadi *et al.* (2023b), $\frac{dz_j^{'}(Q)}{dQ} = p - v - \frac{dH_1(Q)}{dQ} - h_2t_2 - \frac{dH_3^j(Q)}{dQ}$, and $\frac{dz_j^{''}(Q)}{dQ} = g - v - \frac{dH_1(Q)}{dQ} - h_2t_2 - h_3t_3 - \frac{dH_4^j(Q)}{dQ}$ (as described by Ghaniabadi *et al.* 2023b, the aforementioned profit, holding cost, and derivative functions are also equivalent to those developed by Ghaniabadi *et al.* 2023a, under the assumption of linear production and demand functions). $\forall n = 1,...,N$, we also define $Q_n^{'} = \left\{Q|0 < Q < x_n, \frac{dz_n^{'}(Q)}{dQ} = 0\right\}$ and $Q_n^{''} = \left\{Q|x_n < Q < \overline{Q}, \frac{dz_n^{''}(Q)}{dQ} = 0\right\}$. We then develop the following proposition to solve the robust optimization of the problem under discrete scenarios, which provides the maximum worst-case profit, and is a tractable solution which examines a limited number of possible solutions.

Proposition. The robust optimization of the problem (to maximize the profit under the worst case scenario) can be solved using the following maximization problem.

$$\max\left\{0, \max_{n=1,...,N}\left[\min_{j=1,...,N} z_j(x_n)\right], \max_{n=1,...,N}\left[\min_{j=1,...,N} z_j\left(Q_n^{'}\right)\right],\right.$$
$$\left.\max_{n=1,...,N}\left[\min_{j=1,...,N} z_j\left(Q_n^{''}\right)\right], \min_{j=1,...,N} z_j\left(\overline{Q}\right)\right\}.$$

Proof. Under each scenario x_j (i.e., if we know scenario x_j will happen), the optimal production quantity is either 0, x_j, $Q_j^{'} = \left\{Q|0 < Q < x_j, \frac{dz_j^{'}(Q)}{dQ} = 0\right\}$, $Q_j^{''} = \left\{Q|x_j < Q < \overline{Q}, \frac{dz_j^{''}(Q)}{dQ} = 0\right\}$, or $\overline{Q}$ (which are the same possible optimal solutions developed by Ghaniabadi *et al.* (2023a) and Ghaniabadi *et al.* (2023b) given linear and nonlinear production and demand functions, respectively, under discrete probability distribution, while assuming a single scenario x_j with a probability of occurring equal to 1). Therefore, irrespective of which scenario is the worst case, we can be sure that the value of production quantity which will provide the maximum worst case profit (i.e., the optimal production quantity for the robust optimization problem) is in the set $Q_{RO} = \left\{0, x_n : \forall n = 1,...,N, Q_n^{'} : \forall n = 1,...,N, Q_n^{"} : \forall n = 1,...,N, \overline{Q}\right\}$. Therefore, for each of these possible solutions (i.e., if we produce an amount of $Q \in Q_{RO}$), the worst case profit can be found by solving the minimization problem $\min_{j=1,...,N} z_j(Q)$. Therefore, the maximum worst case profit must be in the set below

$$\left\{ \min_{j=1,...,N} z_j(0), \left[\min_{j=1,...,N} z_j(x_n)\right] : \forall n = 1,...,N, \left[\min_{j=1,...,N} z_j\left(Q_n^{'}\right)\right] : \forall n = 1,...,N, \right.$$
$$\left. \left[\min_{j=1,...,N} z_j\left(Q_n^{"}\right)\right] : \forall n = 1,...,N, \min_{j=1,...,N} z_j\left(\overline{Q}\right) \right\}.$$

Since $\forall j = 1,...,N$, $z_j(0) = 0$, we have $\min_{j=1,...,N} z_j(0) = 0$. Hence, the above set can be simplified as follows.

$$\left\{ 0, \left[\min_{j=1,...,N} z_j(x_n)\right] : \forall n = 1,...,N, \left[\min_{j=1,...,N} z_j\left(Q_n^{'}\right)\right] : \forall n = 1,...,N, \right.$$
$$\left. \left[\min_{j=1,...,N} z_j\left(Q_n^{"}\right)\right] : \forall n = 1,...,N, \min_{j=1,...,N} z_j\left(\overline{Q}\right) \right\}.$$

Consequently, the maximum worst case profit can be obtained by solving the maximization problem below

$$\max \left\{ 0, \left[\min_{j=1,...,N} z_j(x_n)\right] : \forall n = 1,...,N, \left[\min_{j=1,...,N} z_j\left(Q_n^{'}\right)\right] : \forall n = 1,...,N, \right.$$
$$\left. \left[\min_{j=1,...,N} z_j\left(Q_n^{"}\right)\right] : \forall n = 1,...,N, \min_{j=1,...,N} z_j\left(\overline{Q}\right) \right\}$$

which is equivalent to

$$\max\left\{0, \max_{n=1,\ldots,N}\left[\min_{j=1,\ldots,N} z_j\left(x_n\right)\right], \max_{n=1,\ldots,N}\left[\min_{j=1,\ldots,N} z_j\left(Q_n^{'}\right)\right],\right.$$
$$\left.\max_{n=1,\ldots,N}\left[\min_{j=1,\ldots,N} z_j\left(Q_n^{''}\right)\right], \min_{j=1,\ldots,N} z_j\left(\overline{Q}\right)\right\}. \blacksquare$$

3.4 Numerical studies

In this section, we first present the numerical studies of the robust problem under the linear cumulative production and demand case, where we utilize the same parameters used by Ghaniabadi *et al.* (2023a) for both synthetic and real datasets, under the discrete distribution. We then present the numerical studies for the nonlinear case, where the Wright's model (Wright 1936) for the cumulative production function and Bass model (Bass 1969) for the cumulative demand function are used. Using the same data studied by Ghaniabadi *et al.* (2023b), the input parameters for the nonlinear case are the same as those for the linear case (Ghaniabadi *et al.* 2023a) except for the parameters of the Wright's and Bass models which are adopted from Ghaniabadi *et al.* (2023b). The corresponding holding cost functions are also the same as the ones used by Ghaniabadi *et al.* (2023a) and Ghaniabadi *et al.* (2023b), for the linear and nonlinear cases, respectively.

For the linear case, the optimal production quantity and the corresponding optimal profit are denoted by Q^*_{RNVH} and $RNVH(Q^*_{RNVH})$, respectively. As in the paper of Ghaniabadi *et al.* (2023a), we denote the optimal production quantity and the optimal expected profit of the stochastic optimization problem by Q^*_{NVH} and $NVH\left(Q^*_{NVH}\right)$, respectively. Since we use the same input parameters as in Ghaniabadi *et al.* (2023a), the values for Q^*_{NVH} and $NVH\left(Q^*_{NVH}\right)$ are also adopted from Ghaniabadi *et al.* (2023a). Moreover, we report the values for $NVH(Q^*_{RNVH})$ which represent the expected profit of the robust solution Q^*_{RNVH} according to the stochastic optimization problem under discrete distribution proposed by Ghaniabadi *et al.* (2023a). In addition, the values for $RNVH(Q^*_{NVH})$ are also reported, representing the profit of the stochastic optimization solution Q^*_{NVH}, according to the robust optimization problem.

Similarly, we also report these values for the nonlinear case: Q^*_{RNVHN} (the optimal production quantity for the robust problem), $RNVHN(Q^*_{RNVHN})$ (the optimal profit for the robust problem), Q^*_{NVHN} (the optimal production quantity for the stochastic problem), $NVHN(Q^*_{NVHN})$ (the optimal expected profit for the stochastic problem), $NVHN(Q^*_{RNVHN})$ (the expected profit of the robust solution), and $RNVHN(Q^*_{NVHN})$ (the robust profit of the stochastic solution). The notation and values of Q^*_{NVHN}, and $NVHN(Q^*_{NVHN})$ are adopted from Ghaniabadi *et al.* (2023b), since we utilize the same input parameters.

The results of the experiments for the synthetic data and the linear case, are given in the Table 3.1. Overall, the values of $RNVH(Q^*_{NVH})$ are considerably lower than those of $RNVH(Q^*_{RNVH})$, which reveals that the solution to the stochastic optimization problem (i.e., Q^*_{NVH}) is not a robust solution. On the other hand, $NVH(Q^*_{RNVH})$ values are significantly lower than $NVH(Q^*_{NVH})$ values, which suggests that the solution to the robust optimization problem results in low expected profits. The optimal robust solution (i.e., Q^*_{RNVH}) for all the instances of the the Table 3.1, are the same. However, this is not necessarily always the case; for example, for a higher value of the unit holding as $h_1, h_2, h_3, h_4 = 0.08$, we have $Q^*_{RNVH} = 0.925$ and $RNVH(Q^*_{RNVH}) = 2.552$.

In the Table 3.2, we have the results for the SKU bb5419c49b, under the linear case, where $t_3 = 24$ hours. In this case, the stochastic solution Q^*_{NVH} results in a loss (negative value for $RNVH(Q^*_{NVH})$), which indicate that Q^*_{NVH} performs poorly in the robust problem, i.e., in the worst case scenario. The robust solution Q^*_{RNVH} also provides significantly less expected profit compared with the stochastic solution Q^*_{NVH}. The results for the same SKU under the linear case and $t_3 = 42$ days, provided in the Table 3.3, demonstrate similar results, although the loss of the stochastic solution Q^*_{NVH} under the worst case (i.e., in the robust problem) becomes even larger, which can be due to a higher holding cost as a result of having a supply chain with a longer duration. In both cases, we can get an optimal robust solution which is different than the ones in the tables, for a larger value of unit holding cost, e.g., under the case of $t_3 = 24$ hours for $h_1, h_2, h_3, h_4 = 0.6$, we have $Q^*_{RNVH} = 0.375$ and $RNVH(Q^*_{RNVH}) = 3.590$, while under the case of $t_3 = 42$ days for $h_1, h_2, h_3, h_4 = 0.009$, we have $Q^*_{RNVH} = 15.476$ and $RNVH(Q^*_{RNVH}) = 91.609$.

For the SKU d17d9135b0, the numerical results are in the Table 3.4, which again demonstrate that the stochastic solution Q^*_{NVH} is not a robust one (although it no longer results in a negative profit in the worst case, as opposed to the results of the SKU bb5419c49b), while the robust solution Q^*_{RNVH} does not provide expected profits close to the optimal one which is Q^*_{NVH}. In this case also even the optimal robust solution remains the same for all instances in the table, for a higher value of unit holding cost, it may not be the case, as in when $h_1, h_2, h_3, h_4 = 0.11$, we have $Q^*_{RNVH} = 5.434$ and $RNVH(Q^*_{RNVH}) = 14.961$.

In the tables 3.1, 3.2, and 3.4, the values of $RNVH(Q^*_{RNVH})$ and $NVH(Q^*_{RNVH})$ are relatively close. Hence, it may be worthwhile to investigate what causes this proximity, especially in regards to the structure of profit function in these cases. Moreover, another analysis can be done to find out the likelihood of having a negative value for $RNVH(Q^*_{NVH})$, which is the case in the tables 3.2 and 3.3.

The results for the nonlinear counterparts of the above case are presented in the Tables 3.5, 3.6, 3.7, and 3.8, respectively. The observations are similar to their corresponding linear cases where the optimal robust solution Q^*_{RNVH} can be too conservative compared with the stochastic solution Q^*_{NVH}, while Q^*_{NVH} does not provide acceptable robust solutions. Moreover, for a sufficiently higher value for the unit holding cost, in the nonlinear case one can also reach solutions which are different from the ones in the corresponding table.

Table 3.1: The numerical results for robust optimization problem under discrete uncertainty set for synthetic data and linear case ($N = 100$).

h_1, h_2, h_3, h_4	Q^*_{RNVH}	$RNVH$ (Q^*_{RNVH})	$NVH(Q^*_{RNVH})$	Q^*_{NVH}	$NVH(Q^*_{NVH})$	$RNVH(Q^*_{NVH})$
10^{-100}	1.017	10.170	10.170	2.757	18.850	8.429
0.00275	1.017	9.907	9.932	2.683	17.823	7.318
0.004125	1.017	9.775	9.814	2.634	17.336	6.829
0.0055	1.017	9.644	9.695	2.596	16.857	6.349
0.006875	1.017	9.513	9.576	2.538	16.386	5.949
0.00825	1.017	9.381	9.458	2.488	15.941	5.567

Table 3.2: The numerical results for robust optimization problem under discrete uncertainty set for real data (SKU bb5419c49b) and linear case ($t_3 = 24$ hours, $N = 5$).

h_1, h_2, h_3, h_4	Q^*_{RNVH}	$RNVH(Q^*_{RNVH})$	$NVH(Q^*_{RNVH})$	Q^*_{NVH}	$NVH(Q^*_{NVH})$	$RNVH(Q^*_{NVH})$
10^{-100}	0.4	9.5740	9.5740	2.0	32.1066	-6.4260
0.000685	0.4	9.5672	9.5693	2.0	32.0324	-6.5446
0.0010275	0.4	9.5637	9.5669	2.0	31.9953	-6.6040
0.00137	0.4	9.5603	9.5646	2.0	31.9582	-6.6633
0.0017125	0.4	9.5569	9.5622	2.0	31.9211	-6.7226
0.002055	0.4	9.5535	9.5599	2.0	31.8840	-6.7819

Table 3.3: The numerical results for robust optimization problem under discrete uncertainty set for real data (SKU bb5419c49b) and linear case ($t_3 = 42$ days, $N = 5$).

h_1, h_2, h_3, h_4	Q^*_{RNVH}	$RNVH(Q^*_{RNVH})$	$NVH(Q^*_{RNVH})$	Q^*_{NVH}	$NVH(Q^*_{NVH})$	$RNVH(Q^*_{NVH})$
10^{-100}	16.8	402.108	402.108	84.0	1348.479	-269.892
0.000685	16.8	378.425	382.197	84.0	1159.550	-537.177
0.0010275	16.8	366.583	372.242	84.0	1065.085	-670.819
0.00137	16.8	354.741	362.287	84.0	970.620	-804.462
0.0017125	16.8	342.899	352.331	84.0	876.156	-938.104
0.002055	16.8	331.058	342.376	71.811	789.644	-765.623

Table 3.4: The numerical results for robust optimization problem under discrete uncertainty set for real data (SKU d17d9135b0), and linear demand case ($N = 5$).

h_1, h_2, h_3, h_4	Q^*_{RNVH}	$RNVH(Q^*_{RNVH})$	$NVH(Q^*_{RNVH})$	Q^*_{NVH}	$NVH(Q^*_{NVH})$	$RNVH(Q^*_{NVH})$
10^{-100}	5.7	36.400	36.400	28.5	48.143	22.401
0.0001085	5.7	36.379	36.380	26.058	47.277	23.071
0.00016275	5.7	36.368	36.370	19.010	47.033	27.633
0.000217	5.7	36.358	36.360	17.1	46.894	28.786
0.00027125	5.7	36.347	36.350	17.1	46.762	28.632
0.0003255	5.7	36.337	36.340	17.1	46.630	28.478

Table 3.5: The numerical results for robust optimization problem under discrete uncertainty set for synthetic data and nonlinear case ($N = 100$).

h_1, h_2, h_3, h_4	Q^*_{RNVHN}	$RNVHN$ (Q^*_{RNVHN})	$NVHN$ (Q^*_{RNVHN})	Q^*_{NVHN}	$NVHN$ (Q^*_{NVHN})	$RNVHN$ (Q^*_{NVHN})
10^{-100}	1.017	10.170	10.170	2.757	18.850	8.429
0.00275	1.017	9.900	9.921	2.717	17.880	7.275
0.004125	1.017	9.765	9.797	2.634	17.414	6.840
0.0055	1.017	9.631	9.673	2.633	16.958	6.271
0.006875	1.017	9.496	9.549	2.596	16.509	5.797
0.00825	1.017	9.361	9.424	2.511	16.071	5.482

Table 3.6: The numerical results for robust optimization problem under discrete uncertainty set for real data (SKU bb5419c49b) and nonlinear case ($t_3 = 24$ hrs, $N = 5$).

h_1, h_2, h_3, h_4	Q^*_{RNVHN}	$RNVHN$ (Q^*_{RNVHN})	$NVHN$ (Q^*_{RNVHN})	Q^*_{NVHN}	$NVHN$ (Q^*_{NVHN})	$RNVHN$ (Q^*_{NVHN})
10^{-100}	0.4	9.5740	9.5740	2.0	32.1066	-6.4260
0.000685	0.4	9.5638	9.5654	2.0	31.9953	-6.6247
0.0010275	0.4	9.5587	9.5612	2.0	31.9396	-6.7241
0.00137	0.4	9.5536	9.5569	2.0	31.8839	-6.8235
0.0017125	0.4	9.5485	9.5526	2.0	31.8282	-6.9229
0.002055	0.4	9.5434	9.5483	2.0	31.7725	-7.0222

Table 3.7: The numerical results for robust optimization problem under discrete uncertainty set for real data (SKU bb5419c49b) and nonlinear case ($t_3 = 42$ days, $N = 5$).

h_1, h_2, h_3, h_4	Q^*_{RNVHN}	$RNVHN$ (Q^*_{RNVHN})	$NVHN$ (Q^*_{RNVHN})	Q^*_{NVHN}	$NVHN$ (Q^*_{NVHN})	$RNVHN$ (Q^*_{NVHN})
10^{-100}	16.8	402.1080	402.108	84.0	1348.479	-269.892
0.000685	16.8	373.591	376.485	84.0	1107.487	-657.996
0.0010275	16.8	359.332	363.673	84.0	986.991	-852.048
0.00137	16.8	345.074	350.862	84.0	866.494	-1046.100
0.0017125	16.8	330.815	338.050	67.2	755.245	-808.021
0.002055	16.8	316.556	325.238	50.4	687.229	-487.621

Table 3.8: The numerical results for robust optimization problem under discrete uncertainty set for real data (SKU d17d9135b0) and nonlinear case ($N = 5$).

h_1, h_2, h_3, h_4	Q^*_{RNVHN}	$RNVHN$ (Q^*_{RNVHN})	$NVHN$ (Q^*_{RNVHN})	Q^*_{NVHN}	$NVHN$ (Q^*_{NVHN})	$RNVHN$ (Q^*_{NVHN})
10^{-100}	5.7	36.400	36.400	28.5	48.143	22.401
0.0001085	5.7	36.351	36.352	17.1	46.411	28.164
0.00016275	5.7	36.326	36.328	17.1	45.907	27.546
0.000217	5.7	36.302	36.304	17.1	45.402	26.928
0.00027125	5.7	36.277	36.280	17.1	44.898	26.310
0.0003255	5.7	36.252	36.255	17.1	44.394	25.692

3.5 Conclusion

This article studies a robust newsvendor problem where holding costs of a supply chain with production, shipment, and two sale seasons (with regular and discount selling prices) are taken into consideration, while the goal is to maximize the profit according to the worst case scenario of the demand, and making a decision accordingly on the optimal production quantity. We proved an optimal and efficient solution process for the problem, under discrete uncertain demand set. We also conducted several numerical experiments leveraging artificial and real demand datasets, under both linear and nonlinear production and demand functions. The numerical analysis illustrates that the stochastic optimization problem under a given probability distribution does not provide profitable solutions compared with the solution of the robust optimization problem, while the optimal robust solution is conservative and yield low expected profit. This result suggests that developing distributionally robust solutions can be a promising future research direction for this problem, where the the goal can be developing robust solutions with acceptable expected profit.

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General Conclusion

In this dissertation, we studied a newsvendor problem with the addition of quantity-and-time-dependent holding costs calculated over the progression of time, in a supply chain consisting of production, shipping, and two selling seasons, first with a regular pricing and then with discount. The first paper, examined linear cumulative production and demand functions of time, considering general piecewise linear and discrete distributions for the uncertain demand in the first selling season, while the second paper took into account general nonlinear production and demand functions of time (with Bass (1969) demand and Wright (1936) production functions, as a special case), and discrete distribution. In both papers, we derive optimal and efficient solution processes, in regards to maximization of the expected profit of the problem, and provide numerical results based on synthetic and real datasets, which illustrate that overall both cases outperform the approximations based on the standard newsvendor problem in terms of the expected profit, while the nonlinear case outperformed the linear one as well. Under longer duration for production and demand phases, and also higher holding costs, the profitability of our models compared with the standard newsvendor model, becomes more significant. In the third paper, we use the same models of the problem in the first two papers, under discrete demand scenarios, while taking into account the goal of optimizing based on the worst possible scenario. The corresponding numerical experiments show that the robust solution is conservative while the solution based on maximizing the expected profit is not robust.

The findings in this dissertation can help managers in manufacturing and purchasing sectors decide on how much of a product to produce or order so that the expected profit of the corresponding supply chain is maximized, while considering the holding costs of the supply chain, and utilizing the available data for the product demand, and also on how the total production and demand may vary over time during production and selling periods, respectively. Moreover, the efficiency of the proposed solution, helps in making timely decisions. A managerial insight implied from this work is that, holding costs should be accurately taken into account when the goal is to make profitable decisions related to production or ordering amount of a product, especially when the supply chain involves high holding costs and long periods. For example, for selling seasons which last for a few weeks or a few months it can be more critical to take into account the holding costs when making decisions on the production quantity, in order to avoid the profit loss incurred as a result of making that production decision without considering the relevant holding costs.

The models and solutions proposed in this dissertation provide optimal solutions for the cases where either the goal is maximizing the expected profit (i.e., the first two articles) or maximizing the profit under the worst case scenario (i.e., the third article). Hence, it may be worthwhile to investigate approaches which can provide solutions that have high expected profits and are not too conservative, while at the same time are robust. Accordingly, a distributionally robust optimization approach can be a promising research avenue in order to diminish the aforementioned limitation of this dissertation.

Some other limitations and possible extensions of this dissertation are outlined as below.

In this dissertation, we assume that the length of the regular selling season is fixed. This is the case for many practical situations, for example, Christmas sales season or the fashion sales for the 4 annual seasons, or fast fashion which can take a few weeks. In the book of Cachon and Terwiesch (2008) also provides an example for O'Neill's wetsuit sales where ordering to receipt can take 3 months (this can be considered equivalent to our production and transportation periods), and the Spring selling period (equivalent to our regular selling season) can take 6 months, from February to July, which is a fixed period. Nevertheless, in case we assume that the length of regular selling season is variable and depending on the inventory amount and the demand function it may have a different duration, the corresponding profit function for this period can be derived using a process similar to our discount season which has a variable length and depends on the leftover amount and the demand function.

Another assumption is that we know how the demand evolves over time during the selling season. Nevertheless, in the nonlinear demand case of our problem, it is possible to extend our model to consider a different cumulative demand function for each particular demand scenario. For instance, for the Bass (1969) demand model, we can extend the model to use a different Bass model for each demand scenario.

Although in the linear demand case of the problem we assume that we have a constant demand rate in the discount selling phase, in the nonlinear case we can have a variable demand rate over this season, since we use a nonlinear cumulative demand function which can have a demand rate which varies over time.

We also have the assumption of having a sufficiently long discount season with a variable length during which all the items are sold. In case in a particular situation this is not relevant and instead we have a shorter and fixed period during which only a portion of the leftover inventory is sold, one can use a process similar to our regular selling season which is also fixed, to model the profit function for this fixed discount season. In case there is excess inventory at the end of the fixed discount season, they can be sold at a single point in time (as in the standard newsvendor model) with a price lower than that of the fixed discount season to a third-party.

The structure of the profit functions can also be analyzed in more depth, especially through establishing the concavity results wherever possible. We have shown the concavity of the problem in the linear demand case under the general distribution, using the second order derivative. Analyzing other profit functions of this dissertation through obtaining and analyzing their second order derivative, can provide valuable insights in regards to the structure of the profit functions.

This dissertation does not provide optimal solutions for the problem under arbitrary continuous probability distributions for the demand. Nevertheless, since we provide efficient solution processes for the problem under the discrete probability distribution for both linear and nonlinear demand cases, we can leverage those solution approaches to solve the problem under any continuous probability distribution, using the Sample Average Approximation (SAA) method. Due to efficiency of the proposed solutions, it is expected that using the SAA approach will result in approximate solutions close to the optimal one, since we will be able to utilize a large number of samples in the problem.

Aside from the optimal solution approaches made in this dissertation, it is also possible to approximately solve the problem by enumeration where one can only consider a limited set of discrete values for the decision variable which is the production quantity. This method can also result in effective approximate solutions for the problem.

We can also take into account an extension of the problem where we have a discrete set of possible prices for the discount season. Depending on each of these prices, we can have a different demand pattern for the discount season. In such a case, we can utilize the already developed solution processes where for each of the possible prices, we can solve a different problem, and select the price which results in the highest optimal expected profit. Since for each price level a different problem must be solved, the computational tractability will also depend on the number of possible price levels in the pricing problem.

In case the firm wants to produce a new product where we have no demand data to model the problem, one can use the historical demand data of the products which have similar characteristics to the new product. Machine learning approaches could be used to predict the demand for such new products based on such similar characteristics of older products.

In our numerical experiments in the third paper, the minimum demand was the worst case scenario in the optimal robust solution. Hence, it can be interesting to see if one can prove whether the optimal robust solution can be obtained only under the minimum demand scenario, especially under the case of linear production and demand, or under Wright's model for production and Bass model for demand. If so, we can have an even more efficient solution for the robust problem under such cases.

It may also be worthwhile to analyze the performance of the developed optimal solution processes in this dissertation in terms of their computational time. Some of the preliminary results in this regard are as follows, using Google Colab (with Intel Xeon CPU, and 13GB of RAM). For the problem in the first paper, for the discrete distribution case an instance with 1000, 2000, 4000, and 10000 number of demand scenarios could be solved in 1.4, 5.1, 21.3, and 151.6 seconds, respectively, while for the GPLD case, an instance with 50, 100, 200, and 400 number of GPLD intervals was solved in 1.5, 5.49, 112.87, and 819.71 seconds, respectively. For the problem in the second paper, under Wright's model for production and Bass model for demand, an instance with 50, 100, 250, and 500 number of demand scenarios could be solved in 5.8, 19.5, 73.4, and 292.6 seconds, respectively. For the problem in the third paper, in the linear case, we could have the solution for an instance with 1000, 2000, 4000, and 10000 number of scenarios in 2.1, 16.8, 87.2, and 261.9 seconds, respectively, while for the nonlinear case (under Wright's model for production and Bass model for demand), an instance with 250, 500, 1000, and 2000 number of scenarios could be solved in 6.5, 21.7, 83.7, and 193.6 seconds, respectively. These computational results also demonstrate the efficiency of the proposed approaches in this dissertation.

We can also study an extension of the problem where we have multiple items with a common capacity constraint involving all items. For the problem in the first two papers, one could leverage our efficient solution under the discrete distribution case, and use the Lagrangian relaxation of the capacity constraint in the profit function, to reach a heuristic solution using the Lagrange multipliers method.

Another possible extension could be considering the updated information on the future demand during the production stage and study how this can affect the problem and how we will need to accordingly adjust the decision on the production quantity. Moreover, during the regular selling season we can get a more realistic view of how the future demand will be in the remaining parts of this season, and hence have production and shipping in parallel with the selling phase, especially if we have relatively fast production and transportation for the corresponding supply chain.

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