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Integrated Production and Transportation Planning

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Résumé

La pratique et la littérature sur la planification logistique intégrée mettent en évidence d'importants potentiels d'économies dans les chaînes d'approvisionnement. Les études sur ces problèmes de prise de décision intégrée se concentrent principalement sur l'aval de la chaîne d'approvisionnement. Cette thèse comble le vide en étudiant le problème intégré de tournées de véhicules pour le transport entrant, de planification de la production et de gestion des stocks. Dans le premier chapitre, nous étudions un modèle général pour le problème de tournées et assemblage (ARP), qui consiste à planifier simultanément l'assemblage d'un produit fini dans une usine et les tournées des véhicules collectant des matières auprès des fournisseurs pour répondre aux exigences de stocks imposées par la production. Chaque fournisseur fournit un composant unique nécessaire à la production du produit final dans l'usine de fabrication. Nous formulons le problème comme un programme linéaire en variables mixtes et nous proposons une matheuristique de décomposition en trois phases qui s'appuie sur la solution itérative de différents sous-problèmes. L'algorithme est flexible et nous montrons comment il peut également être utilisé pour résoudre deux problèmes de distribution bien connus liés à l'ARP: le problème de tournées et de production (PRP) et le problème de tournées et de gestion des stocks (IRP). En particulier, sur les instances multi-véhicules à grande échelle, le nouvel algorithme surpasse les heuristiques spécialisées de pointe pour ces deux problèmes.

Dans le deuxième chapitre, nous étendons le champ d'application pour considérer le cas où chaque fournisseur peut fournir un sous-ensemble des composants nécessaires au produit final et où certains composants peuvent être obtenus auprès de plusieurs fournisseurs. Nous proposons une formulation de programmation en nombres entiers mixtes du problème et proposons plusieurs familles d'inégalités valides pour renforcer la relaxation de programmation linéaire. Nous proposons deux nouveaux algorithmes pour séparer les contraintes d'élimination des sous-circuits pour les solutions fractionnaires. Les inégalités et les procédures de séparation sont utilisées dans un algorithme de séparation et de coupe. Des expériences de calcul sur un grand nombre d'instances générées aléatoirement montrent que les inégalités valides et les nouvelles procédures de séparation améliorent considérablement les performances de l'algorithme.

Dans le troisième chapitre, nous étudions un autre problème pratique et complexe se posant dans le contexte de la planification logistique intégrée. Les nombreuses études sur ces problèmes supposent des durées de production et de planification des itinéraires identiques. Nous présentons des modèles de programmation mathématique et des méthodes de résolution qui ne reposent pas sur cette hypothèse. Par conséquent, nous considérons la possibilité d'avoir différentes durées de production et de planification des tournées. En plus, nous considérons la production de différents types de produit. Nous développons des modèles et des algorithmes de résolution exacts pour optimiser simultanément la production, les tournées de transport, ainsi que les décisions d'expédition et de gestion des stocks.

Mots-clés

Logistique, assemblage, production, gestion des stocks, distribution, tournées, heuristiques, décomposition matheuristique, inégalités valides, méthodes de résolution exactes, séparation et coupes

Méthodes de recherche

Recherche opérationnelle, programmation mathématique

Abstract

The practice and the literature on integrated logistics planning highlight a significant potential for cost savings in supply chains. The studies on these integrated decision-making problems are mostly focused on the downstream supply chain. This thesis fills the gap by studying the problem of integrated inbound transportation routing, production, and inventory planning. In the first chapter, we study a general model for the assembly routing problem (ARP), which consists of simultaneously planning the assembly of a finished product at a plant and the routing of vehicles collecting materials from suppliers to meet the inventory requirements imposed by the production. Each supplier provides a unique component necessary for the production of the final product at the manufacturing plant. We formulate the problem as a mixed-integer linear program and we propose a three-phase decomposition matheuristic that relies on the iterative solution of different subproblems. The algorithm is flexible and we show how it can also be used to solve two well-known outbound distribution problems related to the ARP: the production routing problem (PRP) and the inventory routing problem (IRP). In particular, on large-scale multi-vehicle instances, the new algorithm outperforms specialized state-of-the-art heuristics for these two problems.

In the second chapter, we extend the scope to consider the case where each supplier may provide a subset of the components necessary for the final product and where some components can be obtained from more than one supplier. We provide a mixed integer programming formulation of the problem and propose several families of valid inequalities to strengthen the linear programming relaxation. We propose two new algorithms to separate the subtour elimination constraints for fractional solutions. The inequalities and separation procedures are used in a branch-and-cut algorithm. Computational experiments on a large set of randomly generated test instances show that both the valid inequalities and the new separation procedures significantly improve the performance of the branch-and-cut algorithm.

In the third chapter, we study another practical and complicated problem in the context of integrated logistics planning. The numerous studies in the literature on these problems all assume identical production and route planning period lengths. We present mathematical programming models and solution methods that do not rely on this assumption. Hence, we consider the possibility of having different production and route planning period lengths. Furthermore, we consider the production of different types of products. We develop models and exact solution algorithms to simultaneously optimize the production setup and quantity, transportation and routing, and shipment and inventory decisions.

Keywords

Logistics, assembly, production, inventory, distribution, routing, heuristics, decomposition matheuristic, valid inequalities, exact solution method, branch-and-cut

Research methods

Operations research, mathematical programming

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List of acronyms

ARP Assembly Routing Problem

BB Branch-and-Bound

BC Branch-and-Cut

CCC Capacity-Cut Constraint

DFJ Dantzig-Fulkerson-Johnson

GFSEC Generalized Fractional Subtour Elimination Constraint

IRP Inventory Routing Problem

JIT Just-In-Time

LB Lower Bound

LP Linear Program

LR Lagrangian Relaxation

LSP Lot-Sizing Problem

MILP Mixed Integer Linear Program

MIP Mixed Integer Program

MTZ Miller-Tucker-Zemlin

PRP Production Routing Problem

SEC Subtour Elimination Constraint

TSP Travelling Salesman Problem

UB Upper Bound

VRP Vehicle Routing Problem

WW Wagner-Whitin

To

Julian & Dylan

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General Introduction

To stay on top of the competition, more and more companies have to integrate their supply chains. The effective implementation of this integration is key in making a difference. A well-integrated supply chain allows for a reduction in wasted time and materials, tightens the coordination between production, warehousing, and shipment planning, and lowers overall costs. The integration requires a large-scale change in the decision-making process across the chain. Every link in the chain benefits through sharing information and working with other members and with customers. Traditional supply chain planning consists of scheduling the sequential processes of production, storage, and distribution. Each process is usually planned and optimized having the decisions from the previous process fixed. Reaching a level of integration that creates synergy throughout the chain requires that the production and transportation decisions be optimized together (Christopher, 1998; Simchi-Levi et al., 2013). Several studies and success stories have shown a significant potential for cost savings in the supply chain by combining production and transportation decisions (Chandra and Fisher, 1994; Arntzen et al., 1995; Viswanathan and Mathur, 1997; Fumero and Vercellis, 1999; Brown et al., 2001; Chen and Vairaktarakis, 2005; Rudberg and Cederborg, 2011; Steinrücke, 2011; Archetti and Speranza, 2016). More specifically, these studies highlight the need for the optimization and management of the entire supply chain as a single entity to obtain cost reduction advantages and hence service enhancements.

Considering all the decision levels in a single framework offers a holistic view of the logistics network planning and provides a starting point for the full integration of the supply chain. Due to the importance of the integration, vendor-managed inventory (VMI) initiatives have become an increasingly effective process and business model to help organizations share risk and information between suppliers and customers. VMI connects suppliers to customers, with the former making the replenishment decisions for products supplied to the latter, based on specific inventory and supply chain policies. VMI is often described as a win-win collaboration to benefit from lower stockouts, reduced uncertainty, and lower costs. Particularly, suppliers save on distribution and production costs as they are able to coordinate demand and combine shipments for different customers. Customers save by allocating the smallest necessary efforts to control and manage inventories. The VMI approach cuts costs by benefiting from economies of scale when a supplier plans the shipments to many retail stores in the downstream of the supply chain.

In this context, different problems have been studied to optimize the necessary processes. Examples include two classical problems in logistics, namely lot-sizing and vehicle routing, which were introduced by Wagner and Whitin (1958), and by Dantzig and Ramser (1959), respectively. The lot-sizing problem (LSP) consists of determining production lot sizes and inventory levels over a given planning horizon. The vehicle routing problem (VRP) consists of designing vehicle routes to make deliveries to customers in each period. Each of these problems has been the subject of numerous studies, yet most of them focus on a single problem and very few address the integration of the two problems. However, focusing on the cost minimization in one sub-process typically leads to higher costs in the other. The problem of simultaneously planning the production at a manufacturing plant and the outbound delivery routing is known in the literature as the production routing problem (PRP) (Archetti et al., 2011; Adulyasak et al., 2015). When the production plan at the factory is given and the decisions concern only the inventory and route planning, the problem is referred to as the inventory routing problem (IRP) (Andersson et al., 2010; Coelho et al., 2013). In contrast, few studies have considered the integration of production planning with inbound transportation for the collection of components from suppliers to assemble a final product. In this thesis, we investigate the problem of achieving this integration in the upstream supply chain. When the manufacturer is the largest player in the chain and has the opportunity to benefit from economies of scale, it can be responsible for organizing the inbound transportation of the various components. Significant gains can be achieved in such a case by integrating production planning with inbound transportation (Carter and Ferrin, 1996).

In a standard supply chain, a manufacturing plant often uses several different components to assemble a final product. These components are typically produced in other plants or purchased from suppliers. If the assembly plant is responsible for organizing the inbound transportation of the various components, then gains can be achieved by integrating the production planning with the inbound vehicle routing. We refer to this problem as the assembly routing problem (ARP). The ARP considers a joint planning problem with a primary manufacturing plant that produces a final product to meet a dynamic but deterministic demand. The factory gets the essential components from diverse suppliers, each providing a subset of the components. The plant coordinates the production scheduling as well as the routing decisions and shipment quantities from the suppliers. The aim is to minimize the total costs of production, inventory and routing subject to certain constraints. The planning is done over a finite and discrete-time horizon. The quantities available at the suppliers are assumed to be known in advance. The factory has a limited capacity for the production and no backlogging or stockouts are allowed. Both the factory and the suppliers can carry inventory. The factory has separate and capacitated inbound and outbound storage areas for the incoming components from suppliers and for the final product, respectively. Each supplier has a global storage capacity for its own components. The plant manages a limited fleet of capacitated vehicles to handle the shipment of components from the suppliers to the factory. Similar to the standard variants of the IRP and PRP, we do not allow a supplier to be visited by more than one vehicle in a specific period (i.e., no split pickups). The objective is to optimize the following decisions simultaneously:

- When to produce at the central manufacturing plant and the quantity to be produced,

- Routing decisions for the planning horizon which include the supplier visit timing and vehicle assignment, and

- The quantity of each component to be transported from the suppliers to the factory.

A solution to this problem, within the planning horizon, gives the production amounts that respect the manufacturing and inventory keeping limitations. It also includes the visit schedules for each supplier and the amount of the components to be shipped to the factory. The visit and shipment schedules satisfies the storage and transportation capacities.

To the best of our knowledge, the problem of jointly optimizing production planning and inbound vehicle routing with a finite horizon and discrete planning periods has only been considered by Hein and Almeder (2016). The authors consider two scenarios. In the first scenario, the plant is allowed to keep the components in stock while in the second scenario, which represents a JIT environment, the components that arrive at the plant must be used immediately in production. They examine both scenarios under the traditional sequential planning approach and under the integrated approach. In the sequential planning process, an LSP is solved first to obtain the production plan for the final product. Then, in the second step, they solve an IRP for the first scenario and one vehicle routing problem (VRP) for each period in the second scenario. Because the authors did not consider the holding cost at the suppliers in their study, the integrated decision-making is entirely focused on the costs associated with the plant. This is appropriate when the suppliers and the assembly plant are separate organizations and the assembly plant is not concerned with the inventory costs at the suppliers. Hence, the gap to integrate the upstream supply by considering the costs at the suppliers is not fully addressed yet.

Integrated logistics planning for manufacturers and their suppliers is a relevant practical problem in several business domains. This thesis is motivated in part by the practice of many U.S. and German auto manufacturers that realized the value of the integrated production and transportation. Fleischmann and Meyr (2003) indicate that in the automotive industry the organization that receives the components is usually responsible for the supply transport. Florian et al. (2011) show that in addition to the direct financial benefits for the supply chain, inbound logistics integration for a German car manufacturer has some further important outcomes such as a reduction in CO_2 emissions. In an application for the Delco Electronics Division of General Motors (GM), Blumenfeld et al. (1987) find that the overall optimization of the inbound transportation resulted in a

26% (2.9 million dollars per year, in 1987's USD) savings potential. Closed loop supply chain is another example in which the collection of the end-of-life products should be coordinated with the disassembly planning (Guide and Van Wassenhove, 2009).

Danese (2006) presents the case of GlaxoSmithKline (GSK), an international pharmaceutical group that extended the VMI approach to its suppliers as a response to the highly competitive and regulated market to benefit from the integrated and coordinated planning process. To benefit from economies of scale in shipping products to the stores, the concept of factory gate pricing (FGP) has emerged in the retail sector (Whiteoak, 1994; Le Blanc et al., 2006; Fernie and Sparks, 2014). Under FGP, the supplier no longer delivers the products to the customer but makes them available at its own factory gate (Le Blanc et al., 2006). This requires the customer to plan and synchronize the pickups from the suppliers to reduce the transportation costs as reported by a number of FGP studies. Examples are Le Blanc et al. (2006) for a large Dutch retail distribution company and Potter et al. (2007) for UK retailers.

Furthermore, we investigate, in this thesis, a generalized PRP which takes into account the fact that the production planning and the route planning period lengths are not necessarily identical. The overall planning horizon may, as a consequence, contain a different number of production and route planning periods. In such cases, the capacity of the production and routing may be expressed in a different time dimension, which creates the need to have a decoupled discretization of the time horizon. To the best of our knowledge, this is the first effort in looking at this problem with this generality. For the lot-sizing part of the formulation, we will consider multiple products and both big-bucket and small-bucket problems (Pochet and Wolsey, 2006). In a big-bucket model, it is possible to produce several different types of items within the same planning period whereas in a small-bucket model only one type of item can be produced in a specific time period. A single manufacturing plant synchronises the production scheduling for these multiple products as well as the routing decisions and shipment quantities to the customers. Demand at the customers is time-varying and predetermined for each product. The aim is to minimize the total costs of production, inventories and distribution routing subject to the limitations of the problem. Storage capacities as well as truck capacities are limited. Backlogging, stockouts, and split deliveries are allowed. Due to the difference in the planning period lengths for the production planning and for the distribution routing, the mathematical models will be different from the basic PRP. In practice, multiple periods of distribution and transportation exist within one production planning period or vice versa. For example, daily distribution routing decisions might have to be made together with weekly production scheduling. Consequently, an important aspect of these multi-period problems is to deal with the different period lengths while properly representing the available capacity.

As the benefits of considering and solving such integrated supply chain planning problems highly depend on the quality and performance of the solution methods, this thesis not only presents new models and frameworks but also aims to develop efficient algorithms to solve these models. This thesis is composed of three papers.

Chapter 1 presents:

Chitsaz, M., Cordeau, J.-F., & Jans, R. (2019). A unified decomposition matheuristic for assembly, production, and inventory routing. INFORMS Journal on Computing, 31(1), 134-152.

In this paper, we present the ARP in detail. Moreover, we develop a heuristic method capable of solving not only this problem but also the IRP and PRP. The algorithm decomposes the problem into three separate subproblems. The first subproblem is a special lot-sizing problem that uses the number of dispatched vehicles to approximate the routing costs in the objective function. A solution to this subproblem provides a given setup schedule. Given this setup schedule, the second subproblem uses a transportation cost approximation associated with each supplier visit, and schedules the visits and determines the shipment quantities. For multi-vehicle instances, a modified model of the second subproblem is employed in this phase to look for possible improvements in finding better supplier visits and shipment volumes. The third subproblem solves a series of separate vehicle routing problems, one for each planning period. This procedure is repeated for a number of iterations to reach a local optimum. The solutions of the routing subproblems are used to update the supplier visit (transportation) cost approximation in the second subproblem. To escape a local optimum, a local branching scheme forces a change in the setup schedule and hence creates entirely new solutions. The entire procedure continues until a stopping condition is met. We introduce many ARP instances on which we report the performance of the algorithm. Moreover, we show the excellent performance of the algorithm on standard data sets for the IRP and PRP. In particular, the algorithm outperforms the state-of-the-art heuristics on large-scale multi-vehicle instances for these problems. In addition, the results confirm the robust behavior of the algorithm in tackling different problems, several data sets, and various sizes of instances. This paper was selected as one of the four finalists in the 2017 best student paper competition of the Canadian Operational Research Society (CORS).

Chapter 2 presents:

Chitsaz, M., Cordeau, J.-F., & Jans, R. (2020). A branch-and-cut algorithm for an assembly routing problem. European Journal of Operational Research, 282(3), 896-910.

In this paper, we generalize the ARP of the first paper (Chapter 1). In that first paper, we assume that every supplier provides a unique component and hence a one-to-one relationship between the supplier and component sets. In this paper, we relax this assumption to consider the case where each supplier potentially provides a subset of the components necessary for the final product and some components are sourced from more than one supplier. Moreover, we develop several new valid inequalities to strengthen the linear programming relaxation of the mixed integer programming formulation of the problem. Three classes of valid inequalities are presented. The first class contains *(l,S,WW)*-type inequalities (Barany et al., 1984; Pochet and Wolsey, 1994). The second one concerns tightening the bounds on the binary and integer decision variables. The last class includes general inequalities for the problem. A novelty in the proposed inequalities, compared to the existing ones in the literature of the LSP, is that some of them use the known supply instead of the known demand. The inequalities are used in a branch-and-cut algorithm. We adapt the unified method proposed in Chapter 1 and apply it to the generalized ARP to obtain high quality feasible solutions as well as cutoff values that can be used to prune branches in our branch-and-cut algorithm. We generate a large test bed consisting of small to large instances with diverse ranges for the number of suppliers, products and planning periods. Finally, we analyze the impact of each class of valid inequalities on the value of the LP relaxation and on the final solution. Our extensive computational experiments show that our valid inequalities notably enhance the performance of the branch-and-cut algorithm.

Chapter 3 presents:

Chitsaz, M., Cordeau, J.-F., & Jans, R. Multi-Product Production Routing Under Decoupled Planning Periods. Under Review at European Journal of Operational Research.

In this paper, we consider a generalized PRP which takes into account the fact that the production planning and the route planning period lengths are not necessarily identical. As a consequence of this assumption, the overall planning horizon may contain a different number of production and route planning periods. This results in two different discretizations of the planning horizon. This practical feature is a major source of complication for supply chain planners. With respect to the production planning aspect, we consider both big-bucket and small-bucket lot-sizing models. These models also consider the production of multiple types of products. We mathematically formulate the problem under different practical scenarios for the production and route planning period lengths. An exact solution method and a heuristic algorithm are proposed to efficiently solve large problem instances with this feature. To assess the effectiveness of our approach, we generate many test instances and perform an extensive computational study.

Finally, we summarize the main contributions of this thesis and point to several potential future research avenues.

Chapter 1

A Unified Decomposition Matheuristic for Assembly, Production and Inventory Routing

Abstract

While the joint optimization of production and outbound distribution decisions in a manufacturing context has been intensively studied in the past decade, the integration of production, inventory and inbound transportation from suppliers has received much less attention despite its practical relevance. This paper aims to fill the gap by introducing a general model for the assembly routing problem (ARP), which consists of simultaneously planning the assembly of a finished product at a plant and the routing of vehicles collecting materials from suppliers to meet the inventory requirements imposed by the production. We formulate the problem as a mixed-integer linear program and we propose a three-phase decomposition matheuristic that relies on the iterative solution of different subproblems. The first phase determines a setup schedule while the second phase optimizes production quantities, supplier visit schedules and shipment quantities. The third phase solves a vehicle routing problem for each period in the planning horizon. The algorithm is flexible and we show how it can also be used to solve two well-known outbound distribution problems related to the ARP: the production routing problem (PRP) and the

inventory routing problem (IRP). Using the same parameter setting for all problems and instances, we obtain 781 new best known solutions out of 2,628 standard IRP and PRP test instances. In particular, on large-scale multi-vehicle instances, the new algorithm outperforms specialized state-of-the-art heuristics for these two problems.

1.1 Introduction

The literature on production planning has paid a lot of attention in the past decade to the integration of lot sizing and outbound transportation decisions. The typical supply chain that is considered consists of a plant that delivers final products to several customers. Considering both the production planning at the plant and the outbound delivery to the customers via routes results in what is called the production routing problem (PRP) Adulyasak et al. (2015). If the production quantities at the plant are assumed to be given and the decisions only relate to the inventory and route planning, the problem is referred to as the inventory routing problem (IRP) (Andersson et al., 2010; Bertazzi et al., 2008; Coelho et al., 2013).

In contrast, only few studies have focused on the integration of production planning with inbound transportation planning. Yet, in a standard supply chain, a plant often uses several different components to assemble a final product. These components are typically produced in other plants or purchased from suppliers. If the assembly plant is responsible for organizing the inbound transportation of the various components, then gains can be achieved by integrating the production planning with the inbound vehicle routing. We refer to this problem as the assembly routing problem (ARP).

The aim of this paper is to introduce a general model for the ARP. We provide a mathematical formulation of the problem which serves as the basis for a decomposition matheuristic that iteratively solves different subproblems. We also explain how the same methodology can solve the related IRP and PRP. Using the same parameter setting for all three problems, this algorithm outperforms existing heuristics on large-scale multi-vehicle instances of the IRP and PRP, obtaining new best known solutions to many standard test instances.

The ARP has many industrial applications in situations where the production plant and several suppliers are owned by the same company, or when the manufacturer is the biggest player in the supply chain and centrally coordinates the inbound logistics decisions. This is a relevant practical problem in several areas. Fleischmann and Meyr (2003) indicate that in the automotive industry the organization that receives the components is usually responsible for the supply transport. Florian et al. (2011) show that in addition to the direct financial benefits for the supply chain, inbound logistics integration for a German car manufacturer has some further important outcomes such as a reduction in CO_2 emissions. In an application for the Delco Electronics Division of General Motors (GM), Blumenfeld et al. (1987) find that the overall optimization of the inbound transportation resulted in a 26% (2.9 million dollars per year, in 1987's USD) savings potential. They propose the use of an approximation method for the routing cost estimation in their studies to reduce the complexity of the problem. Implementing their solution package, GM of Canada reports savings of approximately 157 thousand USD in four months. Danese (2006) presents the case of GlaxoSmithKline (GSK), an international pharmaceutical group that extended the vendor managed inventory (VMI) approach to its suppliers as a response to the highly competitive and regulated market to benefit from the integrated and coordinated planning process.

Other cases where the buyer is responsible for the transportation are incorporated in several Incoterms, which are often used to clearly define the contractual responsibilities of the buyer and seller in international commercial transactions. Several of these terms consider the cases where the buyer is responsible for the transportation costs and risks. The Incoterm EXW (Ex Works) indicates a situation in which the seller makes the goods available, typically at the factory or a warehouse, and the buyer is responsible for the further transportation. In maritime transport, the Incoterms like FOB (Free On Board) for sea transport or inland waterway transport and FCA (Free Carrier) for roll-on/roll-off or container traffic, address the situation where the seller is responsible for the costs and risks up to when the goods are delivered to the ship at the named port of shipment. Then, it is the buyer who is responsible for the costs and risks from that point onwards.

In the retail sector, the concept of factory gate pricing (FGP) has emerged (Whiteoak, 1994; Le Blanc et al., 2006; Fernie and Sparks, 2014). Under FGP, the supplier no longer delivers the products to the customer but makes them available at its own factory gate

(Le Blanc et al., 2006). This requires the customer to plan and synchronize the pickups from the suppliers to reduce the transportation costs as reported by a number of FGP studies. Examples are Le Blanc et al. (2006) for a large Dutch retail distribution company and Potter et al. (2007) for UK retailers. Potter et al. (2007), Whiteoak (1994) and Fernie and Sparks (2014) report success in increasing the product flow while at the same time reducing distance for Tesco, ASDA and Sainsbury's retailers.

To the best of our knowledge, the problem of jointly optimizing production planning and inbound vehicle routing with a finite horizon and discrete planning periods has only been considered by Hein and Almeder (2016). They study the case of multiple components and products, and consider two scenarios. In the first scenario, components can be kept at the plant, whereas the second scenario considers a JIT environment assuming that the components that arrive at the plant must be used immediately in production. Furthermore, the holding cost at the suppliers is not considered in their specific study. Consequently, the combined decision making is entirely centered on the plant costs without taking the suppliers' cost into account.

Motivated by the above-mentioned applications and to fill the gap in the literature, we study for the first time the problem of the integrated inbound transportation, production and inventory planning in a finite planning horizon with the standard basic assumptions similar to the IRP and PRP. This is the first contribution of this paper. Second, we present a unified decomposition matheuristic capable of solving not only the ARP, but also the IRP and PRP. Also, we propose several cost update mechanisms to approximate the routing cost and, as our sensitivity analysis indicates, using a mix of two update mechanisms improves the quality of the solutions. Third, we report the results of extensive computational experiments on more than four thousand instances for these three problems, including standard data sets for the IRP and PRP. The results indicate that our algorithm outperforms the state-of-the-art heuristics on the large-scale multi-vehicle IRP and PRP instances. Finally, further analyses demonstrate the robust behavior of the algorithm.

The remainder of the paper is organized as follows. We provide a short literature review on the integration of production planning with outbound and inbound transportation in Section 1.2 in order to better position our problem with respect to the existing literature. Then, we define the ARP and express it mathematically in Section 1.3. We describe the decomposition matheuristic in Section 1.4. We present the algorithm implementation, the benchmark algorithms and the results of extensive computational experiments on all data sets in Section 1.5. Finally, Section 1.6 concludes the paper.

1.2 Literature Review

The majority of the research on the integrated production planning and outbound routing problem, which is most commonly referred to as the PRP, considers a finite time horizon with discrete planning periods. The associated models are typically formulated as mixed integer linear programs. Chandra (1993) was the first to address this problem by assuming a fixed cost for the warehouse orders, which in terms of modeling is similar to the production setup cost; Chandra (1993) studies a problem with an uncapacitated order size and an unlimited number of capacitated vehicles. Later, Chandra and Fisher (1994) define the same multi-commodity version of the problem in a more formal way, this time by considering the production setup costs. Several studies on this problem (Boudia et al., 2007; Boudia and Prins, 2009; Bard and Nananukul, 2009, 2010; Adulyasak et al., 2014a,b; Absi et al., 2015) consider one capacitated production plant producing a single product for multiple customers with inventory costs and inventory capacities both at the plant and customers. The plant is responsible for fulfilling the deterministic demand of the customers during the planning periods. The production setup cost is considered to be constant over the periods. A limited number of homogeneous and capacitated vehicles is also considered to perform the shipments from the plant to the customers. The multi-commodity version of the problem was studied by Fumero and Vercellis (1999) and Armentano et al. (2011). Lei et al. (2006) is the only study that considers multiple production plants producing one single final product and they assume a heterogeneous fleet of vehicles. The studies of Solyalı et al. (2009) and of Archetti et al. (2011) do not assume a capacity for the production. The state-of-the-art heuristic algorithms for the PRP are the adaptive large neighborhood search (ALNS) of Adulyasak et al. (2014b) and the matheuristic of Absi et al. (2015). For the IRP, the heuristic of Archetti et al. (2012) is the best performing algorithm for single-vehicle instances and the matheuristic of Archetti et al. (2017) is the best algorithm for multi-vehicle instances.

There are some studies that consider the optimization of the inbound transportation and inventory decisions without considering the production planning at the central plant. Inspired by the automotive parts supply chain, Moin et al. (2011) and Mjirda et al. (2014) study a multi-period, multi-supplier problem with a single assembly plant in which each supplier provides a distinct part type. Popken (1994) and Berman and Wang (2006) study a single period (static) multicommodity inbound logistics problem with three sets of nodes: origin nodes or suppliers, a destination node, and transshipment terminal nodes. In their model, the origin-destination commodity flows pass through the paths of the network using at most one terminal node, but the vehicle routes are not considered explicitly.

Some studies address inbound vehicle routing in JIT/lean production systems to coordinate the material inflow with the production rate. Vaidyanathan et al. (1999) and, later, Patel and Patel (2013) and Satoglu and Sahin (2013) investigate the delivery of parts in a central warehouse to the stations of an assembly line on a JIT basis. The quantity delivered per trip should meet the demand for the duration of the trip. As a result, vehicles will have no idle time between trips and inventories at the demand points are minimized. Qu et al. (1999) and Sindhuchao et al. (2005) consider the joint replenishment of multiple items in an inbound material-collection system for a central warehouse under the assumption of an infinite planning horizon. They do not take into account the vehicle capacity and storage space limit. Chuah and Yingling (2005) consider these two assumptions and study a JIT supply pickup problem for an automotive assembly plant with a restricted set of possible discrete frequencies. They also assume time windows at the suppliers. Stacey et al. (2007) and Natarajarathinam et al. (2012) offer new heuristics for the same problem. Ohlmann et al. (2007) expand the work of Chuah and Yingling (2005) by assuming general visit frequencies. They allow suppliers on the same route to have different pickup frequencies so that not every supplier is visited every time. Jiang et al. (2010) study a JIT parts supply problem in the automobile industry to minimize the inventory and transportation costs under storage space limit and common frequency routing assumptions. Yücel et al. (2013) consider a bilevel optimization problem for transporting specimens from a number of geographically dispersed sites to the processing facility of a clinical testing company. At the first level they maximize the daily processed amount while at the second level they minimize the daily transportation cost. Dong and Turnquist (2015) investigate a similar problem to design the inbound material collection routes. They consider pick-up frequency and spatial design as joint decisions to minimize total inventory and transportation costs with a single-level objective function. Lamsal et al. (2016) study a deterministic sugarcane harvest logistics problem in Brazil. The decisions to make are the harvest rate at the geographically dispersed fields and the truck assignment schedule to pick up the loads to minimize the time between the cutting of the sugar cane in the field and the crushing at the mill. They consider the constraint that the mill should never run out of raw material. Francis et al. (2006) study a variation of the periodic vehicle routing problem (PVRP) in which service frequency is a decision of the model. This brings more flexibility for the system's operator.

The problem of integrating inbound transportation with the production and inventory decisions is also gaining attention. Almost all of the research on this problem, with the exception of the previously mentioned study by Hein and Almeder (2016), considers an infinite planning horizon in a continuous time framework and uses mixed integer nonlinear programming models. This problem is referred to in the literature as the economic lot and supply scheduling problem (ELSSP) and was introduced by Liske and Kuhn (2009). Extending the economic order quantity (EOQ) assumptions, the ELSSP aims at finding synchronized cyclic production and routing patterns. Other studies on this problem include Kuhn and Liske (2011), Kuhn and Liske (2014), Bae et al. (2014), and Chen and Sarker (2014).

1.3 Problem Definition and Formulation

We consider a many-to-one assembly system where n suppliers, represented by the set $N_s = \{1, ..., n\}$, each provide a unique component necessary for the production of a final product at the central plant, denoted by node 0. The planning horizon comprises a finite number of discretized time periods, represented by the set $T = \{1, .., l\}$. The component supply, s_{it}, at each supplier $i \in N_s$ in each period $t \in T$ is predetermined over the planning horizon. The production system has to satisfy the external demand, d_t, for the final product at the plant in each period $t \in T$ without stockouts while respecting the plant's production capacity, which is given by C. Both the suppliers and the plant can hold inventory. Each supplier $i \in N_s$ has a storage capacity L_i for its components. The plant provides a shared storage with capacity L for the components and has a separate outbound storage capacity K for the final product. A fleet of m homogeneous vehicles, each with a capacity of Q, is available to perform shipments from the suppliers to the plant using routes that start and end at the plant. We suppose throughout that the components delivered to the plant in period $t \in T$ can be used for production in the same period.

We assume that one unit of each component is needed to make one unit of the final product. Note that in basic assembly structures, it is possible to define the units of measurement of the components so as to satisfy this assumption without loss of generality (see Pochet and Wolsey, 2006, chap. 13). Obviously, the unit components may not have identical sizes. Therefore, we consider that each component has a unit size of b_i. This size will be taken into account in the vehicle capacity and plant storage area for components. We consider a unit production cost u and setup cost f at the plant level. The unit holding costs of h_i and r_i are imposed for the inventory of component i at its supplier and at the plant, respectively. The inventory of the final product incurs a unit holding cost of r_0 at the plant. When a vehicle travels from location i to j it entails a period-independent cost of c_{ij}.

In the ARP, the following decisions should be optimized simultaneously for each period:

1. whether or not to produce the final product at the plant and the quantity to be produced;
2. the quantity to be shipped from the suppliers to the plant, and;
3. which suppliers to visit, in what order and by which vehicle.

To model the ARP we define a complete undirected graph $G = (N, E)$, and assume that the triangular inequality holds. Let $N = N_s \cup \{0\}$ be the set of nodes, and $E = \{(i, j) : i, j \in N, i < j\}$ be the set of edges. Since we assume a one-to-one relationship between suppliers and components, N_s also represents the set of components and $i = 0$ the final product. For each period $t \in T$, we let the binary variable y_t take value 1 if and only if production takes place at the plant and we let p_t denote the production quantity. Let I_{it} represent the inventory of component i at supplier $i \in N_s$ at the end of period t.

Define F_{it} as the inventory of component $i \in N_s$ or of the final product $i = 0$ at the plant at the end of period t. Let q_{it} indicate the shipment quantity from supplier i to the plant in period t. The variable x_{ijt} represents the number of times a vehicle traverses the edge $(i,j) \in E$ in period $t \in T$. Since we define the model on an undirected network, x_{ijt} is a binary variable for $i > 0$ and may take values in $\{0,1,2\}$ for $i = 0$. The binary supplier visit variable z_{it} takes value 1 if and only if a supplier $i \in N_s$ is visited in period t, and the integer variable z_{0t} indicates the number of vehicles dispatched from the plant in period t. Table 1.1 presents a summary of the notation.

Table 1.1: ARP notation

Sets:	
N	Set of nodes, indexed by $i \in \{0,...,n\}$, where 0 represents the plant and $N_s = N \setminus \{0\}$ is the set of suppliers. Note that since there is a one-to-one relationship between nodes and items, N also represents the set of components and the final product.
E	Set of edges, $E = \{(i,j) : i,j \in N, i < j\}$.
T	Set of time periods, indexed by $t \in \{1,..,l\}$.
$E(S)$	Set of edges $(i,j) \in E$ such that $i,j \in S$, where $S \subseteq N$ is a given set of nodes.
$\delta(S)$	Set of edges incident to a node set S, $\delta(S) = \{(i,j) \in E : i \in S, j \notin S \text{ or } i \notin S, j \in S\}$.
Parameters:	
f, u	Fixed setup and unit production costs, respectively.
h_i	Unit holding cost at node $i \in N_s$.
r_i	Unit holding cost of component/final product $i \in N$ at the plant.
c_{ij}	Transportation cost between nodes i and j, $(i,j) \in E$.
C, Q	Production and vehicle capacity, respectively.
m	Fleet size.
s_{it}	Component supply at node $i \in N_s$ in period t.
b_i	Unit size of component $i \in N_s$.
d_t	Demand for the final product in period t.
L_i	Inventory capacity for the components at node $i \in N$.
L	Shared inventory capacity for the components at the plant.
K	Inventory capacity for the final product (at the plant).
I_{i0}	Initial inventory available at node $i \in N_s$.
F_{i0}	Initial inventory of component/final product $i \in N$ at the plant.
Decision variables:	
p_t	Production quantity in period t at the plant.
y_t	Equals to 1 if there is a setup at the plant in period t, 0, otherwise.
I_{it}	Inventory of component i at node $i \in N_s$ at the end of period t.
F_{it}	Inventory of component/final product $i \in N$ at the plant at the end of period t.
x_{ijt}	Number of times a vehicle traverses the edge $(i,j) \in E$ in period t.
z_{it}	Equals to 1 if node $i \in N_s$ is visited in period t, 0, otherwise.
z_{0t}	Number of vehicles dispatched from the plant in period t.
q_{it}	Quantity shipped from node $i \in N_s$ to the plant in period t.

Using this notation, the ARP can be formulated as the following mixed integer program ($\mathcal{M}_{ARP}$).

$$(\mathcal{M}_{ARP})\ \ \min \sum_{t\in T} \Big(up_t + fy_t + \sum_{i\in N_s} h_i I_{it} + \sum_{i\in N} r_i F_{it} + \sum_{(i,j)\in E} c_{ij} x_{ijt} \Big) \tag{1.1}$$

s.t.

$$F_{i,t-1} + q_{it} = p_t + F_{it} \quad \forall i \in N_s, \forall t \in T \tag{1.2}$$

$$F_{0,t-1} + p_t = d_t + F_{0t} \quad \forall t \in T \tag{1.3}$$

$$I_{i,t-1} + s_{it} = q_{it} + I_{it} \quad \forall i \in N_s, \forall t \in T \tag{1.4}$$

$$p_t \leq C y_t \quad \forall t \in T \tag{1.5}$$

$$\sum_{i\in N_s} b_i F_{it} \leq L \quad \forall t \in T \tag{1.6}$$

$$F_{0t} \leq K \quad \forall t \in T \tag{1.7}$$

$$I_{it} \leq L_i \quad \forall i \in N_s, \forall t \in T \tag{1.8}$$

$$z_{0t} \leq m \quad \forall t \in T \tag{1.9}$$

$$b_i q_{it} \leq Q z_{it} \quad \forall i \in N_s, \forall t \in T \tag{1.10}$$

$$\sum_{(j,j')\in\delta(i)} x_{jj't} = 2z_{it} \quad \forall i \in N, \forall t \in T \tag{1.11}$$

$$Q \sum_{(i,j)\in E(S)} x_{ijt} \leq \sum_{i\in S} (Q z_{it} - b_i q_{it}) \quad \forall S \subseteq N_s, |S| \geq 2, \forall t \in T \tag{1.12}$$

$$z_{0t} \in \mathbb{Z} \quad \forall t \in T \tag{1.13}$$

$$F_{0t}, p_t \geq 0, y_t \in \{0,1\} \quad \forall t \in T \tag{1.14}$$

$$I_{it}, F_{it}, q_{it} \geq 0, z_{it} \in \{0,1\} \quad \forall i \in N_s, \forall t \in T \tag{1.15}$$

$$x_{ijt} \in \{0,1\} \quad \forall (i,j) \in E : i \neq 0, \forall t \in T \tag{1.16}$$

$$x_{0it} \in \{0,1,2\} \quad \forall i \in N_s, \forall t \in T. \tag{1.17}$$

The objective function (1.1) minimizes the total production, setup and holding costs in addition to the transportation costs. The holding cost includes component inventory at the suppliers and plant as well as the final product inventory at the plant. The inventory flow balance for the components and the final product at the plant is imposed through constraints (1.2) and (1.3). Constraints (1.4) ensure the inventory flow balance at each supplier. Constraints (1.5) force a setup at the plant for each period in which production takes place. They also impose the production capacity. Constraints (1.6) and (1.7) represent the storage capacity for the components and final product at the plant. The storage capacity for the components at each supplier is imposed by constraints (1.8). Constraints (1.9) limit the fleet size. Constraints (1.10) force a vehicle visit whenever components are shipped from a supplier to the plant. The maximum component shipment quantity from each supplier in each period is also limited by the vehicle capacity. Constraints (1.11) are the degree constraints. Constraints (1.12) are the subtour elimination constraints (SECs) and they also impose the vehicle capacity. These constraints are the modified version of the VRP capacity-cut constraints (Toth and Vigo, 2002; Lysgaard et al., 2004; Iori et al., 2007), and are referred to as generalized fractional subtour elimination constraints (GF-SEC) (Adulyasak et al., 2014a) in the context of the PRP. Contardo et al. (2012) introduce similar inequalities for the two-echelon capacitated location-routing problem that consider the variable flow through satellite nodes to define capacity cuts.

It is easy to show that the ARP is NP-hard since the VRP is a special case of it. Note that the ARP and PRP are not special cases of each other. Moreover, the ARP and PRP are not mirror problems and one cannot simply exchange customers and suppliers. In the ARP, we consider two separate storage areas at the plant for the components (inbound storage) and the final product (outbound storage), respectively. This results in inventory balance constraints for both the components and the final product at the plant. In the ARP, one unit of each component is required for producing one unit of the final product. In contrast, in the PRP, only the final product is represented. Another difference is that in the ARP it may be necessary to visit a supplier to avoid exceeding the maximum storage capacity (overflow). However, in the PRP, one prevents the stockout at the customers/retailers. For the same reasons, although the IRP is a special case of the PRP (where the production rates are predetermined and given), it is not a special case of the ARP.

1.4 A Decomposition Matheuristic

In this section we present a unified decomposition matheuristic for the ARP, which can also be applied to the PRP and the IRP. We explain the algorithm in the context of the ARP and its adaptation for the other two problems is explained in Section 1.5.2 and in

Appendix A.

Our algorithm decomposes the $\mathcal{M}_{ARP}$ model into three separate subproblems. The first subproblem, $\mathcal{M}_y$, is a special lot-sizing problem that determines a setup schedule by using the number of dispatched vehicles to calculate an approximation of the routing costs (Section 1.4.1). Considering a given setup schedule, the second subproblem, $\mathcal{M}_z$, uses a transportation cost approximation (σ_{it}) associated with each visit to supplier i, and chooses the node visits and shipment quantities (Section 1.4.2). For multi-vehicle instances, a modified model ($\mathcal{M}_z^{\mathcal{R}}$) is employed in this phase to look for possible improvements in node visits and shipments. Finally, the third subproblem solves a series of separate vehicle routing problems (Section 1.4.3), one for each period t (VRP_t). The solutions of the routing subproblems are then used to update the transportation cost approximation (σ_{it}) in the $\mathcal{M}_z$ subproblem (Section 1.4.4). This procedure is repeated for a number of iterations to reach a local optimum. Then, a local branching scheme is used to change the setup schedule and explore other parts of the feasible solution space, looking for better solutions (Section 1.4.5). The entire procedure continues until a stopping condition is met (Section 1.4.6).

Our algorithm shares similarities with the decomposition-based heuristic developed by Absi et al. (2015) for the PRP. However, there are also important differences between the two algorithms. The method of Absi et al. (2015) uses a two-phase approach where in the first phase it fixes y_t, p_t and q_{it} decisions. In our algorithm, this is done in two separate phases: it fixes the y_t decisions at the end of the first phase, then finds p_t and q_{it} in the second phase. Our method also prevents the same solution to appear twice by adding diversification constraints (Section 1.4.5) to cut the current node visit pattern in the next iteration and to cut the current setup schedule in order to diversify the search. We also implement two transportation cost approximation mechanisms. Finally, for the diversification, Absi et al. (2015) employ a random transportation cost perturbation mechanism while we change the setup schedule. An overview of our three-phase decomposition heuristic is presented in Algorithm 1.

Algorithm 1: CCJ-DH

```
Initialize $\sigma_{it}$
repeat
  if first iteration or diversification step then
    if diversification step then
      Cut the current setup schedule from $\mathcal{M}_y$ subproblem
      Reset aggregate fleet capacity for all periods
    end if
    Solve $\mathcal{M}_y \to y_t$ (and $p_t, z_{it}, q_{it}$)
    Fix $y_t$ decisions
  else
    Solve $\mathcal{M}_z$ with fixed $y_t \to p_t, z_{it}, q_{it}$
  end if
  Solve $VRP_t$ subproblems with fixed $z_{it}, q_{it} \to x_{ijt}$
  Select transportation cost update mechanism $\to \sigma_{it}$
  if all $VRP_t$ solutions are feasible then
    Update incumbent solution
    if (effective aggregate fleet capacity is reduced in some periods and
        after a minimum number of iterations and
        for a minimum quality of the current solution) then
      repeat
        Solve $\mathcal{M}_z^{\mathcal{R}}$ with fixed $z_{it} \to p_t, z_{it}, q_{it}$
        Solve $VRP_t$ subproblems with fixed $z_{it}, q_{it} \to x_{ijt} \to \sigma_{it}$
        Update incumbent solution
      until the stopping condition is met
    end if
  else
    Decrease effective aggregate fleet capacity for the periods with infeasible $VRP_t$
  end if
  Cut the current node visit pattern from $\mathcal{M}_z$ subproblem
until the stopping condition is met
return incumbent solution
```

1.4.1 Phase 1: The $\mathcal{M}_y$ Subproblem

The $\mathcal{M}_y$ subproblem aims to generate a good setup schedule by solving a simplified problem in which we use an approximate transportation cost based on the number of vehicles dispatched from the plant. To this end, we update the original objective function (1.1) with the following:

$$\min \sum_{t \in T} \Big(up_t + fy_t + \sum_{i \in N_s} h_i I_{it} + \sum_{i \in N} r_i F_{it} + \sigma_{0t} z_{0t} \Big). \tag{1.18}$$

We consider a cost (σ_{0t}) for each dispatched vehicle in each period (Section 1.4.4). With this modification, constraints (1.11)-(1.12) become redundant and they are replaced with the following constraints which impose an aggregate fleet capacity:

$$\sum_{i \in N_s} b_i q_{it} \leq Q z_{0t} \quad \forall t \in T. \tag{1.19}$$

We define the $\mathcal{M}_y$ subproblem with the objective function (1.18) subject to constraints (1.2)-(1.10), (1.13)-(1.15) and (1.19). This model yields a setup schedule in the first iteration and whenever a diversification step is performed. Adding a diversification constraint LBI_y (Section 1.4.5) prevents the same setup schedule to appear when we solve the model again. As a by-product, the solution to this model specifies the shipment quantity variables, q_{it}. Based on these shipment quantities, we can deduce the corresponding node visit variables. Therefore, whenever solving the $\mathcal{M}_y$ subproblem, the execution of phase 2 is not needed as the shipment quantities and node visits are already specified so we can skip phase 2 and immediately go to phase 3 in which the VRP_t subproblems (line 13 of Algorithm 1) are solved.

1.4.2 Phase 2: $\mathcal{M}_z$ and $\mathcal{M}_z^R$ Subproblems

In the second phase, the focus is on obtaining proper node visit decisions and shipment quantities. Using the solution found in the first phase, the binary decisions y_t are fixed in constraints (1.5) of the $\mathcal{M}_{ARP}$ subproblem. We approximate the transportation cost in the objective function using the node visit variables z_{it}, which results in the following objective function:

$$\min \sum_{t \in T} \Big(u p_t + \sum_{i \in N_s} h_i I_{it} + \sum_{i \in N} r_i F_{it} + \sum_{i \in N_s} \sigma_{it} z_{it} \Big). \tag{1.20}$$

We assume a cost (σ_{it}) for each node visit in each period (Section 1.4.4). With the removal of variables x_{ijt} and z_{0t} as well as constraints (1.9) and (1.11)-(1.13), it is no longer possible to enforce the vehicle capacity. However, by adding constraint $\sum_{i \in N_s} b_i q_{it} \leq mQ$ for every period t we can preserve the aggregate fleet capacity. Since split pickups are not allowed, we may not be able to find a feasible VRP solution for a certain period in phase 3 because the different quantities (q_{it}) to be shipped cannot be packed in the available vehicles. Therefore, as in Absi et al. (2015), we use the following constraints to impose a smaller aggregate fleet capacity ($0 \leq \lambda_t \leq 1$):

$$\sum_{i \in N_s} b_i q_{it} \leq \lambda_t m Q \quad \forall t \in T. \tag{1.21}$$

The $\mathcal{M}_z$ subproblem minimizes the objective function (1.20) subject to constraints (1.2)-(1.8), (1.10), (1.14)-(1.15) and (1.21). In the single-vehicle case ($m = 1$) and unlimited vehicle case ($m = n$), a modification of the aggregate fleet capacity is not necessary since a feasible VRP solution can always be found in phase 3 for each period; in these cases, lines 19-21 of Algorithm 1 are not executed. When the routing subproblem cannot find a feasible solution for a certain period, we reduce the λ_t for that period (line 25). Next, the $\mathcal{M}_z$ subproblem is solved with the reduced capacity (line 11), and based on this solution the VRP_t subproblems are solved (line 13). If all VRP_t solutions are feasible, we update the incumbent solution. Since we have reduced some λ_t, this yields some unused aggregate fleet capacity. To explore the possible benefits from the unutilized capacity, we solve a modified $\mathcal{M}_z$ subproblem. Let $\mathcal{R}_{kt}$ be the set of suppliers visited by vehicle k in period t. We replace constraints (1.21) with the following constraints for the periods where $\lambda_t < 1$ in the $\mathcal{M}_z$ subproblem:

$$\sum_{i \in \mathcal{R}_{kt}} b_i q_{it} \leq Q \quad \forall k \in \{1, ..., m\}, \forall t \in T | \lambda_t < 1. \tag{1.22}$$

Each constraint (1.22) relates to a vehicle that is used in a period with $\lambda_t < 1$. Then, we fix the node visit decisions z_{it} for these periods and obtain the $\mathcal{M}_z^{\mathcal{R}}$ subproblem (line 19 of Algorithm 1). Using the $\mathcal{M}_z^{\mathcal{R}}$ subproblem the algorithm can directly impose the vehicle capacity for each route, while we avoid the vehicle-indexed formulation which requires many more binary node visit variables for every vehicle as well as continuous quantity variables. We repeatedly solve $\mathcal{M}_z^{\mathcal{R}}$ with an updated approximation of the transportation cost (lines 18-22 of Algorithm 1) until the stopping criterion specified in Section 1.4.6 is met.

1.4.3 Phase 3: VRP_t Subproblems

Following each solution of the $\mathcal{M}_y$, $\mathcal{M}_z$ and $\mathcal{M}_z^{\mathcal{R}}$ subproblems, we fix for each time period the current node visit decisions $\bar{z}_{it}$ and the shipment amounts $\bar{q}_{it}$. Therefore, we have to solve one VRP for each period. As discussed in the previous section, this routing problem can be infeasible for one or several periods. To solve this subproblem we use the tabu search heuristic of Cordeau et al. (1997), which allows violations of the vehicle capacity constraints through a penalty cost in the objective function. Based on this (possibly infeasible) solution, we update the transportation costs for the next iteration. To reduce the computing time in the tabu search heuristic, we implement the following formula to specify the number of available vehicles for each VRP_t subproblem:

$$\bar{m}_t = \min\left\{m, \max\left\{1, \left\lceil \frac{2}{Q} \sum_{i \in N_s | z_{it}=1} b_i \bar{q}_{it} \right\rceil - 1\right\}\right\} \quad \forall t \in T. \tag{1.23}$$

The intuition behind this equation relies on the observation that in a solution to the capacitated vehicle routing problem if two routes are loaded less than half of the vehicle capacity, they can be merged as a single route that respects the vehicle capacity. This transformation results in a new solution with a smaller or equal cost assuming that the triangle inequality holds. Therefore, an optimal solution cannot include more than one less-than-half loaded route. We prove the validity of this upper bound and analyze its impact in Appendix A. Moreover, to control the running time of the heuristic, we set the number of tabu search iterations $\iota^{VRP} = \iota^V \sqrt{\bar{m}_t \sum_{i \in N_s | z_{it}=1} \bar{z}_{it}}$, for every period t, where ι^V is a parameter in our algorithm. To spend more time on promising solutions, we use a linearly varying value for the tabu search coefficient ι^V in the $[\iota^V_{\min}, \iota^V_{\max}]$ interval. When the previous solution is more than g (%) away from the incumbent, we let $\iota^V = \iota^V_{\min}$, and when it is better than or equal to the incumbent solution, we set $\iota^V = \iota^V_{\max}$.

1.4.4 Node Visit and Vehicle Dispatch Costs

We tested three mechanisms to update the node visit costs for the next iteration. Having a complete solution at hand, the first mechanism (Marginal) approximates the node visit costs as follows: If node i is visited in the current solution, then we set $\sigma_{it} = (c_{i_p i} + c_{i i_s}) - c_{i_p i_s}$, where i_p and i_s are the predecessor and successor of node i in its current route. If node i is currently not served in period t, then we set σ_{it} equal to the cost of the cheapest insertion into an existing route. This is based on the assumption that when a node i is eliminated from its route, an acceptable route can be obtained by connecting the predecessor and successor nodes. Similarly, when inserting node i in a certain period

t, an acceptable route can be obtained by the best insertion among all the routes in that period. Hence, σ_{it} can be seen as the (estimated) marginal transportation cost for visiting node i in period t. This marginal cost updating procedure is also used by Absi et al. (2015).

The second mechanism (TSP-share) splits the TSP cost of each route in each period over its nodes proportional to their direct shipment cost. Let c_{kt}^{TSP} be the route cost of vehicle k in period t, and $\mathcal{R}_{kt}$ be the set of suppliers visited by vehicle k in period t. We define

$$\sigma_{it} = c_{kt}^{TSP} \frac{c_{0i}}{\sum_{j \in \mathcal{R}_{kt}} c_{0j}} \quad \forall k \in \{1, ..., m\}, \forall t \in T, \forall i \in \mathcal{R}_{kt},$$
$$\sigma_{it} = \min_{k} \left\{ (c_{kt}^{TSP} + c_{kt}^{i}) \frac{c_{0i}}{\sum_{j \in \mathcal{R}_{kt} \cup \{i\}} c_{0j}} \right\} \quad \forall t \in T, \forall i \in N_s | z_{it} = 0,$$

where c_{kt}^{i} is equal to the cheapest insertion cost for non-visited node i into vehicle route k in period t. The last mechanism (VRP-share) divides the entire transportation cost of a certain period among the visited nodes proportional to their direct shipment cost. Let $c_t^{VRP} = \sum_{k=1}^{m} c_{kt}^{TSP}$ be the total transportation cost in period t, and $\mathcal{R}_t$ be the set of suppliers visited in period t. We define

$$\sigma_{it} = c_t^{VRP} \frac{c_{0i}}{\sum_{j \in \mathcal{R}_t} c_{0j}} \quad \forall t \in T, \forall i \in \mathcal{R}_t,$$
$$\sigma_{it} = (c_t^{VRP} + c_t^{i}) \frac{c_{0i}}{\sum_{j \in \mathcal{R}_t \cup \{i\}} c_{0j}} \quad \forall t \in T, \forall i \in N_s | z_{it} = 0,$$

where c_t^{i} is equal to the cheapest insertion cost for a non-visited node i into the available vehicle routes in period t.

The first and second mechanisms generally return better results than the last one. Our initial experiments revealed that by switching between the first two mechanisms, after using each for ι^U iterations, we generally get better results compared to using any one of them alone (line 14 of Algorithm 1). The maximum improvement by this hybrid update mechanism in the average solution cost is 1.9% compared to the marginal cost mechanism, 1.7% compared to the TSP-share mechanism, and 4.5% compared to the VRP-share mechanism. We report results with this mixed mechanism in Section 1.5.4.

Throughout the algorithm, we fix the vehicle dispatch cost $\sigma_{0t} = \sum_{i \in N_s} \bar{\sigma}_{it} / m$, where $\bar{\sigma}_{it}$ represents the initial node visit transportation cost. The performance analysis of using the three updating mechanisms as well as different initial node visit costs is presented in Appendix A.

1.4.5 Local Branching Inequalities

To diversify the search, we rely on two types of inequalities inspired by the local branching approach of Fischetti and Lodi (2003). Fischetti et al. (2004) apply these inequalities as diversification constraints for a telecommunication network design problem. The first type of inequality, LBI_z, is specific to the $\mathcal{M}_z$ subproblem and ensures that we do not return to a node visit pattern (and hence solution) we obtained before. The inequality

$$\sum_{i,t|\bar{z}_{it}=1} (1 - z_{it}) + \sum_{i,t|\bar{z}_{it}=0} z_{it} \geq r \tag{1.24}$$

forces at least r node visit variables to change value compared to the current solution. By varying r we can force different numbers of node visit changes in the next iteration of our algorithm. Our experiments show that if we let $r > 1$ the algorithm reaches a better solution in a shorter time compared to the case of $r = 1$. However, large values of r may remove some good quality solutions. We choose two different values for r. When the algorithm returns a better solution value compared to the previous iteration, we let $r = 1$ to allow the algorithm to search the entire neighborhood of the current solution. In case a worse solution value (compared to the previous iteration) is obtained, we let $r = l$, where l is the number of periods in the planning horizon. We add one inequality to the $\mathcal{M}_z$ subproblem at each iteration. Because these inequalities slow down the solution of the $\mathcal{M}_z$ subproblem, we remove all the previous LBI_z inequalities when the setup schedule is changed (by means of the diversification mechanism), and we continue adding new ones in future iterations.

The second type of inequality, LBI_y, is specific to the $\mathcal{M}_y$ subproblem and forces the model to obtain a new setup schedule. Therefore, it is used as a means of diversification. The inequality

$$\sum_{t|\bar{y}_t=1} (1 - y_t) + \sum_{t|\bar{y}_t=0} y_t \geq 1 \tag{1.25}$$

forces at least one of the binary setup schedule variables to change value. We add one inequality to the $\mathcal{M}_y$ subproblem each time we execute the diversification procedure and we keep these inequalities until the end of the algorithm. Adulyasak et al. (2014b) use this latter type of inequality to generate new setup schedules in their ALNS.

1.4.6 Stopping Conditions

The stopping condition for the overall algorithm (line 28 of Algorithm 1) is a maximum number of iterations, ι^A. To terminate the search for a local optimum within a specific setup schedule and introduce a diversification step (line 3 of Algorithm 1), we consider two stopping conditions. The search procedure stops after a maximum number of local search iterations, ι^L, or after a number of iterations without incumbent solution improvement, ι^N. Whenever one of these stopping conditions is met, the algorithm stops the local search, adds the associated LBI_y and solves the $\mathcal{M}_y$ subproblem to find another setup schedule (lines 4-9 of Algorithm 1). We allow the algorithm to use the $\mathcal{M}_z^{\mathcal{R}}$ subproblem only when it has performed at least ι^s iterations. This is to avoid wasting time with the very first solutions. The algorithm also runs the $\mathcal{M}_z^{\mathcal{R}}$ subproblem only for the cases where the current solution obtained from the $\mathcal{M}_z$ subproblem (and subsequent VRP_t subproblems) is close enough to the incumbent solution. More specifically, if the gap is less than g (%) the algorithm starts using the $\mathcal{M}_z^{\mathcal{R}}$ subproblem to fix some vehicle routes as explained in Section 1.4.2. The $\mathcal{M}_z^{\mathcal{R}}$ subproblem is allowed to be run until a maximum of $\iota^{\mathcal{R}}$ iterations is reached or until at any iteration it fails to return a solution with a gap less than $g^{\mathcal{R}}$ (%) from the incumbent solution. This condition corresponds to line 22 of Algorithm 1. The specific setting for the algorithm parameters and stopping conditions will be presented in the next section.

1.5 Computational Experiments

We test our algorithm on three different problems, the IRP, the PRP and the ARP, with a total of 4,068 instances. The IRP data sets were generated by Archetti et al. (2007) for the single-vehicle case and were later adapted to the multi-vehicle case by Coelho and Laporte (2013a) and by Desaulniers et al. (2015). The PRP data sets were introduced by Archetti et al. (2011) and by Boudia et al. (2005). We introduce the ARP instances in Section 1.5.1. Appendix A provides an overview of all the problem data sets.

We consider the same parameter setting when applying our algorithm to all data sets. The maximum number of algorithm iterations ι^A is set to 200 and the number of local search iterations ι^L is set to 80. The maximum number of non-improving iterations ι^N is set to 60. A maximum number of $\mathcal{M}_z^{\mathcal{R}}$ subproblem iterations ($\iota^{\mathcal{R}}$) equal to 10 is considered. The values of the minimum and maximum tabu search iteration coefficients ($\iota_{\min}^V$ and $\iota_{\max}^V$) are 100 and 500, respectively. We let the algorithm switch between the marginal and TSP-share mechanisms every 7 iterations (ι^U). The values of g and $g^{\mathcal{R}}$ are set to 3% and 0.3%, respectively. We set the initial node visit cost equal to $c_{0i}/2$, where c_{0i} is the cost of the edge between the plant and node i. We explain the details of the parameter setting procedure in Appendix A.

1.5.1 ARP Test Instances

We use the PRP data sets of Archetti et al. (2011) as a base for developing our ARP data sets. For each test instance, the number of nodes, their position and the distance function as well as the number of time periods and vehicles have been kept the same as in the corresponding Archetti et al. (2011) instance. Note that the nodes are suppliers in the ARP, but represent customers in the IRP and PRP.

As in Archetti et al. (2011), we consider an unlimited production capacity. The component supply at each node, s_{it}, is constant and equal to the amount used for the demand in Archetti et al. (2011). The demand at the central plant, d_t, is set equal to the average amount of all the suppliers' production rates. We randomly generate an integer number according to a uniform distribution in the interval [1,10] for each component's size, b_i. Then, to adjust the vehicle capacity (Q) we multiply the values given in Archetti et al. (2011) by a factor of 10. We set the component inventory capacities at the suppliers (L_i) the same as the retailers' capacities presented in Archetti et al. (2011). We assume an uncapacitated storage for the components at the plant. We consider a uniformly distributed random integer between 2 to 4 times the product demand of a period as the storage limit, K. The unit component holding cost at the suppliers, h_i, is set the same as in Archetti et al. (2011). The unit component holding cost at the plant, r_i, is set equal to a uniform random integer over the $[h_i, 2h_i]$ interval. To generate the unit product holding cost, r_0, we select a uniformly distributed random integer over the interval $[\sum_{i \in N_s} r_i, 2\sum_{i \in N_s} r_i]$. The initial inventory of the components at the suppliers, I_{i0}, is set equal to the amount that Archetti et al. (2011) established for the customers. The initial inventory of the final product at the plant, F_{00}, is set randomly in the interval from 0 to the demand of two periods ($[0, 2d_t]$). To avoid infeasibility and meet the final product demand, we need to have enough initial component inventory at the plant. Therefore, we set for each component i the initial inventory F_{i0} equal to $\max\{0, \sum_{t \in T}(d_t - s_{it}) - I_{i0} - F_{00}\}$. Table 1.2 presents an overview of the ARP instance parameters.

Table 1.2: ARP test instances

	Set 1	Set 2	Set 3
# of instances (SA‡)	480	480	480
# of components (SA‡): n	14	50	100
# of periods (SA‡): l	6	6	6
# of suppliers (SA‡): n	14	50	100
# of trucks (SA‡): m	1	UL†	UL†
Component supply: s_{it}		SA‡	
Production capacity: C		SA‡	
Demand (final product): d_t		$(\sum_{i=1}^{n} s_{it})/n$	
Item size: b_i		UDRI†† $[1, 10]$	
Vehicle capacity: Q		SA‡ by a factor of 10	
Supplier inventory capacity: L_i		SA‡	
Plant inventory capacity for components: L		UL†	
Plant inventory capacity for final product: K		UDRI†† $[2d_t, 4d_t]$	
Supplier initial inventory: I_{i0}		SA‡	
Plant initial inventory of components: F_{i0}		$\max\{0, \sum_{t \in T}(d_t - s_{it}) - I_{i0} - F_{00}\}$	
Plant initial inventory of final product: F_{00}		UDRI†† $[0, 2d_t]$	
Supplier and plant x,y coordinates		SA‡	
Unit production cost: u		SA‡	
Production setup cost: f		SA‡	
Unit transportation cost		SA‡	
Travel distance		SA‡	
Supplier unit holding cost: h_i		SA‡	
Plant unit component holding cost: r_i		UDRI†† $[h_i, 2h_i]$	
Plant unit final product holding cost: r_0		UDRI†† $[\sum_{i \in N_s} r_i, 2\sum_{i \in N_s} r_i]$	

† UL: Unlimited
‡ SA: The same as Archetti et al. (2011)
†† UDRI: Uniformly Distributed Random Integer

1.5.2 Algorithm Implementation

Some modules of the algorithm become redundant for some problems or data sets. The main modules of the algorithm are the $\mathcal{M}_y, \mathcal{M}_z$ and $\mathcal{M}_z^R$ subproblems and VRP_t subproblems. The aim of the $\mathcal{M}_y$ subproblem is to find proper setup schedules. Therefore, this module is not applicable in the case of the IRP. The module with which we find node visit schedules, the $\mathcal{M}_z$ subproblem, is relevant and necessary for all data sets and problems. The $\mathcal{M}_z^{\mathcal{R}}$ subproblem is only required for the data sets with a limited number of vehicles ($1 < m < n$). We present the $\mathcal{M}_y$ subproblem for the PRP and $\mathcal{M}_z$ subproblems for the IRP and PRP in Appendix A. We solve the $\mathcal{M}_y$, $\mathcal{M}_z$ and $\mathcal{M}_z^{\mathcal{R}}$ subproblems with CPLEX 12.6. Because all problems take the routing decisions into account, we must solve the VRP_t subproblems in every case.

1.5.3 Benchmark Algorithms

Since the ARP is a new problem, there is no algorithm to use as a benchmark. Consequently, we developed two lower bounding procedures as a basis for comparison. Furthermore, we validate the quality of our algorithm by applying it to the IRP and PRP standard test instances. For the IRP and PRP, we select the state-of-the-art algorithms as basis for comparison. Some of these are exact algorithms which we include for two reasons: to show the difference in running times and to consider their best found solutions in our comparison. We set the acronyms for each algorithm (including ours) by the authors' family name initials, followed by the implemented method identifier. For example, BC stands for branch-and-cut algorithm. Note that SV and MV in the data set names refer to single-vehicle and multi-vehicle instances, respectively.

It is difficult to make comparisons between different platforms and algorithms. It becomes more complicated when different numbers of threads are used. Therefore, we report the running times for each benchmark algorithm as it was presented in the original paper. To have an approximation of the speed of each employed platform, we additionally report a time adjustment factor for each benchmark algorithm using the CPU marks presented in PassMark®CPU marks (www.cpubenchmark.net/cpu_list.php). Table 1.3 provides the list of benchmark algorithms, their running platform, number of threads, time adjustment factor and solver version. Since some of the algorithms for the IRP are applied to only a subset of the instances, we provide more details in Table 1.4 on the number of instances each algorithm was applied to.

Table 1.3: Benchmark algorithms, the running platforms and standard MILP solver for the IRP and PRP data sets

Prob	Reference	Name	Sol	CPU	#Thread	TAF	Solver
IRP	Archetti et al. (2007)	ABLS-BC	E	Pentium IV 2.8GHz	Def	323	CPLEX 9.0
	Coelho and Laporte (2013b)	CL-BC	E	Xeon 2.67GHz	6	7,518	CPLEX 12.3
	Archetti et al. (2017)	ABS-H	HM	Xeon W3680, 3.33GHz	8	9,211	CPLEX 12.5
	Avella et al. (2017)	ABW-BC	E	Core i7-2620, 2.70GHz	1	3,825	Xpress 7.6
	Desaulniers et al. (2015)	DRC-BPC	E	Core i7-2600 3.4GHz	1	8,220	CPLEX 12.2
	Archetti et al. (2012)	ABHS-H	H	Intel Dual Core 1.86GHz	Def	2,288	CPLEX 10.1
	Coelho et al. (2012)	CCL-ALNS	M	Intel T7700, 2.4GHz	Def	1,419	-
	Adulyasak et al. (2014a)	ACJ-ALNS-1000	M	2.10GHz Duo CPU PC	Def	6,340	CPLEX 12.3
PRP	Archetti et al. (2011)	ABPS-BC ABPS-H	E H	AMD Athlon 64, 2.89GHz Intel Core 2, 2.40GHz	Def 1	437 1,440	CPLEX 10.1 -
	Boudia and Prins (2009)	BP-MA	M	2.30GHz PC	1	3,298	-
	Bard and Nananukul (2009)	BN-TS	M	2.53GHz PC	1	3,538	-
	Armentano et al. (2011)	ASL-TS	M	Pentium IV 2.8GHz	1	323	-
	Adulyasak et al. (2014b)	ACJ-ALNS-500 ACJ-ALNS-1000	M M	2.10GHz Duo CPU PC	Def	6,340	CPLEX 12.2
	Absi et al. (2015)	AADF-MS AADF-DMS AADF-VRP AADF-MTSP	H H H H	Xeon 2.67GHz PC	Def	7,518	CPLEX 12.1
	Solyalı and Süral (2017)	SS-H	H	2.40GHz PC	12	3,538	CPLEX 12.5
Both	This paper	CCJ-DH	H	Xeon X5650 2.67GHz	1	7,518	CPLEX 12.6

Note. Prob: Problem, Sol: Solution approach, E: Exact, M: Metaheuristic, H: Heuristic, Def: Default
TAF: Time adjustment factor according to the CPU marks presented in PassMark®(accessed: 14 July 2017)

1.5.4 Computational Results for the IRP and PRP Data Sets

Tables 1.5, 1.6 and 1.7 present the computational results and comparison between our algorithm, CCJ-DH, and the benchmark algorithms. Table 1.5 presents the average gap of the different algorithms applied to the IRP and PRP data sets. We calculate the percentage gap for each solution to each instance with respect to the previous best known solution so far (not including CCJ-DH). Then, for each class and number of vehicles (m) of a data set, we calculate the average gaps of the different algorithms.

Table 1.6 presents the number of best solutions found by different algorithms. Because for some small instances it is possible that CCJ-DH finds the same previous best found solution, we also present the number of new best solutions (NBS) in the last column of this table. Table 1.7 shows the average running times (in seconds) of the different algorithms.

For the SV-I1 data set, the exact BC algorithms (ABLS-BC and CL-BC) solved all the

Table 1.4: Number of instances each benchmark algorithm is applied to for the IRP and PRP data sets

Name of the Algorithm					ABLS-BC	CL-BC	ABS-H	ABW-BC	DRC-BPC	ABHS-H	CCL-ALNS	ABPS-BC	ABPS-H	BP-MA	BN-TS	ASL-TS	ACJ-ALNS-500	ACJ-ALNS-1000	AADF-MS	AADF-DMS	AADF-VRP	AADF-MTSP	SS-H	CCJ-DH
Prob	Set	m	Class	Size	E	E	HM	E	E	H	M	E	H	M	M	M	M	M	H	H	H	H	H	H
IRP	SV-I1	1	-	160	160	160	-	-	-	160	160	-	-	-	-	-	-	-	-	-	-	-	-	160
	MV-I1	2	-	160	-	160	160	50	158	-	-	-	-	-	-	-	-	150	-	-	-	-	-	160
		3	-	160	-	160	160	50	160	-	-	-	-	-	-	-	-	150	-	-	-	-	-	160
		4	-	160	-	160	160	50	160	-	-	-	-	-	-	-	-	-	-	-	-	-	-	160
		5	-	158	-	158	158	49	158	-	-	-	-	-	-	-	-	-	-	-	-	-	-	158
	SV-I2	1	-	60	-	60	-	-	-	60	-	-	-	-	-	-	-	-	-	-	-	-	-	60
	MV-I2	2	-	60	-	40	60	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	60
		3	-	60	-	40	60	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	60
		4	-	60	-	-	60	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	60
		5	-	60	-	-	60	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	60
PRP	SV-A1	1	1	120	-	-	-	-	-	-	-	120	120	-	-	-	120	120	120	120	-	-	120	120
		1	2	120	-	-	-	-	-	-	-	120	120	-	-	-	120	120	120	120	-	-	120	120
		1	3	120	-	-	-	-	-	-	-	120	120	-	-	-	120	120	120	120	-	-	120	120
		1	4	120	-	-	-	-	-	-	-	116	120	-	-	-	120	120	120	120	-	-	120	120
	MV-A2	UL	1	120	-	-	-	-	-	-	-	-	120	-	-	-	120	120	-	-	120	120	120	120
		UL	2	120	-	-	-	-	-	-	-	-	120	-	-	-	120	120	-	-	120	120	120	120
		UL	3	120	-	-	-	-	-	-	-	-	120	-	-	-	120	120	-	-	120	120	120	120
		UL	4	120	-	-	-	-	-	-	-	-	120	-	-	-	120	120	-	-	120	120	120	120
	MV-A3	UL	1	120	-	-	-	-	-	-	-	-	120	-	-	-	120	120	-	-	120	120	120	120
		UL	2	120	-	-	-	-	-	-	-	-	120	-	-	-	120	120	-	-	120	120	120	120
		UL	3	120	-	-	-	-	-	-	-	-	120	-	-	-	120	120	-	-	120	120	120	120
		UL	4	120	-	-	-	-	-	-	-	-	120	-	-	-	120	120	-	-	120	120	120	120
	MV-B1	5	-	30	-	-	-	-	-	-	-	-	-	30	30	30	30	30	-	-	30	30	30	30
	MV-B2	9	-	30	-	-	-	-	-	-	-	-	-	30	30	30	30	30	-	-	30	30	30	30
	MV-B3	13	-	30	-	-	-	-	-	-	-	-	-	30	30	30	30	-	-	-	30	30	30	30
Total				2628	160	938	878	199	636	220	160	476	1440	90	90	90	1530	1800	480	480	1050	1050	1530	2628

Note. E: Exact, H: Heuristic, M: Metaheuristic, MV: Multi-Vehicle, Prob: Problem, SV: Single-Vehicle, UL: Unlimited

instances to optimality. ABHS-H and CCL-ALNS are the state-of-the-art heuristics on this data set. They were able to find 125 and 72 optimal solutions, respectively, and finished with small gaps. Our algorithm was able to find 31 of the optimal solutions. The average gap of our algorithm on this data set is 1.62%, which is higher than the gap of the other algorithms. For the MV-I1 data set, the state-of-the-art heuristic algorithm is ABS-H. It was applied to all the instances in this set and obtained 261 best solutions with average gaps ranging from 0.21% to 1.5%. ACJ-ALNS-1000 found 26 best upper bounds (BUBs) in total with gaps of more than 7%. CCJ-DH obtained solutions with an average gap of 2.4% to 2.75% and found 126 best solutions among which it was successful to obtain 66 new best solutions.

For the SV-I2 data set, results are available for the CL-BC and ABHS-H algorithms. The first algorithm (which is a BC) spent on average more than 64,000 seconds to solve the instances in the set and obtained 30 BUBs. This algorithm has an average gap of more

Table 1.5: Average gaps by different algorithms applied to IRP and PRP data sets (%)

Name of the Algorithm					ABLS-BC	CL-BC	ABS-H	ABW-BC	DRC-BPC	ABHS-H	CCL-ALNS	ABPS-BC	ABPS-H	BP-MA	BN-TS	ASL-TS	ACJ-ALNS-500	ACJ-ALNS-1000	AADF-MS	AADF-DMS	AADF-VRP	AADF-MTSP	SS-H	CCJ-DH
Prob	Set	m	Class	Size	E	E	HM	E	E	H	M	E	H	M	M	M	M	M	H	H	H	H	H	H
IRP	SV-I1	1	-	160	**0**	**0**	-	-	-	0.05	0.46	-	-	-	-	-	-	-	-	-	-	-	-	1.62
	MV-I1	2	-	160	-	**0**	0.21	1.3	20.2	-	-	-	-	-	-	-	-	7.01	-	-	-	-	-	2.6
		3	-	160	-	**0.27**	0.67	2.63	18.41	-	-	-	-	-	-	-	-	7.2	-	-	-	-	-	2.4
		4	-	160	-	5.32	**1.34**	3.75	17.05	-	-	-	-	-	-	-	-	-	-	-	-	-	-	2.55
		5	-	158	-	10.53	**1.5**	4.08	14.85	-	-	-	-	-	-	-	-	-	-	-	-	-	-	2.75
	SV-I2	1	-	60	-	10.91	-	-	-	**0.27**	-	-	-	-	-	-	-	-	-	-	-	-	-	3.51
	MV-I2	2	-	60	-	61.32	0.12	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	**-1.82**
		3	-	60	-	106.28	0	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	**-2.86**
		4	-	60	-	-	0	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	**-4.73**
		5	-	60	-	-	0	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	**-4.9**
PRP	SV-A1	1	1	120	-	-	-	-	-	-	-	**0**	2.21	-	-	-	1.73	1.7	0.09	0.13	-	-	0.03	0.24
		1	2	120	-	-	-	-	-	-	-	**0**	0.3	-	-	-	0.36	0.36	0.01	0.02	-	-	**0**	0.03
		1	3	120	-	-	-	-	-	-	-	**0**	3.65	-	-	-	9.1	8.43	0.57	0.72	-	-	0.18	1.49
		1	4	120	-	-	-	-	-	-	-	**0**	0.9	-	-	-	0.96	0.96	0.04	0.07	-	-	0.05	0.13
	MV-A2	UL	1	120	-	-	-	-	-	-	-	-	2.08	-	-	-	1.14	1.13	-	-	0.17	1.09	0.05	**-0.06**
		UL	2	120	-	-	-	-	-	-	-	-	0.38	-	-	-	0.17	0.17	-	-	0.05	0.09	**0.01**	**0.01**
		UL	3	120	-	-	-	-	-	-	-	-	3.5	-	-	-	3.76	3.52	-	-	0.99	2.73	0.09	**-0.2**
		UL	4	120	-	-	-	-	-	-	-	-	1.25	-	-	-	0.2	0.19	-	-	0.1	0.46	0.06	**-0.03**
	MV-A3	UL	1	120	-	-	-	-	-	-	-	-	2.28	-	-	-	1.06	1	-	-	0.22	1.87	**0.07**	0.18
		UL	2	120	-	-	-	-	-	-	-	-	0.35	-	-	-	0.31	0.3	-	-	0.2	0.3	**0.18**	**0.18**
		UL	3	120	-	-	-	-	-	-	-	-	3.66	-	-	-	3.83	3.65	-	-	1.22	3.8	**0.15**	1.07
		UL	4	120	-	-	-	-	-	-	-	-	1.34	-	-	-	0.4	0.38	-	-	0.3	0.75	**0.11**	**0.11**
	MV-B1	5	-	30	-	-	-	-	-	-	-	-	-	14.55	7.7	5.37	1.15	1.04	-	-	1.65	1.41	**0.03**	0.78
	MV-B2	9	-	30	-	-	-	-	-	-	-	-	-	13.26	12.91	8.71	0.97	0.96	-	-	0.85	1.32	**0.07**	1.21
	MV-B3	13	-	30	-	-	-	-	-	-	-	-	-	16.09	19.95	10.29	1.61	-	-	-	0.36	1.93	2.24	**0.18**

Note. The best average gap at each row is presented with the bold font.

than 10.9%. The ABHS-H heuristic spent an average computing time of 3,630 seconds and obtained 31 BUBs. The UBs obtained by this algorithm are generally of high quality, resulting in an average gap of 0.27%. CCJ-DH spent about 6,700 seconds on average for the instances in this set and ended up with an average gap of around 3.5%.

For the MV-I2 data set there are two algorithms to compare with: CL-BC and ABS-H. Because the size of the instances and the number of available vehicles are larger compared to the MV-I1 data set, the CL-BC algorithm was not able to solve the instances with $m = 4$ and 5, and $n = 200$. This algorithm left average gaps of more than 61% and 106% for the instances with $m = 2$ and 3, respectively and found only 8 BUBs (among the instances with $m = 2$) while spending 86,400 seconds on every instance in the set. ABS-H was also successful on this data set by finding 38 BUBs. CCJ-DH outperformed the two existing approaches on this data set, finding 194 new best solutions which counts for more than 80% of the instances in this data set. Our algorithm obtained average gaps between -1.82% and -4.9%. The larger the number of nodes and the number of vehicles, the better the results obtained by CCJ-DH compared to ABS-H. This is an interesting result since

Table 1.6: Number of BUBs found by different algorithms applied to IRP and PRP data sets

Name of the Algorithm					ABLS-BC	CL-BC	ABS-H	ABW-BC	DRC-BPC	ABHS-H	CCJ-ALNS	ABPS-BC	ABPS-H	BP-MA	BN-TS	ASL-TS	ACJ-ALNS-500	ACJ-ALNS-1000	AADF-MS	AADF-DMS	AADF-VRP	AADF-MTSP	SS-H	CCJ-DH	
Prob	Set	m	Class	Size	E	E	HM	E	E	H	M	E	H	M	M	M	M	M	H	H	H	H	H	H	NBS
IRP	SV-I1	1	-	160	**160**	**160**	-	-	-	125	72	-	-	-	-	-	-	-	-	-	-	-	-	31	0
	MV-I1	2	-	160	-	**158**	98	22	84	-	-	-	-	-	-	-	-	14	-	-	-	-	-	28	0
		3	-	160	-	**142**	74	2	93	-	-	-	-	-	-	-	-	12	-	-	-	-	-	31	13
		4	-	160	-	**108**	50	5	95	-	-	-	-	-	-	-	-	-	-	-	-	-	-	29	22
		5	-	158	-	77	39	6	**102**	-	-	-	-	-	-	-	-	-	-	-	-	-	-	38	31
	SV-I2	1	-	60	-	30	-	-	-	**31**	-	-	-	-	-	-	-	-	-	-	-	-	-	0	0
	MV-I2	2	-	60	-	8	12	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	**40**	40
		3	-	60	-	0	15	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	**45**	45
		4	-	60	-	-	9	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	**51**	51
		5	-	60	-	-	2	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	**58**	58
PRP	SV-A1	1	1	120	-	-	-	-	-	-	-	**119**	6	-	-	-	1	1	81	70	-	-	71	19	0
		1	2	120	-	-	-	-	-	-	-	**120**	5	-	-	-	0	0	81	68	-	-	71	20	0
		1	3	120	-	-	-	-	-	-	-	**120**	2	-	-	-	0	0	52	44	-	-	67	5	0
		1	4	120	-	-	-	-	-	-	-	**116**	14	-	-	-	1	1	85	76	-	-	70	32	0
	MV-A2	UL	1	120	-	-	-	-	-	-	-	-	0	-	-	-	0	0	-	-	9	2	10	**101**	99
		UL	2	120	-	-	-	-	-	-	-	-	4	-	-	-	2	2	-	-	12	8	**49**	47	46
		UL	3	120	-	-	-	-	-	-	-	-	0	-	-	-	0	0	-	-	2	2	27	**92**	89
		UL	4	120	-	-	-	-	-	-	-	-	0	-	-	-	0	1	-	-	10	6	29	**75**	74
	MV-A3	UL	1	120	-	-	-	-	-	-	-	-	9	-	-	-	0	0	-	-	5	0	28	**79**	79
		UL	2	120	-	-	-	-	-	-	-	-	24	-	-	-	0	0	-	-	17	4	29	**46**	46
		UL	3	120	-	-	-	-	-	-	-	-	7	-	-	-	0	0	-	-	8	0	**77**	28	28
		UL	4	120	-	-	-	-	-	-	-	-	16	-	-	-	0	0	-	-	17	3	38	**46**	46
	MV-B1	5	-	30	-	-	-	-	-	-	-	-	-	0	0	1	0	1	-	-	0	0	**26**	2	2
	MV-B2	9	-	30	-	-	-	-	-	-	-	-	-	0	0	0	0	1	-	-	2	1	**26**	0	0
	MV-B3	13	-	30	-	-	-	-	-	-	-	-	-	0	0	0	0	-	-	-	9	3	6	**12**	12
Total (All Instances)				2628	160	683	299	35	374	156	72	475	87	0	0	1	4	33	299	258	91	29	624	955	781
Total (LSMV[†] Instances)				1290	0	8	38	0	0	0	0	0	60	0	0	1	2	5	0	0	91	29	345	722	715

Note. The largest number of obtained BUBs at each row is presented with the bold font. NBS: New best solutions.
[†] Large-scale multi-vehicle.

ABS-H is a specialized algorithm for the multi-vehicle IRP.

For the SV-A1 data set, there are five algorithms available in the benchmark set that were applied to all the instances: ABPS-BC, ABPS-H, ACJ-ALNS with 500 and 1000 iterations, AADF-MS, AADF-DMS and SS-H. Among the heuristic and metaheuristic algorithms, the specialized algorithms of AADF-MS, AADF-DMS and SS-H are the best performing ones. ABPS-H, ACJ-ALNS (with 500 and 1000 iterations) and SS-H are the only benchmark algorithms that were applied to all three data sets of Archetti et al. (2011). While ABPS-H generally obtained better results than ACJ-ALNS for SV-A1 with almost negligible computing times, both are outperformed by CCJ-DH in terms of the number of BUBs and average gaps.

There are five sophisticated heuristic or metaheuristic algorithms available for the

Table 1.7: Average running time of different algorithms applied to IRP and PRP data sets (seconds)

Name of the Algorithm					ABLS-BC	CL-BC	ABS-H	ABW-BC	DRC-BPC	ABHS-H	CCJ-ALNS	ABPS-BC	ABPS-H	BP-MA	BN-TS	ASL-TS	ACJ-ALNS-500	ACJ-ALNS-1000	AADF-MS	AADF-DMS	AADF-VRP	AADF-MTSP	SS-H	CCJ-DH
Prob	Set	m	Class	Size	E	E	HM	E	E	H	M	E	H	M	M	M	M	M	H	H	H	H	H	H
IRP	SV-I1	1	-	160	620	19	-	-	-	459	498	-	-	-	-	-	-	-	-	-	-	-	-	48
	MV-I1	2	-	160	-	4099	1259	2729	4121	-	-	-	-	-	-	-	-	34	-	-	-	-	-	68
		3	-	160	-	15319	1585	3467	4124	-	-	-	-	-	-	-	-	39	-	-	-	-	-	69
		4	-	160	-	23884	800	3600	3862	-	-	-	-	-	-	-	-	-	-	-	-	-	-	74
		5	-	158	-	28244	914	3600	3680	-	-	-	-	-	-	-	-	-	-	-	-	-	-	65
	SV-I2	1	-	60	-	64509	-	-	-	3630	-	-	-	-	-	-	-	-	-	-	-	-	-	6668
	MV-I2	2	-	60	-	86400	4066	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	5657
		3	-	60	-	86400	4540	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	4209
		4	-	60	-	-	4257	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	5132
		5	-	60	-	-	4418	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	5527
PRP	SV-A1	1	1	120	-	-	-	-	-	-	-	445	†	-	-	-	5	9	251	243	-	-	5	18
		1	2	120	-	-	-	-	-	-	-	11	†	-	-	-	5	9	214	210	-	-	5	18
		1	3	120	-	-	-	-	-	-	-	81	†	-	-	-	5	9	237	233	-	-	5	17
		1	4	120	-	-	-	-	-	-	-	527	†	-	-	-	5	9	217	218	-	-	5	18
	MV-A2	UL	1	120	-	-	-	-	-	-	-	-	11	-	-	-	29	50	-	-	23	315	5	400
		UL	2	120	-	-	-	-	-	-	-	-	12	-	-	-	28	50	-	-	23	288	14	344
		UL	3	120	-	-	-	-	-	-	-	-	9	-	-	-	24	43	-	-	26	335	16	309
		UL	4	120	-	-	-	-	-	-	-	-	11	-	-	-	26	44	-	-	26	330	25	434
	MV-A3	UL	1	120	-	-	-	-	-	-	-	-	188	-	-	-	136	249	-	-	86	514	324	2125
		UL	2	120	-	-	-	-	-	-	-	-	217	-	-	-	125	221	-	-	76	497	51	1947
		UL	3	120	-	-	-	-	-	-	-	-	168	-	-	-	107	191	-	-	75	509	350	1461
		UL	4	120	-	-	-	-	-	-	-	-	181	-	-	-	108	189	-	-	87	507	125	2213
	MV-B1	5	-	30	-	-	-	-	-	-	-	-	-	173	331	317	298	481	-	-	551	1653	2464	3559
	MV-B2	9	-	30	-	-	-	-	-	-	-	-	-	1108	976	1148	1405	1570	-	-	2054	9483	7487	9811
	MV-B3	13	-	30	-	-	-	-	-	-	-	-	-	4098	2492	3926	5794	-	-	-	4197	19270	16365	15891

† The computing times are negligible.

MV-A2 and MV-A3 data sets. Due to the size of the instances ($n = 50$ and 100, $l = 6$), no exact algorithm has yet been applied to these sets. The results presented in Tables 1.5 and 1.6 show that our algorithm and SS-H outperform all other algorithms on these two data sets both in total number of BUBs and average gaps. Over all the eight subclasses of MV-A2 and MV-A3, our algorithm provides an equal or better performance with respect to the gap for six subclasses compared to SS-H. Furthermore, our algorithm found 514 BUBs, while SS-H found 287 BUBs. Our algorithm was able to improve the overall previous best known solutions obtained by other benchmark algorithms on MV-A1, MV-A3 and MV-A4.

Seven different algorithms were tested on the MV-B1, MV-B2, and MV-B3 data sets: BP-MA, BN-TS, ASL-TS, ACJ-ALNS with 500 and 1000 iterations, AADF-VRP, AADF-MTSP and SS-H. On MV-B1 and MV-B2, SS-H is the best performing algorithm. However, on the MV-B3 which includes the largest PRP instances, CCJ-DH is the best algorithm with an average gap of 0.18%. SS-H returned a large gap of 2.24% on this data set. Overall, SS-H and CCJ-DH are the best performing algorithms (non-dominated ones) on these three data sets. The average gap of CCJ-DH on all the 90 instances in these data sets is 0.72%, performing better than SS-H with an overall average gap of 0.78%.

On all IRP and PRP data sets with 2,628 instances, CCJ-DH was able to find 955 BUBs out of which 781 are new best solutions. Our algorithm shows consistent performance especially on the large-scale multi-vehicle instances of both IRP and PRP. For this family of instances, CCJ-DH successfully obtains improved solutions compared to the previous BUBs found in the literature by the specialized algorithms. Among the 1,290 large-scale multi-vehicle instances of IRP and PRP data sets (240 instances of MV-I2, 960 instances of MV-A2 and MV-A3 and 90 instances of MV-B1, MV-B2 and MV-B3), CCJ-DH found 715 new best solutions. The algorithm also finished with the best or one of the best average gaps among the other benchmark algorithms. Moreover, CCJ-DH is the only algorithm that has been applied to all the IRP and PRP data sets. ACJ-ALNS-1000 is the only other algorithm that has been applied to both the IRP and PRP problems. This metaheuristic was developed specifically for the PRP (Adulyasak et al., 2014b) and was next applied to a limited set of multi-vehicle IRP instances (Adulyasak et al., 2014a). The results in Table 1.5 indicate that CCJ-DH obtains improved gaps compared to ACJ-ALSN-1000 in all the tested classes, except for MV-B2.

In the existing algorithms for the IRP and PRP, we observe imbalances between the CPU times. Because we worked with the same parameters for all problems and data sets, it was impossible to find one setting that led to similar CPU times for all classes compared to the state-of-the-art algorithms.

1.5.5 Computational Results for the ARP Data Sets

On the ARP data sets, we compare our algorithm against a truncated BC method implemented in C++ with the CPLEX callable library and a time limit of 12 hours. In the $\mathcal{M}_{ARP}$ model, we include another type of SEC (Archetti et al., 2011) in addition to constraints (1.12), to strengthen the LP relaxation of $\mathcal{M}_{ARP}$:

$$\sum_{(i,j)\in E(S)} x_{ijt} \leq \sum_{i\in S} z_{it} - z_{et} \quad \forall S \subseteq N_s, |S| \geq 2, \forall e \in S, \forall t \in T. \tag{1.26}$$

We add SECs dynamically through the search whenever they are violated. To this end, we use the CVRP package of Lysgaard et al. (2004) for separation. Moreover, we add the following valid inequalities together with constraints (1.19) a priori to the model:

$$z_{it} \leq z_{0t} \quad \forall i \in N_s, \forall t \in T, \tag{1.27}$$

$$x_{ijt} \leq z_{it} \quad and \quad x_{ijt} \leq z_{jt} \quad \forall (i,j) \in E(N_s), \forall t \in T. \tag{1.28}$$

Our initial experiments showed that when relaxing GFSEC, i.e. constraints (1.12), from the $\mathcal{M}_{ARP}$ model, CPLEX is able to solve the resulting MIP for large ARP instances in an average of 60 seconds. However, the integral solution may have subtours in each period. Therefore, we also implement another lower bounding method for the ARP instances. We iteratively add GFSEC cutting planes for the violated subtours and re-solve the new MIP (MIP-CP). Note that at each iteration the solution time grows significantly due to the newly added SECs and the marginal benefit of adding them becomes smaller. We observed that after five hours this method is no longer able to effectively improve the solutions (lower bound) for the MV-C2 and MV-C3 data sets. Because the BC method is able to solve the MV-C1 instances to optimality in a very short time, we did not apply the MIP-CP method to these instances.

Table 1.8 presents the performance of CCJ-DH on the ARP data sets. Columns five and six in this table show the number of upper bounds (UBs) and best upper bounds (BUBs) obtained by the BC method, respectively. The next column presents the average gap (%) of the BC UBs with respect to (w.r.t.) BUBs (found either by BC or CCJ-DH). The rest of the columns for the BC method show the number of optimal solutions, the number of best lower bounds (BLBs) found either by BC or MIP-CP, the average gap (%) of the BC method (compared to its own LB), and the average gap (%) of its lower bounds (LBs) w.r.t. BLBs (found either by BC or MIP-CP). The two columns for MIP-CP present the number of BLBs and the average gap (%) of its LBs w.r.t. BLBs. Note that MIP-CP does not produce a feasible solution. The four columns for CCJ-DH show the solution time, number of BUBs, and the average gap (%) of its solutions w.r.t. BUBs and BLBs, respectively.

The BC method is able to solve every instance in the SV-C1 data set in less than 44 seconds, but for the other two data sets it reaches the time limit of 12 hours. It finds 304 feasible solutions for MV-C2 within the time limit among which only 22 are better UBs compared to CCJ-DH. On the MV-C3 data set, the BC method is unable to find any feasible solution in the time limit. On all classes of the MV-C2 data set and the third class of the MV-C3 data set, the BC method finds more BLBs and better average lower bounds compared to MIP-CP. Although the BC method finds more BLBs for the first two classes of the MV-C3 data set, MIP-CP reaches better average BLBs. On the fourth class of the MV-C3 data set, MIP-CP finds more BLBs and better lower bounds compared to the BC method. MIP-CP proved to be efficient in obtaining 167 BLBs for the MV-C3 data set. The difference in the performance of these two lower bounding methods is due to the different approaches to eliminate the subtours. The BC method has to deal with the fractional node visits and hence should include many more (fractional) SECs in the LP model along the search tree. However, MIP-CP is restricted to integral node visits and consequently only needs the SECs to eliminate the integral subtours (disconnected components) and enforce the vehicle capacity. The results show that the fractional SECs applied in the BC method generally return better lower bounds on the instances with large transportation cost (the third class of each data set). On the small test instances, our heuristic provides good quality solutions, with an average gap between 0.3% and 1.1%, compared to the optimal solutions of a specialized BC approach. On the medium and large size instances, our algorithm generally provides high quality solutions compared to the best lower bounds found either by the BC approach or a specialized lower bound algorithm. These average gaps vary between 0.9% and 2.4%, except for the third class (with very large transportation cost) of MV-C2 and MV-C3, for which the gaps are close to 6% and 10%.

We further discuss the behavior of our algorithm in Appendix A. All instances, detailed solutions and results can be found at
`http://chairelogistique.hec.ca/en/scientific-data/`.

1.6 Summary and Conclusion

This study fills a gap in the literature by introducing a MILP model for the integrated production, inventory and inbound routing problem. Although some similarities between the PRP and ARP exist, fundamental differences arise in the nature of the problem and in the modeling such as the presence of inventory of both the final product and the compo-

Table 1.8: CCJ-DH performance on ARP data sets.

				BC†							MIP-CP‡		CCJ-DH			
Set	m	Class	Size	# UB	# BUB	Gap-UB BUB	# Opt	# BLB	Gap CPLEX	Gap-LB BLB	# BLB	Gap-LB BLB	CPU (sec)	# BUB	Gap-UB BUB	Gap-UB BLB
SV-C1	1	1	120	120	120	0	120	120	0	0	-	-	43	23	0.3	0.3
	1	2	120	120	120	0	120	120	0	0	-	-	41	19	0.26	0.26
	1	3	120	120	120	0	120	120	0	0	-	-	42	19	1.07	1.07
	1	4	120	120	120	0	120	120	0	0	-	-	34	16	0.48	0.48
MV-C2	UL	1	120	73	3	47.32	0	114	48.01	0.21	6	0.56	603	117	0	1.54
	UL	2	120	76	7	47.35	0	117	47.91	0.01	3	0.61	592	113	0.01	1.54
	UL	3	120	60	1	62.98	0	114	64.61	0.19	6	2.56	468	119	0	5.98
	UL	4	120	95	11	26.85	0	67	27.39	0.14	53	0.18	914	109	0.02	0.88
MV-C3	UL	1	120	0	0	100	0	95	100	0.56	25	0.32	2967	120	0	2.4
	UL	2	120	0	0	100	0	93	100	0.47	27	0.3	2932	120	0	2.39
	UL	3	120	0	0	100	0	92	100	0.56	28	1.14	1971	120	0	9.81
	UL	4	120	0	0	100	0	33	100	1	87	0.11	4213	120	0	1.58

Note. BC: Branch-and-cut algorithm, MIP-CP: Cutting plane method with sequential MIPs.

† With a time limit of 12 hours and maximum 30 GB memory. The algorithm finds optimal solution for SV-C1 in less than 44 seconds for any instance in the set, and it reaches the time limit for both MV-C2 and MV-C3.

‡ With a time limit of 5 hours.

nents at the plant. We present a compact formulation for the ARP ($\mathcal{M}_{ARP}$) and developed many test instances for this problem as well as an efficient heuristic algorithm. On the small test instances, our heuristic provides good quality solutions, compared to the optimal solutions of a specialized BC approach. On the medium and large size instances, our algorithm generally provides high quality solutions compared to the best obtained lower bounds either by the BC approach or a specialized lower bound algorithm, with the exception of the data sets with the high transportation cost.

We further test this algorithm on other problems of the same nature where the routing decisions are integrated with inventory management (and production planning): the IRP and the PRP. We consider standard data sets from the literature. These data sets include 2,628 instances ranging from small to very large-scale ones. We compare our results to those from the current state-of-the-art algorithms. Our algorithm presents acceptable results on the small data sets and outperforms specialized state-of-the-art algorithms for the large-scale multi-vehicle instances. We also outperform the only other algorithm that has been applied to both the IRP and PRP problems. Moreover, we show that the algorithm finds good quality solutions with different transportation cost update mechanisms as well as different initial node visit costs. We believe this shows the robustness of our decomposition approach. One of the most important contributions of this paper is the design of a unified algorithm that can be applied to different data sets of different problems

(ARP, PRP and IRP) with the same parameter setting.

References

Absi, N., Archetti, C., Dauzère-Pérès, S., and Feillet, D. (2015). A two-phase iterative heuristic approach for the production routing problem. *Transportation Science*, 49(4):784–795.

Adulyasak, Y., Cordeau, J.-F., and Jans, R. (2014a). Formulations and branch-and-cut algorithms for multivehicle production and inventory routing problems. *INFORMS Journal on Computing*, 26(1):103–120.

Adulyasak, Y., Cordeau, J.-F., and Jans, R. (2014b). Optimization-based adaptive large neighborhood search for the production routing problem. *Transportation Science*, 48(1):20–45.

Adulyasak, Y., Cordeau, J.-F., and Jans, R. (2015). The production routing problem: A review of formulations and solution algorithms. *Computers & Operations Research*, 55:141–152.

Andersson, H., Hoff, A., Christiansen, M., Hasle, G., and Løkketangen, A. (2010). Industrial aspects and literature survey: Combined inventory management and routing. *Computers & Operations Research*, 37(9):1515–1536.

Archetti, C., Bertazzi, L., Hertz, A., and Speranza, M. G. (2012). A hybrid heuristic for an inventory routing problem. *INFORMS Journal on Computing*, 24(1):101–116.

Archetti, C., Bertazzi, L., Laporte, G., and Speranza, M. G. (2007). A branch-and-cut algorithm for a vendor-managed inventory-routing problem. *Transportation Science*, 41(3):382–391.

Archetti, C., Bertazzi, L., Paletta, G., and Speranza, M. G. (2011). Analysis of the maximum level policy in a production-distribution system. *Computers & Operations Research*, 38(12):1731–1746.

Archetti, C., Boland, N., and Speranza, M. (2017). A matheuristic for the multi-vehicle inventory routing problem. *INFORMS Journal on Computing*, 29(3):377–387.

Armentano, V. A., Shiguemoto, A., and Løkketangen, A. (2011). Tabu search with path relinking for an integrated production–distribution problem. *Computers & Operations Research*, 38(8):1199–1209.

Avella, P., Boccia, M., and Wolsey, L. A. (2017). Single-period cutting planes for inventory routing problems. *Transportation Science (Articles in Advance)*.

Bae, H., Moon, I., and Yun, W. (2014). Economic lot and supply scheduling problem: a time-varying lot sizes approach. *International Journal of Production Research*, 52(8):2422–2435.

Bard, J. F. and Nananukul, N. (2009). The integrated production–inventory–distribution–routing problem. *Journal of Scheduling*, 12(3):257–280.

Bard, J. F. and Nananukul, N. (2010). A branch-and-price algorithm for an integrated production and inventory routing problem. *Computers & Operations Research*, 37(12):2202–2217.

Berman, O. and Wang, Q. (2006). Inbound logistic planning: minimizing transportation and inventory cost. *Transportation Science*, 40(3):287–299.

Bertazzi, L., Savelsbergh, M., and Speranza, M. G. (2008). Inventory routing. In Golden, B. L., Raghavan, S., and Wasil, E. A., editors, *The Vehicle Routing Problem: Latest Advances and New Challenges*, pages 49–72. Springer Science & Business Media.

Blumenfeld, D. E., Burns, L. D., Daganzo, C. F., Frick, M. C., and Hall, R. W. (1987). Reducing logistics costs at General Motors. *Interfaces*, 17(1):26–47.

Boudia, M., Louly, M. A. O., and Prins, C. (2005). Combined optimization of production and distribution. In *Proceedings of the International Conference on Industrial Engineering and Systems Management, IESM*, volume 5.

Boudia, M., Louly, M. A. O., and Prins, C. (2007). A reactive grasp and path relinking for a combined production–distribution problem. *Computers & Operations Research*, 34(11):3402–3419.

Boudia, M. and Prins, C. (2009). A memetic algorithm with dynamic population management for an integrated production–distribution problem. *European Journal of Operational Research*, 195(3):703–715.

Chandra, P. (1993). A dynamic distribution model with warehouse and customer replenishment requirements. *Journal of the Operational Research Society*, 44(7):681–692.

Chandra, P. and Fisher, M. L. (1994). Coordination of production and distribution planning. *European Journal of Operational Research*, 72(3):503–517.

Chen, Z. and Sarker, B. R. (2014). An integrated optimal inventory lot-sizing and vehicle-routing model for a multisupplier single-assembler system with JIT delivery. *International Journal of Production Research*, 52(17):5086–5114.

Chuah, K. H. and Yingling, J. C. (2005). Routing for a just-in-time supply pickup and delivery system. *Transportation Science*, 39(3):328–339.

Coelho, L. C., Cordeau, J.-F., and Laporte, G. (2012). The inventory-routing problem with transshipment. *Computers & Operations Research*, 39(11):2537–2548.

Coelho, L. C., Cordeau, J.-F., and Laporte, G. (2013). Thirty years of inventory routing. *Transportation Science*, 48(1):1–19.

Coelho, L. C. and Laporte, G. (2013a). A branch-and-cut algorithm for the multi-product multi-vehicle inventory-routing problem. *International Journal of Production Research*, 51(23-24):7156–7169.

Coelho, L. C. and Laporte, G. (2013b). The exact solution of several classes of inventory-routing problems. *Computers & Operations Research*, 40(2):558–565.

Contardo, C., Hemmelmayr, V., and Crainic, T. G. (2012). Lower and upper bounds for the two-echelon capacitated location-routing problem. *Computers & Operations Research*, 39(12):3185–3199.

Cordeau, J.-F., Gendreau, M., and Laporte, G. (1997). A tabu search heuristic for periodic and multi-depot vehicle routing problems. *Networks*, 30(2):105–119.

Danese, P. (2006). The extended VMI for coordinating the whole supply network. *Journal of Manufacturing Technology Management*, 17(7):888–907.

Desaulniers, G., Rakke, J. G., and Coelho, L. C. (2015). A branch-price-and-cut algorithm for the inventory-routing problem. *Transportation Science*, 50(3):1060–1076.

Dong, Z. and Turnquist, M. (2015). Combining service frequency and vehicle routing for managing supplier shipments. *Transportation Research Part E: Logistics and Transportation Review*, 79:231–243.

Fernie, J. and Sparks, L. (2014). *Logistics and Retail Management: Emerging Issues and New Challenges in the Retail Supply Chain*. Kogan Page Publishers.

Fischetti, M. and Lodi, A. (2003). Local branching. *Mathematical Programming*, 98(1-3):23–47.

Fischetti, M., Polo, C., and Scantamburlo, M. (2004). A local branching heuristic for mixed-integer programs with 2-level variables, with an application to a telecommunication network design problem. *Networks*, 44(2):61–72.

Fleischmann, B. and Meyr, H. (2003). Planning hierarchy, modeling and advanced planning systems. *Handbooks in Operations Research and Management Science*, 11:455–523.

Florian, M., Kemper, J., Sihn, W., and Hellingrath, B. (2011). Concept of transport-oriented scheduling for reduction of inbound logistics traffic in the automotive industries. *CIRP Journal of Manufacturing Science and Technology*, 4(3):252–257.

Francis, P., Smilowitz, K., and Tzur, M. (2006). The period vehicle routing problem with service choice. *Transportation Science*, 40(4):439–454.

Fumero, F. and Vercellis, C. (1999). Synchronized development of production, inventory, and distribution schedules. *Transportation Science*, 33(3):330–340.

Hein, F. and Almeder, C. (2016). Quantitative insights into the integrated supply vehicle routing and production planning problem. *International Journal of Production Economics*, 177:66–76.

Iori, M., Salazar-González, J.-J., and Vigo, D. (2007). An exact approach for the vehicle routing problem with two-dimensional loading constraints. *Transportation Science*, 41(2):253–264.

Jiang, Z., Huang, Y., and Wang, J. (2010). Routing for the milk-run pickup system in automobile parts supply. In *Proceedings of the 6th CIRP-Sponsored International Conference on Digital Enterprise Technology*, pages 1267–1275. Springer.

Kuhn, H. and Liske, T. (2011). Simultaneous supply and production planning. *International Journal of Production Research*, 49(13):3795–3813.

Kuhn, H. and Liske, T. (2014). An exact algorithm for solving the economic lot and supply scheduling problem using a power-of-two policy. *Computers & Operations Research*, 51:30–40.

Lamsal, K., Jones, P. C., and Thomas, B. W. (2016). Sugarcane harvest logistics in Brazil. *Transportation Science*, 51(2):771 – 789.

Le Blanc, H. M., Cruijssen, F., Fleuren, H. A., and De Koster, M. (2006). Factory gate pricing: An analysis of the dutch retail distribution. *European Journal of Operational Research*, 174(3):1950–1967.

Lei, L., Liu, S., Ruszczynski, A., and Park, S. (2006). On the integrated production, inventory, and distribution routing problem. *IIE Transactions*, 38(11):955–970.

Liske, T. and Kuhn, H. (2009). The economic lot and supply scheduling problem under a power-of-two policy. In *Operations Research Proceedings 2008*, pages 215–220. Springer Science & Business Media.

Lysgaard, J., Letchford, A. N., and Eglese, R. W. (2004). A new branch-and-cut algorithm for the capacitated vehicle routing problem. *Mathematical Programming*, 100(2):423–445.

Mjirda, A., Jarboui, B., Macedo, R., Hanafi, S., and Mladenović, N. (2014). A two phase variable neighborhood search for the multi-product inventory routing problem. *Computers & Operations Research*, 52:291–299.

Moin, N. H., Salhi, S., and Aziz, N. (2011). An efficient hybrid genetic algorithm for the multi-product multi-period inventory routing problem. *International Journal of Production Economics*, 133(1):334–343.

Natarajarathinam, M., Stacey, J., and Sox, C. (2012). Near-optimal heuristics and managerial insights for the storage constrained, inbound inventory routing problem. *International Journal of Physical Distribution & Logistics Management*, 42(2):152–173.

Ohlmann, J., Fry, M., and Thomas, B. (2007). Route design for lean production systems. *Transportation Science*, 42(2):352–370.

Patel, D. and Patel, M. (2013). Design and development of an internal milk-run material supply system in automotive industry. *International Journal of Application or Innovation in Engineering & Management (IJAIEM)*, 2(8):233–235.

Pochet, Y. and Wolsey, L. A. (2006). *Production Planning by Mixed Integer Programming*. Springer Science & Business Media.

Popken, D. A. (1994). An algorithm for the multiattribute, multicommodity flow problem with freight consolidation and inventory costs. *Operations Research*, 42(2):274–286.

Potter, A., Mason, R., and Lalwani, C. (2007). Analysis of factory gate pricing in the U.K. grocery supply chain. *International Journal of Retail & Distribution Management*, 35(10):821–834.

Qu, W. W., Bookbinder, J. H., and Iyogun, P. (1999). An integrated inventory–transportation system with modified periodic policy for multiple products. *European Journal of Operational Research*, 115(2):254–269.

Satoglu, S. and Sahin, I. (2013). Design of a just-in-time periodic material supply system for the assembly lines and an application in electronics industry. *The International Journal of Advanced Manufacturing Technology*, 65(1-4):319–332.

Sindhuchao, S., Romeijn, H. E., Akçali, E., and Boondiskulchok, R. (2005). An integrated inventory-routing system for multi-item joint replenishment with limited vehicle capacity. *Journal of Global Optimization*, 32(1):93–118.

Solyalı, O. and Süral, H. (2017). A multi-phase heuristic for the production routing problem. *Computers & Operations Research*, 87:114–124.

Solyalı, O., Süral, H., Neogy, S., Das, A., and Bapat, R. (2009). A relaxation based solution approach for the inventory control and vehicle routing problem in vendor managed systems. *Modeling, Computation and Optimization, World Scientific, Singapore*, pages 171–189.

Stacey, J., Natarajarathinam, M., and Sox, C. (2007). The storage constrained, inbound inventory routing problem. *International Journal of Physical Distribution & Logistics Management*, 37(6):484–500.

Toth, P. and Vigo, D. (2002). An overview of vehicle routing problems. In Toth, P. and Vigo, D., editors, *The Vehicle Routing Problem*, pages 1–26. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, USA.

Vaidyanathan, B., Matson, J., Miller, D., and Matson, J. (1999). A capacitated vehicle routing problem for just-in-time delivery. *IIE Transactions*, 31(11):1083–1092.

Whiteoak, P. (1994). The realities of quick response in the grocery sector: a supplier viewpoint. *International Journal of Physical Distribution & Logistics Management*, 24(10):33–39.

Yücel, E., Salman, F. S., Gel, E. S., Örmeci, E. L., and Gel, A. (2013). Optimizing specimen collection for processing in clinical testing laboratories. *European Journal of Operational Research*, 227(3):503–514.

Chapter 2

A Branch-and-Cut Algorithm for an Assembly Routing Problem

Abstract

We consider an integrated planning problem that combines production, inventory and inbound transportation decisions in a context where several suppliers each provide a subset of the components necessary for the production of a final product at a central plant. We provide a mixed integer programming formulation of the problem and propose several families of valid inequalities to strengthen the linear programming relaxation. We propose two new algorithms to separate the subtour elimination constraints for fractional solutions. The inequalities and separation procedures are used in a branch-and-cut algorithm. Computational experiments on a large set of generated test instances show that both the valid inequalities and the new separation procedures significantly improve the performance of the branch-and-cut algorithm.

2.1 Introduction

The literature on integrated planning in manufacturing industries highlights a significant potential for cost savings in the supply chain by combining production and transportation decisions (Viswanathan and Mathur, 1997; Fumero and Vercellis, 1999; Chen and Vairaktarakis, 2005; Archetti and Speranza, 2016). The problem of simultaneously planning the production at a plant and the outbound delivery routing is known in the literature as the production routing problem (PRP) (Archetti et al., 2011; Adulyasak et al., 2015). When the production plan at the plant is given and the decisions concern only the inventory and route planning, the problem is referred to as the inventory routing problem (IRP) (Andersson et al., 2010; Coelho et al., 2013). There exist many models and solution algorithms for these two problems. In contrast, few studies have considered the integration of production planning with inbound transportation for the collection of components from suppliers to assemble a final product.

When the assembly plant is responsible for organizing the inbound transportation of the various components, significant gains can be achieved by integrating production planning with inbound transportation (Carter and Ferrin, 1996). Automotive industry examples are studied in Blumenfeld et al. (1987) and Florian et al. (2011) for US and German manufacturers. Fernie and Sparks (2014) indicate that in the retail industry the logistics system should be effectively integrated with the suppliers. More specifically, they highlight the need for the optimization and management of the entire supply chain of retailers to be a single entity to obtain cost reduction advantages and service enhancements. Closing the supply chain loop is another example where the collection of the end-of-life products should be coordinated with the disassembly planning (Guide and Van Wassenhove, 2009).

We study the assembly routing problem (ARP) which considers a joint planning problem with a central plant that produces a final product to satisfy a dynamic but deterministic demand. The plant collects the necessary components from several suppliers, each providing a subset of the components. The plant coordinates the scheduling of the production as well as the routing decisions and shipment quantities from the suppliers. The aim is to minimize the total costs of production, inventory and routing subject to several types of capacity constraints. The planning is done over a finite and discrete time horizon. The quantities available at the suppliers are assumed to be known in advance. The plant has a limited capacity for the production and no backlogging or stockouts are allowed. Both the plant and the suppliers can carry inventory. The plant has separate and capacitated inbound and outbound storage areas for the incoming components from suppliers and for the final product, respectively. Each supplier has a global storage capacity for its own components. The plant manages a limited fleet of capacitated vehicles to handle the shipment of components from the suppliers to the plant.

Similar to the basic variants of the IRP and PRP, we do not allow a supplier to be visited by more than one vehicle in a specific period (i.e., no split pickups).

Some studies in the literature consider the optimization of the inbound transportation and inventory decisions without taking the production planning at the central plant into account. Popken (1994) and Berman and Wang (2006) study a single-period inbound logistics problem. They consider a multicommodity network with the origin (suppliers), destination (plant), and transshipment terminal nodes. The origin-destination commodity flows are supposed to be optimally routed through this network using at most one terminal node. The cost function includes the transportation and pipeline inventory costs for all supplier-plant pairs. The optimization of the inventory decisions together with the explicit inbound vehicle routes through multiple planning periods is studied in Moin et al. (2011) and Mjirda et al. (2014). Considering the automotive parts supply chain, these studies investigate the case of a single assembly plant for which multiple suppliers each provide a distinct part type.

A number of studies investigate the coordination of the inbound vehicle routes with the production rate in a just-in-time (JIT) environment where no end-period inventory exists in the planning horizon. Vaidyanathan et al. (1999) and Satoglu and Sahin (2013) study the parts delivery to an assembly line with the objective of minimizing the material handling equipment requirements in a central warehouse. Qu et al. (1999) and Sindhuchao et al. (2005) study the joint replenishment of multiple items in an inbound material-collection system for a central warehouse under the assumption of an infinite planning horizon. Chuah and Yingling (2005), Ohlmann et al. (2007), Stacey et al. (2007) and Natarajarathinam et al. (2012) consider a JIT supply pickup problem for an automotive assembly plant to minimize the inventory and transportation costs. Jiang et al. (2010) study a similar problem taking the storage space limit into account. Yücel et al. (2013) consider the problem of transporting specimens from different sites to the central processing facility of a clinical testing company. Lamsal et al. (2016) study a sugarcane harvest logistics problem in Brazil that requires the continuous operation of the production mill. Therefore, the inbound flow of raw material should never terminate.

One observes that the ARP includes a lot-sizing substructure with additional inventory constraints together with the distribution routing decisions in each period. Similar to the ARP, an inventory substructure exists in the uncapacitated LSP with inventory bounds which is well-studied in the literature. This problem was first introduced by Love (1973). Atamtürk and Küçükyavuz (2008) propose an $O(n^2)$ dynamic programming algorithm. Van Den Heuvel and Wagelmans (2008) show that the problem is equivalent to the LSP with a remanufacturing option, the LSP with production time windows, and the LSP with cumulative capacities. Di Summa and Wolsey (2010) consider a variable upper bound on the initial inventory and give valid inequalities and extended formulations to describe the convex hull. More recently, Hwang and van den Heuvel (2012) and Phouratsamay et al. (2018) study this problem and propose polynomial and pseudo-polynomial algorithms for different cost structures. Akbalik et al. (2015) study the multi-item LSP with stationary production capacity, time-dependent inventory bounds and concave costs as well as a global capacitated storage space for all the items. They show that the problem is NP-hard even when each item has stationary and identical production cost and capacity over periods. Also, other integrated problems such as the IRP (Archetti et al., 2007; Solyalı and Süral, 2011; Avella et al., 2015), maritime IRP (Agra et al., 2013), and PRP (Archetti et al., 2011; Adulyasak et al., 2014) consider bounded inventory in the problem structure. Due to the inventory structure similarity, the feasible sets of these integrated problems are related to each other. Although there are certain similarities between the ARP and these problems, they possess a distribution lot-sizing structure whereas the ARP is based on an assembly structure. The difference in the lot-sizing structure makes the feasible set of the ARP different particularly because of the given rate of the supply at the suppliers, and the fact that the suppliers and the production plant are connected via a routing structure.

To the best of our knowledge, there are two papers that studied a problem close to the one being addressed in this paper. A general case with multiple components and products is introduced by Hein and Almeder (2016). The authors consider two scenarios. In the first scenario, the plant is allowed to keep the components in stock while in the second scenario, which represents a JIT environment, the components that arrive at the plant must be used immediately in production. They examine both scenarios under the traditional sequential planning approach and under the integrated approach. In the sequential planning process, an LSP is solved first to obtain the production plan for the final product. Then, in the second step, they solve an IRP for the first scenario and one vehicle routing problem (VRP) for each period in the second scenario. The computational experiments are performed on randomly generated instances with either 4 suppliers, 8 components, 3 final products, and 5 periods or 6 suppliers, 12 components, 4 final products, and 10 periods. They report cost savings of up to 12% with the integrated planning approach compared to the classical sequential approach. According to this study, one may expect a higher potential for cost savings in the JIT scenario when applying the integrated approach. Because the authors did not consider the holding cost at the suppliers in their study, the integrated decision making is entirely focused on the costs associated with the plant. This is appropriate when the suppliers and the assembly plant are separate organizations and the assembly plant is not concerned with the inventory costs at the suppliers.

In the case where both the suppliers and the assembly plant belong to the same firm, one should ideally take into account the suppliers' inventory costs and capacities in the integrated decision making process. Chitsaz et al. (2019) study the case with multiple components and one final product but consider the inventory costs and storage capacity of the suppliers as well as a component storage area at the plant. They assume that every supplier provides a unique component. Consequently, a one-to-one relationship exists between the suppliers and components. The authors develop a three-phase decomposition-based matheuristic that iteratively solves different subproblems. They apply their algorithm not only to the ARP, but also to the IRP and the PRP with the same parameter setting. The computational experiments show that this algorithm returns high quality solutions for the ARP instances and outperforms existing heuristics on large-scale multi-vehicle instances of the IRP and PRP. The algorithm finds new best-known solutions to many standard test instances of these two problems.

We extend the model of Chitsaz et al. (2019) to consider the case where each supplier may provide a subset of the components necessary for the final product and some components can be obtained from more than one supplier. This is the first contribution of this paper. Second, we develop several new valid inequalities to strengthen the linear programming (LP) relaxation of the mixed integer programming formulation of the problem. Although several of the proposed inequalities are inspired from existing lot sizing inequalities, a novelty is that some of the inequalities use the known supply instead of the known demand. Third, we present novel algorithms to efficiently separate the subtour elimination constraints for the LP solutions that contain fractional routes, which can be adapted for other vehicle routing problems with the same feature. The inequalities and separation procedures are used in a branch-and-cut algorithm (BC). We generate a large test bed consisting of small to large instances with diverse ranges for the number of suppliers, products and planning periods. Finally, we analyze the impact of each class of valid inequalities on the value of the LP relaxation and on the final solution. Our extensive computational experiments show that both the valid inequalities and the new separation procedures notably enhance the performance of the branch-and-cut algorithm.

The remainder of the paper is organized as follows. We formally define the ARP and express it mathematically in Section 2.2. Section 2.3 is devoted to the presentation of the inequalities and to the proof of their validity. In Section 2.4, we present the upper bound generation procedure. To separate the subtour elimination constraints for our multi-period VRP, we present two heuristic algorithms in Section 2.5. The generation of the test instances and computational experiments are presented in Section 2.6. Finally, Section 2.7 concludes the paper.

2.2 Problem Definition and Mathematical Formulation

We consider a many-to-one assembly system with n suppliers represented by the set $N = \{1, ..., n\}$. The planning horizon includes l discrete time periods forming the set $T = \{1, .., l\}$. To produce the final product, k distinct components, represented by the set $K = \{1, ..., k\}$, are required. We extend the basic ARP introduced in Chitsaz et al. (2019) by assuming that each supplier i may provide a subset of the components $K_i \subseteq K$, where $K = \bigcup_i K_i$. Moreover, each component k can be provided by a subset of suppliers $N_k \subseteq N$, where $N = \bigcup_k N_k$. We define the problem on a complete undirected graph with the node set $N^+ = N \cup \{0\}$, where 0 represents the plant, and the edge set $E = \{(i,j) : i,j \in N^+, i < j\}$. We let $K^+ = K \cup \{0\}$ represent the set of all items, where 0 represents the final product. The suppliers as well as the central plant each have a global storage area for the components and may have some component inventory at hand at the beginning of the planning horizon. Moreover, the central plant has a separate storage space for the final product. A fleet of m homogeneous vehicles, each with a capacity of Q, is available to transport the components from the suppliers to the plant.

The decisions to make include whether or not to produce the final product and the quantity to be produced at the plant in each period, the supplier visit schedule and order in each vehicle route, and the shipment quantities from the suppliers to the plant. The manufacturing plant needs to minimize the production, inventory and transportation costs simultaneously for the entire planning horizon. The complete list of notations is presented in Table 2.1.

Table 2.1: ARP notation list

Sets:	
N^+	Set of nodes, $N^+ = \{0, ..., n\}$, where 0 represents the plant, and $N = N^+ \setminus \{0\}$ represents the set of suppliers.
E	Set of edges, $E = \{(i,j) : i,j \in N^+, i < j\}$.
K	Set of components indexed by $k \in \{1, ..., \lvert K \rvert\}$. We let $K^+ = K \cup \{0\}$.
K_i	Set of available components at supplier $i \in N$, $K_i \subseteq K$.
N_k	Set of suppliers that provide component $k \in K$, $N_k \subseteq N$.
T	Set of time periods, indexed by $t \in T = \{1, .., l\}$.
$E(S)$	Set of edges $(i,j) \in E$ such that $i,j \in S$, where $S \subseteq N^+$ is a given set of nodes.
$\delta(S)$	Set of edges incident to a node set S, $\delta(S) = \{(i,j) \in E : i \in S, j \notin S \text{ or } i \notin S, j \in S\}$.
Decision variables:	
p_t	Production quantity in period t at the plant.
y_t	Equal to 1 if there is production at the plant in period t, 0 otherwise.
I_{ikt}	Inventory of component $k \in K_i$ at supplier $i \in N$ at the end of period t.
I_{0kt}	Inventory of component or final product $k \in K^+$ at the plant at the end of period t.
x_{ijt}	Number of times a vehicle traverses the edge $(i,j) \in E$ in period t.
z_{it}	Equal to 1 if node $i \in N$ is visited in period t, 0 otherwise.
z_{0t}	Number of vehicles dispatched from the plant in period t.
q_{ikt}	Shipment quantity of component $k \in K$ from node $i \in N_k$ to the plant in period t.
Parameters:	
f, u	Fixed setup and unit production costs, respectively.
h_{ik}	Unit holding cost of item k at the plant or at supplier $i \in N+$.
c_{ij}	Transportation cost between nodes i and j, $(i,j) \in E$.
m	Fleet size.
C, Q	Production and vehicle capacity, respectively.
s_{ikt}	Supply of component $k \in K$ at node $i \in N_k$ in period t.
$s_{ikt_1t_2}$	Cumulative supply of component $k \in K$ at node $i \in N_k$ from period t_1 to period t_2 (inclusive), $t_1, t_2 \in T, t_1 \leq t_2$.
b_k	Unit size of component $k \in K$.
d_t	Demand for the final product at the plant in period t.
$d_{t_1t_2}$	Cumulative demand for the final product at the plant from period t_1 to period t_2 (inclusive), $t_1, t_2 \in T, t_1 \leq t_2$.
L_i	Global inventory capacity at supplier $i \in N$ for the components $k \in K_i$.
L	Global inventory capacity at the plant for the components $k \in K$.
L_0	Inventory capacity at the plant for the final product.
I_{ik0}	Initial inventory of component $k \in K$ available at supplier $i \in N_k$.
I_{0k0}	Initial inventory of component or final product $k \in K^+$ available at the plant.

A compact formulation for the ARP can be written as the following $\mathcal{M}_{ARP}$ model:

$$(\mathcal{M}_{ARP})\ \min \sum_{t \in T} \Big(up_t + fy_t + \sum_{k \in K^+} h_{0k} I_{0kt} + \sum_{i \in N} \sum_{k \in K_i} h_{ik} I_{ikt} + \sum_{(i,j) \in E} c_{ij} x_{ijt} \Big) \tag{2.1}$$

s.t.

$$I_{00,t-1} + p_t = d_t + I_{00t} \quad \forall t \in T \tag{2.2}$$

$$I_{0k,t-1} + \sum_{i \in N_k} q_{ikt} = p_t + I_{0kt} \quad \forall k \in K, \forall t \in T \tag{2.3}$$

$$I_{ik,t-1} + s_{ikt} = q_{ikt} + I_{ikt} \quad \forall i \in N, \forall k \in K_i, \forall t \in T \tag{2.4}$$

$$p_t \le Cy_t \quad \forall t \in T \tag{2.5}$$

$$I_{00t} \le L_0 \quad \forall t \in T \tag{2.6}$$

$$\sum_{k \in K} b_k I_{0kt} \le L \quad \forall t \in T \tag{2.7}$$

$$\sum_{k \in K_i} b_k I_{ikt} \le L_i \quad \forall i \in N, \forall t \in T \tag{2.8}$$

$$z_{0t} \le m \quad \forall t \in T \tag{2.9}$$

$$\sum_{k \in K_i} b_k q_{ikt} \le Q z_{it} \quad \forall i \in N, \forall t \in T \tag{2.10}$$

$$\sum_{(j,j') \in \delta(i)} x_{jj't} = 2 z_{it} \quad \forall i \in N^+, \forall t \in T \tag{2.11}$$

$$Q \sum_{(i,j) \in E(S)} x_{ijt} \le \sum_{i \in S} \Big(Q z_{it} - \sum_{k \in K_i} b_k q_{ikt} \Big) \quad \forall S \subseteq N, |S| \ge 2, \forall t \in T \tag{2.12}$$

$$p_t \ge 0, y_t \in \{0,1\}, z_{0t} \in \mathbb{Z} \quad \forall t \in T \tag{2.13}$$

$$I_{0kt} \ge 0 \quad \forall k \in K^+, \forall t \in T \tag{2.14}$$

$$I_{ikt}, q_{ikt} \ge 0 \quad \forall i \in N, \forall k \in K_i, \forall t \in T \tag{2.15}$$

$$x_{ijt} \in \{0,1\} \quad \forall (i,j) \in E : i \ne 0, \forall t \in T \tag{2.16}$$

$$x_{0it} \in \{0,1,2\}, z_{it} \in \{0,1\} \quad \forall i \in N, \forall t \in T. \tag{2.17}$$

The objective function (2.1) minimizes the total production, setup, inventory, and transportation costs. The inventory costs include both component inventories at the suppliers and at the plant, as well as the final product at the plant. The set of constraints (2.2) ensures the final product inventory flow while constraints (2.3) do the same for each component at the plant. Constraints (2.4) guarantee the inventory flow balance for each component at each supplier. Constraints (2.5) force a setup at the plant in each period where production takes place. They also impose a maximum limit on the production quantity. Constraints (2.6) consider the storage capacity of the final product at the plant. Constraints (2.7) impose the shared storage capacity of the components at the plant. The shared storage capacity of components at each supplier is enforced by constraints (2.8). Constraints (2.9) impose the limit on the fleet size. Constraints (2.10) force a vehicle visit whenever components are shipped from a certain node to the plant. The total component shipment quantity from each supplier in each period will also be limited by the vehicle capacity. Constraints (2.11) are the degree constraints. Constraints (2.12) are the subtour elimination constraints (SEC).

These constraints are the modified version of the VRP capacity-cuts (Toth and Vigo, 2002; Iori et al., 2007). They require each route to be connected to the plant and the total shipments on each route to not exceed the vehicle capacity. There exists an exponential number of these constraints. They are referred to in the literature as generalized fractional subtour elimination constraints (GFSEC) (Adulyasak et al., 2014). Constraints (2.13)-(2.17) are domain constraints.

2.3 Strengthening the LP Relaxation Bound

We present valid inequalities to improve the LP relaxation of $\mathcal{M}_{ARP}$. Moreover, we present the links between these inequalities and related polyhedral studies in the literature. The polyhedral structure of the LSP and VRP has been researched extensively. Barany et al. (1984) give a complete linear description of the convex hull of the solutions for the uncapacitated LSP. Pochet (1988), Miller et al. (2000), and Atamtürk and Muñoz (2004) present inequalities for the capacitated LSP with unlimited storage capacity. Atamtürk and Küçükyavuz (2005) investigate the polyhedral structure of the lot-sizing problem with inventory bounds and fixed costs. The polyhedral study of multi-echelon LSP with intermediate demands is given in Zhang et al. (2012). The uncapacitated LSP is a special case of fixed charge network design (Van Roy and Wolsey, 1985). Gendron et al. (1999) and Kucukyavuz (2005) study polyhedral approaches for capacitated multicommodity network design and fixed-charge network flow problems, respectively. Chouman et al. (2016) present cut-set-based inequalities for multicommodity capacitated fixed-charge network design problems. Similarly, many polyhedral studies are presented in the literature for different variants of the VRP. Cornuejols and Harche (1993) and Ralphs et al. (2003) study the capacitated variant and Belenguer et al. (2000) investigate the split delivery VRP.

Three classes of valid inequalities are presented to improve the LP relaxation bound for the $\mathcal{M}_{ARP}$ model. The first class contains *(l,S,WW)*-type inequalities. The second one concerns the bounds on the variables. We present the proof of the propositions in Appendix B. The last class includes general inequalities for the ARP. Propositions 1, 2 and 7 present inequalities derived from the particular structure of the underlying LSP for each component k (Pochet and Wolsey, 2006). These inequalities take advantage of the aggregated available inventory of each component k at the suppliers (that provide component k) and the production plant for each period $t \in T$.

2.3.1 *(l,S,WW)*-Type Inequalities

The *(l,S)* inequalities were introduced in Barany et al. (1984) and provide the convex hull of the single-item uncapacitated LSP. In the (l,S) inequalities, l refers to a period ($l \leq |T|$) where T is the number of periods, and S is a subset of periods $\{1,...,l\}$ not necessarily connected ($S \subseteq \{1,...,l\}$) such as periods $\{1,3,7\}$ when $l = 10$. For a numerical example, we refer to Pochet and Wolsey (2006), pp. 122-123. Although there is an exponential number of these constraints for a general cost structure, Pochet and Wolsey (1994) showed that under the Wagner-Whitin (WW) cost condition it is sufficient to consider only $O(l^2)$ inequalities to describe the convex hull of the single item uncapacitated lot sizing problem which are referred to as (l,S,WW) inequalities. The WW non-speculative cost structure requires the sum of unit production and inventory costs in every period to be larger than or equal to the unit production cost in the next period. Therefore, when the unit production costs are the same for all periods, the WW cost condition holds because the inventory costs are nonnegative. We first present the known (l,S,WW) inequalities applied to the lot sizing structure (2.2) and (2.5):

$$\sum_{e=t_1}^{t_2} p_e \leq I_{00t_2} + \sum_{e=t_1}^{t_2} d_{et_2} y_e \quad \forall t_1, t_2 \in T, t_1 \leq t_2. \tag{2.18}$$

These inequalities link the production and setup variables at the plant with the predetermined downstream demand in order to improve the LP relaxation lower bound. Next, we derive three new families of valid inequalities for the ARP. The new inequalities are inspired from the standard (l, S, WW) inequalities, but present some novelties. In Proposition 2.1, we develop new inequalities that link the production and setup variables at the plant with the known upstream supply. The structure of the proof (given in Appendix B) follows a similar structure as for the (l, S) inequalities (Pochet and Wolsey, 2006), but with an inverted logic as it takes into account the known supply at the suppliers. Moreover, in Propositions 2.2 and 2.3 we propose new inequalities linking the shipment quantities and node visit variables with the given supply and demand, respectively. The novelty in the structure of these constraints is that, for a given period, the shipment variables are defined for each supplier-component combination, whereas the supplier visit variables are only related to the supplier. There is no setup-type constraint in the model that directly links each component shipment variable to its supplier visit variable. This is different from a traditional lot-sizing structure.

Proposition 2.1. *Inequalities*

$$\sum_{e=t_1}^{t_2} p_e \leq I_{0k,t_1-1} + \sum_{i \in N_k} I_{ik,t_1-1} + \sum_{e=t_1}^{t_2} \sum_{i \in N_k} s_{ikt_1e} y_e \quad \forall k \in K, \forall t_1, t_2 \in T, t_1 \leq t_2 \tag{2.19}$$

are valid for the $\mathcal{M}_{ARP}$.

Notice that although both inequalities (2.18) and (2.19) provide bounds on the total production quantities, the first set of inequalities considers the cumulative demand and the remaining product inventory at the last period (t_2) while the second set of inequalities takes the cumulative component supply and the available inventory at the beginning of the first period (t_1) into account.

Proposition 2.2. *Inequalities*

$$\sum_{e=t_1}^{t_2} q_{ike} \leq I_{ik,t_1-1} + \sum_{e=t_1}^{t_2} s_{ikt_1e} z_{ie} \quad \forall i \in N, \forall k \in K_i, \forall t_1, t_2 \in T, t_1 \leq t_2 \tag{2.20}$$

are valid for the $\mathcal{M}_{ARP}$.

Proposition 2.3. *Inequalities*

$$\sum_{e=t_1}^{t_2} \sum_{i \in N_k} q_{ike} \leq I_{00t_2} + I_{0kt_2} + \sum_{e=t_1}^{t_2} d_{et_2} \sum_{i \in N_k} z_{ie} \quad \forall k \in K, \forall t_1, t_2 \in T, t_1 \leq t_2 \tag{2.21}$$

are valid for the $\mathcal{M}_{ARP}$.

Both inequalities (2.20) and (2.21) provide bounds on the total shipment quantities. The first set of inequalities considers the cumulative component supply and the available inventory at the beginning of the first period (t_1) at each supplier while the second set of inequalities takes the cumulative demand and the remaining product and component inventory at the plant in the last period (t_2) into account.

2.3.2 Bounds on Variables

The bounds we propose in this subsection are linked to the cut-set type inequalities. Atamtürk and Küçükyavuz (2005) observe that *(l,S)* inequalities may not cut off fractional LP extreme solutions for lot-sizing with inventory bounds and fixed costs if for the subset of periods S incoming or outgoing inventory is at capacity. They introduce cut-set type inequalities to enforce one production setup for a certain number of periods. We introduce inequalities that are both a generalization and an extension of the cut-set type inequalities. We generalize the cut-set type inequalities to provide integer lower bounds on the number of required production setups from period $e = 1$ to $t \in T$ (Proposition 2.4). We further extend these cut-set type inequalities to enforce integer lower bounds on the number of vehicles dispatched (Proposition 2.5), and supplier visits from period $e = 1$ to $t \in T$ (Propositions 2.6-2.7).

Let Q_{it} (measured in required space) be a parameter equal to the sum of cumulative supply of components and the initial inventory of the components at supplier i minus its available storage capacity, i.e.,

$$Q_{it} = \sum_{k \in K_i} b_k(s_{ik1t} + I_{ik0}) - L_i.$$

Proposition 2.4. *Inequalities*

$$\left\lceil \frac{\max\left\{0, d_{1t} - I_{000}, (\sum_{k\in K} b_k I_{0k0} + \sum_{i\in N} \max\{0, Q_{it}\} - L) / \sum_{k\in K} b_k\right\}}{\min\{C, \max_{e\in\{1,\dots,t\}}\{d_e\} + L_0\}} \right\rceil \leq \sum_{e=1}^{t} y_e \quad \forall t \in T \tag{2.22}$$

are valid for $\mathcal{M}_{ARP}$.

Notice that $\sum_{k\in K} b_k$ in the last expression of the LHS of the inequalities (2.22) represents the total required space by the components which are required to produce one unit of the final product. Next, we present valid inequalities for the lower bound on the total number of necessary vehicles dispatched from period $e = 1$ to t.

Proposition 2.5. *Inequalities*

$$\left\lceil \frac{1}{Q} \max\left\{ \sum_{k\in K} b_k \max\{0, d_{1t} - I_{000} - I_{0k0}\}, \sum_{i\in N} \max\{0, Q_{it}\} \right\} \right\rceil \leq \sum_{e=1}^{t} z_{0e} \quad \forall t \in T \tag{2.23}$$

are valid for $\mathcal{M}_{ARP}$.

Next, we present valid inequalities for a lower bound on the total number of necessary node visits from period $e = 1$ to t in the following proposition.

Proposition 2.6. *Inequalities*

$$\left\lceil \frac{\max\{0, Q_{it}\}}{\min\left\{Q, L_i + \max_{e\in\{1,\dots,t\}}\{\sum_{k\in K_i} b_k s_{ike}\}, \sum_{k\in K_i} b_k (I_{ik0} + s_{ik1t})\right\}} \right\rceil \leq \sum_{e=1}^{t} z_{ie} \quad \forall i \in N, \forall t \in T \tag{2.24}$$

are valid for $\mathcal{M}_{ARP}$.

At any supplier, when the initial inventories plus the cumulative supply of components in the first t periods exceed the storage capacity, inequalities (2.24) provide a lower bound on the number of required visits to that supplier during these periods. The cumulative shipments from the supplier in the first t periods is limited first by the vehicle capacity, second by the available storage plus the maximum total component supply in any of those periods, and third by the sum of the initial inventories and the total supply of all components during these periods.

Proposition 2.7. *Inequalities*

$$\left\lceil \frac{\max\{0, d_{1t} - I_{000} - I_{0k0}\}}{\min\left\{\frac{Q}{b_k}, \max_{i\in N_k}\{I_{ik0} + s_{ik1t}\}\right\}} \right\rceil \leq \sum_{e=1}^{t} \sum_{i\in N_k} z_{ie} \quad \forall k \in K, \forall t \in T \tag{2.25}$$

are valid for $\mathcal{M}_{ARP}$.

For the periods whose cumulative demand cannot be satisfied from the initial product inventory and in the case where the initial inventory of a given component is not sufficient for the production, inequalities (2.25) force visits to the nodes which supply that specific component. The cumulative shipments of a component from any of the associated suppliers in the first t periods is limited not only by the vehicle capacity but also by the maximum of the initial inventory of that component plus the total supply of the component from those suppliers in the same periods. It is possible to state inequalities (2.24)-(2.25) for the edge variables (x_{ijt}) instead of node visits (z_{it}). This leads to identical constraints due to the degree constraints (2.11).

2.3.3 General Inequalities

Without the SECs (2.12) added a priori to the model (e.g., as in the case of a BC algorithm), it may happen that the plant would not be connected to the other visited nodes in certain periods. In these cases, the following inequalities impose a positive value on the number of dispatched vehicles and hence on the degree of the plant if any node is visited in the same period:

$$z_{it} \leq z_{0t} \quad \forall i \in N, \forall t \in T. \tag{2.26}$$

Another type of SEC is Dantzig-Fulkerson-Johnson (DFJ), which can be represented for the $\mathcal{M}_{ARP}$ as follows:

$$\sum_{(i,j)\in E(S)} x_{ijt} \leq \sum_{i\in S} z_{it} - z_{et} \quad \forall S \subseteq N, |S| \geq 2, \forall e \in S, \forall t \in T. \tag{2.27}$$

DFJ inequalities are referred to in the literature as connectivity constraints (Laporte, 1986), infeasible-path constraints (Ascheuer et al., 2000; Iori et al., 2007), or clique constraints (Bektaş and Gouveia, 2014). They were first proposed by Dantzig et al. (1954) for the travelling salesman problem (TSP). These inequalities imply that the number of edges that can be chosen from the set of all edges with both endpoints in a subset of nodes S cannot be more than $|S| - 1$. The cardinality of these inequalities is exponential and thus they cannot be added a priori to the model in practical applications. Both GFSECs and DFJs can be added to the model at the same time. Observe that DFJs do not impose the vehicle capacity. Archetti et al. (2007) and Archetti et al. (2018) employ DFJ constraints for the IRP, and Archetti et al. (2011) and Adulyasak et al. (2014) use them for the PRP. The following inequalities enforce node visits for each edge traversal:

$$x_{ijt} \leq z_{it} \quad and \quad x_{ijt} \leq z_{jt} \quad \forall (i,j) \in E(N), \forall t \in T. \tag{2.28}$$

Inequalities (2.26) and (2.28) are used by Archetti et al. (2007) for the IRP, and by Archetti et al. (2011) and Adulyasak et al. (2014) for the PRP. Inequalities (2.28) are special cases of DFJs for node pairs (Gendreau et al., 1998), which can be added to the model a priori due to their polynomial cardinality.

2.4 Generating Upper Bounds

We adapted the unified matheuristic proposed in Chitsaz et al. (2019) and applied it to the generalized ARP, where each supplier provides a subset of the components, to obtain high quality feasible solutions as well as cutoff values that can be used to prune branches in our BC algorithm. This matheuristic (CCJ-DH) works by decomposing the problem into three separate subproblems and solving them iteratively. The first subproblem is a special LSP which determines a setup schedule with an approximation of the total transportation cost using the number of dispatched vehicles. The second subproblem returns node visits and shipment quantities. The latter employs another approximation of the total transportation cost using the node visit transportation cost. Finally, the third subproblem considers a separate VRP for each period t.

The solutions of the routing subproblems are used to update the node visit cost approximation in the second subproblem for the next iteration. This procedure is repeated to reach a local optimum. Then, a change in the setup schedule is imposed to explore other parts of the feasible solution space and diversify the search. The algorithm uses diversification constraints (Fischetti et al., 2004) to generate both new setup schedules using the first subproblem, and new node visit patterns using the second subproblem. The method terminates when a stopping condition is met. We present the detailed adaptation of CCJ-DH in Appendix B.

2.5 Separating Fractional Multi-Period Subtour Elimination Constraints

Subtour elimination constraints (2.12) belong to the family of capacity-cut constraints (CCC) which were developed for the capacitated VRP (Toth and Vigo, 2002; Iori et al., 2007). The RHS of these constraints represents the number of vehicles required to serve the subset of nodes for which the inequality is applied. Depending on how the RHS is computed, different classes of this set of constraints can be obtained. The direct use of the fractional RHS results in the *fractional capacity inequalities*. This class of capacity constraints can be separated by solving a series of max-flow or min-cut problems in polynomial time (Semet et al., 2014). The next three classes of CCCs need specific algorithms and their separation is known to be NP-complete (Augerat, 1995). When the RHS is rounded up, one obtains the *rounded capacity inequalities*. Using the optimal value of the bin-packing problem (where the weights of the items are equal to the shipment sizes and the bin capacity is equivalent to the vehicle capacity) in the RHS results in the *weak capacity inequalities*. Finally, computing the minimum number of required vehicles results in *global capacity constraints* and gives the tightest form.

Unlike the other types of CCCs, the quantities in the RHS of GFSECs are not given parameters but node visit (z_{it}) and shipment quantity (q_{ikt}) variables. For the non-vehicle index formulations of the IRP and the PRP, GFSECs are necessary to maintain the vehicle capacity of each route. To the best of our knowledge, there is no exact algorithm to separate GFSECs in polynomial time and it is not known whether separating GFSECs is NP-hard or not. Instead, a weak form of them (with $z_{it} = 1$) is usually separated using separation procedures designed for the TSP and VRP CCCs. Most of the BC algorithms in the IRP and the PRP literature use the separation procedure of Padberg and Rinaldi (1991) or heuristics that are included in the CVRPSEP package of Lysgaard et al. (2004). The procedures of Padberg and Rinaldi (1991) and Lysgaard et al. (2004) were originally developed for the TSP and the VRP, respectively. The algorithm of Padberg and Rinaldi (1991) is used by Archetti et al. (2007, 2011); Solyalı and Süral (2011); Avella et al. (2015) and Archetti et al. (2018). The CVRPSEP package is used by Adulyasak et al. (2014). If a violated inequality is found by one of these procedures, one has to check whether the corresponding GFSEC is violated or not (Solyalı and Süral, 2011). In Appendix B, we present two examples for the LP solutions to the routing problem containing fractional values for the node visit (z_{it}) and edge traversal (x_{ijt}) variables. One example shows the case where a non-violated subtour elimination constraint is returned. The other example demonstrates the case where a violated subtour elimination constraint cannot be identified when the weak GFSEC is separated. Note that Contardo et al. (2012) propose a polynomial time max-flow algorithm to separate the fractional capacity cuts for the two-echelon capacitated location-routing problem. This suggests that it might be possible to do the same for GFSECs.

The separation problem for GFSECs in the ARP is to find a subset of nodes $S \subseteq N$ with cardinality greater than or equal to 2 ($|S| \geq 2$) for which the corresponding constraint is violated by the fractional solution. In each period t, the non-zero z^* and x^* values of the optimal LP solution form a subgraph $G^t(N^t, E^t)$. Each node in G^t has a shipment volume of $\sum_{k \in K_i} b_k q^*_{ikt}$. In order to define the separation problem, let the binary variable v_i be equal to 1 if and only if node $i \in N^t$ is selected and binary variable w_{ij} be equal to 1 if and only if edge $(i,j) \in E^t$ is chosen. We formulate the GFSECs separation problem for each period t as follows:

$$(\mathcal{S}^t_{GFSEC}) \quad \min \sum_{i \in N^t} (Qz^*_{it} - \sum_{k \in K_i} b_k q^*_{ikt}) v_i - Q \sum_{(i,j) \in E(N^t)} x^*_{ijt} w_{ij} \tag{2.29}$$

s.t.

$$\sum_{i \in N^t} v_i \geq 2 \tag{2.30}$$

$$w_{ij} \leq v_i \quad \forall (i,j) \in E^t \tag{2.31}$$

$$w_{ij} \leq v_j \quad \forall (i,j) \in E^t \tag{2.32}$$

$$v_i, w_{ij} \in \{0,1\} \quad \forall i \in N^t, \forall (i,j) \in E^t. \tag{2.33}$$

Since G^t is defined for $(i,j) \in E^t$, it may not be a complete subgraph nor a connected one. Observe that any feasible solution to this problem which has a strictly negative value returns one or more violated GFSECs. Notice that unlike the separation problem for the VRP CCCs, this problem is independent of the plant's (depot's) adjacent edges (x_{0it}). Moreover, the problem S^t_{GFSEC} is separable over the disconnected elements of the subgraph of period t, as was first implemented by Laporte et al. (1985) for the VRP under capacity and distance constraints.

To separate violated GFSECs with fractional node degrees, we propose two heuristics which can also be adapted for other vehicle routing problems. We define $e = (i_e, j_e) \in E^t$, the index of edges in the subgraph edge set of period t. We initialize sets $\Omega_1, ..., \Omega_{|E^t|}$ indexed by ϵ, and populate each Ω_ϵ with edge $\epsilon \in E^t$. We define $\Phi(\Omega_\epsilon)$ as the set of nodes corresponding to all the edges in Ω_ϵ. Let $C_i = Qz^*_{it} - \sum_{k \in K_i} b_k q^*_{ikt}$ represent the node cost and $C^e = Q \sum_{(i,j) \in E(N^t)} x^*_{ijt}$ the edge gain. The first algorithm (Algorithm $\mathcal{A}1$) finds violated GFSECs (for each period t) by adding to set Ω_ϵ the edge e which has the least marginal cost ($C_{i_e} + C_{j_e} - C^e$), not necessarily a negative cost, at each iteration. We only check for $e > \epsilon$ to force every initial set Ω_ϵ to deal with a different subset of edges. Otherwise, different sets eventually may end up with the same result. Notice that the last set, $\Omega_{|E^t|}$, will not examine other edges.

Algorithm 2: GFSEC Separation Procedure: $\mathcal{A}1$

Initialize $|E^t|$ sets Ω_ϵ, for all $\epsilon \in E^t$
for all $\epsilon \in \{1, ..., |E^t|\}$ **do**
  for all $e \in E^t \setminus \Omega_\epsilon, e > \epsilon$ **do**
    $e^* = \arg\min_e \{C_{i_e} + C_{j_e} - C^e\}$
    $\Omega_\epsilon \leftarrow \Omega_\epsilon \cup \{e^*\}$
    if $\Phi(\Omega_\epsilon)$ introduces a violated GFSEC **and** $\Phi(\Omega_\epsilon)$ is not found yet **then**
      Add $\Phi(\Omega_\epsilon)$ to the list of violated GFSECs
    end if
  end for
end for
return the list of violated GFSECs

The second algorithm (Algorithm $\mathcal{A}2$) has a similar structure as $\mathcal{A}1$ with the difference that it terminates the search procedure for each set Ω_ϵ when the set returns the first violated GFSEC and then proceeds to the next set. Moreover, Algorithm $\mathcal{A}2$ does not accept the node sets which have (node) overlap with the violated GFSECs found earlier in the current call of the algorithm. Because every violated GFSEC needs to have at least two nodes, there is an explicit upper bound of $|N^t|/2$ on the number of violated GFSECs that $\mathcal{A}2$ returns for each period t.

2.6 Computational Experiments

The experiments were performed on the Calcul Québec computing infrastructure with Intel Xeon X5650 @ 2.67 GHz processors and a memory limit of 25 GB. The BC procedure is implemented in C++ using the CPLEX 12.6 callable library. All experiments are performed in sequential form using one thread. The algorithm applies the valid inequalities at the root node and adds GFSECs and DFJs at each node of the search tree as cutting planes whenever they are violated by more than 0.1 unit. To separate GFSECs, we either use CVRPSEP, $\mathcal{A}1$ or $\mathcal{A}2$. When a violated GFSEC is found, the BC method also adds the corresponding DFJ. In our experiments we set a time limit of one hour both for the BC and for CCJ-DH. We run the BC experiments with and without the CCJ-DH cutoff values to measure the performance of both methods in providing upper bounds.

We introduce a diverse set of instances to better study and evaluate the performance of the BC. We present the test bed generation procedure for the ARP in Section 2.6.1. We analyze the performance of CCJ-DH on the new instances in Section 2.6.2. We report the sensitivity analysis of the effect of valid inequalities on the LP relaxation of the $\mathcal{M}_{ARP}$ model, and the performance of the BC in Section 2.6.3. The performance analysis of the BC with different separation procedures is presented in Section 2.6.4. In Appendix B, we report the performance of the BC on the existing large instances of Chitsaz et al. (2019) and compare our results with the two lower bounding methods presented in that paper.

2.6.1 ARP Tests Instances

Two out of three ARP data sets introduced in Chitsaz et al. (2019) include instances with 50 and 100 suppliers, all with 6 periods. Therefore, they are too large to be solved by our exact algorithm. Moreover, those instances only consider the case where every supplier provides a unique component. To cover the general case of the ARP presented in this paper, and to test the BC on different sizes of instances, we generated three new classes of instances. The first class includes instances where each supplier provides a unique component type. The second class represents the case where each supplier provides a subset of components. The third class corresponds to the situation in which one single component is offered by all suppliers. Each class includes data sets with five different planning horizons ranging from 4 to 12 periods with a step of two. For each planning horizon we consider eight different numbers of suppliers, increasing by steps of 3. For each combination of the number of planning periods and suppliers we randomly generated five instances. Overall, 600 instances are generated for three classes, five planning horizons, eight numbers of suppliers, and five instances per category. As a result, the test bed includes small to large size instances. The rest of the specifications for the ARP instances are developed similar to the practices of Archetti et al. (2011) for the PRP. Table 2.2 presents an overview of the ARP instance parameters.

Table 2.2: ARP test instances*

Class	1	2	3
Number of instances	200	200	200
Number of periods: l		4 to 12	
Number of suppliers: n (for $l = 4$)		18 to 39	
Number of suppliers: n (for $l = 6$)		15 to 36	
Number of suppliers: n (for $l = 8$)		12 to 33	
Number of suppliers: n (for $l = 10$)		9 to 30	
Number of suppliers: n (for $l = 12$)		6 to 27	
Number of components: k	n	$0.4n$	1
Number of vehicles: m		UL‡	
Vehicle capacity: Q		$2 \max_i L_i$	
Demand (final product): $d_t = d$		Constant and UDRI†† $[50, 100]$	
Production capacity: C		UDRI†† $[d, 3d]$	
Component supply: $s_{ikt} = s_{ik}$		Constant and UDRI†† $[5, 0.5d]$	
Component size: b_k		UDRI†† $[1, 2]$	
Plant inventory capacity for final product: L_0		UDRI†† $[2d, 3d]$	
Plant inventory capacity for components: L		$\sum_{i \in N} L_i$	
Supplier inventory capacity: L_i		$\sum_{k \in K_i} b_k (I_{ik0} + 2s_{ik})$	
Plant initial inventory of final product: I_{000}		UDRI†† $[0, 1.5d]$	
Plant initial inventory of components: I_{0k0}		UDRI†† $[I_k^{*\dagger}, I_k^{*\dagger} + 0.5d]$	
Supplier initial inventory: I_{ik0}		UDRI‡† $[0, d]$	
Unit production cost: u		$h_{00}/5$	
Production setup cost: f		$150u$	
Plant unit final product holding cost: h_{00}		UDRI†† $[\sum_{k \in K} h_{0k}, 1.5 \sum_{k \in K} h_{0k}]$	
Plant unit component holding cost: h_{0k}		$\max_i h_{ik}$	
Supplier unit holding cost for each component: h_{ik}		UDRI†† $[1, 5]$	
Supplier and plant x,y coordinates		UDRI†† $[0, 1000]$	
Travel distance		SA‡†	
Unit transportation cost		1	

* Adapted from Chitsaz et al. (2019)

† $I_k^* = \max\{0, l(d - \sum_{i \in N_k} s_{ik}) - I_{000}\}$, ‡ Unlimited, †† Uniformly Distributed Random Integer,

‡† Similar to Archetti et al. (2011)

Table 2.3: Summary of the CCJ-DH results

Data Set	#	#BUB	CPU	Gap UB† (%)	Gap LB‡ (%)
Class 1					
Not Optimal	51	43	248.9	-59.04	2.74
Optimal	149	1	119.6	1.19	1.19
Total	200	44	152.6	-14.17	1.59
Class 2					
Not Optimal	81	66	2963.1	-62.24	3.62
Optimal	119	4	1786.3	1.22	1.22
Total	200	70	2262.9	-24.48	2.2
Class 3					
Not Optimal	29	13	90.8	-15.54	2.86
Optimal	171	5	44.1	1.55	1.55
Total	200	18	50.9	-0.93	1.74

† Gap UB = (UB_{CCJ-DH} - UB_{BC}) / UB_{BC}
‡ Gap LB = (UB_{CCJ-DH} - LB_{BC}) / LB_{BC}

2.6.2 Performance of the Heuristic

Table 2.3 shows the performance of the adapted CCJ-DH on different classes of the new ARP instances compared to the BC when using the best-bound node selection strategy and algorithm $\mathcal{A}1$ for separating fractional subtours, and with the imposed time limit of one hour. The second column in this table presents the number of instances (#). The rest of the columns show the number of best upper bounds (#BUB) found by CCJ-DH, the average solution time (CPU), and the gaps of the heuristic solution with respect to the upper bound (Gap UB) and lower bound (Gap LB) obtained by the BC, respectively. The results highlight the fact that the instances of the second class need significantly more computing time. In these instances, each supplier provides multiple components. There are consequently more shipment variables (q_{ikt}), which results in a larger lot-sizing part compared to the instances in the two other classes. For the instances that are not solved to optimality by BC (larger instances), the matheuristic finds 122 best upper bounds (BUB) out of 161 instances (all classes). For these instances, CCJ-DH is able to improve the UBs found by the BC by 59%, 62.2% and 15.5% on average for the instances in the first, second and third class, respectively. For the instances solved to optimality, the heuristic provides high quality solutions within 1.2%, 1.2% and 1.6% of the optimal solution for the first, second and third class, respectively.

2.6.3 Analysis of Valid Inequalities

To evaluate the effect of applying valid inequalities, we solve the LP relaxation of the $\mathcal{M}_{ARP}$ model where the SECs (2.12) are relaxed. We present in Table 2.4 the average LP solution times and values when no valid inequality is added to the model (None), and compare it with the cases where known valid inequalities (Known) from the literature (i.e., (2.18), (2.26)-(2.27)), or all valid inequalities (All) (i.e., (2.18)-(2.27)) are added to the model. Each row in this table shows the results for a period-supplier size combination. For the ease of comparison, the LP solution values are presented as a percentage of the BUB (LP%) for each instance. The average LP solution values without the valid inequalities vary in the range 63% to 65.9% for different classes and this range increases to 70.8% to 76.9% when the known inequalities are added and further to 88.7% to 90.2% with all valid inequalities added to the model. This is a significant improvement which is obtained at the expense of longer LP solution times. The average CPU times grow by a factor of 34, 22 and 10 for the instances in the first, second and third class, respectively when comparing the formulation without the valid inequalities to the formulation with all inequalities. We present details on the average LP solution values with and without considering each valid inequality type in the model in Appendix B.

We also compare the effect of the valid inequalities on the BC performance. In Table 2.5, we report a summary of the results on the performance of the BC when the default or the best-bound node selection strategies are employed, and either no inequality (None), only known inequalities (Known) or all inequalities (All) are applied. In all of these experiments we used algorithm $\mathcal{A}1$ to separate SECs (2.12) and (2.27). This table presents the number of optimal solutions (#Opt), CPU time, the average lower bound values as a percentage of the upper bound obtained by the BC without applying the CCJ-DH cutoffs (%UB) and as a percentage of the BUB (%BUB) for each BC scenario and each class. To calculate the BUB for each BC scenario, we considered the upper bounds obtained by either that BC scenario or CCJ-DH.

The results indicate that the BC returns better results, in terms of the number of optimal solutions, average solution time, and optimality gap, when all inequalities are applied and the best-bound node selection strategy is selected. The BC returns better %UB

Table 2.4: Effect of valid inequalities on LP solution

	Class 1							Class 2							Class 3						
	Set	None		Known		All		Set	None		Known		All		Set	None		Known		All	
l/n	Size	CPU	LP%	CPU	LP%	CPU	LP%	Size	CPU	LP%	CPU	LP%	CPU	LP%	Size	CPU	LP%	CPU	LP%	CPU	LP%
4/18	5	0.004	60.4	0.016	69.9	0.022	86.6	5	0.01	71.9	0.02	82	0.042	92.8	5	0	68.1	0.01	70.9	0.012	92.5
4/21	5	0.01	57.2	0.028	70.3	0.03	86.3	5	0.012	69	0.026	77.2	0.066	89.7	5	0	66.5	0.01	68.9	0.012	90.6
4/24	5	0.004	56.5	0.032	68.9	0.038	86.3	5	0.016	64.6	0.044	78.9	0.074	91.3	5	0.002	64.7	0.01	68.5	0.018	92.9
4/27	5	0	59.1	0.034	70.4	0.05	86.6	5	0.02	66.7	0.066	81.5	0.122	92.9	5	0.006	65.3	0.018	68	0.028	94.3
4/30	5	0.01	62.1	0.066	76.6	0.058	91	5	0.034	68.7	0.12	80.9	0.196	92.6	5	0.008	67	0.022	71	0.018	93.9
4/33	5	0.004	61	0.084	73.7	0.076	89.7	5	0.04	69.4	0.12	80.7	0.246	92.3	5	0.008	64.6	0.022	68.9	0.03	92.9
4/36	5	0.01	61.2	0.094	72.5	0.1	87.9	5	0.052	65.6	0.194	77.8	0.294	91.7	5	0.002	61.5	0.03	67.8	0.032	92.3
4/39	5	0.008	53.9	0.112	64.2	0.13	83.3	5	0.074	55.2	0.362	70.6	0.478	88.4	5	0.006	46.1	0.034	53.9	0.078	88.7
6/15	5	0.01	67.5	0.014	79.5	0.044	92.4	5	0.012	72.9	0.032	82.4	0.092	92.7	5	0.002	70.4	0.016	74	0.016	92
6/18	5	0.002	65.8	0.032	74.2	0.056	89	5	0.012	63.1	0.046	77.9	0.148	90.6	5	0.006	69.3	0.024	73.2	0.03	89.9
6/21	5	0.008	56.4	0.068	72.4	0.106	87.4	5	0.026	73.1	0.104	79.5	0.258	90.9	5	0.01	63.6	0.03	69.6	0.034	88.2
6/24	5	0.006	60.3	0.05	74.3	0.114	90	5	0.034	72.8	0.152	84.2	0.434	93.2	5	0.004	65.9	0.028	68.8	0.032	88.4
6/27	5	0.006	63.5	0.078	76.4	0.154	91.3	5	0.056	56.7	0.262	76.1	0.428	89.7	5	0.006	67.3	0.032	71.9	0.042	91
6/30	5	0.01	60.5	0.15	74.7	0.194	89.8	5	0.09	59.8	0.29	73.7	0.75	90.3	5	0.01	60.9	0.05	67.3	0.046	90.5
6/33	5	0.016	55.9	0.176	69.7	0.264	88	5	0.116	59.4	0.566	76.7	1.1	90.7	5	0.01	65.5	0.072	69	0.056	87.1
6/36	5	0.014	54	0.154	74	0.31	89.7	5	0.206	53.8	0.72	75.6	0.952	91.8	5	0.01	60.3	0.074	70.2	0.088	89.3
8/12	5	0.01	69.7	0.01	79.3	0.062	91.7	5	0.008	73.7	0.034	84	0.082	92.1	5	0.002	73.4	0.016	74.9	0.018	91
8/15	5	0.002	68.9	0.016	79.5	0.092	91.5	5	0.018	71.1	0.064	83.5	0.256	92.6	5	0.008	65.8	0.024	72.7	0.032	89.3
8/18	5	0.01	64.6	0.032	79.3	0.118	92.2	5	0.02	76.4	0.09	82.9	0.386	92.2	5	0.002	71.5	0.038	76.3	0.044	89.8
8/21	5	0.01	62.7	0.078	75.5	0.228	88.4	5	0.042	63	0.154	78.2	0.692	90.2	5	0.006	67.7	0.038	71.1	0.044	87.9
8/24	5	0.012	65.4	0.18	77.7	0.33	90.4	5	0.048	58	0.262	73.4	0.98	88.7	5	0.012	63.5	0.058	68.1	0.076	85.3
8/27	5	0.012	66.6	0.206	80	0.408	91.2	5	0.082	52.3	0.37	71.1	0.88	90.1	5	0.01	71.5	0.044	74.7	0.064	89.3
8/30	5	0.018	61.3	0.166	74.5	0.34	89.7	5	0.178	60.6	0.604	79.4	2.258	91.9	5	0.016	70.6	0.096	74.8	0.108	88
8/33	5	0.022	63	0.36	74.4	0.614	86.9	5	0.242	63.8	1.17	79.6	3.008	91.9	5	0.01	65.4	0.122	73.3	0.17	87.4
10/9	5	0.008	67	0.012	83.1	0.074	93.5	5	0.006	69.6	0.014	80.1	0.074	90.6	5	0.004	66	0.016	73	0.032	88.8
10/12	5	0	67.3	0.02	78.8	0.126	92	5	0.01	62	0.038	74.1	0.204	88.3	5	0.004	64.2	0.022	70.6	0.044	85.8
10/15	5	0.002	64.5	0.066	79.4	0.21	90.7	5	0.02	60.8	0.096	77.2	0.454	90.1	5	0.006	67.3	0.036	73.8	0.068	87.4
10/18	5	0.012	68.2	0.124	80.8	0.37	90.8	5	0.034	51.8	0.158	70.4	1.136	90.6	5	0.008	63	0.048	67.9	0.092	84
10/21	5	0.014	67.3	0.132	80.7	0.52	91.7	5	0.056	65	0.252	79.2	2.104	90.8	5	0.008	65.7	0.086	67.7	0.126	85.6
10/24	5	0.016	64.2	0.238	77	0.682	89.9	5	0.136	59	0.502	74.5	3.27	90.8	5	0.01	65.8	0.078	70.3	0.158	86.1
10/27	5	0.02	64.6	0.298	74.9	0.886	87.8	5	0.174	62.2	0.906	77.6	5.01	89.8	5	0.018	67.7	0.108	72.1	0.178	87.1
10/30	5	0.026	62.8	0.382	74.4	1.094	88.2	5	0.278	52.6	1.296	66.2	8.244	82.5	5	0.018	66.3	0.142	72.5	0.244	86.9
12/6	5	0.004	71.2	0.01	83.3	0.058	93.1	5	0	70.5	0.01	79.7	0.042	89.3	5	0.002	70.4	0.01	74.3	0.032	88.2
12/9	5	0	63.8	0.014	76	0.126	88.5	5	0.008	68.7	0.022	77.7	0.108	89.8	5	0.004	69.5	0.024	74.1	0.098	87.6
12/12	5	0.006	61	0.028	78.4	0.244	91.1	5	0.01	65.2	0.046	76.1	0.474	89.3	5	0.008	67.6	0.042	72.2	0.106	85.8
12/15	5	0.008	66.2	0.106	82.4	0.452	93	5	0.03	55.3	0.144	74	1.352	90.6	5	0.008	68.7	0.052	71.8	0.09	84.5
12/18	5	0.012	68.6	0.216	80.7	0.726	91.6	5	0.048	52.4	0.284	72.4	2.012	85.4	5	0.01	65.7	0.066	70.8	0.134	86.2
12/21	5	0.016	63.9	0.274	74.5	1.096	87.9	5	0.084	52.5	0.42	62.5	4.228	82.9	5	0.012	65.2	0.1	70.4	0.196	85.8
12/24	5	0.022	66.2	0.262	79.5	1.474	90.6	5	0.124	56.5	0.688	73.6	5.074	88.2	5	0.012	66.3	0.138	72.4	0.212	87.4
12/27	5	0.03	56.8	0.334	77.7	1.474	91.1	5	0.188	54.6	0.702	73.1	9.02	89.6	5	0.02	60.1	0.17	69.7	0.286	86.9
Total	200	0.010	63.0	0.119	76.1	0.339	89.7	200	0.066	63.0	0.286	76.9	1.426	90.2	200	0.008	65.9	0.051	70.8	0.081	88.7

Note. l/n: Number of periods/Number of suppliers with the default node selection strategy on all classes of instances. This highlights the fact that without applying CCJ-DH cutoffs, the default node selection strategy performs better than the best-bound. By comparing %UB and %BUB for each node selection strategy and each class, one observes the effect of applying CCJ-DH cutoffs within the BC. The best-bound node selection strategy results in better average lower bounds and consequently better results for %BUB.

On the instances of the first class, applying all inequalities and the best-bound node selection strategy enables the BC to obtain 149 (out of 200) optimal solutions in an average of 1422 seconds compared to 52 optimal solutions when known inequalities are employed, and only 8 optimal solutions when no valid inequality is considered. On the harder instances of the second class, the BC finds 119 optimal solutions within the time limit when all inequalities are added to the model while it is able to find 64 optimal solutions with known inequalities and only 5 optimal solutions without the valid inequalities. The same difference in the performance of the BC exists on the instances of the third class where 171 optimal solutions are found with all valid inequalities compared to 107 optimal solutions with known inequalities, and 14 optimal solutions without the valid inequalities. Overall, compared to the cases with no or only known inequalities, using all inequalities in BC with both node selection strategies notably increases the number of optimal solutions and significantly improves the %UB and %BUB for all classes. These results show that our new valid inequalities make a substantial difference in the success of the BC.

The detailed results for the same scenarios of the BC are presented in Tables 2.6 and 2.7. Similarly, in all of these experiments we used algorithm $\mathcal{A}1$ to separate SECs (2.12) and (2.27). These tables present CPU, %UB, and %BUB for every period-supplier combination group of each instance class. The number of instances (out of five) that are not solved to optimality is specified in parentheses within the %BUB figures.

2.6.4 Analysis of Different Separation Procedures

In Table 2.8, we present the performance of the BC with all valid inequalities added when the CVRPSEP package, $\mathcal{A}1$ and $\mathcal{A}2$ are applied to separate SECs (2.12) and (2.27). We used

Table 2.5: Summary of the results of the BC with the default and the best-bound node selection strategies, and with and without the valid inequalities on different instance classes*

Node	Valid	Class 1					Class 2					Class 3				
Selection	Ineq.	Size	#Opt	CPU	%UB	%BUB	Size	#Opt	CPU	%UB	%BUB	Size	#Opt	CPU	%UB	%BUB
Default	None	200	11	3157	69.6	96.7	200	5	3234	65.4	95.2	200	22	3045	79.6	95.9
	Known	200	51	2576	86.3	96.8	200	44	2729	83.9	95.2	200	107	1912	96.1	97.5
	All	200	103	1980	91.2	99	200	69	2420	85	97.9	200	155	1205	98.3	99.5
Best-Bound	None	200	8	3207	56.5	97.3	200	5	3260	36.9	96.3	200	14	3098	64.5	96.6
	Known	200	52	2578	57.3	97.3	200	64	2418	61.8	96.3	200	107	1872	89.8	98.1
	All	200	149	1422	84.7	99.4	200	119	1976	74.4	98.7	200	171	938	97.4	99.8

* Separation procedure used for all BC scenarios: algorithm $\mathcal{A}1$

Size: Number of instances, None: With no inequality, Known: With known inequalities (2.18), (2.26) and (2.27),

All: With all inequalities (2.18)-(2.27)

the best-bound node selection strategy for all these experiments. In this table we report CPU, %BUB and the number of instances that are not solved to optimality (inside the parentheses) for each combination of the period-supplier setting. One observes that both of our separation procedures outperform the CVRPSEP package by enabling the BC to find more optimal solutions within the time limit. The results in this table suggest that the BC is capable of closing the optimality gap for many more period-supplier combinations in each class with a better solution time when it uses $\mathcal{A}1$ and $\mathcal{A}2$ compared to when it employs the CVRPSEP package. Furthermore, the BC with $\mathcal{A}2$ is performing better on larger instances compared to the case with $\mathcal{A}1$. This is why we use $\mathcal{A}2$ in our BC when we apply it to solve the large ARP instances of Chitsaz et al. (2019) presented in Appendix B. The BC is capable of solving instances with up to 4 periods and 33 nodes, 6 periods and 30 nodes, 8 periods and 27 nodes, 10 periods and 24 nodes, and 12 periods and 21 nodes within the time limit.

Moreover, in Table 2.9 we present more details on the BC performance. For each SEC separation procedure and for each class, this table shows #Opt, the average number of explored nodes in the search tree (#Node), the average number of added GFSECs (GFS), the average amount of violation for the added GFSECs (AV^{GFS}), the average number of added DFJs (DFJ), the average amount of violation for the added DFJs (AV^{DFJ}), and information about the number of cuts that are added automatically by CPLEX: cover cuts (Cover), flow cover cuts (Flow), clique cuts (Clique), mixed integer rounding cuts (MIR), flow path cuts (Path), implied bound cuts (ImplBd), zero-half cuts (ZeroHalf), and lift-and-project cuts (LiftProj). The results indicate that for each class the BC has to explore

Table 2.6: Detailed results of the BC with the best-bound node selection strategy, and with and without the valid inequalities*

Set	Class 1									Class 2									Class 3								
	CPU			%UB			%BUB			CPU			%UB			%BUB			CPU			%UB			%BUB		
l/n	None	Known	All	None	Known	All	None	Known	All	None	Known	All	None	Known	All	None	Known	All	None	Known	All	None	Known	All	None	Known	All
4/18	2778	1782	265	98.4	99.7	100	99.1$^{(4)}$	99.7$^{(1)}$	100	3296.8	2214.9	622.9	96.3	99.1	100	98.1$^{(5)}$	99.1$^{(2)}$	100	1483.2	54.9	27.8	99	100	100	99$^{(2)}$	100	100
4/21	3296	1558	317	77.7	98.4	100	98.2$^{(5)}$	99.3$^{(2)}$	100	3295.9	1582.5	892.5	95.8	99.8	100	97.6$^{(5)}$	99.8$^{(2)}$	100	2237.9	699	151.7	97.8	100	100	98.2$^{(3)}$	100	100
4/24	3294	3299	750	91.3	70.8	100	97.8$^{(5)}$	97.6$^{(5)}$	100	3292.2	3026.1	1156.4	77.5	96.9	100	97.4$^{(5)}$	97.9$^{(4)}$	100	3297.4	126.1	30.2	91.3	100	100	98.7$^{(5)}$	100	100
4/27	3295	2943	741	88.7	93.7	100	97.7$^{(5)}$	98.5$^{(3)}$	100	3297	2687.4	587.2	73.7	96.6	100	97.4$^{(5)}$	97.2$^{(4)}$	100	3292.8	346.4	72.7	77.4	100	100	98.2$^{(5)}$	100	100
4/30	3295	3295	812	76	57.4	100	97.8$^{(5)}$	97.6$^{(5)}$	100	3295.3	3298.1	1494.1	77.9	79.8	100	97.1$^{(5)}$	95.7$^{(5)}$	100	3291.5	2153.4	294.1	97.9	98.4	100	98.2$^{(5)}$	99.4$^{(2)}$	100
4/33	3295	3295	1374	80.6	46.8	99.8	97.7$^{(5)}$	96.8$^{(5)}$	99.8$^{(1)}$	3294.2	3297.9	2178.9	43.6	60.1	99.9	97$^{(5)}$	94.5$^{(5)}$	99.9$^{(1)}$	3294.8	1829	509.7	83	96.4	100	97.2$^{(5)}$	99.2$^{(2)}$	100
4/36	3292	3296	2663	39.5	32.3	75.6	96.5$^{(5)}$	97$^{(5)}$	98.8$^{(4)}$	3293.9	3296.2	2343.1	25.4	41.3	79.7	95.9$^{(5)}$	94.4$^{(5)}$	99.1$^{(3)}$	3293.2	3183.6	1229.3	59.8	79.4	99.1	95.4$^{(5)}$	96.7$^{(4)}$	99.5$^{(1)}$
4/39	3295	3292	2716	41	19	48.1	95.3$^{(5)}$	95.4$^{(5)}$	97.5$^{(4)}$	3294.8	3294.4	3293.7	13	33.2	65	94.2$^{(5)}$	91.5$^{(5)}$	97.5$^{(5)}$	3298.4	2793.9	1407.4	9.1	88.6	99.3	93.8$^{(5)}$	95$^{(4)}$	99.3$^{(1)}$
6/15	3296	996	450	97.9	99.9	100	99.4$^{(5)}$	99.9$^{(1)}$	100	3296.6	963.6	424.1	89.9	99.9	100	97.5$^{(5)}$	99.9$^{(1)}$	100	2955.4	1002.5	285.9	99.2	99.8	100	99.2$^{(4)}$	99.8$^{(1)}$	100
6/18	3295	2675	562	91.8	98.2	100	98$^{(5)}$	98.9$^{(3)}$	100	3296.4	2134.9	818.3	58.3	97.7	100	97.2$^{(5)}$	98.1$^{(3)}$	100	3293.6	467.3	100.9	96.5	100	100	98.5$^{(5)}$	100	100
6/21	3295	3290	830	65.6	44.2	100	96.7$^{(5)}$	97$^{(5)}$	100	3295.6	3297.7	1514.7	60	95.2	100	96.8$^{(5)}$	96.4$^{(5)}$	100	3295	1351.7	221.8	91.7	99.8	100	97$^{(5)}$	99.8$^{(2)}$	100
6/24	3297	3297	1050	41.7	57	100	97.6$^{(5)}$	96.7$^{(5)}$	100	3297.1	3296.3	2320.3	13.6	35.8	99.9	96.7$^{(5)}$	94.4$^{(5)}$	99.9$^{(1)}$	3294.4	1724.6	311.6	81.2	97.5	100	96$^{(5)}$	97.5$^{(2)}$	100
6/27	3296	3295	1092	62.9	36.9	100	97.8$^{(5)}$	95.7$^{(5)}$	100	3295.4	3295.3	2292.9	22.4	57.1	99.8	96.6$^{(5)}$	95.2$^{(5)}$	99.8$^{(1)}$	3296.2	1530.5	420.2	74.1	98.9	100	95.8$^{(5)}$	99.7$^{(1)}$	100
6/30	3293	3296	1639	40.9	39.7	99.5	97.2$^{(5)}$	96.4$^{(5)}$	99.5$^{(2)}$	3296.5	3296.4	2862	26	5.3	71.1	95.5$^{(5)}$	92.4$^{(5)}$	97.7$^{(4)}$	3297	2098.8	606.4	37.6	98.4	100	95.4$^{(5)}$	98.8$^{(2)}$	100
6/33	3296	3295	3297	45.4	32.4	46.3	96.1$^{(5)}$	95.2$^{(5)}$	98.1$^{(5)}$	3295.9	3295.9	3293.9	11.9	5.7	39	94.3$^{(5)}$	90.8$^{(5)}$	97.1$^{(5)}$	3294.7	3297.3	1948.7	34.4	91.4	99.8	95.4$^{(5)}$	95.7$^{(5)}$	99.8$^{(2)}$
6/36	3297	3294	3297	40.1	24.8	34.5	94.2$^{(5)}$	93.2$^{(5)}$	96.7$^{(5)}$	3292.6	3295.5	3296.8	29.7	29.2	24.2	93.7$^{(5)}$	92.4$^{(5)}$	96.8$^{(5)}$	3297.7	3295.1	2665.7	30.2	86.1	97.6	94.6$^{(5)}$	93.5$^{(5)}$	98.6$^{(4)}$
8/12	3295	117	78	79.5	100	100	99.3$^{(5)}$	100	100	3297.2	440.1	570	78.6	100	100	98.3$^{(5)}$	100	100	2683.7	692.8	663.7	99.6	100	100	99.6$^{(4)}$	100	100
8/15	3296	978	252	76.5	99.9	100	98.8$^{(5)}$	99.9$^{(1)}$	100	3293.8	893	1074.1	50.8	100	100	97.5$^{(5)}$	100	100	3294.7	187.5	272.1	97.8	100	100	98.1$^{(5)}$	100	100
8/18	3297	2712	962	66.6	88	100	98.1$^{(5)}$	98.4$^{(4)}$	100	3294.1	2886	1446.2	0	77.8	100	97.2$^{(5)}$	98.3$^{(4)}$	100	3292.4	1735.9	292.1	77.8	99.4	100	97.5$^{(5)}$	99.7$^{(2)}$	100
8/21	3295	3293	1037	51.1	33.3	100	97.4$^{(5)}$	97.2$^{(5)}$	100	3294.2	2805.9	1591.1	14.9	93.8	100	97.5$^{(5)}$	98$^{(4)}$	100	3297.7	2705.7	652	80.6	97.1	100	97.2$^{(5)}$	98.6$^{(4)}$	100
8/24	3295	3295	1141	56.8	27.6	100	97.4$^{(5)}$	96.8$^{(5)}$	100	3292.4	3293.7	3138	11.8	53.3	78.6	96.5$^{(5)}$	95.4$^{(5)}$	98.3$^{(4)}$	3293.8	2593.5	1053.4	47.1	75.7	100	95.7$^{(5)}$	97.3$^{(3)}$	100
8/27	3293	3296	1807	30	25.7	100	96.8$^{(5)}$	95.7$^{(5)}$	100	3299.2	3297	3297.1	34.7	42.2	46.2	95.7$^{(5)}$	94.8$^{(5)}$	97.4$^{(5)}$	3293.8	2457.5	1196.3	50.8	77.8	99.9	96.3$^{(5)}$	98.2$^{(3)}$	99.9$^{(1)}$
8/30	3297	3295	2850	32.4	38.1	57.7	95.8$^{(5)}$	94.4$^{(5)}$	98.7$^{(4)}$	3297.3	3298.5	3296.5	17.2	12.8	20.6	95.1$^{(5)}$	93.2$^{(5)}$	96.4$^{(5)}$	3296.3	3221.5	1828.6	11.1	40	63.9	94.6$^{(5)}$	96.4$^{(3)}$	98.6$^{(2)}$
8/33	3297	3293	3296	14.7	12.8	13.1	93.4$^{(5)}$	93.8$^{(5)}$	96.4$^{(5)}$	3293.5	3296	3297.4	12.1	12	19.6	93.1$^{(5)}$	92.2$^{(5)}$	95.9$^{(5)}$	3295.4	3292.6	2804.9	27.9	74.4	97.5	94.1$^{(5)}$	96.1$^{(5)}$	99.3$^{(4)}$
10/9	2738	208	237	99.7	100	100	99.7$^{(3)}$	100	100	3296.6	93.5	120.7	95.6	100	100	97.5$^{(5)}$	100	100	961.8	671.9	86	99.8	99.8	100	99.8$^{(1)}$	99.8$^{(1)}$	100
10/12	3293	724	437	59.5	98.1	100	99.1$^{(5)}$	99.6$^{(1)}$	100	3296.6	162.5	274.5	72.9	100	100	97.4$^{(5)}$	100	100	3294.4	679.7	275.2	78.9	99.9	100	98.1$^{(5)}$	99.9$^{(1)}$	100
10/15	3293	2831	511	65.2	95	100	97.7$^{(5)}$	99.3$^{(3)}$	100	3296.7	513.4	1522.4	17.3	100	100	97.7$^{(5)}$	100	100	3294	777.9	467.7	97.8	100	100	97.9$^{(5)}$	100$^{(1)}$	100
10/18	3297	3296	745	59.8	38.9	100	97.3$^{(5)}$	97.5$^{(5)}$	100	3292.5	3172.4	1914.6	8.4	97.7	100	96.6$^{(5)}$	99$^{(4)}$	100	3294.9	2548.3	615.5	58.4	99.8	100	96.2$^{(5)}$	99.8$^{(2)}$	100
10/21	3297	3296	1104	59	52.8	100	97$^{(5)}$	97$^{(5)}$	100	3295.2	2805.3	2944.4	0	78.5	79.5	97$^{(5)}$	97.7$^{(4)}$	98.6$^{(4)}$	3296.6	3294.5	817.1	50.2	94.7	100	94.4$^{(5)}$	96.1$^{(5)}$	100
10/24	3296	3297	2477	30.5	9.8	70.2	96.6$^{(5)}$	95.6$^{(5)}$	98.9$^{(2)}$	3295.5	3297.9	3293.5	0	8.2	27.8	95.4$^{(5)}$	92.9$^{(5)}$	97.4$^{(5)}$	3297.9	2692.3	1164.6	15.3	93.7	100	93.9$^{(5)}$	94.7$^{(4)}$	100
10/27	3291	3294	3073	28.7	28.1	37.5	94.7$^{(5)}$	95.1$^{(5)}$	97.2$^{(4)}$	3296	3292.6	3292.3	0	0	0	93.3$^{(5)}$	91.8$^{(5)}$	95.1$^{(5)}$	3293.8	3292.5	2403.7	22.8	94.3	99.5	95$^{(5)}$	96.3$^{(5)}$	99.5$^{(2)}$
10/30	3298	3296	3294	7.4	6.8	13.5	94.1$^{(5)}$	94.3$^{(5)}$	97.2$^{(5)}$	3296.2	3297.8	3293.7	5.4	5.4	23.9	92.3$^{(5)}$	91.3$^{(5)}$	95.3$^{(5)}$	3294	3294.3	2755.4	0	35.7	82.9	93.7$^{(5)}$	94$^{(5)}$	98.8$^{(3)}$
12/6	1241	6	10	99.9	100	100	99.9$^{(1)}$	100	100	1889.7	8.8	17.5	100	100	100	100	100	100	1554.4	4.8	14.8	99.6	100	100	99.7$^{(2)}$	100	100
12/9	2906	354	281	59.8	100	100	99$^{(4)}$	100	100	3292.1	155.2	317.5	57.6	100	100	97.4$^{(5)}$	100	100	3297.6	128.9	144.6	99.1	100	100	99.2$^{(5)}$	100	100
12/12	3293	140	606	36.4	100	100	98.9$^{(5)}$	100	100	3292.4	214.2	562.2	51.2	100	100	97.6$^{(5)}$	100	100	3293.4	303.4	313.2	91	100	100	97.9$^{(5)}$	100	100
12/15	3295	2727	686	60	77	100	98.2$^{(5)}$	99.4$^{(4)}$	100	3292.2	1381.9	1902.7	0	80	80	97.2$^{(5)}$	99.7$^{(1)}$	99.7$^{(1)}$	3292.9	2663.8	860.8	76.9	98	100	96.3$^{(5)}$	98.7$^{(3)}$	100
12/18	3295	3298	969	32.2	41	100	97.9$^{(5)}$	97.6$^{(5)}$	100	3293.6	2642.7	2581.4	0	40	60	96.6$^{(5)}$	98$^{(3)}$	98.9$^{(2)}$	3296.6	2067.8	1061.7	77.5	98.6	100	96.1$^{(5)}$	99.6$^{(2)}$	100
12/21	3297	3296	2686	41.2	37.6	100	97.9$^{(5)}$	97$^{(5)}$	100$^{(1)}$	3296.5	3298.1	3296.1	9.2	29	26.5	96.2$^{(5)}$	96.3$^{(5)}$	97.2$^{(5)}$	3294.8	3024.3	1837.1	25	95	100	94.4$^{(5)}$	96.6$^{(4)}$	100$^{(1)}$
12/24	3295	3296	3206	34	20.8	66.8	96.5$^{(5)}$	95$^{(5)}$	98.8$^{(4)}$	3293.2	3296.9	3296.5	15.2	0	25.5	94.4$^{(5)}$	92.7$^{(5)}$	95.5$^{(5)}$	3296.3	3293.7	2375.3	21.5	25.2	79.7	93.6$^{(5)}$	93.6$^{(5)}$	98.9$^{(3)}$
12/27	3297	3294	3295	0	8.8	25.6	94.1$^{(5)}$	94.4$^{(5)}$	97$^{(5)}$	3296.3	3290.9	3293.6	7.6	7.6	7.7	93$^{(5)}$	91.9$^{(5)}$	94.8$^{(5)}$	3295	3294.3	3297.2	13.8	56.8	77.5	93.1$^{(5)}$	93.8$^{(5)}$	98.3$^{(5)}$
Total	3207	2578	1422	56.5	57.3	84.7	97.3$^{(192)}$	97.3$^{(148)}$	99.4$^{(51)}$	3260	2417.7	1975.6	36.9	61.8	74.4	96.3$^{(195)}$	96.3$^{(136)}$	98.7$^{(81)}$	3097.7	1871.8	938.4	64.5	89.8	97.4	96.6$^{(186)}$	98.1$^{(93)}$	99.8$^{(29)}$

* Separation procedure used for all BC scenarios: algorithm $\mathcal{A}1$

l/n: Number of periods/number of suppliers, None: With no inequality, Known: With known inequalities (2.18), (2.26) and (2.27), All: With all inequalities (2.18)-(2.27)

The numbers in parentheses present the number of instances out of five that are not solved to optimality within the time limit

Table 2.7: Detailed results of the BC with the default node selection strategy, and with and without the valid inequalities*

Set	Class 1									Class 2									Class 3								
	CPU			%UB			%BUB			CPU			%UB			%BUB			CPU			%UB			%BUB		
l/n	None	Known	All	None	Known	All	None	Known	All	None	Known	All	None	Known	All	None	Known	All	None	Known	All	None	Known	All	None	Known	All
4/18	2855	1175	438	98.9	99.8	100	$98.9^{(4)}$	$99.8^{(1)}$	100	3295	2199.1	862.6	96.3	99.1	99.9	$96.8^{(5)}$	$99.1^{(2)}$	$99.9^{(1)}$	1331.6	56.6	13.5	99.7	100	100	$99.7^{(1)}$	100	100
4/21	3298	1509	524	95.9	99.7	100	$96.6^{(5)}$	$99.7^{(1)}$	100	3296.5	1646.9	288.3	97.5	99.6	100	$97.5^{(5)}$	$99.6^{(2)}$	100	2294	680.1	215.2	97.4	100	100	$97.9^{(3)}$	100	100
4/24	3293	3294	834	96.6	98.3	100	$97.2^{(5)}$	$98.3^{(5)}$	100	3292	2893.2	1118.5	94.5	97.5	99.9	$96^{(5)}$	$97.5^{(4)}$	$99.9^{(1)}$	3297.7	65.1	32.1	96.4	100	100	$97.7^{(5)}$	100	100
4/27	3298	2329	1218	96.1	98.4	100	$96.9^{(5)}$	$98.7^{(3)}$	100	3297.5	3244.6	1142.2	94.2	96.2	99.8	$96.3^{(5)}$	$96.4^{(4)}$	$99.8^{(1)}$	2952.6	470.6	48.6	95.6	100	100	$97.4^{(4)}$	100	100
4/30	3295	3294	1599	96.5	96.1	99.7	$97.5^{(5)}$	$96.7^{(5)}$	$99.7^{(2)}$	3297.8	3297.1	1677.6	74.4	92.6	99.1	$95.6^{(5)}$	$93.9^{(5)}$	$99.3^{(2)}$	2994.9	2804.2	248.6	94.1	98.5	100	$97.3^{(4)}$	$99.1^{(3)}$	100
4/33	3296	3291	1198	74.3	95.5	99.5	$96.6^{(5)}$	$96^{(5)}$	$99.5^{(1)}$	3297.9	3297.1	3026.9	76.1	91.9	99.1	$95.2^{(5)}$	$92.9^{(5)}$	$99.1^{(4)}$	3296.2	2047.8	208.5	95	99.3	100	$97^{(5)}$	$99.3^{(2)}$	100
4/36	3294	3299	2950	56.8	48.9	97.7	$95.7^{(5)}$	$95.1^{(5)}$	$98.2^{(4)}$	3296.5	3297.4	3296.5	63.3	90.1	78.9	$94.3^{(5)}$	$92.9^{(5)}$	$98.1^{(5)}$	3294.7	3294.9	936.6	80.1	95.5	99.5	$94.8^{(5)}$	$96.5^{(5)}$	$99.5^{(1)}$
4/39	3292	3296	3294	47.9	72.2	83.5	$94.6^{(5)}$	$94.5^{(5)}$	$97.9^{(5)}$	3297.9	3296	3299.4	56.4	83.9	83.5	$92.5^{(5)}$	$89.9^{(5)}$	$96.6^{(5)}$	3295.8	2530.1	1546.1	50.3	92.4	98.5	$92.7^{(5)}$	$94.5^{(3)}$	$98.5^{(1)}$
6/15	2172	851	718	99.4	99.7	99.9	$99.4^{(3)}$	$99.7^{(1)}$	$99.9^{(1)}$	3297.7	1508	906.8	94.4	99.1	100	$95.8^{(5)}$	$99.1^{(2)}$	100	2878.3	617.8	243.2	99.1	100	100	$99.1^{(3)}$	100	100
6/18	3293	2248	633	98.1	99.4	100	$98.1^{(5)}$	$99.4^{(3)}$	100	3297.6	2906	2102.7	94.7	97.2	99.8	$95.9^{(5)}$	$97.2^{(4)}$	$99.8^{(1)}$	3296.5	383.3	59.6	97.8	100	100	$97.9^{(5)}$	100	100
6/21	3292	3290	3028	94.1	96.5	99.7	$95.9^{(5)}$	$96.7^{(5)}$	$99.7^{(4)}$	3292.1	3297.7	2840	88.5	95.2	98.7	$95.5^{(5)}$	$95.6^{(5)}$	$98.9^{(4)}$	3297.4	1250.3	156	94.7	100	100	$96.5^{(5)}$	$100^{(1)}$	100
6/24	3295	3291	1867	73.8	95.3	99.7	$96.6^{(5)}$	$96.1^{(5)}$	$99.7^{(2)}$	3296.2	3297.7	3254.4	52.3	90.5	98.4	$95.1^{(5)}$	$92.4^{(5)}$	$98.4^{(4)}$	3293.6	1877.4	617.5	93.1	97.4	100	$95^{(5)}$	$97.4^{(2)}$	100
6/27	3296	3296	2563	65.9	93.7	99.5	$97.2^{(5)}$	$95.1^{(5)}$	$99.7^{(3)}$	3297.5	3297.9	3298.1	55.4	91.1	98.1	$95.7^{(5)}$	$93.5^{(5)}$	$98.1^{(5)}$	3296.3	1854.6	375.7	90.2	99.6	100	$95.4^{(5)}$	$99.6^{(1)}$	100
6/30	3298	3291	2458	70.5	87.6	95.5	$96.2^{(5)}$	$95.1^{(5)}$	$98.7^{(3)}$	3297.9	3296.8	2969.5	50.9	85.3	92.3	$93.7^{(5)}$	$90.6^{(5)}$	$96.4^{(4)}$	3297.7	2086.8	570.5	70.2	99	100	$94.3^{(5)}$	$99.1^{(2)}$	100
6/33	3295	3298	3296	41.5	70.4	78.4	$95.6^{(5)}$	$93.7^{(5)}$	$97.4^{(5)}$	3294.1	3297.8	3295	33.1	48.6	61.9	$93^{(5)}$	$90.2^{(5)}$	$95.5^{(5)}$	3295.3	3295.2	1532.2	83.1	93.7	99.7	$95.5^{(5)}$	$94.3^{(5)}$	$99.7^{(2)}$
6/36	3295	3298	3298	55.1	64.1	71.8	$93.4^{(5)}$	$91.2^{(5)}$	$95.8^{(5)}$	3292.5	3297.9	3297.4	49	69.8	72.3	$93.6^{(5)}$	$92^{(5)}$	$96.1^{(5)}$	3290.9	3297.5	2488.2	51.9	87.6	98.9	$94^{(5)}$	$91.1^{(5)}$	$98.9^{(3)}$
8/12	3003	159	71	99.2	100	100	$99.2^{(4)}$	100	100	3296.6	1320.5	929.8	97.6	99.6	99.9	$97.8^{(5)}$	$99.6^{(1)}$	$99.9^{(1)}$	2239	485.7	776.6	99.9	100	100	$99.9^{(2)}$	100	$100^{(1)}$
8/15	3298	1246	707	98	100	100	$98.3^{(5)}$	$100^{(1)}$	100	3296.8	2499.9	1317.4	93.1	99	99.9	$95.4^{(5)}$	$99^{(3)}$	$99.9^{(1)}$	3293.4	146.2	375.5	97.1	100	100	$97.5^{(5)}$	100	100
8/18	3298	3298	1927	91.7	98.5	99.9	$97.4^{(5)}$	$98.5^{(5)}$	$99.9^{(2)}$	3294.9	3297.8	2467.6	90.9	94.2	99	$94.9^{(5)}$	$95.5^{(5)}$	$99^{(3)}$	3295.4	2079.7	372.7	96	99.9	100	$96.9^{(5)}$	$99.9^{(1)}$	100
8/21	3298	3298	2378	61	96	99.9	$96.3^{(5)}$	$96.7^{(5)}$	$99.9^{(2)}$	3293.9	3297.7	3247.3	35.5	93.3	98.4	$95.7^{(5)}$	$95.6^{(5)}$	$99^{(4)}$	3295.2	2739.4	951.7	94.6	97.7	100	$96.5^{(5)}$	$97.8^{(4)}$	100
8/24	3298	3298	1826	61.4	96.5	99.6	$96.3^{(5)}$	$96.5^{(5)}$	$99.6^{(2)}$	3292.2	3297.7	3296.2	45	86.6	95.9	$95.1^{(5)}$	$92.4^{(5)}$	$96.8^{(5)}$	3293.7	2246.6	1442.3	55.2	95.1	99.7	$94.9^{(5)}$	$96.2^{(3)}$	$99.7^{(1)}$
8/27	3298	3298	3298	29.6	93.6	97.8	$95.9^{(5)}$	$95^{(5)}$	$98.5^{(5)}$	3290.8	3295.2	3293.6	74.4	79.1	66.3	$94.9^{(5)}$	$93.9^{(5)}$	$96.6^{(5)}$	3292.3	2387.4	1641.2	90.9	97	99.9	$95.4^{(5)}$	$97.6^{(3)}$	$99.9^{(1)}$
8/30	3296	3298	3295	32.1	71.4	86.3	$95.2^{(5)}$	$93^{(5)}$	$98^{(5)}$	3290.9	3297.7	3296.5	24.7	68.8	74.4	$94.4^{(5)}$	$92.8^{(5)}$	$95.8^{(5)}$	3296.8	3297.3	2481.2	68.1	89.7	88.4	$93.6^{(5)}$	$93.5^{(5)}$	$98.2^{(3)}$
8/33	3293	3298	3296	15	57.8	54.1	$93.1^{(5)}$	$93^{(5)}$	$95.3^{(5)}$	3292.8	3293.7	3290.9	58.1	38.6	54.8	$92.9^{(5)}$	$91.1^{(5)}$	$95.5^{(5)}$	3294.2	3297.8	3032.6	53.6	75.9	96.1	$93.3^{(5)}$	$94.2^{(5)}$	$98.8^{(3)}$
10/9	2678	179	191	99.5	100	100	$99.5^{(4)}$	100	100	3297.7	77.5	104.2	97.5	100	100	$97.5^{(5)}$	100	100	1029.2	691.6	155.3	99.7	99.7	100	$99.7^{(1)}$	$99.7^{(1)}$	100
10/12	3297	1098	976	98.9	99.9	100	$98.9^{(5)}$	$99.9^{(1)}$	$100^{(1)}$	3297.9	950.9	848.3	95.6	100	100	$96.3^{(5)}$	$100^{(1)}$	$100^{(1)}$	2767.6	704.9	705.8	98.1	99.6	99.7	$98.1^{(4)}$	$99.6^{(1)}$	$99.7^{(1)}$
10/15	3297	2609	892	95.1	99.4	100	$96.8^{(5)}$	$99.4^{(3)}$	100	3298.1	2874.3	3067.1	95.4	99.5	99.3	$96.4^{(5)}$	$99.5^{(3)}$	$99.3^{(4)}$	3296.2	1235.7	887.9	97.3	99.9	100	$97.6^{(5)}$	$99.9^{(1)}$	$100^{(1)}$
10/18	3295	3297	2416	87.9	97.6	99.9	$96.5^{(5)}$	$97.6^{(5)}$	$99.9^{(3)}$	3296.4	3295.9	3294.1	51	96	98.4	$94.5^{(5)}$	$96.3^{(5)}$	$98.7^{(5)}$	3294.8	2192.1	1170.9	92.8	98.9	99.9	$94.9^{(5)}$	$98.9^{(2)}$	$99.9^{(1)}$
10/21	3294	3296	2282	72.9	95.7	99.8	$96.3^{(5)}$	$96.5^{(5)}$	$99.8^{(2)}$	3297.8	3292.5	3292.6	17.4	76.9	78.2	$95.2^{(5)}$	$95.6^{(5)}$	$96.8^{(5)}$	3294.9	3296	1784.8	83.2	95.4	100	$93.1^{(5)}$	$95.6^{(5)}$	100
10/24	3295	3294	3276	44.5	55.5	95	$95.7^{(5)}$	$95^{(5)}$	$97.7^{(4)}$	3298.5	3294.2	3294.2	16.9	80.8	82.6	$93.6^{(5)}$	$90.3^{(5)}$	$95.2^{(5)}$	3296.2	2695.9	2477.4	52.5	92.8	99.8	$92.5^{(5)}$	$93.2^{(4)}$	$99.8^{(2)}$
10/27	3296	3296	3294	18.9	68.7	49.2	$93.6^{(5)}$	$93.2^{(5)}$	$96.4^{(5)}$	3291	3297.9	3292.8	48.4	33.4	34.5	$92.6^{(5)}$	$90.8^{(5)}$	$94.3^{(5)}$	3290.8	3297.6	3050.5	20.9	93.9	98.8	$94.3^{(5)}$	$95.4^{(5)}$	$98.8^{(4)}$
10/30	3291	3291	3298	24	33.7	36.8	$93.5^{(5)}$	$92.5^{(5)}$	$96.2^{(5)}$	3294.4	3298.3	3295.1	33.3	57.2	23.9	$92.5^{(5)}$	$90.9^{(5)}$	$94.9^{(5)}$	3294.1	3298	3272	17.5	85.8	82.9	$92.6^{(5)}$	$91.9^{(5)}$	$98.2^{(4)}$
12/6	1439	7	11	99.5	100	100	$99.5^{(2)}$	100	100	817.2	10	15.8	100	100	100	100	100	100	1471.6	6.5	12.6	99.5	100	100	$99.5^{(2)}$	100	100
12/9	2080	377	469	98.6	100	100	$98.7^{(2)}$	100	100	3297.9	158	379.6	97.5	100	100	$97.6^{(5)}$	100	100	2977.1	169.2	341.2	99.1	100	100	$99.1^{(4)}$	100	100
12/12	3298	844	810	98.6	100	100	$98.9^{(5)}$	$100^{(1)}$	100	3294.8	1383.9	2255.1	93.8	99.5	99.1	$95.9^{(5)}$	$99.5^{(1)}$	$99.3^{(2)}$	3295.5	767.8	957.2	97.5	100	100	$97.7^{(5)}$	100	100
12/15	3299	2745	1930	76.5	99.4	99.9	$97.3^{(5)}$	$99.4^{(4)}$	$99.9^{(2)}$	3295.2	3060.6	2652.8	52.9	98.1	98.7	$95.5^{(5)}$	$98.4^{(4)}$	$98.7^{(3)}$	3295.1	2274.8	1053.1	94.2	99.6	100	$95.2^{(5)}$	$99.6^{(2)}$	100
12/18	3298	3296	2728	23.5	91	98.6	$97^{(5)}$	$96.6^{(5)}$	$99^{(4)}$	3292.2	3297.6	3297.2	0	91.6	95.6	$94.9^{(5)}$	$95.1^{(5)}$	$96.7^{(5)}$	3296.2	2852.8	2242.9	92.6	97.8	99.6	$95.5^{(5)}$	$98.1^{(3)}$	$99.6^{(3)}$
12/21	3298	3292	3298	60	91.4	97.8	$97.1^{(5)}$	$96.1^{(5)}$	$98.3^{(5)}$	3297.7	3297.9	3297.6	17.9	44.6	69.2	$94.1^{(5)}$	$94^{(5)}$	$94^{(5)}$	3297.8	3098.8	3115.9	25.7	94.6	98.9	$92.2^{(5)}$	$95.8^{(4)}$	$99.1^{(3)}$
12/24	3298	3296	3298	20.5	44.8	75.2	$95.9^{(5)}$	$94.2^{(5)}$	$97.5^{(5)}$	3297.7	3297.7	3294.5	48.5	49.3	43.6	$93.6^{(5)}$	$91.2^{(5)}$	$94.7^{(5)}$	3296.8	3297.8	3296.7	49.4	87.7	95	$92.5^{(5)}$	$92.5^{(5)}$	$96.1^{(5)}$
12/27	3296	3291	3300	17.5	43.9	34.8	$93.4^{(5)}$	$93.1^{(5)}$	$95.9^{(5)}$	3291.2	3298.1	3291.1	59.9	41.9	8.7	$92.9^{(5)}$	$91.5^{(5)}$	$94.4^{(5)}$	3294.2	3298.1	3295.9	20.5	82.3	77.2	$91.6^{(5)}$	$91.6^{(5)}$	$97.5^{(5)}$
Total	3157	2576	1980	69.6	86.3	91.2	$96.7^{(189)}$	$96.8^{(149)}$	$99^{(97)}$	3233.5	2728.9	2419.7	65.4	83.9	85	$95.2^{(195)}$	$95.2^{(156)}$	$97.9^{(131)}$	3044.8	1911.8	1204.6	79.6	96.1	98.3	$95.9^{(178)}$	$97.5^{(93)}$	$99.5^{(45)}$

* Separation procedure used for all BC scenarios: algorithm $\mathcal{A}1$

l/n: Number of periods/number of suppliers, None: With no inequality, Known: With known inequalities (2.18), (2.26) and (2.27), All: With all inequalities (2.18)-(2.27)

The numbers in parentheses present the number of instances out of five that are not solved to optimality within the time limit

Table 2.8: Performance of the BC with different separation procedures*

Set	Class 1						Class 2						Class 3					
	CVRPSEP		A1		A2		CVRPSEP		A1		A2		CVRPSEP		A1		A2	
l/n	CPU	%BUB	CPU	%BUB	CPU	%BUB	CPU	%BUB	CPU	%BUB	CPU	%BUB	CPU	%BUB	CPU	%BUB	CPU	%BUB
4/18	1446	99.9$^{(1)}$	265	100	444	100	1304	99.8$^{(1)}$	623	100	830	99.9$^{(1)}$	80	100	28	100	29	100
4/21	959	99.6$^{(2)}$	317	100	123	100	832	99.8$^{(1)}$	893	100	990	100$^{(1)}$	236	100	152	100	84	100
4/24	1981	99.7$^{(2)}$	750	100	942	100	2089	99.7$^{(3)}$	1156	100	1277	100$^{(1)}$	48	100	30	100	29	100
4/27	1984	99.9$^{(2)}$	741	100	190	100	1472	100$^{(1)}$	587	100	617	100	137	100	73	100	42	100
4/30	2500	99.5$^{(4)}$	812	100	311	100	1838	99.4$^{(2)}$	1494	100	1187	100$^{(1)}$	530	100$^{(1)}$	294	100	247	100
4/33	2876	99.4$^{(3)}$	1374	99.8$^{(1)}$	772	99.5$^{(1)}$	2726	98.8$^{(3)}$	2179	99.9$^{(1)}$	2054	99.7$^{(2)}$	399	100	510	100	94	100
4/36	3298	97.5$^{(5)}$	2663	98.8$^{(4)}$	2715	99.1$^{(4)}$	2901	98.2$^{(4)}$	2343	99.1$^{(3)}$	1821	99.3$^{(2)}$	1059	99.7$^{(1)}$	1229	99.5$^{(1)}$	743	99.6$^{(1)}$
4/39	3298	96.2$^{(5)}$	2716	97.5$^{(4)}$	2230	98.7$^{(3)}$	3294	96.9$^{(5)}$	3294	97.5$^{(5)}$	3298	98.8$^{(5)}$	1669	99$^{(2)}$	1407	99.3$^{(1)}$	983	99.5$^{(1)}$
6/15	755	99.9$^{(1)}$	450	100	724	100$^{(1)}$	1557	99.8$^{(2)}$	424	100	252	100	697	100$^{(1)}$	286	100	487	100
6/18	1976	99.6$^{(2)}$	562	100	483	100	1363	99.9$^{(1)}$	818	100	946	99.9$^{(1)}$	296	100	101	100	105	100
6/21	3295	98.1$^{(5)}$	830	100	974	100	2673	99.2$^{(3)}$	1515	100	1539	100	2034	99.8$^{(2)}$	222	100	257	100
6/24	3106	99.7$^{(4)}$	1050	100	1445	99.9$^{(1)}$	3078	99.2$^{(4)}$	2320	99.9$^{(1)}$	2519	99.8$^{(3)}$	2855	99.4$^{(4)}$	312	100	273	100
6/27	2848	99.2$^{(4)}$	1092	100	805	100	2765	98.6$^{(3)}$	2293	99.8$^{(1)}$	1530	99.2$^{(1)}$	1847	99.9$^{(2)}$	420	100	124	100
6/30	2510	98.7$^{(4)}$	1639	99.5$^{(2)}$	1517	99.2$^{(2)}$	2854	96.3$^{(4)}$	2862	97.7$^{(4)}$	2740	97.7$^{(4)}$	2120	99.2$^{(3)}$	606	100	241	100
6/33	3297	97.9$^{(5)}$	3297	98.1$^{(5)}$	3298	98.4$^{(5)}$	3298	95.8$^{(5)}$	3294	97.1$^{(5)}$	3296	97.2$^{(5)}$	3297	99$^{(5)}$	1949	99.8$^{(2)}$	1148	99.9$^{(1)}$
6/36	3295	95.8$^{(5)}$	3297	96.7$^{(5)}$	3293	97.3$^{(5)}$	3297	96.4$^{(5)}$	3297	96.8$^{(5)}$	3297	97.2$^{(5)}$	2639	98.2$^{(4)}$	2666	98.6$^{(4)}$	2032	99.6$^{(2)}$
8/12	176	100	78	100	80	100	882	99.9$^{(1)}$	570	100	327	100	777	100$^{(1)}$	664	100	973	100$^{(1)}$
8/15	520	100	252	100	175	100	1640	99.7$^{(2)}$	1074	100	1100	99.9$^{(1)}$	1073	100$^{(1)}$	272	100	229	100
8/18	2029	99.5$^{(3)}$	962	100	1076	99.9$^{(1)}$	2188	99.7$^{(2)}$	1446	100	1358	99.7$^{(1)}$	1135	100	292	100	218	100
8/21	2977	99.1$^{(4)}$	1037	100	845	100	2366	99.4$^{(3)}$	1591	100	1785	100$^{(1)}$	2475	99$^{(3)}$	652	100	709	100
8/24	2305	98.2$^{(3)}$	1141	100	793	100	3295	97.4$^{(5)}$	3138	98.3$^{(4)}$	2994	98.4$^{(4)}$	2856	99$^{(4)}$	1053	100	1145	99.9$^{(1)}$
8/27	3296	98.4$^{(5)}$	1807	100	1767	99.7$^{(1)}$	3297	95.7$^{(5)}$	3297	97.4$^{(5)}$	3295	97.4$^{(5)}$	1542	99.3$^{(1)}$	1196	99.9$^{(1)}$	1045	99.9$^{(1)}$
8/30	3297	98$^{(5)}$	2850	98.7$^{(4)}$	2843	99$^{(4)}$	3296	96$^{(5)}$	3296	96.4$^{(5)}$	3297	96.5$^{(5)}$	2725	96.9$^{(4)}$	1829	98.6$^{(2)}$	1863	99.3$^{(2)}$
8/33	3297	95.4$^{(5)}$	3296	96.4$^{(5)}$	3298	97.2$^{(5)}$	3296	95$^{(5)}$	3297	95.9$^{(5)}$	3298	96.2$^{(5)}$	3291	97.3$^{(5)}$	2805	99.3$^{(4)}$	2288	99.4$^{(3)}$
10/9	415	100	237	100	471	100	489	100	121	100	516	100	209	100	86	100	120	100
10/12	697	100	437	100	716	100	795	99.9$^{(1)}$	275	100	273	100	322	100	275	100	222	100
10/15	1503	99.8$^{(1)}$	511	100	290	100	2641	99.3$^{(2)}$	1522	100	1374	100$^{(1)}$	746	99.9$^{(1)}$	468	100	726	100$^{(1)}$
10/18	2803	98.7$^{(4)}$	745	100	602	100	2520	99.7$^{(3)}$	1915	100	2468	99.9$^{(2)}$	2858	100$^{(2)}$	615	100	652	100
10/21	2728	97.8$^{(4)}$	1104	100	978	100	2914	98.1$^{(4)}$	2944	98.6$^{(4)}$	2895	98.6$^{(4)}$	2221	98.5$^{(3)}$	817	100	568	100
10/24	3296	97.5$^{(5)}$	2477	98.9$^{(2)}$	2130	99.2$^{(2)}$	3292	97.1$^{(5)}$	3294	97.4$^{(5)}$	3294	98$^{(5)}$	1802	99.7$^{(1)}$	1165	100	594	100
10/27	3297	96$^{(5)}$	3073	97.2$^{(4)}$	2775	97.7$^{(3)}$	3294	93.6$^{(5)}$	3292	95.1$^{(5)}$	3298	95.3$^{(5)}$	3250	99.1$^{(4)}$	2404	99.5$^{(2)}$	2391	99.6$^{(3)}$
10/30	3297	96.1$^{(5)}$	3294	97.2$^{(5)}$	3298	97.4$^{(5)}$	3298	94.3$^{(5)}$	3294	95.3$^{(5)}$	3298	95.1$^{(5)}$	3294	96.9$^{(5)}$	2755	98.8$^{(3)}$	1866	99.3$^{(2)}$
12/6	24	100	10	100	13	100	18	100	17	100	22	100	14	100	15	100	12	100
12/9	862	100$^{(1)}$	281	100	399	100	777	99.7$^{(1)}$	318	100	246	100	804	100	145	100	196	100
12/12	925	99.9$^{(1)}$	606	100	312	100	891	100	562	100	538	100	492	100	313	100	378	100
12/15	1510	99.7$^{(1)}$	686	100	420	100	2607	98.6$^{(3)}$	1903	99.7$^{(1)}$	2542	99.5$^{(1)}$	2992	99.3$^{(4)}$	861	100	822	100
12/18	2610	99.7$^{(2)}$	969	100	824	100	2841	97.6$^{(4)}$	2581	98.9$^{(2)}$	2613	98.4$^{(3)}$	2754	98.2$^{(4)}$	1062	100	844	100
12/21	3069	99$^{(4)}$	2686	100$^{(1)}$	2142	99.9$^{(1)}$	3292	96.4$^{(5)}$	3296	97.2$^{(5)}$	3298	97$^{(5)}$	3297	97.2$^{(5)}$	1837	100$^{(1)}$	1910	100
12/24	3297	97.4$^{(5)}$	3206	98.8$^{(4)}$	3063	98.9$^{(3)}$	3294	93.6$^{(5)}$	3296	95.5$^{(5)}$	3295	95.2$^{(5)}$	3296	95.3$^{(5)}$	2375	98.9$^{(3)}$	2440	98.4$^{(3)}$
12/27	3295	95.7$^{(5)}$	3295	97$^{(5)}$	3298	97$^{(5)}$	3300	92.3$^{(5)}$	3294	94.8$^{(5)}$	3298	94.7$^{(5)}$	3291	96.1$^{(5)}$	3297	98.3$^{(5)}$	3119	98.4$^{(4)}$
Total	2264	98.6$^{(122)}$	1422	99.4$^{(51)}$	1322	99.5$^{(52)}$	2347	98$^{(123)}$	1976	98.7$^{(81)}$	1973	98.7$^{(95)}$	1688	99.1$^{(83)}$	938	99.8$^{(29)}$	806	99.8$^{(26)}$

* Best-bound node selection strategy is used for all these experiments

l/n: Number of periods/number of suppliers,

The numbers in parentheses present the number of instances that are not solved to optimality within the time limit

Table 2.9: Summary of added SECs and CPLEX cuts for different classes of instances when different separation procedures are applied*

Sep	Class	Size	#Opt	#Node	GFS	AV^{GFS}	DFJ	AV^{DFJ}	Cover	Flow	Clique	MIR	Path	ImplBd	ZeroHalf	LiftProj
CVRPSEP	1	200	78	7016	561.3	0.4	3432.3	0.62	172.2	254.2	19.2	745.9	26.1	69.9	295.9	17.8
	2	200	77	2898	209.1	0.4	1607.3	0.75	156.1	628.5	1.4	2010.5	89	377.4	151.7	24.4
	3	200	117	4452	562.3	0.42	4753.7	0.76	120.4	232.2	3.3	661.1	2.2	68.4	137.7	22.2
	Total	600	272	4768	442.2	0.41	3252.6	0.71	149.5	373.8	7.9	1146.4	39.5	173.6	194.4	21.5
A1	1	200	149	3940	981.2	0.29	4528	0.4	96.6	133.1	16.1	349.8	8	44.1	93.2	16.2
	2	200	119	2295	1024.9	0.24	3958.7	0.37	99.6	359.9	1.3	1034.8	39.3	253.7	68.3	17.5
	3	200	171	1887	748.9	0.22	3839.1	0.42	56.5	114.1	3.3	359	0.8	39.7	45.4	13.4
	Total	600	439	2707	918.3	0.25	4108.6	0.4	84.3	202.4	6.9	581.2	16	112.5	69	15.7
A2	1	200	148	5013	432.1	0.21	1473	0.44	127.8	187.6	18.1	510.3	13.2	58.2	168	14.7
	2	200	105	1962	349.3	0.18	1148.5	0.43	110	419.1	1.4	1320.2	45.2	304.4	79.6	17.6
	3	200	174	2047	305.9	0.19	1481.8	0.48	78.2	173.5	3.3	535.9	1	50.1	70.5	13.5
	Total	600	427	3007	362.4	0.2	1367.7	0.45	105.3	260.1	7.6	788.8	19.8	137.5	106	15.3

* Best-bound node selection strategy is used for all these experiments
Sep: Separation procedure

many more nodes and finds fewer optimal solutions when it employs the CVRPSEP package compared to when it uses one of the proposed separation procedures. Another observation is that the average violation amount of the SECs (both GFSECs and DFJs) found by the CVRPSEP package is higher than the ones found by the other separation procedures. The reason is that CVRPSEP is not able to find violated SECs in the initial stages of the search tree because the node visit values are small in a fractional solution. In other words, because the CVRPSEP package is not effective on the initial fractional solutions, the BC explores more different node visit patterns within the search tree. The same is also true for other types of cuts that are generated by CPLEX. Overall, the performance of the BC when it uses one of the proposed separation algorithms, $\mathcal{A}1$ or $\mathcal{A}2$, is better than when it employs CVRPSEP.

The results in Tables 2.5-2.9 indicate that instances in the second class are generally harder and it takes longer for the BC method to solve them (higher average CPUs and lower %UB and %BUB). Within the specified time limit, the BC obtains fewer optimal solutions for the instances in this class compared to when it is applied to the instances in the first and the third class. Instances in the third class are relatively easier to solve compared to the other ones. The BC method obtains the largest number of optimal solutions and lowest average gaps for the instances in this class within the smallest average solution time.

2.7 Summary

We generalized the assumptions of the assembly routing problem (ARP) to the case where each supplier may provide a subset of the components necessary for production. We presented a mixed integer linear programming model for this problem. We also developed many randomly generated test instances for this problem, for which we obtained good quality upper bounds by adapting the matheuristic of Chitsaz et al. (2019). To solve the problem to optimality, we proposed several types of valid inequalities and analyzed their performance with respect to the LP solution value of the model. Based on the valid inequalities, we proposed a branch-and-cut algorithm and performed extensive experiments to analyze different aspects of the algorithm. In addition, we have developed two algorithms to separate multi-period fractional capacity cut constraints and compared their efficiency with the state-of-the-art separation procedures of Lysgaard et al. (2004) for the single-period VRPs.

Our extensive computational experiments indicate that applying our newly developed valid inequalities significantly improves the performance of the branch-and-cut algorithm. Furthermore, the performance of the branch-and-cut algorithm is substantially enhanced when it employs any of our new separation procedures compared to the case when it uses the separation procedures offered in Lysgaard et al. (2004).

An interesting avenue for future research on the ARP is to compare different reformulations. The ARP is an integrated problem that considers lot-sizing (with an assembly structure) and capacitated vehicle routing problems at the same time. Beside the standard formulation for the LSP, it is possible to consider echelon stock, facility location, and shortest path, among others (Pochet and Wolsey, 2006). Available formulations for the VRP (Toth and Vigo, 2014) are standard, single-/two-/multi-commodity formulations as well as path-based formulations. These result in a large number of promising possibilities to present reformulations for the ARP.

References

Adulyasak, Y., Cordeau, J.-F., and Jans, R. (2014). Formulations and branch-and-cut algorithms for multivehicle production and inventory routing problems. *INFORMS Journal on Computing*, 26(1):103–120.

Adulyasak, Y., Cordeau, J.-F., and Jans, R. (2015). The production routing problem: A review of formulations and solution algorithms. *Computers & Operations Research*, 55:141–152.

Agra, A., Andersson, H., Christiansen, M., and Wolsey, L. (2013). A maritime inventory routing problem: Discrete time formulations and valid inequalities. *Networks*, 62(4):297–314.

Akbalik, A., Penz, B., and Rapine, C. (2015). Capacitated lot sizing problems with inventory bounds. *Annals of Operations Research*, 229(1):1–18.

Andersson, H., Hoff, A., Christiansen, M., Hasle, G., and Løkketangen, A. (2010). Industrial aspects and literature survey: Combined inventory management and routing. *Computers & Operations Research*, 37(9):1515–1536.

Archetti, C., Bertazzi, L., Laporte, G., and Speranza, M. G. (2007). A branch-and-cut algorithm for a vendor-managed inventory-routing problem. *Transportation Science*, 41(3):382–391.

Archetti, C., Bertazzi, L., Paletta, G., and Speranza, M. G. (2011). Analysis of the maximum level policy in a production-distribution system. *Computers & Operations Research*, 38(12):1731–1746.

Archetti, C., Christiansen, M., and Speranza, M. G. (2018). Inventory routing with pickups and deliveries. *European Journal of Operational Research*, 268(1):314–324.

Archetti, C. and Speranza, M. G. (2016). The inventory routing problem: the value of integration. *International Transactions in Operational Research*, 23(3):393–407.

Ascheuer, N., Fischetti, M., and Grötschel, M. (2000). A polyhedral study of the asymmetric traveling salesman problem with time windows. *Networks*, 36(2):69–79.

Atamtürk, A. and Küçükyavuz, S. (2005). Lot sizing with inventory bounds and fixed costs: Polyhedral study and computation. *Operations Research*, 53(4):711–730.

Atamtürk, A. and Küçükyavuz, S. (2008). An o (n2) algorithm for lot sizing with inventory bounds and fixed costs. *Operations Research Letters*, 36(3):297–299.

Atamtürk, A. and Muñoz, J. C. (2004). A study of the lot-sizing polytope. *Mathematical Programming*, 99(3):443–465.

Augerat, P. (1995). *Approche polyèdrale du problème de tournées de véhicules*. PhD thesis, Institut National Polytechnique de Grenoble-INPG.

Avella, P., Boccia, M., and Wolsey, L. A. (2015). Single-item reformulations for a vendor managed inventory routing problem: Computational experience with benchmark instances. *Networks*, 65(2):129–138.

Barany, I., Van Roy, T., and Wolsey, L. A. (1984). Uncapacitated lot-sizing: The convex hull of solutions. In Korte, B. and Ritter, K., editors, *Mathematical programming at Oberwolfach II*, volume 22 of *Mathematical Programming Studies*, pages 32–43. Springer, Berlin, Heidelberg.

Bektaş, T. and Gouveia, L. (2014). Requiem for the Miller–Tucker–Zemlin subtour elimination constraints? *European Journal of Operational Research*, 236(3):820–832.

Belenguer, J.-M., Martinez, M., and Mota, E. (2000). A lower bound for the split delivery vehicle routing problem. *Operations Research*, 48(5):801–810.

Berman, O. and Wang, Q. (2006). Inbound logistic planning: minimizing transportation and inventory cost. *Transportation Science*, 40(3):287–299.

Blumenfeld, D. E., Burns, L. D., Daganzo, C. F., Frick, M. C., and Hall, R. W. (1987). Reducing logistics costs at General Motors. *Interfaces*, 17(1):26–47.

Carter, J. R. and Ferrin, B. G. (1996). Transportation costs and inventory management: Why transportation costs matter. *Production and Inventory Management Journal*, 37(3):58.

Chen, Z.-L. and Vairaktarakis, G. L. (2005). Integrated scheduling of production and distribution operations. *Management Science*, 51(4):614–628.

Chitsaz, M., Cordeau, J.-F., and Jans, R. (2019). A unified decomposition matheuristic for assembly, production and inventory routing. *INFORMS Journal on Computing*, 31(1):134–152.

Chouman, M., Crainic, T. G., and Gendron, B. (2016). Commodity representations and cut-set-based inequalities for multicommodity capacitated fixed-charge network design. *Transportation Science*, 51(2):650–667.

Chuah, K. H. and Yingling, J. C. (2005). Routing for a just-in-time supply pickup and delivery system. *Transportation Science*, 39(3):328–339.

Coelho, L. C., Cordeau, J.-F., and Laporte, G. (2013). Thirty years of inventory routing. *Transportation Science*, 48(1):1–19.

Contardo, C., Hemmelmayr, V., and Crainic, T. G. (2012). Lower and upper bounds for the two-echelon capacitated location-routing problem. *Computers & Operations Research*, 39(12):3185–3199.

Cornuejols, G. and Harche, F. (1993). Polyhedral study of the capacitated vehicle routing problem. *Mathematical Programming*, 60(1-3):21–52.

Dantzig, G. B., Fulkerson, R., and Johnson, S. (1954). Solution of a large-scale traveling-salesman problem. *Operations Research*, 2(4):393–410.

Di Summa, M. and Wolsey, L. A. (2010). Lot-sizing with stock upper bounds and fixed charges. *SIAM Journal on Discrete Mathematics*, 24(3):853–875.

Fernie, J. and Sparks, L. (2014). *Logistics and Retail Management: Emerging Issues and New Challenges in the Retail Supply Chain*. Kogan Page Publishers.

Fischetti, M., Polo, C., and Scantamburlo, M. (2004). A local branching heuristic for mixed-integer programs with 2-level variables, with an application to a telecommunication network design problem. *Networks*, 44(2):61–72.

Florian, M., Kemper, J., Sihn, W., and Hellingrath, B. (2011). Concept of transport-oriented scheduling for reduction of inbound logistics traffic in the automotive industries. *CIRP Journal of Manufacturing Science and Technology*, 4(3):252–257.

Fumero, F. and Vercellis, C. (1999). Synchronized development of production, inventory, and distribution schedules. *Transportation Science*, 33(3):330–340.

Gendreau, M., Laporte, G., and Semet, F. (1998). A branch-and-cut algorithm for the undirected selective traveling salesman problem. *Networks*, 32(4):263–273.

Gendron, B., Crainic, T. G., and Frangioni, A. (1999). Multicommodity capacitated network design. In Sansò, B. and Soriano, P., editors, *Telecommunications network planning*, pages 1–19. Springer.

Guide, V. D. R. and Van Wassenhove, L. N. (2009). OR FORUM-The evolution of closed-loop supply chain research. *Operations Research*, 57(1):10–18.

Hein, F. and Almeder, C. (2016). Quantitative insights into the integrated supply vehicle routing and production planning problem. *International Journal of Production Economics*, 177:66–76.

Hwang, H.-C. and van den Heuvel, W. (2012). Improved algorithms for a lot-sizing problem with inventory bounds and backlogging. *Naval Research Logistics (NRL)*, 59(3-4):244–253.

Iori, M., Salazar-González, J.-J., and Vigo, D. (2007). An exact approach for the vehicle routing problem with two-dimensional loading constraints. *Transportation Science*, 41(2):253–264.

Jiang, Z., Huang, Y., and Wang, J. (2010). Routing for the milk-run pickup system in automobile parts supply. In *Proceedings of the 6th CIRP-Sponsored International Conference on Digital Enterprise Technology*, pages 1267–1275. Springer.

Kucukyavuz, S. (2005). *Polyhedral approaches to capacitated fixed-charge network flow problems*. PhD thesis, University of California, Berkeley.

Lamsal, K., Jones, P. C., and Thomas, B. W. (2016). Sugarcane harvest logistics in Brazil. *Transportation Science*, 51(2):771 – 789.

Laporte, G. (1986). Generalized subtour elimination constraints and connectivity constraints. *Journal of the Operational Research Society*, 37(5):509–514.

Laporte, G., Nobert, Y., and Desrochers, M. (1985). Optimal routing under capacity and distance restrictions. *Operations Research*, 33(5):1050–1073.

Love, S. F. (1973). Bounded production and inventory models with piecewise concave costs. *Management Science*, 20(3):313–318.

Lysgaard, J., Letchford, A. N., and Eglese, R. W. (2004). A new branch-and-cut algorithm for the capacitated vehicle routing problem. *Mathematical Programming*, 100(2):423–445.

Miller, A. J., Nemhauser, G. L., and Savelsbergh, M. W. (2000). On the capacitated lot-sizing and continuous 0–1 knapsack polyhedra. *European Journal of Operational Research*, 125(2):298–315.

Mjirda, A., Jarboui, B., Macedo, R., Hanafi, S., and Mladenović, N. (2014). A two phase variable neighborhood search for the multi-product inventory routing problem. *Computers & Operations Research*, 52:291–299.

Moin, N. H., Salhi, S., and Aziz, N. (2011). An efficient hybrid genetic algorithm for the multi-product multi-period inventory routing problem. *International Journal of Production Economics*, 133(1):334–343.

Natarajarathinam, M., Stacey, J., and Sox, C. (2012). Near-optimal heuristics and managerial insights for the storage constrained, inbound inventory routing problem. *International Journal of Physical Distribution & Logistics Management*, 42(2):152–173.

Ohlmann, J., Fry, M., and Thomas, B. (2007). Route design for lean production systems. *Transportation Science*, 42(2):352–370.

Padberg, M. and Rinaldi, G. (1991). A branch-and-cut algorithm for the resolution of large-scale symmetric traveling salesman problems. *SIAM review*, 33(1):60–100.

Phouratsamay, S.-L., Kedad-Sidhoum, S., and Pascual, F. (2018). Two-level lot-sizing with inventory bounds. *Discrete Optimization*, 30:1–19.

Pochet, Y. (1988). Valid inequalities and separation for capacitated economic lot sizing. *Operations Research Letters*, 7(3):109–115.

Pochet, Y. and Wolsey, L. A. (1994). Polyhedra for lot-sizing with Wagner-Whitin costs. *Mathematical Programming*, 67(1-3):297–323.

Pochet, Y. and Wolsey, L. A. (2006). *Production Planning by Mixed Integer Programming*. Springer Science & Business Media.

Popken, D. A. (1994). An algorithm for the multiattribute, multicommodity flow problem with freight consolidation and inventory costs. *Operations Research*, 42(2):274–286.

Qu, W. W., Bookbinder, J. H., and Iyogun, P. (1999). An integrated inventory–transportation system with modified periodic policy for multiple products. *European Journal of Operational Research*, 115(2):254–269.

Ralphs, T. K., Kopman, L., Pulleyblank, W. R., and Trotter, L. E. (2003). On the capacitated vehicle routing problem. *Mathematical Programming*, 94(2):343–359.

Satoglu, S. and Sahin, I. (2013). Design of a just-in-time periodic material supply system for the assembly lines and an application in electronics industry. *The International Journal of Advanced Manufacturing Technology*, 65(1-4):319–332.

Semet, F., Toth, P., and Vigo, D. (2014). Classical exact algorithms for the capacitated vehicle routing problem. In Toth, P. and Vigo, D., editors, *Vehicle routing: problems, methods, and applications*, pages 37–57. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA.

Sindhuchao, S., Romeijn, H. E., Akçali, E., and Boondiskulchok, R. (2005). An integrated inventory-routing system for multi-item joint replenishment with limited vehicle capacity. *Journal of Global Optimization*, 32(1):93–118.

Solyalı, O. and Süral, H. (2011). A branch-and-cut algorithm using a strong formulation and an a priori tour-based heuristic for an inventory-routing problem. *Transportation Science*, 45(3):335–345.

Stacey, J., Natarajarathinam, M., and Sox, C. (2007). The storage constrained, inbound inventory routing problem. *International Journal of Physical Distribution & Logistics Management*, 37(6):484–500.

Toth, P. and Vigo, D. (2002). An overview of vehicle routing problems. In Toth, P. and Vigo, D., editors, *The Vehicle Routing Problem*, pages 1–26. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, USA.

Toth, P. and Vigo, D. (2014). *Vehicle routing: problems, methods, and applications*. SIAM.

Vaidyanathan, B., Matson, J., Miller, D., and Matson, J. (1999). A capacitated vehicle routing problem for just-in-time delivery. *IIE Transactions*, 31(11):1083–1092.

Van Den Heuvel, W. and Wagelmans, A. P. (2008). Four equivalent lot-sizing models. *Operations Research Letters*, 36(4):465–470.

Van Roy, T. J. and Wolsey, L. A. (1985). Valid inequalities and separation for uncapacitated fixed charge networks. *Operations Research Letters*, 4(3):105–112.

Viswanathan, S. and Mathur, K. (1997). Integrating routing and inventory decisions in one-warehouse multiretailer multiproduct distribution systems. *Management Science*, 43(3):294–312.

Yücel, E., Salman, F. S., Gel, E. S., Örmeci, E. L., and Gel, A. (2013). Optimizing specimen collection for processing in clinical testing laboratories. *European Journal of Operational Research*, 227(3):503–514.

Zhang, M., Küçükyavuz, S., and Yaman, H. (2012). A polyhedral study of multiechelon lot sizing with intermediate demands. *Operations Research*, 60(4):918–935.

Chapter 3

Multi-Product Production Routing Under Decoupled Planning Periods

Abstract

We consider an integrated optimization problem including the production, inventory, and outbound transportation decisions where a central plant fulfills the demand for several final products at its customers. More specifically, we investigate cases where the production planning and routing period lengths are not the same, e.g., days vs. shifts. Thus, we consider the fact that two different discretizations of the planning horizon exist in the decision-making process. This practical feature is a major source of complication for supply chain planners. With respect to the production planning aspect, we consider both big-bucket and small-bucket lot-sizing models. We mathematically formulate the problem under different practical scenarios for the production and route planning period lengths. An exact solution method, as well as heuristic algorithms, are proposed to efficiently solve large problem instances with this feature. To assess the effectiveness of our approach, we generate many test instances and perform an extensive computational study.

3.1 Introduction

A major task in the supply chain planning process is the coordination of the production plan with the distribution and delivery plans. This entails integrating production scheduling with other important functions of the supply chain such as inventory management, shipment planning, and vehicle routing. Many studies in the literature, including Blumenfeld et al. (1987); Chandra and Fisher (1994); Chen and Vairaktarakis (2005) and Archetti and Speranza (2016), among others, report a significant cost saving potential by coordinating these activities. The problem that arises from the integration of the production and route planning processes is referred to in the literature as the production routing problem (PRP) (Adulyasak et al., 2015).

We investigate in this paper a generalized PRP which takes into account the fact that the production planning and the route planning period lengths are not necessarily identical. The overall planning horizon may, as a consequence, contain a different number of production and route planning periods. For the lot-sizing part of the formulation, we will consider both big-bucket and small-bucket problems. Furthermore, we consider several different products. A single plant coordinates the production scheduling for these multiple products as well as the routing decisions and shipment quantities to the customers. The customers have a time-varying and predetermined demand for each product. The aim is to minimize the total costs of production, inventories and distribution routing subject to the limitations of the problem. The plant has a limited capacity for the production. No backlogging or stockouts are allowed at the plant or at the customers. Both the plant and the customers can carry inventory from one period to the next. The plant, as well as the customers, each have a global storage capacity. The plant manages a limited fleet of capacitated vehicles to handle the shipment of products to the customers and split deliveries are not allowed.

The mathematical models used to solve real-life cases can be different due to the practical conditions which vary from one company to another. One such practical issue, in particular, is the difference in the planning period *lengths* for the production planning and the distribution routing. In such cases, the capacity of the production and routing may be expressed in a different time dimension, which creates the need to have a decoupled discretization of the time horizon. In practice, in some cases, multiple periods of distribution and transportation exist within one production planning period, e.g., the production planning period is one week whereas the routing is done on a daily basis. Conversely, in some other cases, the distribution planning is done using daily truck dispatches, but the production planning is performed on a shift-basis, where one day contains multiple shifts. Consequently, an important aspect of these multi-period problems is to deal with the different period lengths while properly representing the available capacity.

The current literature on the PRP and its variants only considers identical production planning and routing period lengths. This is in many cases an abstraction of the problem in the real world. We investigate the problem of coordinating the production and the routing decisions in a decoupled planning horizon. To the best of our knowledge, this is the first paper looking at this problem in this generality. This is the first contribution of this paper. Next, we present mathematical programming formulations for the problem. Third, we present a unified reformulation for which we develop cutting planes to improve the linear programming relaxation of the original formulation. Fourth, we show how to extend and enhance a state-of-the-art heuristic for the single-product PRP (Chitsaz et al., 2019) to the multi-product PRP (MP-PRP). Based on these advancements, we present an exact solution algorithm to solve MP-PRP. Finally, we show the significant impact of our cutting planes through extensive computational experiments.

The remainder of the paper is organized as follows. We present a review of the related literature in Section 3.2 in order to position our study with respect to the existing literature. Then, we formally define the problem and express it mathematically in Section 3.3. We present a reformulation for the problem in Section 3.4, which we use to prove new valid inequalities in Section 3.5. In Section 3.6, we describe the adaptation of a state-of-the-art heuristic to obtain good quality upper bounds for the problem, and further, we show how to enhance the method. The generation of the test instances and computational experiments are presented in Section 3.7. Finally, Section 3.8 concludes the paper.

3.2 Review of the Related Literature

Adulyasak et al. (2015) provide a comprehensive survey on the PRP including a review of different formulation schemes, various solution techniques, and algorithmic and computational issues. The literature reveals that the PRP has received a rapidly growing interest in the operations research and management community. The majority of the studies focus on the development of heuristic algorithms for this complex problem. Absi et al. (2015); Solyalı and Süral (2017) and Chitsaz et al. (2019) develop multi-phase mixed integer linear programming (MILP)-based heuristics for the single-product PRP. Qiu et al. (2018a) study another single-product PRP with pickups and deliveries in the context of reverse logistics and remanufacturing.

We focus in this literature review on the related issues of the presence of multiple products and the length of the planning period. In the literature on the lot-sizing problem (LSP) (Pochet and Wolsey, 2006), several different assumptions are made with respect to the length of the planning periods for multi-product problems. Typically, a distinction is made between small- and big-bucket models. In the basic big-bucket model, it is assumed that several types of products can be made on a shared resource within one time period, and no sequencing of products is done within a time period. The production of a product in a given period requires a specific setup. All products made in a specific time period can be used to satisfy demand at the end of the same time period. Big-bucket models typically have time periods in the order of a day to a week or even a month. The small-bucket models, on the other hand, assume that at most one type of product can be produced within one time period. A start-up occurs when a machine is set up for a new product which was not produced in the previous period. Typically, the small-bucket models include short production periods of a shift or a day. Within the small-bucket models, a further distinction is made between the Discrete Lot-sizing and Scheduling Problem and the Continuous Setup Lot-sizing Problem. In the former, one imposes that if there is production in a period, it must be at full capacity, whereas in the latter the production quantity can take any value up to the capacity limit.

In the following, we give examples from the literature on the application of big-bucket models in production and distribution planning. Glover et al. (1979) develop a computer-based integrated model for the production, distribution, and inventory planning at Agrico Chemical Company with a 12-month planning horizon and monthly time periods. Martin et al. (1993) optimize production, inventory, and distribution in a multi-plant system for the Flat Glass Products group of Libbey-Owens-Ford over 12 one-month planning periods. De Matta and Guignard (1994b) describe a big-bucket model with a planning horizon consisting of 52 one-week periods. They study the effects of production loss during setup in dynamic production scheduling for process industries producing several products on non-identical flexible processors. Hahn et al. (2000) present the coordinated production planning and scheduling activities among supply chain members of the Hyundai Motor Company at Ulsan, Korea. The company prepares a master production schedule with monthly time periods on a six-month rolling horizon basis. Next, they develop daily production and distribution schedules for each month to make the deliveries possible in one week and not more than 15 days as promised. Brown et al. (2001) study the cost minimization of integrated production, inventory, and distribution plans for the cereal and convenience foods business of Kellogg with weekly periods in a 30-week planning horizon. Çetinkaya et al. (2009) develop a cost-minimization model for integrated production and shipment planning for the Frito-Lay North American plant in Irving, Texas in a finite planning horizon of 12 weeks each representing one period. Neves-Moreira et al. (2019) propose an optimization framework to minimize the total production, inventory and transportation costs in a European meat processing center that produces and distributes multiple meat products among its store chain within working shifts of 8 hours and a break of 1 hour between shifts.

Similarly, some studies from the literature employed small-bucket planning periods for the production planning and scheduling. De Matta and Guignard (1994a) consider the manufacturing operations of a tile company with several production lines. The planning horizon spans over six months and up to the entire year with planning periods of one week for the bottleneck stage. Jans and Degraeve (2004) study the production planning problem at the Solideal group which is one of the major manufacturers and distributors of industrial tires worldwide. The authors report that the production start-ups only take place at the beginning of the morning shifts due to the limited availability of the qualified personnel and adequate supervision throughout the day. The planning period used is one day within a planning horizon of up to 30 days. Silva and Magalhaes (2006) study a production planning problem to minimize the number of tool changeovers while meeting the required due dates at an acrylic fibers production firm in the textile industry. In this study, the planning horizon is divided into four or five weeks with days as planning periods. Marinelli et al. (2007) consider a rolling horizon of one week consisting of five working days (periods) followed by two days off for a capacitated lot-sizing and scheduling problem with parallel machines and shared buffers in a packaging company producing yogurt.

Almost all of the literature on the MP-PRP focuses on the big-bucket LSP as the underlying production model. Chandra and Fisher (1994) were the first to study the effect of the coordination between the production planning and the vehicle routing to minimize the total costs of production, inventories, and transportation. Fumero and Vercellis (1999) study an MP-PRP variant in which split delivery to the customers is allowed. They propose a Lagrangian relaxation approach to solve the problem. Armentano et al. (2011) propose a tabu search with path relinking approach for the problem. Belo-Filho et al. (2015) investigate the coordinated production and distribution of perishable goods. They propose an adaptive large neighborhood search (ALNS) algorithm for the problem. Brahimi and Aouam (2016) study the problem with the possibility of backordering. They develop a solution procedure consisting of a relax-and-fix heuristic and a local search algorithm. Motivated by the industrial gas supply chains, Zhang et al. (2017) introduce an MP-PRP with multiple production capacity levels (modes) in a continuous production environment. They propose an iterative MILP-based heuristic that works with a restricted set of candidate routes at each iteration. The method dynamically updates the set of candidate routes for the next iteration. Miranda et al. (2018) study a rich MP-PRP arising in the context of a Brazilian furniture manufacturer. They consider many practical problem limitations such as sequence-dependent setup times, a heterogeneous fleet of vehicles, and customer time windows and deadlines. They propose a two-phase MILP-based iterative heuristic for the problem. There is only one recent study by Qiu et al. (2018b) on the integration of the small-bucket LSP and the vehicle routing problem (VRP). They assume that the production period and routing period have equal lengths. The authors present a MILP to model the problem and provide valid inequalities to tighten the linear programming (LP) relaxation of the proposed model. They further use these inequalities in a branch-and-cut (BC) algorithm.

3.3 Problem Definition and Mathematical Formulation

We first present common problem assumptions and definitions in Section 3.3.1. Next, we mathematically define the variables and constraints of the problem in Section 3.3.2. Finally, we describe specific big- and small-bucket model constraints in Sections 3.3.3 and 3.3.4, respectively.

3.3.1 Common Assumptions and Definitions

We consider a one-to-many production system where a central plant, denoted by node 0, provides several products for different customers, represented by the set $\mathcal{N}$. We let $\mathcal{N}^+ = \mathcal{N} \cup \{0\}$ represent the set of all nodes including the customers and the central plant. Let $\mathcal{E} = \{(i,j) : i,j \in \mathcal{N}^+, i < j\}$ be the set of all edges connecting the plant and the customers together. We represent by $\mathcal{K}$ the set of all products. In the classical production routing problem, the planning horizon comprises a finite number of discretized time planning periods with an equal length for the production and routing periods (Figure 3.1).

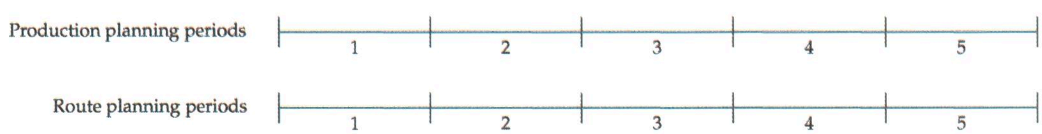


Figure 3.1: Planning horizon with equal period lengths

As indicated in the introduction, we will consider integrated planning problems where the production and routing periods do not necessarily have the same length. We assume that the production and the route planning period lengths can be written as an integer multiple of the *micro period* length, which is defined as the smaller planning period length between the production and the route planning periods. We denote by $\pi \in \mathbb{N}$ and $\rho \in \mathbb{N}$ the integer multiples of the micro period length for the production and the route planning period lengths, respectively. According to the definition, either the production or the route planning period length is equal to the micro period length. Consequently, when the planning period lengths are different, either π or ρ is equal to 1 and the other is strictly greater than 1. In case both planning period lengths are identical, then $\pi = \rho = 1$. Let $\mathcal{T} = \{1, ..., |\mathcal{T}|\}$ be the set of micro periods. We assume that $|\mathcal{T}|$ is divisible by π and ρ. We denote the set of production planning periods by $\mathcal{T}^{\pi} = \{1, ..., |\mathcal{T}|/\pi\}$. Likewise, we represent the set of route planning periods by $\mathcal{T}^{\rho} = \{1, ..., |\mathcal{T}|/\rho\}$. Figure 3.2 shows the situation where the production planning period length is larger than the routing period length, whereas Figure 3.3 represents the inverse situation.

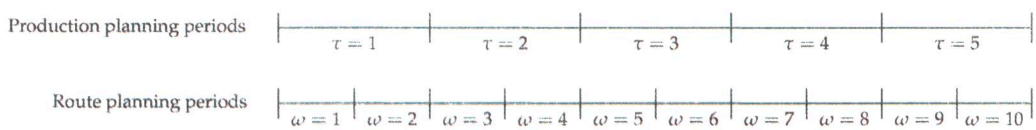


Figure 3.2: Longer production planning period lengths ($|\mathcal{T}| = 10, \pi = 2, \rho = 1, \tau \in \mathcal{T}^{\pi}, \omega \in \mathcal{T}^{\rho}$)

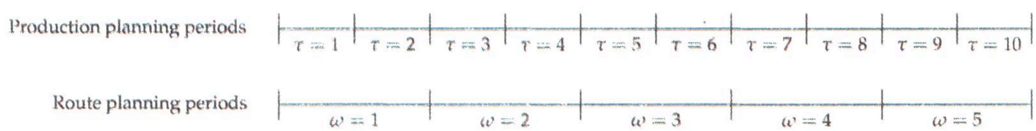


Figure 3.3: Longer route planning period lengths ($|\mathcal{T}| = 10, \pi = 1, \rho = 2, \tau \in \mathcal{T}^{\pi}, \omega \in \mathcal{T}^{\rho}$)

Product availability for shipment. In most production environments and for practical limitations, the production in each period is typically only available for shipment in the next period. This is because the shipments in the same period are already fixed, trucks and drivers are determined and planned to be dispatched. This situation is illustrated in Figure 3.4 for the case of equal production and route planning periods each equivalent to one day of operation. We index the route planning periods with one period shift/lag. Then, we consider the case that the production in each period is available for shipment in the next routing period which is indexed the same as the current production period. Figure 3.5 presents the case with longer production period. In this case, when we ship in period $\omega = 3$ or $\omega = 4$, products made in $\tau = 1, 2$ are available for shipment. Figure 3.6 presents the case with longer routing period in which the shipment in period $\omega = 2$ can include products made in production periods $\tau = 1, 2, 3, 4$.

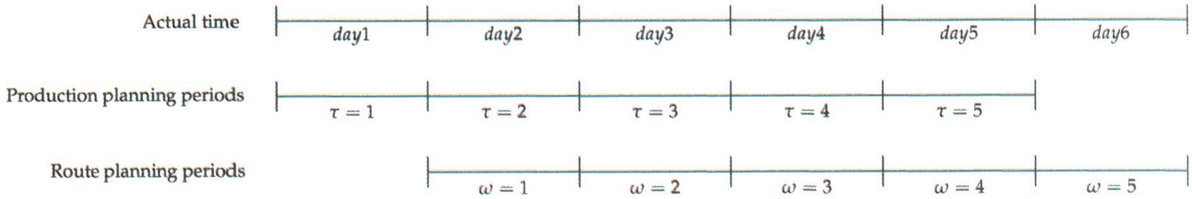


Figure 3.4: Product availability for shipment with equal period lengths ($|\mathcal{T}| = 5, \pi = 1, \rho = 1, \tau \in \mathcal{T}^{\pi}, \omega \in \mathcal{T}^{\rho}$)

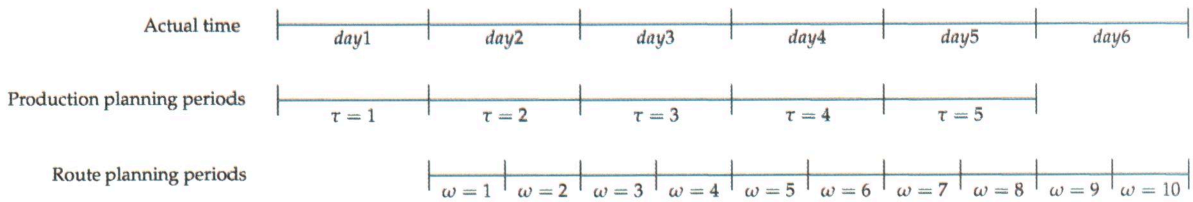


Figure 3.5: Product availability for shipment with longer production planning period lengths ($|\mathcal{T}| = 10, \pi = 2, \rho = 1, \tau \in \mathcal{T}^{\pi}, \omega \in \mathcal{T}^{\rho}$)

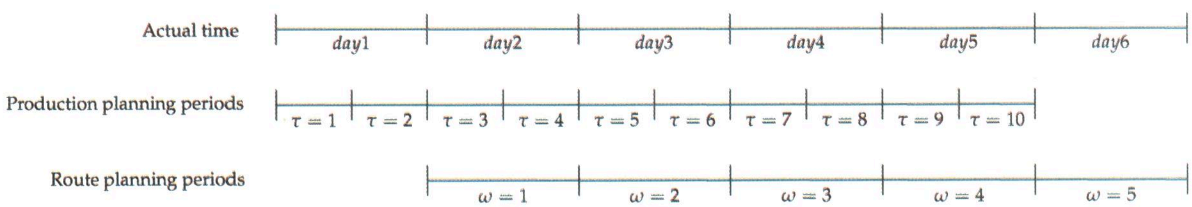


Figure 3.6: Product availability for shipment with longer route planning period lengths ($|\mathcal{T}| = 10, \pi = 1, \rho = 2, \tau \in \mathcal{T}^{\pi}, \omega \in \mathcal{T}^{\rho}$)

A one-period backward graphical shift in the routing period, makes Figures 3.4 to 3.6 equivalent to Figures 3.1 to 3.3, respectively. Therefore, without loss of generality, the entire production in any period is available for distribution in the period with the same index if the period lengths are equal. If the production period length is larger, the production in any period τ is available for distribution period $\omega = \pi\tau - 1$. If the routing period length is larger, the production in any period τ is available for distribution period $\omega = \lfloor \tau/\rho \rfloor + 1$. This choice of planning period indexing makes it possible to present formulations similar to those in many studies in the literature of the production routing problem (Archetti et al., 2011; Absi et al., 2015; Adulyasak et al., 2014).

Demand. We consider that the demand period length is equal to the route planning period length. Each customer $i \in \mathcal{N}$ has a predetermined demand $d_{ik\omega}$ for each product $k \in \mathcal{K}$ in each period $\omega \in \mathcal{T}^{\rho}$.

Production. The production system has to satisfy the demand for all products at every customer in each demand period without stockouts while respecting the plant's production capacity, which is given by C. We denote by θ_k the necessary capacity consumption to produce one unit of product $k \in \mathcal{K}$. The production of every product $k \in \mathcal{K}$ at the plant in a certain period imposes a fixed setup cost f_k.

Distribution. We consider b_k as the unit size of product $k \in \mathcal{K}$. A limited number of homogeneous vehicles, m, each with a capacity of Q, is available to perform shipments from the plant to the customers using routes that start and end at the plant. When a vehicle travels from location $i \in \mathcal{N}^+$ to $j \in \mathcal{N}^+$ a period-independent routing cost of c_{ij} is incurred.

Inventory bookkeeping. We consider the inventory bookkeeping at the plant to be aligned with the micro periods. When the production planning period length is smaller, this assumption is intuitive (Figure 3.3). For the case where the routing period length is smaller (Figure 3.7), during any production period, we have multiple route planning periods and thus it is possible to ship products from the plant within each routing period. Therefore, the level of the products' inventory at the plant may change during the production planning periods. Consequently, when the routing periods are smaller, micro period inventory level tracking enables a precise calculation of the inventory cost at the plant. We let I_{0k0} and I_{ik0} denote the initial inventory of product k at the plant and at the customer i, respectively. The cost at the plant of carrying one unit of product k over to the next micro period is h_{0k}. The cost at customer i to keep one unit of product k in the inventory in one route planning period is h_{ik}. Each customer $i \in \mathcal{N}$ has a global storage capacity L_i. The plant provides a shared storage with the capacity L_0 for all products.

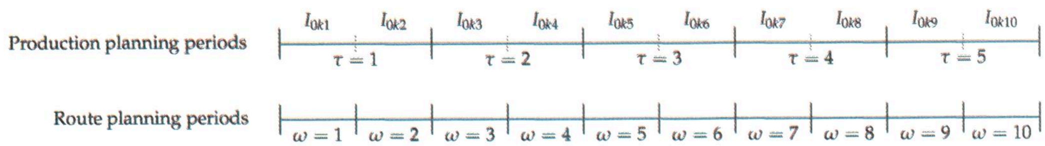


Figure 3.7: Inventory bookkeeping periods for the longer production planning period lengths ($|\mathcal{T}| = 10, \pi = 2, \rho = 1, \tau \in \mathcal{T}^\pi, \omega \in \mathcal{T}^\rho$)

3.3.2 Common Variables and Constraints

For each period $\tau \in \mathcal{T}^\pi$, we let the binary variable $y_{k\tau}$ take value 1 if and only if product $k \in \mathcal{K}$ is produced at the plant and we let $p_{k\tau}$ denote the production quantity. Let I_{0kt} and $I_{ik\omega}$ represent the inventory of product $k \in \mathcal{K}$ at the end of period $t \in \mathcal{T}$ at the plant, and at the end of period $\omega \in \mathcal{T}^\rho$ at the customer $i \in \mathcal{N}$, respectively. Let $q_{ik\omega}$ indicate the shipment quantity of product $k \in \mathcal{K}$ from the plant to the customer i in period $\omega \in \mathcal{T}^\rho$. The variable $x_{ij\omega}$ represents the number of times a vehicle traverses the edge $(i,j) \in \mathcal{E}$ in period $\omega \in \mathcal{T}^\rho$. The binary variable $z_{i\omega}$ takes value 1 if and only if a customer $i \in \mathcal{N}$ is visited in period $\omega \in \mathcal{T}^\rho$. The integer variable $z_{0\omega}$ indicates the number of vehicles dispatched from the plant in period $\omega \in \mathcal{T}^\rho$. The domain of the variables is imposed by constraints (3.1)-(3.6):

$$p_{k\tau} \geq 0, y_{k\tau} \in \{0,1\} \qquad \forall k \in \mathcal{K}, \forall \tau \in \mathcal{T}^\pi, \tag{3.1}$$

$$I_{0kt} \geq 0 \qquad \forall k \in \mathcal{K}, \forall t \in \mathcal{T}, \tag{3.2}$$

$$I_{ik\omega} \geq 0, q_{ik\omega} \geq 0 \qquad \forall i \in \mathcal{N}, \forall k \in \mathcal{K}, \forall \omega \in \mathcal{T}^\rho, \tag{3.3}$$

$$z_{0\omega} \in \mathbb{Z} \qquad \forall \omega \in \mathcal{T}^\rho, \tag{3.4}$$

$$z_{i\omega} \in \{0,1\}, x_{0i\omega} \in \{0,1,2\} \qquad \forall i \in \mathcal{N}, \forall \omega \in \mathcal{T}^\rho, \tag{3.5}$$

$$x_{ij\omega} \in \{0,1\} \qquad \forall (i,j) \in \mathcal{E} : i \neq 0, \forall \omega \in \mathcal{T}^\rho. \tag{3.6}$$

Constraints (3.7)-(3.9) provide the inventory flow balance at the plant. The production and the shipment variables are simultaneously present only during specific micro periods as presented in constraints (3.7). The cases are *(i)* the first micro period ($t \bmod \pi = 1$) of each large production period ($\pi > 1$ and $\rho = 1$), and *(ii)* the last micro period ($t \bmod \rho = 0$) of each large routing period ($\pi = 1$ and $\rho > 1$). Note that in case we have equal lengths for the production and routing periods, these are the only constraints needed. In the rest of the micro periods of the large production periods (t mod $\pi \neq 1, \pi > 1$ and $\rho = 1$), it is only necessary to balance the product inventory and the shipments as in constraints (3.8). Moreover, no shipment will be possible until the last micro period of the large routing periods (t mod $\rho \neq 0, \pi = 1$ and $\rho > 1$). Thus, constraints (3.9) keep track of the inventory at the plant for such cases:

$$I_{0k,t-1} + p_{k\tau} = \sum_{i \in \mathcal{N}} q_{ik\omega} + I_{0kt}$$

$$\forall k \in \mathcal{K}, \forall t \in \mathcal{T}, (t \bmod \pi = 1, \rho = 1) \vee (\pi = 1, t \bmod \rho = 0), \tau = (t-1)/\pi + 1, \omega = t/\rho \quad (3.7)$$

$$I_{0k,t-1} = \sum_{i \in \mathcal{N}} q_{ik\omega} + I_{0kt} \qquad \forall k \in \mathcal{K}, \forall t \in \mathcal{T}, t \bmod \pi \neq 1, \rho = 1, \omega = t \quad (3.8)$$

$$I_{0k,t-1} + p_{k\tau} = I_{0kt} \qquad \forall k \in \mathcal{K}, \forall t \in \mathcal{T}, \pi = 1, t \bmod \rho \neq 0, \tau = t. \quad (3.9)$$

The inventory balance constraints at the customers can be written as

$$I_{ik,\omega-1} + q_{ik\omega} = d_{ik\omega} + I_{ik\omega} \qquad \forall i \in \mathcal{N}, \forall k \in \mathcal{K}, \forall \omega \in \mathcal{T}^{\rho}. \quad (3.10)$$

Constraints (3.11) set the fleet size for each routing period. Constraints (3.12) enforce a vehicle to visit a node in case of a shipment to that node. The storage capacity at the plant and at the customers is imposed by constraints (3.13) and (3.14), respectively:

$$z_{0\omega} \leq m \qquad \forall \omega \in \mathcal{T}^{\rho} \quad (3.11)$$

$$\sum_{k \in \mathcal{K}} b_k q_{ik\omega} \leq Q z_{i\omega} \qquad \forall i \in \mathcal{N}, \forall \omega \in \mathcal{T}^{\rho} \quad (3.12)$$

$$\sum_{k \in \mathcal{K}} b_k I_{0kt} \leq L_0 \qquad \forall t \in \mathcal{T} \quad (3.13)$$

$$\sum_{k \in \mathcal{K}} b_k I_{ik\omega} \leq L_i \qquad \forall i \in \mathcal{N}, \forall \omega \in \mathcal{T}^{\rho}. \quad (3.14)$$

Let $\mathcal{E}(\mathcal{A})$ be the set of edges $(i,j) \in \mathcal{E}$ such that $i, j \in \mathcal{A}$, where $\mathcal{A} \subseteq \mathcal{N}$ is a given subset of nodes. Consider $\delta(\mathcal{A})$ as the set of edges incident to a node set $\mathcal{A}$, $\delta(\mathcal{A}) = \{(i,j) \in \mathcal{E} : i \in \mathcal{A}, j \notin \mathcal{A} \text{ or } i \notin \mathcal{A}, j \in \mathcal{A}\}$. The routing constraints include the node degree requirements (3.15) and the generalized vehicle routing capacity cuts (3.16) to eliminate the subtours and to impose the vehicle capacity. We refer to the latter set of constraints as the generalized fractional subtour elimination constraints (GFSEC) (Adulyasak et al., 2014):

$$\sum_{(j,j')\in\delta(i)} x_{jj'\omega} = 2z_{i\omega} \qquad \forall i \in \mathcal{N}^+, \forall \omega \in \mathcal{T}^p \tag{3.15}$$

$$Q \sum_{(i,j)\in\mathcal{E}(\mathcal{A})} x_{ij\omega} \leq \sum_{i\in\mathcal{A}} (Qz_{i\omega} - \sum_{k\in\mathcal{K}} b_k q_{ik\omega}) \qquad \forall \mathcal{A} \subseteq \mathcal{N}, |\mathcal{A}| \geq 2, \forall \omega \in \mathcal{T}^p. \tag{3.16}$$

3.3.3 MP-PRP with Big-Bucket Lot-Sizing and Scheduling

The big-bucket LSP assumes the possibility of producing several products in the same period on one shared resource with limited capacity (Trigeiro et al., 1989). Constraints (3.17) impose the global production capacity for each production period. The setup for each product is triggered by constraints (3.18) when its production takes place in any production period:

$$\sum_{k\in\mathcal{K}} \theta_k p_{k\tau} \leq C \qquad \forall \tau \in \mathcal{T}^\pi \tag{3.17}$$

$$\theta_k p_{k\tau} \leq C y_{k\tau} \qquad \forall k \in \mathcal{K}, \forall \tau \in \mathcal{T}^\pi. \tag{3.18}$$

The objective is to minimize the total cost of setups, inventory (at the plant and at the customers), and transportation as follows:

$$\min \sum_{k\in\mathcal{K}} (\sum_{\tau\in\mathcal{T}^\pi} f_k y_{k\tau} + \sum_{t\in\mathcal{T}} h_{0k} I_{0kt} + \sum_{\omega\in\mathcal{T}^p} \sum_{i\in\mathcal{N}} h_{ik} I_{ik\omega}) + \sum_{\omega\in\mathcal{T}^p} \sum_{(i,j)\in\mathcal{E}} c_{ij} x_{ij\omega}. \tag{3.19}$$

The mixed integer linear program for the PRP with a big-bucket lot-sizing structure, $\mathcal{M}^B_{MP-PRP}$, is to minimize the objective function (3.19), subject to constraints (3.1)-(3.18).

3.3.4 MP-PRP with Small-Bucket Lot-Sizing and Scheduling

The small-bucket (continuous) LSP assumes that only one product can be made in every production period (Loparic et al., 2003). We let the binary variable $w_{k\tau}$ be the start-up variable for product k in period τ with an associated start-up cost, g_k. We consider the start-up for product k when it is not produced in period $\tau - 1$, and the machine is set up to produce it in period τ (Pochet and Wolsey, 2006):

$$w_{k\tau} \in \{0, 1\} \qquad \forall k \in K, \forall \tau \in \mathcal{T}^\pi \tag{3.20}$$

The start-up variables are modeled in constraints (3.21). Constraints (3.22) enforce the requirement that we can only produce one product in any production period. Constraints (3.23) impose the initial values for the setup variables:

$$w_{k\tau} \geq y_{k\tau} - y_{k,\tau-1} \qquad \forall k \in K, \forall \tau \in \mathcal{T}^{\pi} \tag{3.21}$$

$$\sum_{k \in K} y_{k\tau} \leq 1 \qquad \forall \tau \in \mathcal{T}^{\pi} \tag{3.22}$$

$$y_{k0} = 0 \qquad \forall k \in K. \tag{3.23}$$

The objective is to minimize the total cost of start-ups, inventory and transportation as follows:

$$\min \sum_{k \in \mathcal{K}} \Big(\sum_{\tau \in \mathcal{T}^{\pi}} g_k w_{k\tau} + \sum_{t \in \mathcal{T}} h_{0k} I_{0kt} + \sum_{\omega \in \mathcal{T}^{\rho}} \sum_{i \in \mathcal{N}} h_{ik} I_{ik\omega} \Big) + \sum_{\omega \in \mathcal{T}^{\rho}} \sum_{(i,j) \in \mathcal{E}} c_{ij} x_{ij\omega}. \tag{3.24}$$

The MP-PRP with a small-bucket (continuous) lot-sizing structure, $\mathcal{M}^{S}_{MP-PRP}$, minimizes the objective function (3.24), subject to constraints (3.1)-(3.16), (3.18), (3.20)-(3.23).

3.4 A Reformulation

Constraints (3.7)-(3.9) impose the assumptions on the product flow at the plant level. However, it is not straightforward to strengthen the formulation and derive valid inequalities based on these constraints. We employ some modeling techniques to present these sets of constraints in a unified manner. The general idea is to reformulate the problem using only the micro periods which result in a formulation with the same number of periods at each level. We define π dummy micro periods for every large production planning period ($\pi \geq 1$). We consider ρ dummy micro periods for every large routing period ($\rho \geq 1$). First, we redefine the product demand and the holding cost (problem parameters) at the customers, **d** and **h**, respectively, on the micro periods (equations (3.25)-(3.26)):

$$\mathbf{d}_{ikt} = d_{ik\omega}, \quad \mathbf{h}_{ikt} = h_{ik} \qquad \forall i \in \mathcal{N}, \forall k \in \mathcal{K}, \forall t \in \mathcal{T}, t \bmod \rho = 0, \omega = t/\rho \tag{3.25}$$

$$\mathbf{d}_{ikt} = 0, \quad \mathbf{h}_{ikt} = 0 \qquad \forall i \in \mathcal{N}, \forall k \in \mathcal{K}, \forall t \in \mathcal{T}, t \bmod \rho \neq 0. \tag{3.26}$$

Figure 3.8 shows an example of how the redefinition works for $\mathcal{T} = 10$ and $\rho = 2$. For all $i \in \mathcal{N}$ and $k \in \mathcal{K}$, we let $\mathbf{d}_{ikt} = 0$ for all $t \in \mathcal{T}$ such that $t \bmod \rho \neq 0$, and we let $\mathbf{d}_{ikt} = d_{ik(t/\rho)}$ for all $t \in \mathcal{T}$ such that $t \bmod \rho = 0$. In addition, we define $\mathbf{d}_{ikt_1t_2}$ as the demand for product $k \in \mathcal{K}$ at customer $i \in \mathcal{N}$ from period t_1 to period t_2 (inclusive), $t_1, t_2 \in \mathcal{T}, t_1 \leq t_2$.

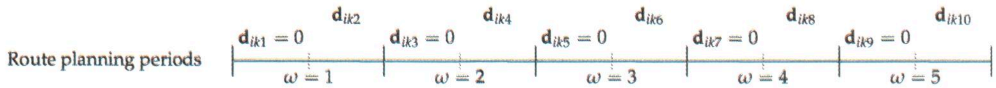


Figure 3.8: Dummy micro periods in the case of longer route planning period lengths ($|\mathcal{T}| = 10, \pi = 1, \rho = 2, \omega \in \mathcal{T}^\rho$)

Next, for each micro period $t \in \mathcal{T}$, we define variables $\mathbf{y}$, $\mathbf{p}$, $\mathbf{q}$, $\mathbf{z}$ and $\mathbf{x}$ similar to y, p, q, z and x, respectively. Furthermore, we define new inventory variables, $\mathbf{I}_{ikt}$ on the micro periods $t \in \mathcal{T}$ only for the customers $i \in \mathcal{N}$ and for all $k \in \mathcal{K}$. Note that the inventory variables of the original formulation (Section 3.3) for the plant, I_{0kt}, are already defined on the micro periods $t \in \mathcal{T}$. The reformulation for the big-bucket MP-PRP can be written as the following $\mathcal{R}^B_{MP-PRP}$ model:

$$(\mathcal{R}^B_{MP-PRP}) \quad \min \sum_{t \in \mathcal{T}} \Big\{ \sum_{k \in \mathcal{K}} (f_k \mathbf{y}_{kt} + h_{0k} I_{0kt} + \sum_{i \in \mathcal{N}} \mathbf{h}_{ikt} \mathbf{I}_{ikt}) + \sum_{(i,j) \in \mathcal{E}} c_{ij} \mathbf{x}_{ijt} \Big\} \tag{3.27}$$

s.t. (3.2),(3.13), and

$$I_{0k,t-1} + \mathbf{p}_{kt} = \sum_{i \in \mathcal{N}} \mathbf{q}_{ikt} + I_{0kt} \quad \forall k \in \mathcal{K}, \forall t \in \mathcal{T} \tag{3.28}$$

$$\mathbf{I}_{ik,t-1} + \mathbf{q}_{ikt} = \mathbf{d}_{ikt} + \mathbf{I}_{ikt} \quad \forall i \in \mathcal{N}, \forall k \in \mathcal{K}, \forall t \in \mathcal{T} \tag{3.29}$$

$$\sum_{k \in \mathcal{K}} \theta_k \mathbf{p}_{kt} \leq C \quad \forall t \in \mathcal{T} \tag{3.30}$$

$$\theta_k \mathbf{p}_{kt} \leq C \mathbf{y}_{kt} \quad \forall k \in \mathcal{K}, \forall t \in \mathcal{T} \tag{3.31}$$

$$\mathbf{z}_{0t} \leq m \quad \forall t \in \mathcal{T} \tag{3.32}$$

$$\sum_{k \in \mathcal{K}} b_k \mathbf{q}_{ikt} \leq Q \mathbf{z}_{it} \quad \forall i \in \mathcal{N}, \forall t \in \mathcal{T} \tag{3.33}$$

$$\sum_{k \in \mathcal{K}} b_k \mathbf{I}_{ikt} \leq L_i \quad \forall i \in \mathcal{N}, \forall t \in \mathcal{T} \tag{3.34}$$

$$\sum_{(j,j') \in \delta(i)} \mathbf{x}_{jj't} = 2 \mathbf{z}_{it} \quad \forall i \in \mathcal{N}^+, \forall t \in \mathcal{T} \tag{3.35}$$

$$Q \sum_{(i,j) \in \mathcal{E}(\mathcal{A})} \mathbf{x}_{ijt} \leq \sum_{i \in \mathcal{A}} (Q \mathbf{z}_{it} - \sum_{k \in \mathcal{K}} b_k \mathbf{q}_{ikt}) \quad \forall \mathcal{A} \subseteq \mathcal{N}, |\mathcal{A}| \geq 2, \forall t \in \mathcal{T} \tag{3.36}$$

$$\mathbf{y}_{kt} = 0 \quad \forall k \in \mathcal{K}, \forall t \in \mathcal{T}, t \bmod \pi \neq 1, \rho = 1 \tag{3.37}$$

$$\mathbf{z}_{it} = 0 \quad \forall i \in \mathcal{N}, \forall t \in \mathcal{T}, \pi = 1, t \bmod \rho \neq 0 \tag{3.38}$$

$$\mathbf{z}_{0t} = 0 \quad \forall t \in \mathcal{T}, \pi = 1, t \bmod \rho \neq 0 \tag{3.39}$$

$$\mathbf{p}_{kt} \geq 0, \mathbf{y}_{kt} \in \{0,1\} \quad \forall k \in \mathcal{K}, \forall t \in \mathcal{T} \tag{3.40}$$

$$\mathbf{I}_{ikt} \geq 0, \mathbf{q}_{ikt} \geq 0 \quad \forall i \in \mathcal{N}, \forall k \in \mathcal{K}, \forall t \in \mathcal{T} \tag{3.41}$$

$$\mathbf{z}_{0t} \in \mathbb{Z} \quad \forall t \in \mathcal{T} \tag{3.42}$$

$$\mathbf{z}_{it} \in \{0,1\}, \mathbf{x}_{0it} \in \{0,1,2\} \quad \forall i \in \mathcal{N}, \forall t \in \mathcal{T} \tag{3.43}$$

$$\mathbf{x}_{ijt} \in \{0,1\} \quad \forall (i,j) \in \mathcal{E} : i \neq 0, \forall t \in \mathcal{T}. \tag{3.44}$$

The objective function (3.27) minimizes the total production, inventory, and transportation costs over the micro periods. Constraints (3.28) and (3.29) impose the product flow balance at the plant and at the customers, respectively. Constraints (3.30) and (3.31) are production capacity constraints. Constraints (3.32)-(3.34) enforce the fleet size, shipment capacity, and storage capacity at the customers. Constraints (3.35)-(3.36) are the node degree and subtour elimination constraints for the micro periods. Constraints (3.37) prevent setups in the micro periods where no production is possible. Constraints (3.38)-(3.39) forbid node visits and vehicle dispatches in the micro periods where no shipment is available. Constraints (3.40)-(3.44) define the domain for the reformulation variables.

Next, for each micro period $t \in \mathcal{T}$, we define variables $\mathbf{w}$ similar to w. The reformulation for the small-bucket MP-PRP, $\mathcal{R}^S_{MP-PRP}$, can be written as follows:

$$(\mathcal{R}^S_{MP-PRP}) \quad \min \sum_{t \in \mathcal{T}} \Big\{ \sum_{k \in \mathcal{K}} (g_k \mathbf{w}_{kt} + h_{0k} I_{0kt} + \sum_{i \in \mathcal{N}} \mathbf{h}_{ikt} \mathbf{I}_{ikt}) + \sum_{(i,j) \in \mathcal{E}} c_{ij} \mathbf{x}_{ijt} \Big\}, \tag{3.45}$$

s.t. (3.2), (3.13), (3.28)-(3.29), (3.31)-(3.44), and

$$\mathbf{w}_{kt} \geq \mathbf{y}_{kt} - \mathbf{y}_{k,t-\pi} \quad \forall k \in K, \forall t \in \mathcal{T}, t \bmod \pi = 1, \rho = 1, \tag{3.46}$$

$$\sum_{k \in \mathcal{K}} \mathbf{y}_{kt} \leq 1 \quad \forall t \in \mathcal{T}, \tag{3.47}$$

$$\mathbf{w}_{kt} = 0 \quad \forall k \in \mathcal{K}, \forall t \in \mathcal{T}, t \bmod \pi \neq 1, \rho = 1, \tag{3.48}$$

$$\mathbf{y}_{k0} = 0 \quad \forall k \in \mathcal{K}, \tag{3.49}$$

$$\mathbf{w}_{kt} \geq 0 \quad \forall k \in \mathcal{K}, \forall t \in \mathcal{T}. \tag{3.50}$$

Constraints (3.46)-(3.47) (together with (3.31)) impose the small-bucket LSP assumptions on the setup and start-up variables. Note that in constraints (3.46), the setup variables in each period t depend on the setup variables in periods t and $t-\pi$. Constraints (3.48) prevent start-ups in the micro periods where no production is possible. Constraints (3.49) force the initial values for the setup variables. Constraints (3.50) define the domain for the start-up variables.

Theorem 3.1. *$\mathcal{R}^B_{MP-PRP}$ and $\mathcal{R}^S_{MP-PRP}$ are valid reformulations for $\mathcal{M}^B_{MP-PRP}$ and $\mathcal{M}^S_{MP-PRP}$, respectively.*

Proof. See Appendix C.

3.5 Valid Inequalities

We develop several valid inequalities to improve the LP relaxation bound of $\mathcal{R}^B_{MP-PRP}$ and $\mathcal{R}^S_{MP-PRP}$. The inequalities in this section are inspired by prior work on similar problems: Archetti et al. (2007) for the IRP; Archetti et al. (2011) and Adulyasak et al. (2014) for the single product PRP; Chitsaz et al. (2020) for the assembly routing problem (ARP) which considers an assembly production structure; and Atamtürk and Küçükyavuz (2005) for the lot-sizing with inventory bounds and fixed costs. First, we present (l,S)-type and cut-set-type inequalities for the the lot-sizing structures of the models. Then, we provide inequalities concerning the distribution and routing structure of the models. The proofs of the propositions are provided in Appendix C.

3.5.1 Inequalities for the Production and Inventory Flow Structures

The *(l,S)* inequalities were introduced in Barany et al. (1984) where l refers to a period ($l \leq |T|$), and S is a subset of periods $\{1,...,l\}$ not necessarily contiguous. Their cardinality is exponential and they are known to provide the convex hull for the single-item uncapacitated LSP. Pochet and Wolsey (1994) showed that when the sum of unit production and inventory costs in every period is larger than or equal to the unit production cost in the next period, it is sufficient to consider only a polynomial subset of these inequalities to describe the convex hull. These inequalities improve the linear relaxation bound of the lot-sizing structure (3.28)-(3.29) and (3.31). Because these two sets of constraints are present in both models, inequalities (3.51) are valid for $\mathcal{R}^{B}_{MP-PRP}$ and $\mathcal{R}^{S}_{MP-PRP}$.

Proposition 3.1.

$$\sum_{e=t_1}^{t_2} \mathbf{p}_{ke} \leq I_{0kt_2} + \sum_{i \in \mathcal{N}} \mathbf{I}_{ikt_2} + \sum_{e=t_1}^{t_2} \Big(\sum_{i \in \mathcal{N}} \mathbf{d}_{iket_2} \Big) \mathbf{y}_{ke} \quad \forall k \in \mathcal{K}, \forall t_1, t_2 \in \mathcal{T}, t_1 \leq t_2 \tag{3.51}$$

are valid for $\mathcal{R}^{B}_{MP-PRP}$, $\mathcal{R}^{S}_{MP-PRP}$.

Next, we present lower bounds for the total number of required production setups ($\mathbf{y}_{kt}$) from period $e = 1$ to $t \in \mathcal{T}$ and for each product $k \in \mathcal{K}$.

Proposition 3.2. *Inequalities*

$$\left\lceil \frac{\max\{0, \sum_{i \in \mathcal{N}} \max\{0, \mathbf{d}_{ik1t} - \mathbf{I}_{ik0}\} - I_{0k0}\}}{C/\theta_k} \right\rceil \leq \sum_{e=1}^{t} \mathbf{y}_{ke} \quad \forall k \in \mathcal{K}, \forall t \in \mathcal{T} \tag{3.52}$$

are valid for $\mathcal{R}^{B}_{MP-PRP}$ *and* $\mathcal{R}^{S}_{MP-PRP}$.

3.5.2 Inequalities for the Distribution and Inventory Flow Structures

Constraints (3.29) and (3.33) form a structure similar to those of constraints (3.28) and (3.31). Therefore, we present new (l, S)-type inequalities in Proposition 3.3.

Proposition 3.3. *Inequalities*

$$\sum_{e=t_1}^{t_2} \mathbf{q}_{ike} \leq \mathbf{I}_{ik,t_2} + \sum_{e=t_1}^{t_2} \mathbf{d}_{iket_2} \mathbf{z}_{ie} \quad \forall i \in \mathcal{N}, \forall k \in \mathcal{K}, \forall t_1, t_2 \in \mathcal{T}, t_1 \leq t_2 \tag{3.53}$$

are valid for $\mathcal{R}^{B}_{MP-PRP}$, $\mathcal{R}^{S}_{MP-PRP}$.

In Propositions 3.4 and 3.5, we present lower bounds for the total number of required vehicle dispatches ($\mathbf{z}_{0t}$), and node visits ($\mathbf{z}_{it}$), respectively, from period $e = 1$ to $t \in \mathcal{T}$.

Proposition 3.4. *Inequalities*

$$\left\lceil \frac{1}{Q} \sum_{i \in \mathcal{N}} \sum_{k \in \mathcal{K}} b_k \max\{0, \mathbf{d}_{ik1t} - \mathbf{I}_{ik0}\} \right\rceil \leq \sum_{e=1}^{t} \mathbf{z}_{0e} \quad \forall t \in \mathcal{T} \tag{3.54}$$

are valid for $\mathcal{R}^{B}_{MP-PRP}$ *and* $\mathcal{R}^{S}_{MP-PRP}$.

Proposition 3.5. *Inequalities*

$$\left\lceil \frac{\sum_{k\in\mathcal{K}} b_k \max\{0, \mathbf{d}_{ik1t} - \mathbf{I}_{ik0}\}}{\min\left\{Q, L_i + \max_{1\leq\theta\leq t}\{\sum_{k\in\mathcal{K}} b_k \mathbf{d}_{ik\theta}\}\right\}} \right\rceil \leq \sum_{e=1}^{t} \mathbf{z}_{ie} \quad \forall i \in \mathcal{N}, \forall t \in \mathcal{T} \tag{3.55}$$

are valid for $\mathcal{R}^B_{MP-PRP}$ and $\mathcal{R}^S_{MP-PRP}$.

One observes that the LHS of inequalities (3.52) and (3.54)-(3.55) includes only problem parameters and hence returns integer values. In addition, we add two more sets of inequalities to improve the routing structure of both models. Inequalities (3.56) require a vehicle dispatch in case a node has to be visited in a certain period. The other set of inequalities, (3.57), is the adaptation of the Dantzig-Fulkerson-Johnson (DFJ) constraints to eliminate infeasible paths and maintain connectivity on the vehicle routes. They were first proposed by Dantzig et al. (1954) for the travelling salesman problem (TSP). These inequalities require that, in an integral solution, the number of edges in any subset of visited nodes is smaller than the cardinality of the set:

$$\mathbf{z}_{it} \leq \mathbf{z}_{0t} \quad \forall i \in \mathcal{N}, \forall t \in \mathcal{T} \tag{3.56}$$

$$\sum_{(i,j)\in E(\mathcal{A})} \mathbf{x}_{ijt} \leq \sum_{i\in\mathcal{A}} \mathbf{z}_{it} - \mathbf{z}_{\alpha t} \quad \forall \mathcal{A} \subseteq \mathcal{N}, |\mathcal{A}| \geq 2, \forall \alpha \in \mathcal{A}, \forall t \in \mathcal{T}. \tag{3.57}$$

The cardinality of these inequalities is exponential and thus they cannot be added a priori to the model in practical applications. These inequalities do not impose the vehicle capacity.

3.6 An Upper Bound Heuristic

To obtain high-quality feasible solutions for the MP-PRP instances, we adapt the unified matheuristic proposed by Chitsaz et al. (2019). The authors applied this algorithm (CCJ-DH) to an assembly routing problem (ARP) where each supplier provides a distinct component. In addition, they applied CCJ-DH on the classic PRP and IRP instances where the plant/depot distributes only one type of product among many customers. In both studies, the authors report small optimality gaps for the solutions obtained by this heuristic especially on the large-scale instances of these problems. Therefore, to obtain high-quality feasible solutions for the MP-PRP instances, we adapt the unified matheuristic proposed in Chitsaz et al. (2019). We pass the solution obtained by this heuristic as cutoff values to our branch-and-cut algorithm.

The underlying idea in this algorithm is to heuristically solve the complex routing part and efficiently communicate the obtained routing costs in the objective function and with the rest of the model. This matheuristic works by decomposing the model into three independent subproblems and solving them iteratively. The first subproblem ($\mathcal{M}_y$) is a special LSP. This subproblem returns a setup schedule using an approximation of the total transportation cost in the objective function based on the number of dispatched vehicles. Using this given setup schedule, the second subproblem ($\mathcal{M}_z$) determines node visits and shipment quantities. In this subproblem, another approximation of the total transportation cost is considered in the objective function: the node visit transportation cost. Finally, the third subproblem considers a separate VRP for each period t. When the routing subproblems are solved, the algorithm updates the node visit cost approximation in the $\mathcal{M}_z$ model for the next iteration. This procedure is repeated to reach a local optimum. Then, the algorithm adds a diversification constraint (Fischetti et al., 2004) to the $\mathcal{M}_y$ model to change the setup schedule to explore other parts of the feasible solution space. The algorithm uses similar diversification constraints to generate new node visit patterns using the $\mathcal{M}_z$ model. The method terminates when a stopping condition is met.

Since we consider the multi-product variant of the PRP, we take this extension into account, compared to CCJ-DH implementation of Chitsaz et al. (2019), in the calculation of product inventories and inventory costs at the customers as well as the total shipment amount from the plant to each customer in all subproblems. However, the existence of multiple products as well as longer planning periods results in much larger subproblems which slow down the solution of the $\mathcal{M}_y$ and $\mathcal{M}_z$ models in this implementation. Efficiently solving these subproblems is a crucial step in the adaptation of CCJ-DH to obtain quality solutions for the MP-PRP variants. To overcome this challenge and to obtain a more efficient algorithm, we enhance the performance of CCJ-DH by adding relevant inequalities from Section 3.5. We add inequalities (3.51)-(3.52) and (3.54) to the $\mathcal{M}_y$ subproblem. Moreover, we incorporate inequalities (3.53) and (3.55) in the $\mathcal{M}_z$ subproblem.

3.7 Computational Experiments

The computational experiments were performed on the Calcul Québec computing infrastructure with Intel Xeon X5650 @ 2.67 GHz processors and a memory limit of 25 GB. The BC procedure is implemented in C++ using the CPLEX 12.7 callable library. All experiments were performed in sequential form using one thread. We consider the best-bound node selection strategy for the BB search tree. We do not change any other CPLEX parameter. The algorithm applies the valid inequalities at the root node and adds GFSECs (3.36) and DFJ (3.57) at each node of the search tree as cutting planes whenever they are violated by more than 0.1 unit. To separate GFSECs, we use algorithm $\mathcal{A}1$ which is presented in Chitsaz et al. (2020). When a violated GFSEC (3.36) is found, the BC method also adds the corresponding DFJ (3.57). In our experiments, we set a time limit of one hour both for the BC method and for CCJ-DH.

3.7.1 MP-PRP Test Bed

Although some studies were conducted on the MP-PRP, there is no standard data set available for this problem. Therefore, we have developed the data sets for each of the extensions of the MP-PRP. The test instances were generated on the basis of the following data:

- micro period planning horizon $|\mathcal{T}|$: 12, 18, 24, 30;
- number of products $|\mathcal{K}|$: 4, 6, 8;
- number of customers $|\mathcal{N}|$ (increasing by steps of 5 for all $|\mathcal{T}|$ values): 5 to 35 for $|\mathcal{T}| = 12$, 5 to 30 for $|\mathcal{T}| = 18$, 5 to 25 for $|\mathcal{T}| = 24$, 5 to 20 for $|\mathcal{T}| = 30$;
- demand at customer i for product k in period t: constant over time, and random integer in the set $\{0, 1, 2\}$;
- storage capacity L_0 at the plant: uniformly distributed random integer (UDRI) in the interval $\left[|\mathcal{T}||\mathcal{K}||\mathcal{N}|/4, |\mathcal{T}||\mathcal{K}||\mathcal{N}|/3\right]$, storage capacity L_i at customer i: UDRI in the interval $\left[|\mathcal{T}||\mathcal{K}|/4, |\mathcal{T}||\mathcal{K}|/3\right]$;
- production capacity C: UDRI in the interval $\left[|\mathcal{T}||\mathcal{K}||\mathcal{N}|/5, |\mathcal{T}||\mathcal{K}||\mathcal{N}|/4\right]$;
- production resource consumption θ_k for product k: random integer in the set $\{1, 2\}$;
- unit size b_k of product k: random integer in the set $\{1, 2\}$;

- truck capacity Q: $[10|\mathcal{K}|, 20|\mathcal{K}|$;
- number of trucks m: $|\mathcal{N}|$;
- initial inventory I_{0k0} of product k at the plant: UDRI in the interval $[0, 3|\mathcal{K}||\mathcal{N}|/2]$, initial inventory I_{ik0} of product k at customer i: UDRI in the interval $[0, 3|\mathcal{K}|/2]$;
- fixed setup/start-up cost f_k and g_k for product k: UDRI in the interval $[5000, 6000]$;
- holding cost h_{0k} of product k in each micro period at the plant: random integer in the set $\{1,2\}$, holding cost h_{ik} of product k in each micro period at customer i: random integer in the set $\{3,4\}$;
- longitude and latitude coordinates of the nodes (plant and the customers): UDRI in the interval $[0, 1500]$, transportation cost c_{ij}: Euclidean distance between nodes (rounded up to the nearest integer).

For each combination of the number of planning periods and customers we randomly generated 5 instances. As a result, the test bed includes medium ($|\mathcal{T}| = 12, |\mathcal{K}| = 4, |\mathcal{N}| = 5$) to very large size ($|\mathcal{T}| = 30, |\mathcal{K}| = 8, |\mathcal{N}| = 20$ or $|\mathcal{T}| = 12, |\mathcal{K}| = 8, |\mathcal{N}| = 35$) instances. Overall, instances are generated with 22 combinations of the planning horizons and numbers of customers, three numbers of product sizes and 5 instances per category. We apply the $\mathcal{R}^B_{MP-PRP}$ and $\mathcal{R}^S_{MP-PRP}$ models for each instance. We consider $\pi = \{1,2,3\}, \rho = 1$ for the $\mathcal{R}^B_{MP-PRP}$ model, and $\pi = 1, \rho = \{1,2,3\}$ for the $\mathcal{R}^S_{MP-PRP}$ model. Note that the case where $\pi = \rho = 1$ corresponds to the case with equal period lengths at the production and routing levels and can be applied for both $\mathcal{R}^B_{MP-PRP}$ and $\mathcal{R}^S_{MP-PRP}$ models. Considering 6 combinations of the π and ρ parameters for both models, our test bed includes 1980 instances (990 instances for each model).

3.7.2 Performance of the Heuristic

We report in Table 3.1 the performance of CCJ-DH with and without the addition of the valid inequalities. The results are presented for both small- and big-bucket models for $\rho = \pi = 1$. Each row in this table corresponds to a combination of the number of planning periods, number of products, and number of customers. In these tables, columns 4 to 12 and 13 to 21 include the results for the small-bucket and big-bucket MP-PRP instances, respectively. Columns four and five show the number of executed CCJ-DH iterations in the time limit for CCJ-DH without applying valid inequalities (None), and for the case where CCJ-DH is equipped with the valid inequalities (All), respectively. Column six presents the percent change in the number of iterations between these two implementations. Columns seven and eight show the average solution time in seconds for CCJ-DH with and without the inequalities, respectively. Column nine presents the percent change in the solution times. Columns 10 and 11 show the average solution values obtained by CCJ-DH without and with applying the valid inequalities, respectively. Column 12 presents the percent change in the average solution values. The same information is provided in columns 13 to 21 for the big-bucket model.

By adding the valid inequalities we were expecting to obtain better solution times. In addition, we also obtained better solution values due to the fact that on average the algorithm is able to perform more iterations in the one-hour time limit. On the small-bucket MP-PRP instances, the average number of iterations is increased by more than 29% and the average computing time is decreased by 34.2%. Moreover, on average, the solution values are improved by 0.4%. On the big-bucket MP-PRP instances, the improvement in the average solution values is 4.0%. This is obtained by a 26.7% increase in the number of iterations while the solution time is decreased by more than 38%. This is a significant improvement in the performance of CCJ-DH which is obtained by incorporating the valid inequalities.

3.7.3 Performance of Valid Inequalities

We further compare the effect of the valid inequalities on the performance of the BC method. In Tables 3.2-3.7, we report a summary of the results on the performance of the BC when we apply no inequality (None) or we employ inequalities (3.51)-(3.57) (All). These tables present CPU times, the average lower bound values as a percentage of the upper bound obtained by the BC without applying CCJ-DH cutoffs (%UB) and as a percentage of the best upper bound (%BUB) for each BC setting. To calculate the best upper bound (BUB) for each BC setting, we considered the upper bounds obtained by either that BC setting or CCJ-DH. When we do not consider the valid inequalities in the BC method (None), we do not apply them in CCJ-DH either. For the case where we include all inequalities in the BC method (All), we apply them in the heuristic cutoff procedure

Table 3.1: Performance of enhanced CCJ-DH with valid inequalities

			Small-Bucket LSP ($\rho = 1$)									Big-Bucket LSP ($\pi = 1$)								
			Iterations			CPU (s)			Avg Solution Value			Iterations			CPU (s)			Avg Solution Value		
l	k	n	None	All	(%)	None	All	(%)	None	All	(%)	None	All	(%)	None	All	(%)	None	All	(%)
12	4	5	200	200	-0.2	248	196	-21.0	34183	34190	0.0	200	200	0.0	80	60	-25.6	34230	34301	0.2
		10	200	200	0.0	574	212	-63.1	40150	40011	-0.3	200	200	0.2	331	92	-72.1	39880	39962	0.2
		15	200	200	0.0	912	267	-70.7	50775	50810	0.1	201	200	-0.4	494	151	-69.3	50331	50132	-0.4
		20	200	200	0.0	1430	294	-79.4	56281	56240	-0.1	200	200	0.0	970	173	-82.1	55796	55658	-0.2
		25	200	200	-0.2	2670	429	-83.9	63327	63562	0.4	200	200	0.0	1265	230	-81.8	62888	62911	0.0
		30	192	200	4.4	3244	515	-84.1	69228	69054	-0.3	200	200	-0.2	2597	397	-84.7	67852	67820	0.0
		35	177	200	13.3	3365	622	-81.5	78314	78079	-0.3	191	200	4.7	3040	653	-78.5	77693	77652	-0.1
	6	5	200	200	0.0	365	241	-34.0	43093	43082	0.0	200	200	0.0	202	167	-17.0	42723	42707	0.0
		10	200	200	0.2	1478	329	-77.7	48757	48891	0.3	200	200	0.0	631	225	-64.4	48410	48525	0.2
		15	200	200	0.0	1718	980	-42.9	56683	56640	-0.1	200	200	0.0	1856	835	-55.0	55924	56452	0.9
		20	190	194	1.9	3170	1611	-49.2	61770	61672	-0.2	197	200	1.4	2814	460	-83.6	61090	60595	-0.8
		25	188	197	4.5	3211	1679	-47.7	67404	67317	-0.1	186	200	7.5	3302	751	-77.3	66360	66248	-0.2
		30	158	188	18.7	3582	2174	-39.3	72819	72740	-0.1	166	200	20.8	3559	1398	-60.7	71737	70892	-1.2
		35	133	186	39.6	3597	2221	-38.3	77567	77207	-0.5	170	200	17.6	3579	1085	-69.7	75884	75480	-0.5
	8	5	200	200	0.0	1450	240	-83.5	53041	52767	-0.5	200	200	0.2	794	215	-72.9	53173	53073	-0.2
		10	184	200	8.9	3562	455	-87.2	63032	62842	-0.3	185	200	7.9	3508	299	-91.5	62413	61482	-1.5
		15	167	200	20.1	3526	861	-75.6	68983	68863	-0.2	149	200	34.3	3589	780	-78.3	68040	67274	-1.1
		20	140	200	43.1	3584	1600	-55.3	80499	79842	-0.8	143	200	40.1	3583	952	-73.4	80092	76515	-4.5
		25	149	200	34.6	3393	1264	-62.7	82656	81318	-1.6	166	200	20.8	3584	662	-81.5	80274	78480	-2.2
		30	144	200	38.7	3081	1701	-44.8	90853	89879	-1.1	132	200	51.5	3588	1178	-67.2	89828	85252	-5.1
		35	123	200	62.9	3597	2118	-41.1	95675	95265	-0.4	127	200	58.3	3596	1602	-55.5	93896	89776	-4.4
18	4	5	193	200	3.4	1529	1081	-29.3	44005	44193	0.4	195	195	0.0	1381	1163	-15.8	47542	47354	-0.4
		10	184	200	8.7	3062	1228	-59.9	50401	50341	-0.1	193	200	3.5	2986	1350	-54.8	50394	50390	0.0
		15	165	200	21.1	3581	2304	-35.7	65459	65661	0.3	173	200	16.0	3588	2542	-29.2	67004	66833	-0.3
		20	168	200	19.2	3598	3093	-14.0	83147	83209	0.1	175	195	11.8	3367	3219	-4.4	84535	84373	-0.2
		25	153	195	27.4	3371	3139	-6.9	91539	90713	-0.9	164	195	18.9	3280	3268	-0.4	91764	91674	-0.1
		30	155	187	20.4	3583	3063	-14.5	105551	105052	-0.5	159	179	12.3	3597	3582	-0.4	107328	106134	-1.1
	6	5	200	200	0.2	2162	1159	-46.4	48619	48686	0.1	200	200	0.0	1184	360	-69.6	47936	47955	0.0
		10	133	189	42.0	3582	1433	-60.0	62216	61944	-0.4	163	188	15.3	3586	1449	-59.6	61406	61210	-0.3
		15	133	190	43.0	3589	2054	-42.8	73024	72573	-0.6	171	200	17.2	3021	1071	-64.5	72569	70633	-2.7
		20	115	200	74.5	3598	2243	-37.7	84906	84728	-0.2	127	186	46.6	3589	1469	-59.1	85001	82481	-3.0
		25	108	182	69.5	3589	2882	-19.7	96786	96242	-0.6	130	200	54.3	3579	1131	-68.4	94421	92337	-2.2
		30	109	185	70.0	3597	3050	-15.2	102082	101396	-0.7	110	200	82.1	3578	1561	-56.4	101295	98367	-2.9
	8	5	186	200	7.5	2764	754	-72.7	61821	61516	-0.5	200	200	0.0	2508	603	-75.9	60828	60521	-0.5
		10	118	200	70.4	3579	927	-74.1	75394	75052	-0.5	129	200	55.0	3594	778	-78.3	74838	73190	-2.2
		15	104	200	92.7	3567	1786	-49.9	86876	86392	-0.6	105	200	91.2	3596	1913	-46.8	87568	83270	-4.9
		20	99	200	101.2	3581	2668	-25.5	99424	98795	-0.6	100	200	98.8	3588	2519	-29.8	101322	95169	-6.1
		25	93	200	114.1	3583	3043	-15.1	111463	110933	-0.5	96	182	89.6	3596	3464	-3.6	114865	105700	-8.0
		30	94	200	113.7	3573	3035	-15.1	126134	124048	-1.7	94	188	100.0	3583	3361	-6.2	126910	119132	-6.1
24	4	5	186	200	7.5	2105	1427	-32.2	45357	45304	-0.1	191	187	-2.1	1836	1617	-11.9	48114	48210	0.2
		10	163	192	17.4	3586	2572	-28.3	70579	70707	0.2	171	195	14.4	3237	2835	-12.4	77565	77060	-0.7
		15	150	178	19.1	3582	3549	-0.9	89644	89110	-0.6	141	170	20.6	3571	3541	-0.8	92714	91905	-0.9
		20	153	187	22.5	3394	3402	0.2	104324	104300	0.0	153	186	21.3	3572	3462	-3.1	109874	109908	0.0
		25	148	182	22.7	3575	3588	0.4	128542	128189	-0.3	164	182	10.9	3556	3569	0.4	132887	131388	-1.1
	6	5	183	200	9.5	3052	1056	-65.4	58642	58960	0.5	200	200	0.0	1368	775	-43.3	58364	58643	0.5
		10	107	186	73.8	3592	2053	-42.9	74421	74210	-0.3	130	181	38.9	3590	2318	-35.4	74195	72214	-2.7
		15	101	172	70.7	3582	2765	-22.8	98203	96800	-1.4	116	165	42.4	3589	2591	-27.8	102216	97108	-5.0
		20	97	164	69.6	3587	3376	-5.9	111720	112166	0.4	109	198	82.0	3579	2355	-34.2	118017	106025	-10.2
		25	85	185	118.6	3004	3243	8.0	129831	130443	0.5	125	200	59.5	3305	2228	-32.6	133399	125202	-6.1
	8	5	126	194	53.2	3560	1706	-52.1	70954	70295	-0.9	152	200	31.6	3042	1993	-34.5	69948	69255	-1.0
		10	101	191	89.9	3589	2012	-43.9	89034	87965	-1.2	107	200	86.6	3589	1683	-53.1	90867	86005	-5.4
		15	112	161	44.4	3348	2838	-15.2	108971	109180	0.2	91	198	116.6	3570	2910	-18.5	114113	103271	-9.5
		20	90	145	60.3	3588	2895	-19.3	129890	129134	-0.6	92	191	107.6	3576	3303	-7.6	132371	121384	-8.3
		25	88	145	63.9	3592	3088	-14.0	149381	149307	0.0	91	187	105.5	3590	3286	-8.5	155235	139698	-10.0
30	4	5	148	169	14.8	3593	3082	-14.2	56334	56214	-0.2	168	172	2.1	3439	3381	-1.7	59314	59069	-0.4
		10	146	180	23.0	3588	3314	-7.6	86405	87148	0.9	154	162	5.3	3580	3452	-3.6	96627	94970	-1.7
		15	138	181	30.9	3584	3578	-0.2	107240	107804	0.5	147	158	7.2	3576	3594	0.5	113827	112225	-1.4
		20	133	168	26.0	3588	3578	-0.3	131404	131804	0.3	141	169	20.1	3585	3585	0.0	139196	137726	-1.1
	6	5	113	179	58.5	3578	2537	-29.1	64532	64005	-0.8	147	189	28.2	3508	2074	-40.9	63748	62687	-1.7
		10	99	166	67.5	3598	3307	-8.1	97895	96699	-1.2	126	186	48.4	3253	2292	-29.5	99542	94685	-4.9
		15	95	176	84.9	3609	2762	-23.5	113507	114587	1.0	95	200	110.1	3583	2200	-38.6	123983	112317	-9.4
		20	109	167	52.8	3590	3587	-0.1	150344	146634	-2.5	119	200	67.5	3300	3097	-6.2	154903	145043	-6.4
	8	5	106	198	87.9	3599	2772	-23.0	82451	82577	0.2	121	198	63.4	3588	2461	-31.4	85282	80352	-5.8
		10	92	169	84.1	3589	3288	-8.4	110313	109752	-0.5	94	181	93.8	3580	2758	-23.0	122004	104552	-14.3
		15	89	92	2.7	3588	2219	-38.1	144806	145270	0.3	91	168	83.6	3575	2240	-37.3	154682	134254	-13.2
		20	89	94	5.9	3587	2495	-30.5	170344	168063	-1.3	89	161	80.5	3585	2687	-25.1	179248	155739	-13.1
Average			144	187	29.6	3107	2046	-34.2	84833	84521	-0.4	152	192	26.7	2953	1810	-38.7	86641	83164	-4.0

Table 3.2: Performance of branch-and-cut algorithm on the big-bucket LSP ($k = 4$)

		$\pi = 1$						$\pi = 2$						$\pi = 3$					
		None			All			None			All			None			All		
l	n	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB
12	5	1059	99.8	99.8	383	100.0	100.0	235	100.0	100.0	184	100.0	100.0	170	100.0	100.0	68	100.0	100.0
	10	3584	93.8	94.6	3587	96.2	96.5	3582	95.2	97.3	3587	97.1	97.1	3581	95.5	95.5	2934	98.3	98.3
	15	3587	28.3	91.1	3589	15.9	93.4	3585	47.6	93.0	3589	15.6	94.3	3585	67.4	94.9	3588	61.3	95.9
	20	3588	25.2	83.1	3589	13.5	92.2	3587	28.8	89.0	3589	10.6	93.3	3586	30.1	89.6	3589	13.7	95.7
	25	3588	15.9	72.7	3589	8.5	89.7	3589	18.3	83.1	3590	11.5	91.5	3589	16.0	86.0	3589	8.5	93.3
	30	3590	7.9	67.2	3589	6.6	89.6	3590	9.6	82.6	3588	9.5	90.9	3590	10.2	85.3	3590	10.4	94.6
	35	3590	3.9	62.1	3590	7.9	88.3	3590	4.8	67.4	3590	8.8	90.4	3591	7.1	72.1	3590	9.3	93.7
18	5	3584	90.2	90.5	2604	98.0	98.0	3212	98.4	98.4	2224	98.9	98.9	2437	98.6	98.6	2187	99.5	99.5
	10	3588	30.8	86.8	3588	17.1	92.6	3586	46.1	92.6	3589	16.8	94.3	3587	50.4	94.7	3589	62.6	94.9
	15	3590	13.7	76.2	3590	11.9	91.2	3590	18.9	88.7	3590	12.0	93.2	3589	23.1	93.6	3590	28.6	96.5
	20	3589	11.7	70.0	3590	5.8	88.7	3589	13.6	78.5	3590	7.8	91.2	3589	15.3	87.9	3590	10.4	93.8
	25	3590	5.4	64.1	3590	7.2	88.8	3590	8.7	74.6	3590	10.0	91.0	3590	7.0	85.3	3590	8.5	94.2
	30	3590	3.9	58.3	3590	5.1	89.0	3590	5.2	67.5	3590	7.5	91.5	3591	6.0	73.3	3590	8.0	94.8
24	5	3587	90.5	94.2	3216	97.9	97.9	3537	97.4	98.0	3187	98.8	98.8	3584	97.4	97.7	3297	99.1	99.1
	10	3589	22.4	81.9	3590	10.0	91.7	3589	27.8	91.7	3590	43.7	93.0	3588	34.1	94.3	3590	15.3	96.4
	15	3590	6.7	66.3	3590	8.9	88.3	3590	10.3	74.4	3590	11.4	91.0	3590	13.2	84.0	3590	12.0	93.7
	20	3590	5.3	61.0	3590	11.1	88.0	3590	6.8	66.5	3590	11.1	89.9	3590	8.3	75.7	3590	12.2	92.1
	25	3590	5.4	53.2	3590	7.6	85.9	3591	7.3	60.7	3590	8.3	88.0	3591	8.9	69.6	3589	9.3	90.6
30	5	3588	60.4	83.8	3590	81.4	96.8	3587	84.6	93.3	3589	96.6	97.7	3588	82.6	94.2	3432	74.3	98.3
	10	3590	12.8	74.1	3590	6.8	88.8	3590	21.0	84.9	3590	9.7	90.8	3589	30.7	90.0	3590	8.3	93.3
	15	3590	5.6	60.8	3590	9.3	87.4	3590	7.7	65.0	3590	9.3	88.4	3590	9.4	71.5	3590	11.2	90.6
	20	3590	5.7	55.8	3589	7.3	86.8	3590	8.1	62.3	3590	7.9	88.5	3590	9.3	69.5	3590	11.5	90.1
Avg		3474	29.3	74.9	3382	28.8	91.3	3417	34.8	82.2	3354	31.9	92.9	3381	37.3	86.5	3316	35.1	95.0

Table 3.3: Performance of branch-and-cut algorithm on the big-bucket LSP ($k = 6$)

		$\pi = 1$						$\pi = 2$						$\pi = 3$					
		None			All			None			All			None			All		
l	n	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB
12	5	686	100.0	100.0	56	100.0	100.0	99	100.0	100.0	21	100.0	100.0	78	100.0	100.0	25	100.0	100.0
	10	3587	96.4	97.2	1315	99.8	99.8	3009	97.3	97.5	1069	99.7	99.7	2733	99.9	99.9	864	99.9	99.9
	15	3588	17.8	79.9	3590	65.0	96.1	3588	53.3	92.8	3590	97.1	97.3	3588	56.6	96.9	3050	82.8	98.1
	20	3589	10.6	77.5	3590	26.1	95.6	3588	14.0	84.8	3590	30.9	96.2	3590	18.2	95.2	3590	47.1	97.1
	25	3590	4.6	52.8	3591	4.4	94.2	3590	7.6	64.8	3590	8.0	94.8	3590	9.4	83.3	3591	8.3	95.3
	30	3590	0.8	49.1	3590	1.6	94.0	3591	5.1	60.0	3590	5.6	94.8	3590	3.3	73.4	3590	6.3	94.9
	35	3591	0.7	42.4	3590	2.6	92.7	3590	2.9	54.0	3590	8.1	93.9	3590	0.0	58.6	3590	3.9	93.7
18	5	3021	92.8	92.8	297	100.0	100.0	1391	99.8	99.8	145	100.0	100.0	1365	100.0	100.0	49	100.0	100.0
	10	3589	24.1	87.0	3590	67.2	97.9	3589	33.8	95.6	3558	97.0	98.4	3588	39.4	95.8	3590	98.7	98.9
	15	3589	7.7	70.2	3590	3.2	95.5	3590	11.1	88.6	3590	5.6	96.0	3590	16.3	93.0	3590	12.8	98.6
	20	3590	5.1	59.4	3590	1.4	92.1	3590	7.6	71.6	3590	1.9	94.1	3590	10.3	84.8	3590	6.7	98.2
	25	3590	3.5	48.8	3590	2.6	91.9	3590	2.1	55.0	3590	4.8	92.3	3590	5.2	61.4	3590	8.0	97.0
	30	3590	0.6	43.6	3590	7.4	91.3	3591	0.8	55.1	3590	3.3	92.4	3591	0.0	59.0	3590	7.3	96.7
24	5	3588	86.1	86.7	2948	99.4	99.4	3588	96.4	96.4	1934	99.9	99.9	3588	96.6	97.1	455	100.0	100.0
	10	3589	12.8	72.0	3590	66.0	96.9	3589	18.7	91.3	3590	96.5	98.3	3589	23.5	93.1	3590	80.8	98.4
	15	3590	3.9	57.3	3590	0.0	91.5	3590	6.7	67.5	3590	2.1	95.3	3590	10.3	77.0	3590	7.5	95.5
	20	3590	3.7	48.1	3590	2.5	92.8	3590	5.7	57.5	3590	5.2	93.5	3591	7.0	62.2	3590	3.9	94.4
	25	3590	2.8	44.5	3590	3.8	92.0	3591	1.8	50.8	3590	5.9	92.0	3591	1.2	55.0	3590	9.8	97.6
30	5	3589	63.1	76.6	1733	100.0	100.0	3589	67.0	94.0	973	100.0	100.0	3589	85.9	97.3	1191	99.6	99.9
	10	3590	6.8	64.3	3590	2.4	93.7	3590	10.6	74.4	3590	7.4	94.5	3590	14.6	81.7	3590	7.1	95.1
	15	3590	3.2	47.2	3589	1.3	93.9	3590	6.2	56.8	3589	2.2	94.3	3590	8.9	63.2	3589	4.5	96.8
	20	3590	3.6	46.4	3589	3.9	90.8	3590	5.1	51.4	3589	3.3	91.5	3590	6.4	57.0	3589	9.2	94.6
Avg		3432	25.0	65.6	3063	34.6	95.1	3305	29.7	75.4	2961	40.2	95.9	3290	32.4	81.1	2867	41.1	97.3

as well. In these tables, a zero value under %UB columns means that no feasible solution (UB) is found by the BC method. The results indicate that the BC performs better, in terms of the average solution time and optimality gap, when all inequalities are applied. Furthermore, in all cases for the planning period length scenarios and the bucket size models, valid inequalities create a significant improvement in the final results (%BUB).

Table 3.4: Performance of branch-and-cut algorithm on the big-bucket LSP ($k = 8$)

		$\pi = 1$						$\pi = 2$						$\pi = 3$					
		None			All			None			All			None			All		
l	n	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB
12	5	618	100.0	100.0	7	100.0	100.0	1586	96.7	96.8	5	100.0	100.0	87	100.0	100.0	4	100.0	100.0
	10	3587	33.0	59.9	170	100.0	100.0	3588	49.7	70.1	139	100.0	100.0	3588	62.4	89.3	123	100.0	100.0
	15	3589	34.7	63.0	2365	99.8	99.8	3589	51.1	74.7	947	100.0	100.0	3589	51.6	88.0	1464	100.0	100.0
	20	3590	7.9	51.2	3590	97.1	97.2	3587	9.0	63.5	3590	96.6	97.0	3588	11.5	72.7	3589	98.9	99.0
	25	3588	3.5	48.4	3590	21.5	95.3	3587	7.3	59.0	3590	38.9	95.6	3588	8.2	65.3	3590	95.2	97.4
	30	3588	2.2	42.3	3590	1.4	94.2	3588	4.9	50.3	3590	0.0	94.7	3588	7.1	62.0	3590	18.4	95.4
	35	3587	0.0	38.0	3590	0.0	93.0	3588	0.9	48.8	3590	1.5	93.2	3588	2.6	58.2	3590	1.8	93.9
18	5	3585	82.8	89.4	34	100.0	100.0	1968	100.0	100.0	11	100.0	100.0	2268	99.7	99.7	15	100.0	100.0
	10	3586	17.4	68.8	2702	99.4	99.4	3587	25.4	85.5	1990	99.7	99.7	3586	30.2	93.9	1364	100.0	100.0
	15	3587	5.6	53.5	3589	40.4	97.6	3587	9.0	71.9	3590	48.8	98.3	3587	11.9	84.4	3147	78.5	98.9
	20	3587	1.8	53.0	3589	1.3	94.9	3588	5.8	63.8	3590	0.0	95.5	3589	9.9	78.6	3589	2.6	95.4
	25	3588	3.6	45.2	3589	2.8	94.9	3588	3.2	51.8	3590	1.9	95.2	3587	7.1	61.1	3589	2.3	95.1
	30	3588	0.8	36.8	3590	1.3	93.0	3588	0.0	46.8	3590	3.3	93.8	3587	0.0	57.0	3590	1.8	93.8
24	5	3586	66.5	84.9	383	100.0	100.0	3585	79.2	92.7	158	100.0	100.0	3585	83.8	89.5	142	100.0	100.0
	10	3586	9.6	62.4	3576	78.9	97.9	3589	13.9	75.8	3437	83.1	99.0	3587	15.9	90.6	3471	99.6	99.6
	15	3589	4.3	50.5	3581	0.0	96.6	3590	5.8	64.0	3586	1.9	96.9	3590	8.5	77.3	3589	7.1	97.9
	20	3590	3.9	47.9	3589	0.0	94.9	3589	4.3	54.8	3590	0.0	95.8	3590	6.1	64.5	3590	0.0	98.1
	25	3589	3.4	43.5	3590	1.4	94.7	3589	5.0	50.2	3590	5.0	95.9	3589	6.4	55.4	3590	2.3	97.7
30	5	3588	30.1	70.3	763	100.0	100.0	3588	73.5	87.7	434	100.0	100.0	3589	86.4	90.7	553	100.0	100.0
	10	3589	5.6	52.3	3590	2.2	96.6	3590	7.8	68.1	3590	3.4	96.8	3589	12.6	77.4	3590	4.8	96.2
	15	3589	4.0	44.0	3590	0.0	96.2	3589	5.6	57.1	3590	1.8	97.3	3589	8.1	64.7	3589	8.3	93.7
	20	3589	3.1	40.1	3590	0.0	94.4	3589	4.6	46.2	3590	1.8	95.9	3589	6.3	53.6	3590	4.6	97.2
	Avg	3453	19.3	56.6	2738	43.1	96.8	3423	25.6	67.2	2608	44.9	97.3	3369	28.9	76.1	2589	51.2	97.7

Table 3.5: Performance of branch-and-cut algorithm on the small-bucket LSP ($k = 4$)

		$\rho = 1$						$\rho = 2$						$\rho = 3$					
		None			All			None			All			None			All		
l	n	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB
12	5	1540	99.7	99.7	1003	99.8	99.8	380	100.0	100.0	74	100.0	100.0	25	100.0	100.0	14	100.0	100.0
	10	3586	85.6	94.0	3588	95.1	95.7	3581	98.2	98.2	3079	98.2	98.2	2418	99.5	99.5	2028	100.0	100.0
	15	3589	40.3	70.1	3590	0.0	92.2	3584	51.2	91.3	3587	52.7	93.5	3581	0.0	93.8	3583	38.3	95.3
	20	3589	0.0	64.1	3590	12.6	91.4	3587	27.0	85.5	3588	15.1	92.7	3584	13.2	93.6	3585	0.0	94.5
	25	3590	0.0	59.5	3591	0.0	88.1	3588	0.0	71.4	3589	0.0	91.1	3586	0.0	91.7	3587	12.5	92.8
	30	3591	0.0	56.0	3590	0.0	85.5	3589	0.0	70.2	3590	3.8	91.1	3588	0.0	83.2	3589	0.0	91.9
	35	3590	0.0	46.6	3590	0.0	84.3	3590	0.0	68.9	3591	0.0	89.8	3589	0.0	79.0	3590	0.0	91.4
18	5	3587	91.6	92.4	3589	97.6	97.6	2752	98.7	98.7	1788	99.7	99.7	1252	99.8	99.8	1256	100.0	100.0
	10	3589	48.5	70.7	3589	16.4	90.9	3585	28.9	88.3	3587	70.1	93.7	3583	73.0	94.1	3584	52.1	94.1
	15	3590	8.1	60.6	3590	2.1	88.1	3587	12.0	77.2	3588	31.6	90.6	3586	28.8	90.6	3586	33.4	92.7
	20	3590	0.0	59.5	3590	0.0	81.0	3589	0.0	70.1	3590	0.0	89.9	3587	14.2	87.1	3588	0.0	90.5
	25	3591	0.0	51.7	3590	0.0	81.5	3590	0.0	68.0	3590	0.0	89.7	3589	0.0	81.2	3589	0.0	91.0
	30	3590	0.0	46.5	3590	0.0	78.3	3591	0.0	65.0	3590	0.0	84.8	3590	0.0	71.9	3590	0.0	89.8
24	5	3588	77.0	81.3	3589	96.3	96.3	3368	97.3	98.0	3001	99.1	99.1	1789	100.0	100.0	1356	99.9	99.9
	10	3589	28.5	70.9	3590	18.9	89.4	3587	62.5	78.5	3589	41.2	91.7	3585	27.2	89.3	3587	26.6	92.9
	15	3590	7.4	57.2	3590	0.0	77.0	3588	21.0	67.4	3590	0.0	86.7	3587	12.7	80.1	3588	28.9	88.2
	20	3590	0.0	47.5	3590	0.0	75.0	3590	0.0	64.2	3591	0.0	84.1	3589	0.0	75.2	3590	0.0	88.9
	25	3590	0.0	44.1	3590	0.0	69.8	3590	0.0	60.0	3590	0.0	74.0	3590	0.0	74.4	3590	0.0	83.3
30	5	3589	13.6	74.6	3590	54.4	92.5	3586	75.6	87.8	3079	77.3	95.7	3456	91.0	94.6	2940	76.2	96.0
	10	3590	7.1	58.9	3591	0.0	73.8	3588	8.9	69.0	3590	4.4	85.6	3587	39.3	82.1	3589	13.0	87.3
	15	3591	0.0	50.7	3590	0.0	72.1	3589	0.0	62.9	3590	4.0	80.5	3588	0.0	73.6	3590	0.0	85.4
	20	3590	0.0	46.3	3591	0.0	67.7	3590	0.0	60.0	3590	0.0	71.4	3590	0.0	68.7	3590	4.9	81.0
	Avg	3496	23.1	63.8	3472	22.4	84.9	3394	31.0	77.3	3275	31.7	89.7	3178	31.8	86.5	3118	31.2	92.1

More specifically, on the big-bucket MP-PRP instances with four products ($k = 4$), employing the valid inequalities improves %BUB on average from 74.9% to 91.3%, 82.2% to 92.9%, and 86.5% to 95.0%, respectively for $\pi = 1$, $\pi = 2$ and $\pi = 3$ (Table 3.2). On the same LSP type MP-PRP instances with six products ($k = 6$), the addition of the valid inequalities increases %BUB on average from 65.6% to 95.1%, 75.4% to 95.9%, and 81.1%

Table 3.6: Performance of branch-and-cut algorithm on the small-bucket LSP ($k = 6$)

		$\rho = 1$						$\rho = 2$						$\rho = 3$					
		None			All			None			All			None			All		
l	n	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB
12	5	3095	98.4	98.4	1247	99.9	99.9	986	100.0	100.0	109	100.0	100.0	127	100.0	100.0	14	100.0	100.0
	10	3583	65.8	68.2	3581	98.2	98.2	3578	88.5	88.7	2360	99.7	99.7	3570	99.4	99.4	272	100.0	100.0
	15	3584	27.2	54.2	3581	50.9	96.5	3580	14.6	77.2	3577	93.2	97.5	3573	68.9	90.0	3577	78.3	98.8
	20	3581	0.0	49.9	3583	0.0	94.2	3578	13.3	70.4	3580	38.8	96.6	3579	47.9	88.9	2981	58.3	97.3
	25	3587	5.3	40.1	3585	0.0	91.6	3582	0.0	62.2	3585	0.0	95.4	3581	25.0	83.2	3581	19.5	96.2
	30	3587	0.0	33.4	3590	0.0	88.3	3580	0.0	60.0	3586	0.0	95.1	3577	0.0	78.5	3588	0.0	97.0
	35	3581	0.0	29.3	3590	0.0	84.4	3580	0.0	57.1	3585	0.0	92.5	3578	0.0	72.9	3583	0.0	95.4
18	5	3577	69.7	72.7	3184	99.5	99.5	3573	91.1	91.5	2049	99.8	99.8	3351	97.5	97.5	168	100.0	100.0
	10	3579	38.4	60.4	3584	56.9	96.2	3576	37.5	70.8	3581	78.0	97.2	3573	41.6	80.3	3463	97.8	98.2
	15	3586	0.0	50.8	3584	0.0	89.2	3583	32.5	71.0	3584	47.0	95.0	3575	49.5	77.1	3579	57.6	96.6
	20	3586	0.0	42.5	3584	0.0	83.9	3583	0.0	60.7	3582	14.3	92.8	3582	10.5	71.4	3580	19.5	95.9
	25	3586	0.0	39.3	3584	0.0	81.0	3581	0.0	56.6	3584	0.0	88.0	3584	0.0	67.7	3582	0.0	94.3
	30	3586	0.0	30.7	3577	0.0	81.5	3591	0.0	52.9	3577	0.0	82.2	3585	0.0	66.3	3576	0.0	93.6
24	5	3580	67.7	71.1	3404	97.3	97.3	3575	84.4	84.4	3193	98.0	98.2	3578	94.3	94.3	1405	99.8	99.8
	10	3584	16.5	52.0	3582	0.0	88.0	3581	39.8	61.3	3581	83.4	94.5	3580	38.0	69.7	3571	96.8	98.0
	15	3585	0.0	44.5	3576	0.0	79.9	3583	18.9	57.2	3578	0.0	87.4	3581	37.3	63.4	3581	17.4	91.6
	20	3584	0.0	42.3	3577	0.0	76.1	3584	0.0	53.9	3577	0.0	79.8	3581	0.0	63.6	3581	13.8	88.3
	25	3585	0.0	35.6	3583	0.0	71.0	3584	0.0	47.2	3583	0.0	74.9	3584	0.0	60.5	3578	0.0	85.5
30	5	3582	35.6	55.4	3579	92.1	93.9	3581	64.8	69.1	3585	94.4	96.3	3579	78.6	81.4	2236	97.8	97.8
	10	3583	0.0	43.9	3587	11.4	73.6	3586	43.5	55.7	3586	0.0	83.7	3586	27.7	60.7	3581	17.9	87.7
	15	3590	0.0	39.3	3582	0.0	71.2	3587	7.3	52.1	3583	0.0	75.6	3580	25.2	58.8	3582	27.8	84.8
	20	3583	0.0	38.5	3582	0.0	64.5	3582	0.0	46.7	3582	0.0	67.6	3580	0.0	57.0	3582	0.0	77.9
Avg		3562	19.3	49.7	3450	27.6	86.4	3463	28.9	65.8	3281	38.5	90.4	3412	38.2	76.5	2920	45.6	94.3

Table 3.7: Performance of branch-and-cut algorithm on the small-bucket LSP ($k = 8$)

		$\rho = 1$						$\rho = 2$						$\rho = 3$					
		None			All			None			All			None			All		
l	n	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB
12	5	3586	80.7	81.2	2276	99.8	99.8	3581	89.5	89.5	1261	100.0	100.0	2823	99.6	99.6	41	100.0	100.0
	10	3588	0.0	59.6	3589	99.1	99.1	3585	17.1	78.1	1372	99.9	99.9	3583	63.1	77.2	180	100.0	100.0
	15	3589	0.0	47.8	3590	97.7	97.7	3587	56.3	69.8	3588	98.6	98.7	3584	29.1	72.9	1529	100.0	100.0
	20	3590	0.0	43.9	3590	37.1	92.2	3588	0.0	65.4	3589	96.6	96.6	3586	0.0	70.5	3375	98.8	98.8
	25	3590	0.0	36.7	3590	0.0	90.8	3589	0.0	59.6	3590	71.4	94.9	3587	40.9	69.4	3588	98.1	98.1
	30	3590	0.0	37.4	3590	17.5	89.1	3590	0.0	52.4	3590	0.0	90.7	3589	13.3	65.4	3589	72.7	94.7
	35	3590	0.0	27.4	3590	0.0	84.9	3590	0.0	52.0	3590	15.2	91.0	3590	0.0	65.6	3589	38.3	95.0
18	5	3588	44.3	61.1	2708	99.9	99.9	3586	73.0	74.1	1208	99.9	99.9	3584	84.7	84.7	305	100.0	100.0
	10	3589	0.0	48.6	3590	57.8	95.6	3587	12.8	65.5	3588	98.3	98.3	3586	47.6	69.5	3106	99.6	99.6
	15	3590	0.0	43.2	3590	0.0	86.9	3588	15.5	57.2	3589	91.9	94.4	3587	36.8	66.7	3588	98.1	98.1
	20	3590	0.0	40.3	3590	0.0	85.5	3589	0.0	60.2	3590	51.3	93.4	3588	11.3	66.5	3588	57.5	96.0
	25	3590	0.0	40.1	3590	0.0	83.6	3590	0.0	53.4	3590	0.0	88.0	3589	0.0	65.1	3590	35.8	94.1
	30	3590	0.0	27.7	3590	0.0	81.3	3590	0.0	48.4	3590	0.0	83.8	3590	0.0	64.0	3590	0.0	91.8
24	5	3588	30.9	54.5	3590	98.1	98.1	3586	41.3	65.8	3153	99.2	99.2	3585	72.1	72.2	1899	99.9	99.9
	10	3590	7.0	44.8	3590	16.1	86.2	3587	9.1	56.1	3590	95.5	95.7	3586	51.7	63.3	3589	97.2	97.2
	15	3590	0.0	39.3	3590	0.0	79.8	3589	8.3	52.9	3590	36.0	89.7	3588	37.0	59.4	3589	71.2	90.7
	20	3590	0.0	38.6	3590	0.0	79.3	3590	0.0	51.9	3590	0.0	80.2	3589	0.0	60.7	3590	17.9	86.1
	25	3590	0.0	39.2	3590	0.0	70.0	3590	0.0	46.5	3590	0.0	77.2	3589	0.0	59.1	3591	0.0	85.0
30	5	3589	34.2	54.1	3590	65.8	92.1	3587	36.4	64.5	3059	99.3	99.3	3587	73.8	73.9	1810	99.6	99.6
	10	3590	0.0	40.0	3590	0.0	75.9	3589	16.9	52.4	3590	66.1	83.1	3588	44.2	56.6	3590	83.5	85.7
	15	3589	0.0	34.0	3590	0.0	65.9	3588	0.0	47.8	3590	0.0	71.2	3587	7.7	54.2	3590	45.3	79.0
	20	3589	0.0	35.5	3590	0.0	60.7	3589	0.0	43.3	3590	0.0	70.1	3588	0.0	53.7	3590	0.0	74.9
Avg		3589	9.0	44.3	3490	31.3	86.1	3588	17.1	59.4	3231	55.4	90.7	3552	32.4	67.7	2841	68.8	93.8

to 97.3%, respectively for $\pi = 1$, $\pi = 2$ and $\pi = 3$ (Table 3.3). On the big-bucket MP-PRP instances with eight products ($k = 8$) which consist of the highest number of products, the implementation of the valid inequalities increases %BUB on average from 56.6% to 96.8%, 67.2% to 97.3%, and 76.1% to 97.7%, respectively for $\pi = 1$, $\pi = 2$ and $\pi = 3$ (Table 3.4). This indicates the substantial impact of applying the valid inequalities.

Similarly, on the small-bucket MP-PRP instances with four products ($k = 4$), employing the valid inequalities improves %BUB on average from 63.8% to 84.9%, 77.3% to 89.7%, and 86.5% to 92.1%, respectively for $\rho = 1$, $\rho = 2$ and $\rho = 3$ (Table 3.5). On the small-bucket instances with six products ($k = 6$), the addition of the valid inequalities increases %BUB on average from 49.7% to 86.4%, 65.8% to 90.4%, and 76.5% to 94.3%, respectively for $\rho = 1$, $\rho = 2$ and $\rho = 3$ (Table 3.6). On the big-bucket instances with eight products ($k = 8$) with the largest number of products, the addition of the valid inequalities increases %BUB on average from 44.3% to 86.1%, 59.4% to 90.7%, and 67.7% to 93.8%, respectively for $\rho = 1$, $\rho = 2$ and $\rho = 3$ (Table 3.7). This indicates the substantial impact of applying the valid inequalities. In Appendix C, we report the lower bound improvements obtained by incorporating the valid inequalities in the small- and big-bucket models. Generally, when the instances are harder to solve (smaller ρ and π), the impact of the inequalities on the lower bound improvement is bigger.

3.7.4 Analysis of the Cost Shares

Finally, we analyze the cost component shares on different MP-PRP instances. Tables 3.8 and 3.9 present the different cost component values and proportions for $\rho = \{1,2,3\}$ and $\pi = \{1,2,3\}$, respectively for the small- and big-bucket LSP instances. In Table 3.8, columns three, 10, and 17 show the total cost values. Columns four to nine present the production, inventory, and the transportation costs and shares (in percent), respectively for $\rho = 1$. Columns 11 to 16 and 18 to 23 do the same for the cases where $\rho = 2$ and $\rho = 3$, respectively. Table 3.9 reports the same information for big-bucket LSP instances. In all cases, the share of the production setup cost decreases when for the same number of periods, the number of customers increases. In most situations, for any number of periods and π (or ρ) combination, the share of the inventory cost, and the share of the transportation cost increase when the number of customers increases. Note that the production costs that are taken into account in the model are the fixed production costs. The variable production costs are not included, since the total demand for all customers needs to be satisfied and hence the total variable production cost represents a fixed amount that is left out of the objective function.

Table 3.8: Cost component values and proportions for small-bucket LSP

		$\rho = 1$							$\rho = 2$							$\rho = 3$						
l	n	Total	Production		Inventory		Transport		Total	Production		Inventory		Transport		Total	Production		Inventory		Transport	
12	5	43346	32273	73.5%	3646	8.1%	7428	18.4%	43669	32273	72.9%	3745	8.2%	7652	18.9%	44212	32273	72.2%	3594	7.9%	8345	20.0%
	10	50581	32780	63.9%	7555	14.3%	10246	21.8%	50879	32780	63.5%	7925	14.9%	10174	21.6%	52214	32780	62.0%	7216	13.2%	12219	24.7%
	15	58771	33160	55.4%	10861	18.1%	14750	26.4%	59427	33160	54.8%	11685	19.2%	14582	25.9%	60249	33160	54.1%	10972	17.9%	16117	28.0%
	20	65918	32788	49.0%	14936	22.0%	18194	29.0%	67142	32788	48.2%	15349	22.2%	19005	29.6%	67604	32788	47.9%	15443	21.9%	19374	30.2%
	25	70732	33066	46.1%	16897	23.4%	20769	30.5%	72047	33066	45.3%	17580	24.0%	21400	30.8%	72604	33066	44.9%	17753	23.8%	21784	31.3%
	30	77224	33166	42.3%	20703	26.4%	23355	31.3%	78886	33166	41.4%	21388	26.7%	24332	31.9%	79423	33166	41.1%	21186	26.0%	25072	32.9%
	35	83517	32816	38.9%	24080	28.3%	26621	32.8%	84989	32816	38.2%	24059	27.8%	28114	33.9%	85562	32816	37.9%	24026	27.5%	28720	34.6%
18	5	51465	32888	63.0%	7781	14.7%	10796	22.2%	51956	32936	62.5%	8072	15.1%	10947	22.4%	53110	33261	61.9%	7323	13.4%	12526	24.7%
	10	62446	32897	51.8%	14587	22.8%	14962	25.5%	62935	32897	51.4%	14442	22.3%	15595	26.3%	63142	32897	51.2%	14843	22.8%	15401	26.0%
	15	74875	32854	43.3%	21881	28.6%	20140	28.1%	75578	32854	42.9%	21930	28.4%	20794	28.7%	75353	32854	43.0%	21110	27.4%	21389	29.6%
	20	88911	32689	36.4%	28301	31.4%	27920	32.2%	89445	32689	36.2%	28200	31.1%	28556	32.8%	89075	32689	36.3%	27124	30.0%	29262	33.6%
	25	99296	33013	32.8%	34436	34.1%	31847	33.0%	100226	33013	32.5%	35083	34.4%	32130	33.0%	99719	33013	32.7%	33676	33.2%	33029	34.1%
	30	110165	32990	29.6%	40596	36.3%	36713	34.1%	111469	33377	29.7%	40757	36.1%	37335	34.3%	111398	33377	29.7%	40129	35.4%	37891	34.9%
24	5	58186	33016	55.8%	11711	19.6%	13460	24.6%	58582	33016	55.4%	12429	20.6%	13137	24.0%	60098	33351	54.4%	11362	18.2%	15586	27.5%
	10	77628	33284	42.3%	22990	29.4%	21421	28.3%	78191	33284	42.0%	23174	29.3%	21759	28.7%	78071	33284	42.1%	22939	29.1%	21848	28.8%
	15	98363	33036	33.1%	35823	36.0%	29505	30.9%	99483	33375	33.1%	36821	36.5%	29286	30.4%	99498	33728	33.3%	35001	34.6%	30902	32.1%
	20	115200	34125	29.1%	44300	37.9%	36909	33.0%	114614	33747	29.0%	43630	37.5%	37237	33.5%	114753	33726	28.9%	42816	36.5%	38544	34.6%
	25	135980	34282	25.0%	57031	41.5%	44667	33.5%	135970	34944	25.4%	56176	40.7%	44917	33.8%	135521	33919	24.7%	56620	41.3%	44981	34.0%
30	5	67599	33073	48.3%	18422	26.6%	16170	25.1%	68019	33073	48.0%	18928	27.2%	16085	24.8%	69663	33791	47.2%	19066	26.0%	18272	26.8%
	10	97866	34012	34.3%	36414	36.6%	27707	29.2%	98690	35412	35.4%	35474	35.2%	27937	29.4%	97996	35057	35.3%	34994	34.8%	28345	29.9%
	15	122554	34973	28.1%	52127	41.4%	36055	30.5%	121843	34349	27.8%	51825	41.7%	35802	30.6%	121253	34638	28.1%	51142	41.1%	35739	30.8%
	20	148833	37500	25.0%	64408	42.5%	46993	32.4%	148201	40102	26.6%	62499	41.5%	45866	31.9%	147073	38239	25.7%	62440	41.8%	46527	32.5%

Table 3.9: Cost component values and proportions for big-bucket LSP

		$\pi = 1$							$\pi = 2$							$\pi = 3$						
l	n	Total	Production		Inventory		Transport		Total	Production		Inventory		Transport		Total	Production		Inventory		Transport	
12	5	43360	32273	73.5%	3335	7.4%	7753	19.1%	43294	32273	73.6%	3331	7.4%	7691	19.0%	43167	32273	73.8%	3475	7.9%	7420	18.3%
	10	49989	32780	64.5%	6619	12.9%	10657	22.7%	50009	32780	64.5%	6637	12.9%	10658	22.6%	50003	32780	64.5%	6759	13.1%	10531	22.5%
	15	57953	33160	56.1%	9727	16.6%	15133	27.3%	57961	33160	56.1%	9728	16.6%	15140	27.3%	57941	33160	56.2%	9823	16.8%	14958	27.0%
	20	64256	32788	50.3%	13111	20.0%	18357	29.7%	64336	32788	50.2%	13134	20.1%	18414	29.7%	64231	32788	50.3%	13123	20.1%	18320	29.6%
	25	69213	33066	47.2%	15006	21.4%	21141	31.5%	69176	33066	47.2%	15151	21.6%	20959	31.3%	69177	33066	47.2%	15137	21.6%	20973	31.2%
	30	74655	33166	43.8%	17920	23.7%	23569	32.5%	74742	33166	43.7%	17837	23.6%	23739	32.7%	74640	33166	43.8%	17768	23.5%	23707	32.7%
	35	80969	32816	40.2%	20296	24.9%	27857	35.0%	80893	32816	40.2%	20239	24.8%	27838	35.0%	80913	32816	40.2%	20176	24.7%	27921	35.0%
18	5	51943	34055	64.4%	6903	13.1%	11185	22.5%	51802	34055	64.7%	7016	13.4%	10865	21.9%	52154	34055	64.3%	7229	13.7%	10964	22.0%
	10	61597	32897	52.5%	13577	21.6%	15122	25.9%	61414	32897	52.6%	13312	21.3%	15205	26.1%	61548	32897	52.5%	13341	21.2%	15309	26.2%
	15	73579	33529	45.1%	19502	26.2%	20547	28.7%	73451	33529	45.2%	19920	26.7%	20003	28.1%	73520	33529	45.1%	19483	26.2%	20508	28.7%
	20	87341	33066	37.6%	26673	30.2%	27602	32.1%	86730	33066	37.9%	26528	30.4%	27136	31.7%	86865	33066	37.9%	25938	29.7%	27861	32.5%
	25	96570	33013	33.9%	32145	33.0%	31413	33.1%	96471	33013	33.9%	32398	33.3%	31127	32.8%	96495	33013	33.9%	32331	33.2%	31152	32.8%
	30	107878	33362	30.7%	38345	35.2%	36171	34.0%	107441	33362	30.9%	38078	35.2%	36001	33.9%	107456	33362	30.9%	37883	35.0%	36211	34.1%
24	5	58703	33752	56.6%	11442	19.2%	13523	24.2%	58601	33752	56.7%	11440	19.2%	13450	24.0%	58655	33752	56.7%	11691	19.6%	13213	23.7%
	10	78427	35936	45.6%	21351	27.3%	21140	27.1%	78426	35936	45.6%	21564	27.5%	20926	26.9%	78464	35936	45.5%	20909	26.7%	21619	27.7%
	15	97428	34063	34.7%	34803	35.4%	28562	29.9%	96433	33723	34.6%	34286	35.3%	28424	30.0%	96250	33723	34.7%	33975	35.1%	28552	30.2%
	20	112439	34923	30.9%	41918	37.1%	35598	32.0%	112490	34923	30.9%	41730	36.9%	35837	32.3%	112607	34923	30.8%	41635	36.8%	36048	32.4%
	25	132096	35711	27.0%	53485	40.4%	42900	32.7%	132198	35711	27.0%	53564	40.4%	42923	32.6%	132138	35711	27.0%	53369	40.3%	43058	32.8%
30	5	67369	34090	50.2%	17681	26.0%	15598	23.9%	67339	34090	50.1%	17443	25.6%	15873	24.2%	67323	34090	50.2%	17373	25.6%	15861	24.2%
	10	98069	37351	37.9%	33407	33.9%	27311	28.1%	98206	37351	37.9%	33483	34.0%	27371	28.2%	98355	37351	37.8%	33042	33.5%	27962	28.7%
	15	119599	36496	30.3%	48146	39.8%	34957	29.8%	120592	36933	30.4%	48075	39.4%	35584	30.1%	119876	36496	30.3%	48332	39.9%	35047	29.8%
	20	146169	43719	29.9%	57433	38.9%	45017	31.2%	146122	43719	30.0%	57715	39.1%	44688	31.0%	146593	43719	29.9%	57793	39.0%	45080	31.1%

Figures (3.9) and (3.10) present a comparison of the cost component share (in percentage) for different numbers of customers and periods $l = 12$ and 30 when small- and big-bucket LSP instances are considered, respectively. These figures show that by increasing the number of planning periods it is possible to schedule the production in such a way that the share of the production setups decreases. Similar tendencies are observed for instances with periods $l = 18$ and 24. The challenge for the practitioners is in designing and developing efficient methods to both obtain feasible solutions and proving the quality of those solutions.

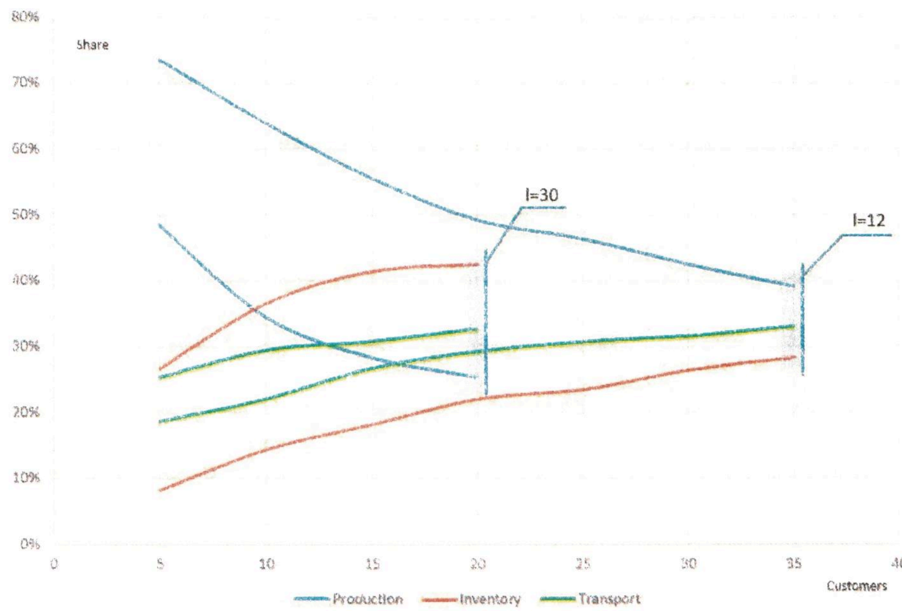


Figure 3.9: Cost share (%) comparison for different number of customers and periods in small-bucket LSP with $\rho = 1$

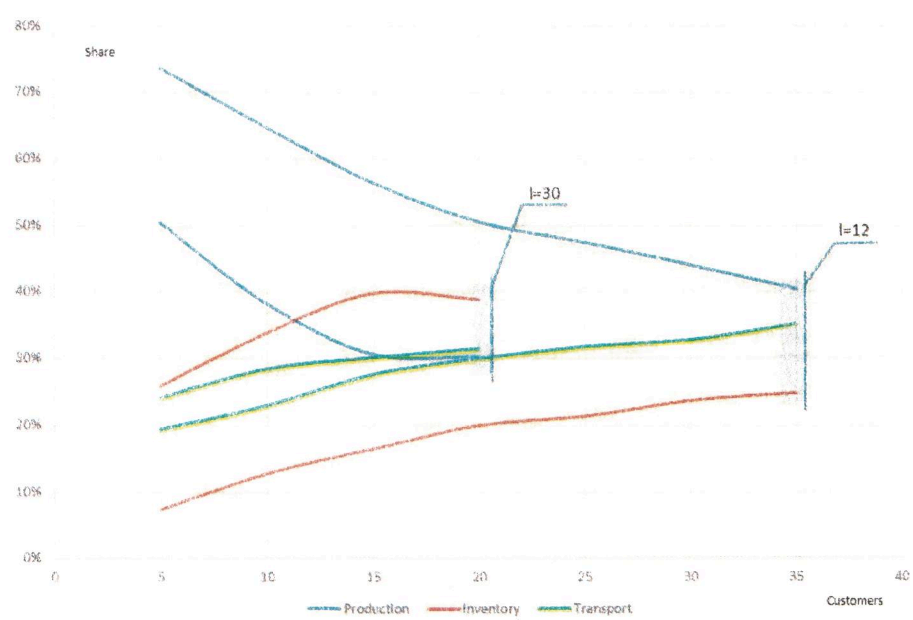


Figure 3.10: Cost share (%) comparison for different number of customers and periods in big-bucket LSP with $\pi = 1$

3.8 Conclusion

While classical production routing problems have received considerable attention from the research community, all studies on this problem and its variants consider identical production and route planning period lengths. In this paper, we have presented formulations for a multi-product production routing problem with the possibility of incorporating different production and route planning period lengths. This is the first attempt in the literature to consider such a practical limitation. We model both big-bucket and small-bucket lot-sizing problems at the production level. Next, we have adapted a state-of-the-art matheuristic to obtain quality solutions for instances of this problem with different numbers of products, planning periods, and customers. We have developed many sets of valid inequalities that exploit the structure of the problem. The effectiveness of the derived valid inequalities within our branch-and-cut algorithm was tested through an extensive set of computational experiments. The availability of an exact algorithm has allowed us to measure the quality of the upper bounding heuristic. We have shown that by including the relevant valid inequalities in the heuristic, significant improvements in terms of the number of iterations, the solution time and quality can be achieved.

We observe that for the same numbers of micro periods, customers and products, the problem can be solved more efficiently when the number of production planning periods or routing periods decreases. One explanation is that in these cases the number of decision variables will quickly decrease in our proposed reformulation model.

References

Absi, N., Archetti, C., Dauzère-Pérès, S., and Feillet, D. (2015). A two-phase iterative heuristic approach for the production routing problem. *Transportation Science*, 49(4):784–795.

Adulyasak, Y., Cordeau, J.-F., and Jans, R. (2014). Formulations and branch-and-cut algorithms for multivehicle production and inventory routing problems. *INFORMS Journal on Computing*, 26(1):103–120.

Adulyasak, Y., Cordeau, J.-F., and Jans, R. (2015). The production routing problem: A review of formulations and solution algorithms. *Computers & Operations Research*, 55:141–152.

Archetti, C., Bertazzi, L., Laporte, G., and Speranza, M. G. (2007). A branch-and-cut algorithm for a vendor-managed inventory-routing problem. *Transportation Science*, 41(3):382–391.

Archetti, C., Bertazzi, L., Paletta, G., and Speranza, M. G. (2011). Analysis of the maximum level policy in a production-distribution system. *Computers & Operations Research*, 38(12):1731–1746.

Archetti, C. and Speranza, M. G. (2016). The inventory routing problem: the value of integration. *International Transactions in Operational Research*, 23(3):393–407.

Armentano, V. A., Shiguemoto, A., and Løkketangen, A. (2011). Tabu search with path relinking for an integrated production–distribution problem. *Computers & Operations Research*, 38(8):1199–1209.

Atamtürk, A. and Küçükyavuz, S. (2005). Lot sizing with inventory bounds and fixed costs: Polyhedral study and computation. *Operations Research*, 53(4):711–730.

Barany, I., Van Roy, T., and Wolsey, L. A. (1984). Uncapacitated lot-sizing: The convex hull of solutions. In Korte, B. and Ritter, K., editors, *Mathematical programming at Oberwolfach II*, volume 22 of *Mathematical Programming Studies*, pages 32–43. Springer, Berlin, Heidelberg.

Belo-Filho, M., Amorim, P., and Almada-Lobo, B. (2015). An adaptive large neighbourhood search for the operational integrated production and distribution problem of perishable products. *International Journal of Production Research*, 53(20):6040–6058.

Blumenfeld, D. E., Burns, L. D., Daganzo, C. F., Frick, M. C., and Hall, R. W. (1987). Reducing logistics costs at General Motors. *Interfaces*, 17(1):26–47.

Brahimi, N. and Aouam, T. (2016). Multi-item production routing problem with backordering: a MILP approach. *International Journal of Production Research*, 54(4):1076–1093.

Brown, G., Keegan, J., Vigus, B., and Wood, K. (2001). The Kellogg company optimizes production, inventory, and distribution. *Interfaces*, 31(6):1–15.

Çetinkaya, S., Üster, H., Easwaran, G., and Keskin, B. B. (2009). An integrated outbound logistics model for Frito-Lay: Coordinating aggregate-level production and distribution decisions. *Interfaces*, 39(5):460–475.

Chandra, P. and Fisher, M. L. (1994). Coordination of production and distribution planning. *European Journal of Operational Research*, 72(3):503–517.

Chen, Z.-L. and Vairaktarakis, G. L. (2005). Integrated scheduling of production and distribution operations. *Management Science*, 51(4):614–628.

Chitsaz, M., Cordeau, J.-F., and Jans, R. (2019). A unified decomposition matheuristic for assembly, production and inventory routing. *INFORMS Journal on Computing*, 31(1):134–152.

Chitsaz, M., Cordeau, J.-F., and Jans, R. (2020). A branch-and-cut algorithm for an assembly routing problem. *European Journal of Operational Research*, 282(3):896–910.

Dantzig, G. B., Fulkerson, R., and Johnson, S. (1954). Solution of a large-scale traveling-salesman problem. *Operations Research*, 2(4):393–410.

De Matta, R. and Guignard, M. (1994a). Dynamic production scheduling for a process industry. *Operations Research*, 42(3):492–503.

De Matta, R. and Guignard, M. (1994b). Studying the effects of production loss due to setup in dynamic production scheduling. *European Journal of Operational Research*, 72(1):62–73.

Fischetti, M., Polo, C., and Scantamburlo, M. (2004). A local branching heuristic for mixed-integer programs with 2-level variables, with an application to a telecommunication network design problem. *Networks*, 44(2):61–72.

Fumero, F. and Vercellis, C. (1999). Synchronized development of production, inventory, and distribution schedules. *Transportation Science*, 33(3):330–340.

Glover, F., Jones, G., Karney, D., Klingman, D., and Mote, J. (1979). An integrated production, distribution, and inventory planning system. *Interfaces*, 9(5):21–35.

Hahn, C. K., Duplaga, E. A., and Hartley, J. L. (2000). Supply-chain synchronization: lessons from Hyundai Motor Company. *Interfaces*, 30(4):32–45.

Jans, R. and Degraeve, Z. (2004). An industrial extension of the discrete lot-sizing and scheduling problem. *IIE transactions*, 36(1):47–58.

Loparic, M., Marchand, H., and Wolsey, L. A. (2003). Dynamic knapsack sets and capacitated lot-sizing. *Mathematical Programming*, 95(1):53–69.

Marinelli, F., Nenni, M. E., and Sforza, A. (2007). Capacitated lot sizing and scheduling with parallel machines and shared buffers: A case study in a packaging company. *Annals of Operations Research*, 150(1):177–192.

Martin, C. H., Dent, D. C., and Eckhart, J. C. (1993). Integrated production, distribution, and inventory planning at libbey-owens-ford. *Interfaces*, 23(3):68–78.

Miranda, P. L., Cordeau, J.-F., Ferreira, D., Jans, R., and Morabito, R. (2018). A decomposition heuristic for a rich production routing problem. *Computers & Operations Research*, 98:211–230.

Neves-Moreira, F., Almada-Lobo, B., Cordeau, J.-F., Guimaraes, L., and Jans, R. (2019). Solving a large multi-product production-routing problem with delivery time windows. *Omega*, 86:154–172.

Pochet, Y. and Wolsey, L. A. (1994). Polyhedra for lot-sizing with Wagner-Whitin costs. *Mathematical Programming*, 67(1-3):297–323.

Pochet, Y. and Wolsey, L. A. (2006). *Production Planning by Mixed Integer Programming*. Springer Science & Business Media.

Qiu, Y., Ni, M., Wang, L., Li, Q., Fang, X., and Pardalos, P. M. (2018a). Production routing problems with reverse logistics and remanufacturing. *Transportation Research Part E: Logistics and Transportation Review*, 111:87–100.

Qiu, Y., Wang, L., Xu, X., Fang, X., and Pardalos, P. M. (2018b). Formulations and branch-and-cut algorithms for multi-product multi-vehicle production routing problems with startup cost. *Expert Systems with Applications*, 98:1–10.

Silva, C. and Magalhaes, J. M. (2006). Heuristic lot size scheduling on unrelated parallel machines with applications in the textile industry. *Computers & Industrial Engineering*, 50(1-2):76–89.

Solyalı, O. and Süral, H. (2017). A multi-phase heuristic for the production routing problem. *Computers & Operations Research*, 87:114–124.

Trigeiro, W. W., Thomas, L. J., and McClain, J. O. (1989). Capacitated lot sizing with setup times. *Management Science*, 35(3):353–366.

Zhang, Q., Sundaramoorthy, A., Grossmann, I. E., and Pinto, J. M. (2017). Multiscale production routing in multicommodity supply chains with complex production facilities. *Computers & Operations Research*, 79:207–222.

General Conclusion

Making integrated supply chain decisions by considering the customers in the downstream supply chain or the suppliers in the upstream supply chain is quite different in nature. Even though these two types of integrated problems show some similarities with respect to the inventory structure, they possess a very different lot-sizing structure. Particularly, the IRP and PRP have a distribution structure (i.e., items are distributed from a central plant/warehouse to many customers), whereas the ARP is based on an assembly structure (i.e., items are sent from many different suppliers to one central plant/warehouse). Furthermore, multiple components are needed to produce one single product in the basic variant of the ARP whereas in the standard IRP and PRP, a single product is distributed in the network. Few studies have considered the integration of production planning with inbound transportation for the collection of components from suppliers to assemble a final product. In this thesis, we mainly focused on filling this gap in the literature by proposing the standard ARP in Chapter 1. We studied the case in which a manufacturer is responsible for organizing the inbound flow of raw material and components necessary for the production from its suppliers. While producing some finished or semi-finished goods, the manufacturer mainly faces two challenging problems concerning the planning of the production of the final product and the supply of input materials. The manufacturer has to optimize its internal processes with respect to inventory, production and set-up costs to meet the customer's demand. In addition, the manufacturer has to simultaneously optimize its supply processes including the collection and transportation of the components from the suppliers. These processes are closely connected because the production volume in each period highly depends on the available level of inventory of components at the plant which itself depends on the level of shipments from the suppliers. We proposed an efficient unified method to find high quality solutions for the instances of this problem in Chapter 1 and developed an exact algorithm to solve the problem instances to optimality in Chapter 2.

Furthermore, in this thesis, we studied a practical and complex case arising in the context of production planning and distribution routing. Very often in real-world logistics management, the planning period lengths for the production and distribution routing are not the same. For example, the distribution planning is done using daily truck dispatches, but the production planning is performed on a shift-basis, where one day contains multiple shifts. There exist many studies in the literature on the classical PRP, most of which consider identical production and route planning period lengths. In Chapter 3, we presented models for a multi-product PRP which accommodates the possibility of having different production and route planning period lengths. In addition, we have adapted our unified method (Chapter 1) to obtain high quality solutions for instances of this problem. In the following, we present an overview of the contributions of this thesis, and possible interesting avenues for future research.

Contributions

This thesis contributes to the literature in several ways. First, it introduces two new problems together with mathematical programming models for them: the assembly routing problem (ARP), and the production routing problem (PRP) under decoupled planning periods. Second, this thesis offers advanced solution methods including a unified heuristic capable of solving several integrated supply chain problems. Finally, the thesis provides many test instances for these new problems, and the results of extensive computational experiments which give the means to evaluate the high quality of the suggested algorithms.

I. Integrated Inbound Transportation, Production and Inventory Planning

Motivated by the many practical applications mentioned in Chapter 1 and to fill the gap in the literature, we studied the ARP. The ARP jointly optimizes the integrated inbound transportation, production and inventory planning in a finite planning horizon with the standard basic assumptions similar to the IRP and PRP. In other words, the problem is a complement to the IRP and PRP which focuses on the downstream supply chain and integrates production planning with inbound routing decisions. In Chapter 1 we introduce the problem and propose a mathematical programming formulation to model this problem. In Chapter 2, we extend the presented model to consider the case where each supplier provides a subset of the components necessary for the final product and some components can be obtained from more than one supplier. This is the first contribution of this thesis.

II. Production and Distribution Routing Under Decoupled Planning Periods

The current literature on the PRP and its variants only considers identical production planning and routing period lengths. This is in many cases an abstraction of the problem in the real world. In Chapter 3, we propose a generalized multi-product problem by considering decoupled planning periods for the production and distribution routing planning. This is a practical operational decision-making problem that has not been studied before due to its complexity. We study the cases in which the production planning subproblem may necessitate the use of small- or big-bucket lot-sizing problem assumptions. These cases include the situations where the route planning period length or production planning period length are larger. We present mathematical programming formulations for the problem and a unified reformulation that can be used to model different production planning and distribution routing period length situations.

III. A Unified Heuristic Method for the ARP, IRP, and PRP

The originality of the models presented in this thesis demands the development of novel algorithms capable of solving them effectively and efficiently. The successful development of such methods requires both a well-designed structure and proper implementation of each part of the method. In Chapter 1, we introduce a novel unified matheuristic capable of solving not only the ARP, but also the IRP and PRP. In Chapter 2, we show how to adapt this method to solve the generalized ARP, and in Chapter 3, we present the adaptation of this method for the PRP instances with decoupled planning periods.

The core idea is to decompose the problem into its natural subproblems and solve these subproblems iteratively until a local optimum is found. We also suggest effective diversification mechanisms. The search continues until a stopping condition, e.g., time limit is reached. To have an efficient solution method, each subproblem has to include a proper cost approximation of the other subproblem. The appropriateness of the approximation depends on the level of the decisions to make at the current subproblem. For example, when in a subproblem the main decisions to make concern the high-level production setups, the average cost of dispatching each truck is sufficient to approximate the total cost of distribution in the objective function. In another subproblem that is concerned with the production volume decisions, the approximate cost of visiting each node (supplier or customer) is needed to accommodate more detailed information about the distribution cost in the objective function.

Another critical component in our unified method is in efficiently communicating between different subproblems. The communication here relates to which information to transmit from one subproblem to another and how to calculate it. An example is in how to assign the transportation cost share of a node, given the sequence of all nodes in the route. We show in detail that different cost approximation mechanisms result in different solution qualities. We propose several cost update mechanisms to approximate the routing cost. The sensitivity analysis indicates that using a mix of two update mechanisms improves the quality of the solutions. Moreover, we propose a scheme to effectively switch between different cost splitting methods. Using the same parameter setting for all problems and instances, we obtain 781 new best known solutions out of 2,628 standard IRP and PRP test instances. In particular, on large-scale multi-vehicle instances, the new algorithm outperforms specialized state-of-the-art heuristics for these two problems.

IV. Valid Inequalities to Strengthen the Mathematical Models

In Chapters 2 and 3, we develop several new valid inequalities to strengthen the linear programming (LP) relaxation of the mixed integer programming formulations for the ARP and for the PRP under decoupled planning periods. Three classes of valid inequalities are presented. The first class contains inequalities for the production and inventory flow structures. These inequalities improve the linear relaxation bound of the lot-sizing part of the models. The second class includes the bounds on the variables which are essentially an extension of the cut-set inequalities for the lot-sizing problem. The last class includes general inequalities. The inequalities are used in a branch-and-cut algorithm. We show the significant impact of these valid inequalities by analyzing the effect of each class on the value of the LP relaxation and on the final solution through extensive computational experiments. In addition, in Chapter 2, we present two algorithms to separate multi-period fractional capacity cut constraints and compared their efficiency with the state-of-the-art separation procedures of Lysgaard et al. (2004) (CVRPSEP) for the single-period VRPs. To the best of our knowledge, there is no exact algorithm to separate these constraints in polynomial time and it is not known whether separating them is NP-hard or not. We show that the performance of the branch-and-cut algorithm is significantly better when it uses one of the proposed separation algorithms compared to the case when it employs CVRPSEP.

V. Development of Various Test Data Sets

We created many small, medium, and large instances for the standard ARP (Chapter 1), the generalized ARP (Chapter 2), and the PRP under decoupled planning periods (Chapter 3). The standard ARP instances include data sets with 14, 50, and 100 suppliers, each with 6 periods. Each of these sets has four classes of instances. The first class includes the normal or standard instances. The second class contains high unit production and setup cost instances. The third class represents the case with high transportation costs. The fourth class includes instances with no supplier inventory costs. Each of the three sets has 480 instances resulting in a total of 1,440 instances.

For the generalized ARP, we generated three classes of instances. The first class includes instances where each supplier provides a unique component type. The second class represents the case where each supplier provides a subset of components. The third class corresponds to the situation in which one single component is offered by all suppliers. Each class includes data sets with five different planning horizons ranging from 4 to 12 periods. For each planning horizon we considered eight different numbers of suppliers. For each combination of the number of planning periods and suppliers we randomly generated five instances. Overall, 600 small to large size instances are generated for three classes, five planning horizons, eight numbers of suppliers, and five instances per category.

Various test instances were generated for the PRP under decoupled planning periods. The test bed includes instances with a different number of periods, products, and customers. We considered four different planning horizons varying from 12 to 30 periods, a different number of products ranging from 4 to 8, and a different number of customers from 5 to 35 (depending on the number of periods). Overall, instances are generated with 22 combinations of the planning horizon and number of customers, three numbers of product sizes and five randomly generated instances per category. Considering six combinations for the period planning length scenarios, this test bed includes 1,980 instances. All instances are available at `http://chairelogistique.hec.ca/en/scientific-data/`.

VI. Extensive Computational Experiments

We report the results of extensive computational experiments on all the generated instances on the standard ARP, the generalized ARP, and the multi-product PRP under decoupled planning periods as well as the results on the standard data sets for the IRP and PRP. Standard IRP instances include various data sets provided by Archetti et al. (2007), Archetti et al. (2012), Coelho and Laporte (2013) and Desaulniers et al. (2015). Overall we consider 1,098 instances in four IRP data sets. The standard PRP instances include six data sets. Archetti et al. (2011) and Boudia et al. (2005) each introduce three data sets with a total number of 1,530 instances. The results indicate that our unified method outperforms the state-of-the-art heuristics on the large-scale multi-vehicle IRP and PRP instances. Further analyses confirm the robust behavior of this algorithm. The computational experiments show that the valid inequalities for the ARP variants and the multi-product PRP under decoupled planning periods notably enhance the performance of the branch-and-cut algorithm.

Future Work

This thesis not only fills important gaps in the theory and practice of the integrated production and distribution routing for supply chains but also opens several doors for future research. The possibilities can be categorized into two major directions, i.e., modeling, and solution methodology. We explain the ideas for each category in the following section.

I. Directions for Future Research on the Modeling

The first possible research direction is to integrate both the upstream and downstream routing decisions for a central manufacturing plant. This is to combine the ARP and PRP (or IRP) together. Several different pick-up (from the suppliers) and delivery (to the customers) strategies can be considered.

- Separate pick-up and delivery routes, where the suppliers and retailers need to be visited in separate routes. This routing policy itself includes three different operational variants.
  - Joint fleet. In this case, the suppliers and the retailers should be visited in separate routes, but the vehicles can be shared meaning that every available vehicle can be used for either pick-ups or deliveries in each planning period. This is relevant when we have trucks that can be used for both pick-ups and deliveries, but it is not favourable or possible to do a joint pick-up and delivery operation e.g., because it is too complex, or because components and end products cannot be in the same truck for safety reasons. As another example, this fits for the cases where the same trucks can carry the different types of trailers needed for shipments from suppliers and to the retailers.
  - Separate fleet. In this case, the fleet for inbound and outbound transportation are separate and the size for each is given in advance. This case is relevant when the type of truck needed for the pick-up and delivery operation is different. This addresses the manufacturing environments where the components and final product cannot be loaded in the same type of truck e.g., auto man-

ufacturing. In this example, the pick-up of the components can be done by regular container trucks and the delivery of the final product, which is a car, should be done by double-deckers.

- Mixed routes, in which the suppliers and retailers may be visited in the same route. In this case, the components and the final product are allowed to be carried in the same truck. This is relevant when we have trucks that can be used for both pick-ups and deliveries.

An interesting extension of the model to include a more tactical decision is to optimize the size of the fleet. Therefore, the number of inbound and outbound trucks will also be a decision variable to be fixed for the entire planning horizon. This extension can be applied for separate and mixed route strategies. In case of separate routes, both variants (joint and separate fleets) can be considered. In the first variant, the total size of the fleet as well as the assignment of vehicles to pick-up/delivery routes will be optimized. In the second variant, the fleet size for the pick-up as well as the delivery operations will be optimized. In the case of mixed routes, the total fleet size will be optimized.

Other possibilities for the future research directions on the modeling side include:

- Considering suppliers' setup and production quantity decisions in the ARP.

- Studying the multi-product variant of the ARP in which different final products require (overlapping) subsets of components.

- Studying the multi-depot variant of the ARP.

Furthermore, an interesting avenue for future research on the ARP is to compare different reformulations. Beside the standard formulation for the LSP, it is possible to consider echelon stock, facility location, and shortest path, among others (Pochet and Wolsey, 2006). Available formulations for the VRP (Toth and Vigo, 2014) are standard, single-/two-/multi-commodity formulations as well as path-based formulations. These possibilities result in a large number of promising reformulations for the ARP.

II. Directions for Future Research on the Solution Methodology

The unified method presented in this thesis has the potential to be applied to many variants of the IRP and PRP with customer time windows, transshipment, and perishable products. Furthermore, this framework can be applied to other integrated supply chain problems that consider distribution routing as part of the decision-making process. Examples include the location routing problem, and the hub location routing problem.

References

Absi, N., Archetti, C., Dauzère-Pérès, S., and Feillet, D. (2015). A two-phase iterative heuristic approach for the production routing problem. *Transportation Science*, 49(4):784–795.

Adulyasak, Y., Cordeau, J.-F., and Jans, R. (2014a). Formulations and branch-and-cut algorithms for multivehicle production and inventory routing problems. *INFORMS Journal on Computing*, 26(1):103–120.

Adulyasak, Y., Cordeau, J.-F., and Jans, R. (2014b). Optimization-based adaptive large neighborhood search for the production routing problem. *Transportation Science*, 48(1):20–45.

Adulyasak, Y., Cordeau, J.-F., and Jans, R. (2015). The production routing problem: A review of formulations and solution algorithms. *Computers & Operations Research*, 55:141–152.

Agra, A., Andersson, H., Christiansen, M., and Wolsey, L. (2013). A maritime inventory routing problem: Discrete time formulations and valid inequalities. *Networks*, 62(4):297–314.

Akbalik, A., Penz, B., and Rapine, C. (2015). Capacitated lot sizing problems with inventory bounds. *Annals of Operations Research*, 229(1):1–18.

Andersson, H., Hoff, A., Christiansen, M., Hasle, G., and Løkketangen, A. (2010). Industrial aspects and literature survey: Combined inventory management and routing. *Computers & Operations Research*, 37(9):1515–1536.

Archetti, C., Bertazzi, L., Hertz, A., and Speranza, M. G. (2012). A hybrid heuristic for an inventory routing problem. *INFORMS Journal on Computing*, 24(1):101–116.

Archetti, C., Bertazzi, L., Laporte, G., and Speranza, M. G. (2007). A branch-and-cut algorithm for a vendor-managed inventory-routing problem. *Transportation Science*, 41(3):382–391.

Archetti, C., Bertazzi, L., Paletta, G., and Speranza, M. G. (2011). Analysis of the maximum level policy in a production-distribution system. *Computers & Operations Research*, 38(12):1731–1746.

Archetti, C., Boland, N., and Speranza, M. (2017). A matheuristic for the multi-vehicle inventory routing problem. *INFORMS Journal on Computing*, 29(3):377–387.

Archetti, C., Christiansen, M., and Speranza, M. G. (2018). Inventory routing with pickups and deliveries. *European Journal of Operational Research*, 268(1):314–324.

Archetti, C. and Speranza, M. G. (2016). The inventory routing problem: the value of integration. *International Transactions in Operational Research*, 23(3):393–407.

Armentano, V. A., Shiguemoto, A., and Løkketangen, A. (2011). Tabu search with path relinking for an integrated production–distribution problem. *Computers & Operations Research*, 38(8):1199–1209.

Arntzen, B. C., Brown, G. G., Harrison, T. P., and Trafton, L. L. (1995). Global supply chain management at digital equipment corporation. *Interfaces*, 25(1):69–93.

Ascheuer, N., Fischetti, M., and Grötschel, M. (2000). A polyhedral study of the asymmetric traveling salesman problem with time windows. *Networks*, 36(2):69–79.

Atamtürk, A. and Küçükyavuz, S. (2005). Lot sizing with inventory bounds and fixed costs: Polyhedral study and computation. *Operations Research*, 53(4):711–730.

Atamtürk, A. and Küçükyavuz, S. (2008). An o (n2) algorithm for lot sizing with inventory bounds and fixed costs. *Operations Research Letters*, 36(3):297–299.

Atamtürk, A. and Muñoz, J. C. (2004). A study of the lot-sizing polytope. *Mathematical Programming*, 99(3):443–465.

Augerat, P. (1995). *Approche polyèdrale du problème de tournées de véhicules*. PhD thesis, Institut National Polytechnique de Grenoble-INPG.

Avella, P., Boccia, M., and Wolsey, L. A. (2015). Single-item reformulations for a vendor managed inventory routing problem: Computational experience with benchmark instances. *Networks*, 65(2):129–138.

Avella, P., Boccia, M., and Wolsey, L. A. (2017). Single-period cutting planes for inventory routing problems. *Transportation Science (Articles in Advance)*.

Bae, H., Moon, I., and Yun, W. (2014). Economic lot and supply scheduling problem: a time-varying lot sizes approach. *International Journal of Production Research*, 52(8):2422–2435.

Barany, I., Van Roy, T., and Wolsey, L. A. (1984). Uncapacitated lot-sizing: The convex hull of solutions. In Korte, B. and Ritter, K., editors, *Mathematical programming at Oberwolfach II*, volume 22 of *Mathematical Programming Studies*, pages 32–43. Springer, Berlin, Heidelberg.

Bard, J. F. and Nananukul, N. (2009). The integrated production–inventory–distribution–routing problem. *Journal of Scheduling*, 12(3):257–280.

Bard, J. F. and Nananukul, N. (2010). A branch-and-price algorithm for an integrated production and inventory routing problem. *Computers & Operations Research*, 37(12):2202–2217.

Bektaş, T. and Gouveia, L. (2014). Requiem for the Miller–Tucker–Zemlin subtour elimination constraints? *European Journal of Operational Research*, 236(3):820–832.

Belenguer, J.-M., Martinez, M., and Mota, E. (2000). A lower bound for the split delivery vehicle routing problem. *Operations Research*, 48(5):801–810.

Belo-Filho, M., Amorim, P., and Almada-Lobo, B. (2015). An adaptive large neighbourhood search for the operational integrated production and distribution problem of perishable products. *International Journal of Production Research*, 53(20):6040–6058.

Berman, O. and Wang, Q. (2006). Inbound logistic planning: minimizing transportation and inventory cost. *Transportation Science*, 40(3):287–299.

Bertazzi, L., Savelsbergh, M., and Speranza, M. G. (2008). Inventory routing. In Golden, B. L., Raghavan, S., and Wasil, E. A., editors, *The Vehicle Routing Problem: Latest Advances and New Challenges*, pages 49–72. Springer Science & Business Media.

Blumenfeld, D. E., Burns, L. D., Daganzo, C. F., Frick, M. C., and Hall, R. W. (1987). Reducing logistics costs at General Motors. *Interfaces*, 17(1):26–47.

Boudia, M., Louly, M. A. O., and Prins, C. (2005). Combined optimization of production and distribution. In *Proceedings of the International Conference on Industrial Engineering and Systems Management, IESM*, volume 5.

Boudia, M., Louly, M. A. O., and Prins, C. (2007). A reactive grasp and path relinking for a combined production–distribution problem. *Computers & Operations Research*, 34(11):3402–3419.

Boudia, M. and Prins, C. (2009). A memetic algorithm with dynamic population management for an integrated production–distribution problem. *European Journal of Operational Research*, 195(3):703–715.

Brahimi, N. and Aouam, T. (2016). Multi-item production routing problem with backordering: a MILP approach. *International Journal of Production Research*, 54(4):1076–1093.

Brown, G., Keegan, J., Vigus, B., and Wood, K. (2001). The Kellogg company optimizes production, inventory, and distribution. *Interfaces*, 31(6):1–15.

Carter, J. R. and Ferrin, B. G. (1996). Transportation costs and inventory management: Why transportation costs matter. *Production and Inventory Management Journal*, 37(3):58.

Çetinkaya, S., Üster, H., Easwaran, G., and Keskin, B. B. (2009). An integrated outbound logistics model for Frito-Lay: Coordinating aggregate-level production and distribution decisions. *Interfaces*, 39(5):460–475.

Chandra, P. (1993). A dynamic distribution model with warehouse and customer replenishment requirements. *Journal of the Operational Research Society*, 44(7):681–692.

Chandra, P. and Fisher, M. L. (1994). Coordination of production and distribution planning. *European Journal of Operational Research*, 72(3):503–517.

Chen, Z. and Sarker, B. R. (2014). An integrated optimal inventory lot-sizing and vehicle-routing model for a multisupplier single-assembler system with JIT delivery. *International Journal of Production Research*, 52(17):5086–5114.

Chen, Z.-L. and Vairaktarakis, G. L. (2005). Integrated scheduling of production and distribution operations. *Management Science*, 51(4):614–628.

Chitsaz, M., Cordeau, J.-F., and Jans, R. (2019). A unified decomposition matheuristic for assembly, production and inventory routing. *INFORMS Journal on Computing*, 31(1):134–152.

Chitsaz, M., Cordeau, J.-F., and Jans, R. (2020). A branch-and-cut algorithm for an assembly routing problem. *European Journal of Operational Research*, 282(3):896–910.

Chouman, M., Crainic, T. G., and Gendron, B. (2016). Commodity representations and cut-set-based inequalities for multicommodity capacitated fixed-charge network design. *Transportation Science*, 51(2):650–667.

Christopher, M. (1998). *Logistics and supply chain management: Strategies for reducing cost and improving service*. Prentice-Hall.

Chuah, K. H. and Yingling, J. C. (2005). Routing for a just-in-time supply pickup and delivery system. *Transportation Science*, 39(3):328–339.

Coelho, L. C., Cordeau, J.-F., and Laporte, G. (2012). The inventory-routing problem with transshipment. *Computers & Operations Research*, 39(11):2537–2548.

Coelho, L. C., Cordeau, J.-F., and Laporte, G. (2013). Thirty years of inventory routing. *Transportation Science*, 48(1):1–19.

Coelho, L. C. and Laporte, G. (2013a). A branch-and-cut algorithm for the multi-product multi-vehicle inventory-routing problem. *International Journal of Production Research*, 51(23-24):7156–7169.

Coelho, L. C. and Laporte, G. (2013b). The exact solution of several classes of inventory-routing problems. *Computers & Operations Research*, 40(2):558–565.

Contardo, C., Hemmelmayr, V., and Crainic, T. G. (2012). Lower and upper bounds for the two-echelon capacitated location-routing problem. *Computers & Operations Research*, 39(12):3185–3199.

Cordeau, J.-F., Gendreau, M., and Laporte, G. (1997). A tabu search heuristic for periodic and multi-depot vehicle routing problems. *Networks*, 30(2):105–119.

Cornuejols, G. and Harche, F. (1993). Polyhedral study of the capacitated vehicle routing problem. *Mathematical Programming*, 60(1-3):21–52.

Danese, P. (2006). The extended VMI for coordinating the whole supply network. *Journal of Manufacturing Technology Management*, 17(7):888–907.

Dantzig, G. B., Fulkerson, R., and Johnson, S. (1954). Solution of a large-scale traveling-salesman problem. *Operations Research*, 2(4):393–410.

Dantzig, G. B. and Ramser, J. H. (1959). The truck dispatching problem. *Management Science*, 6(1):80–91.

De Matta, R. and Guignard, M. (1994a). Dynamic production scheduling for a process industry. *Operations Research*, 42(3):492–503.

De Matta, R. and Guignard, M. (1994b). Studying the effects of production loss due to setup in dynamic production scheduling. *European Journal of Operational Research*, 72(1):62–73.

Desaulniers, G., Rakke, J. G., and Coelho, L. C. (2015). A branch-price-and-cut algorithm for the inventory-routing problem. *Transportation Science*, 50(3):1060–1076.

Di Summa, M. and Wolsey, L. A. (2010). Lot-sizing with stock upper bounds and fixed charges. *SIAM Journal on Discrete Mathematics*, 24(3):853–875.

Dong, Z. and Turnquist, M. (2015). Combining service frequency and vehicle routing for managing supplier shipments. *Transportation Research Part E: Logistics and Transportation Review*, 79:231–243.

Fernie, J. and Sparks, L. (2014). *Logistics and Retail Management: Emerging Issues and New Challenges in the Retail Supply Chain*. Kogan Page Publishers.

Fischetti, M. and Lodi, A. (2003). Local branching. *Mathematical Programming*, 98(1-3):23–47.

Fischetti, M., Polo, C., and Scantamburlo, M. (2004). A local branching heuristic for mixed-integer programs with 2-level variables, with an application to a telecommunication network design problem. *Networks*, 44(2):61–72.

Fleischmann, B. and Meyr, H. (2003). Planning hierarchy, modeling and advanced planning systems. *Handbooks in Operations Research and Management Science*, 11:455–523.

Florian, M., Kemper, J., Sihn, W., and Hellingrath, B. (2011). Concept of transport-oriented scheduling for reduction of inbound logistics traffic in the automotive industries. *CIRP Journal of Manufacturing Science and Technology*, 4(3):252–257.

Francis, P., Smilowitz, K., and Tzur, M. (2006). The period vehicle routing problem with service choice. *Transportation Science*, 40(4):439–454.

Fumero, F. and Vercellis, C. (1999). Synchronized development of production, inventory, and distribution schedules. *Transportation Science*, 33(3):330–340.

Gendreau, M., Laporte, G., and Semet, F. (1998). A branch-and-cut algorithm for the undirected selective traveling salesman problem. *Networks*, 32(4):263–273.

Gendron, B., Crainic, T. G., and Frangioni, A. (1999). Multicommodity capacitated network design. In Sansò, B. and Soriano, P., editors, *Telecommunications network planning*, pages 1–19. Springer.

Glover, F., Jones, G., Karney, D., Klingman, D., and Mote, J. (1979). An integrated production, distribution, and inventory planning system. *Interfaces*, 9(5):21–35.

Guide, V. D. R. and Van Wassenhove, L. N. (2009). OR FORUM-The evolution of closed-loop supply chain research. *Operations Research*, 57(1):10–18.

Hahn, C. K., Duplaga, E. A., and Hartley, J. L. (2000). Supply-chain synchronization: lessons from Hyundai Motor Company. *Interfaces*, 30(4):32–45.

Hein, F. and Almeder, C. (2016). Quantitative insights into the integrated supply vehicle routing and production planning problem. *International Journal of Production Economics*, 177:66–76.

Hwang, H.-C. and van den Heuvel, W. (2012). Improved algorithms for a lot-sizing problem with inventory bounds and backlogging. *Naval Research Logistics (NRL)*, 59(3-4):244–253.

Iori, M., Salazar-González, J.-J., and Vigo, D. (2007). An exact approach for the vehicle routing problem with two-dimensional loading constraints. *Transportation Science*, 41(2):253–264.

Jans, R. and Degraeve, Z. (2004). An industrial extension of the discrete lot-sizing and scheduling problem. *IIE transactions*, 36(1):47–58.

Jiang, Z., Huang, Y., and Wang, J. (2010). Routing for the milk-run pickup system in automobile parts supply. In *Proceedings of the 6th CIRP-Sponsored International Conference on Digital Enterprise Technology*, pages 1267–1275. Springer.

Kucukyavuz, S. (2005). *Polyhedral approaches to capacitated fixed-charge network flow problems*. PhD thesis, University of California, Berkeley.

Kuhn, H. and Liske, T. (2011). Simultaneous supply and production planning. *International Journal of Production Research*, 49(13):3795–3813.

Kuhn, H. and Liske, T. (2014). An exact algorithm for solving the economic lot and supply scheduling problem using a power-of-two policy. *Computers & Operations Research*, 51:30–40.

Lamsal, K., Jones, P. C., and Thomas, B. W. (2016). Sugarcane harvest logistics in Brazil. *Transportation Science*, 51(2):771 – 789.

Laporte, G. (1986). Generalized subtour elimination constraints and connectivity constraints. *Journal of the Operational Research Society*, 37(5):509–514.

Laporte, G., Nobert, Y., and Desrochers, M. (1985). Optimal routing under capacity and distance restrictions. *Operations Research*, 33(5):1050–1073.

Le Blanc, H. M., Cruijssen, F., Fleuren, H. A., and De Koster, M. (2006). Factory gate pricing: An analysis of the dutch retail distribution. *European Journal of Operational Research*, 174(3):1950–1967.

Lei, L., Liu, S., Ruszczynski, A., and Park, S. (2006). On the integrated production, inventory, and distribution routing problem. *IIE Transactions*, 38(11):955–970.

Liske, T. and Kuhn, H. (2009). The economic lot and supply scheduling problem under a power-of-two policy. In *Operations Research Proceedings 2008*, pages 215–220. Springer Science & Business Media.

Loparic, M., Marchand, H., and Wolsey, L. A. (2003). Dynamic knapsack sets and capacitated lot-sizing. *Mathematical Programming*, 95(1):53–69.

Love, S. F. (1973). Bounded production and inventory models with piecewise concave costs. *Management Science*, 20(3):313–318.

Lysgaard, J., Letchford, A. N., and Eglese, R. W. (2004). A new branch-and-cut algorithm for the capacitated vehicle routing problem. *Mathematical Programming*, 100(2):423–445.

Marinelli, F., Nenni, M. E., and Sforza, A. (2007). Capacitated lot sizing and scheduling with parallel machines and shared buffers: A case study in a packaging company. *Annals of Operations Research*, 150(1):177–192.

Martin, C. H., Dent, D. C., and Eckhart, J. C. (1993). Integrated production, distribution, and inventory planning at libbey-owens-ford. *Interfaces*, 23(3):68–78.

Miller, A. J., Nemhauser, G. L., and Savelsbergh, M. W. (2000). On the capacitated lot-sizing and continuous 0–1 knapsack polyhedra. *European Journal of Operational Research*, 125(2):298–315.

Miranda, P. L., Cordeau, J.-F., Ferreira, D., Jans, R., and Morabito, R. (2018). A decomposition heuristic for a rich production routing problem. *Computers & Operations Research*, 98:211–230.

Mjirda, A., Jarboui, B., Macedo, R., Hanafi, S., and Mladenović, N. (2014). A two phase variable neighborhood search for the multi-product inventory routing problem. *Computers & Operations Research*, 52:291–299.

Moin, N. H., Salhi, S., and Aziz, N. (2011). An efficient hybrid genetic algorithm for the multi-product multi-period inventory routing problem. *International Journal of Production Economics*, 133(1):334–343.

Natarajarathinam, M., Stacey, J., and Sox, C. (2012). Near-optimal heuristics and managerial insights for the storage constrained, inbound inventory routing problem. *International Journal of Physical Distribution & Logistics Management*, 42(2):152–173.

Neves-Moreira, F., Almada-Lobo, B., Cordeau, J.-F., Guimaraes, L., and Jans, R. (2019). Solving a large multi-product production-routing problem with delivery time windows. *Omega*, 86:154–172.

Ohlmann, J., Fry, M., and Thomas, B. (2007). Route design for lean production systems. *Transportation Science*, 42(2):352–370.

Padberg, M. and Rinaldi, G. (1991). A branch-and-cut algorithm for the resolution of large-scale symmetric traveling salesman problems. *SIAM review*, 33(1):60–100.

Patel, D. and Patel, M. (2013). Design and development of an internal milk-run material supply system in automotive industry. *International Journal of Application or Innovation in Engineering & Management (IJAIEM)*, 2(8):233–235.

Phouratsamay, S.-L., Kedad-Sidhoum, S., and Pascual, F. (2018). Two-level lot-sizing with inventory bounds. *Discrete Optimization*, 30:1–19.

Pochet, Y. (1988). Valid inequalities and separation for capacitated economic lot sizing. *Operations Research Letters*, 7(3):109–115.

Pochet, Y. and Wolsey, L. A. (1994). Polyhedra for lot-sizing with Wagner-Whitin costs. *Mathematical Programming*, 67(1-3):297–323.

Pochet, Y. and Wolsey, L. A. (2006). *Production Planning by Mixed Integer Programming*. Springer Science & Business Media.

Popken, D. A. (1994). An algorithm for the multiattribute, multicommodity flow problem with freight consolidation and inventory costs. *Operations Research*, 42(2):274–286.

Potter, A., Mason, R., and Lalwani, C. (2007). Analysis of factory gate pricing in the U.K. grocery supply chain. *International Journal of Retail & Distribution Management*, 35(10):821–834.

Qiu, Y., Ni, M., Wang, L., Li, Q., Fang, X., and Pardalos, P. M. (2018a). Production routing problems with reverse logistics and remanufacturing. *Transportation Research Part E: Logistics and Transportation Review*, 111:87–100.

Qiu, Y., Wang, L., Xu, X., Fang, X., and Pardalos, P. M. (2018b). Formulations and branch-and-cut algorithms for multi-product multi-vehicle production routing problems with startup cost. *Expert Systems with Applications*, 98:1–10.

Qu, W. W., Bookbinder, J. H., and Iyogun, P. (1999). An integrated inventory–transportation system with modified periodic policy for multiple products. *European Journal of Operational Research*, 115(2):254–269.

Ralphs, T. K., Kopman, L., Pulleyblank, W. R., and Trotter, L. E. (2003). On the capacitated vehicle routing problem. *Mathematical Programming*, 94(2):343–359.

Rudberg, M. and Cederborg, O. (2011). Aps for tactical planning in a steel processing company. *Industrial Management & Data Systems*, 111(4):608–628.

Satoglu, S. and Sahin, I. (2013). Design of a just-in-time periodic material supply system for the assembly lines and an application in electronics industry. *The International Journal of Advanced Manufacturing Technology*, 65(1-4):319–332.

Semet, F., Toth, P., and Vigo, D. (2014). Classical exact algorithms for the capacitated vehicle routing problem. In Toth, P. and Vigo, D., editors, *Vehicle routing: problems, methods, and applications*, pages 37–57. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA.

Silva, C. and Magalhaes, J. M. (2006). Heuristic lot size scheduling on unrelated parallel machines with applications in the textile industry. *Computers & Industrial Engineering*, 50(1-2):76–89.

Simchi-Levi, D., Chen, X., and Bramel, J. (2013). *The logic of logistics: theory, algorithms, and applications for logistics management*. Springer Science & Business Media.

Sindhuchao, S., Romeijn, H. E., Akçali, E., and Boondiskulchok, R. (2005). An integrated inventory-routing system for multi-item joint replenishment with limited vehicle capacity. *Journal of Global Optimization*, 32(1):93–118.

Solyalı, O. and Süral, H. (2011). A branch-and-cut algorithm using a strong formulation and an a priori tour-based heuristic for an inventory-routing problem. *Transportation Science*, 45(3):335–345.

Solyalı, O. and Süral, H. (2017). A multi-phase heuristic for the production routing problem. *Computers & Operations Research*, 87:114–124.

Solyalı, O., Süral, H., Neogy, S., Das, A., and Bapat, R. (2009). A relaxation based solution approach for the inventory control and vehicle routing problem in vendor managed systems. *Modeling, Computation and Optimization, World Scientific, Singapore*, pages 171–189.

Stacey, J., Natarajarathinam, M., and Sox, C. (2007). The storage constrained, inbound inventory routing problem. *International Journal of Physical Distribution & Logistics Management*, 37(6):484–500.

Steinrücke, M. (2011). An approach to integrate production-transportation planning and scheduling in an aluminium supply chain network. *International Journal of Production Research*, 49(21):6559–6583.

Toth, P. and Vigo, D. (2002). An overview of vehicle routing problems. In Toth, P. and Vigo, D., editors, *The Vehicle Routing Problem*, pages 1–26. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, USA.

Toth, P. and Vigo, D. (2014). *Vehicle routing: problems, methods, and applications*. SIAM.

Trigeiro, W. W., Thomas, L. J., and McClain, J. O. (1989). Capacitated lot sizing with setup times. *Management Science*, 35(3):353–366.

Vaidyanathan, B., Matson, J., Miller, D., and Matson, J. (1999). A capacitated vehicle routing problem for just-in-time delivery. *IIE Transactions*, 31(11):1083–1092.

Van Den Heuvel, W. and Wagelmans, A. P. (2008). Four equivalent lot-sizing models. *Operations Research Letters*, 36(4):465–470.

Van Roy, T. J. and Wolsey, L. A. (1985). Valid inequalities and separation for uncapacitated fixed charge networks. *Operations Research Letters*, 4(3):105–112.

Viswanathan, S. and Mathur, K. (1997). Integrating routing and inventory decisions in one-warehouse multiretailer multiproduct distribution systems. *Management Science*, 43(3):294–312.

Wagner, H. M. and Whitin, T. M. (1958). Dynamic version of the economic lot size model. *Management Science*, 5(1):89–96.

Whiteoak, P. (1994). The realities of quick response in the grocery sector: a supplier viewpoint. *International Journal of Physical Distribution & Logistics Management*, 24(10):33–39.

Yücel, E., Salman, F. S., Gel, E. S., Örmeci, E. L., and Gel, A. (2013). Optimizing specimen collection for processing in clinical testing laboratories. *European Journal of Operational Research*, 227(3):503–514.

Zhang, M., Küçükyavuz, S., and Yaman, H. (2012). A polyhedral study of multiechelon lot sizing with intermediate demands. *Operations Research*, 60(4):918–935.

Zhang, Q., Sundaramoorthy, A., Grossmann, I. E., and Pinto, J. M. (2017). Multiscale production routing in multicommodity supply chains with complex production facilities. *Computers & Operations Research*, 79:207–222.

Appendix A – A Unified Decomposition Matheuristic for Assembly, Production and Inventory Routing

Overview of Problem Data Sets

We test our algorithm on three different problems: the IRP, the PRP and the ARP. Each problem has its own set of different instances. IRP instances include four data sets. The first set (SV-I1) is provided by Archetti et al. (2007) and contains a total of 160 single-vehicle IRP instances. This data set includes instances with 5 to 50 nodes and 3 periods (100 instances) and instances with 5 to 30 nodes and 6 periods (60 instances). Coelho and Laporte (2013a) and Desaulniers et al. (2015) further adapt the SV-I1 data set and construct new instances (second data set) by dividing the fleet capacity equally between the number of vehicles. They consider $m = 2, 3, 4$ and 5 vehicles for each SV-I1 instance and develop four new multi-vehicle IRP instances (MV-I1). Dividing the vehicle capacity by 5 made two of the instances infeasible. Therefore, instead of 640 they have 638 instances in this set. The third IRP data set (SV-I2) includes bigger single-vehicle instances presented by Archetti et al. (2012). This data set includes 60 instances with 6 periods and 50, 100 and 200 nodes (20 instances for each). Similar to the second data set, Coelho and Laporte (2013b) adapt the SV-IRP instances of the third set and developed the fourth multi-vehicle

IRP data set (MV-I2) which includes 240 instance. Therefore, we consider a total of 1,098 instances in four IRP data sets.

PRP instances include six data sets. Archetti et al. (2011) and Boudia et al. (2005) each introduce three data sets. Each set of Archetti et al. (2011) has 480 instances including four classes of randomly generated problem instances. The first set provides single-vehicle PRP instances (SV-A1). The other two sets include multi-vehicle instances (MV-A2 and MV-A3). Sets SV-A1, MV-A2 and MV-A3 have 14, 50, 100 customers, respectively, each with 6 periods. Each of these sets has four classes of instances. The first class includes the normal or standard instances. The second class contains high production unit and setup cost instances. The third class represents the case with high transportation costs (by multiplying the customer coordinates of the first class by a factor of 5). Finally, the fourth class includes instances with no retailer inventory costs. Boudia et al. (2005) present 30 test instances in each of their three sets: sets MV-B1, MV-B2 and MV-B3 which have 50, 100 and 200 customers, respectively, each with 20 periods. Accordingly, the total number of instances in these six PRP data sets is 1,530.

We adapt the ARP instances from the PRP data sets of Archetti et al. (2011). ARP instances include three data sets (SV-C1, MV-C2 and MV-C3) with a total number of 1,440 instances. Consequently, we are solving a total of 4,068 instances of the IRP, PRP and ARP problems. Table A.1 provides an overview of the main characteristics of these instances.

Parameter Setting

We chose a random but varied subset of instances from the entire test bed of problems to calibrate the parameters of the algorithm. For the IRP data sets of Archetti et al. (2007) and Archetti et al. (2012), with a total number of 1,098 instances, we randomly chose two instances for each combination of the fleet size (m), period (l) and inventory cost level (h) which resulted in a total of 60 instances: 40 instances from Archetti et al. (2007) and 20 instances from Archetti et al. (2012). From the PRP data sets of Archetti et al. (2011), we randomly selected four instances from each class of instances, resulting in 16 instances from each data set and a total of 48 instances. From the PRP data sets of Boudia et al. (2005), we randomly chose four instances from MV-B1 and MV-B2, resulting in 8

Table A.1: Overview of the benchmark data sets for the IRP, PRP and ARP

Problem	Reference	Set name	Size	l	n	m	d_i	C	L_0	L_i	I_0	I_i	Q
IRP	Archetti et al. (2007)	SV-I1	100	3	5 to 50	1	C	-	UL	C	V	V	C
			60	6	5 to 50	1	C	-	UL	C	V	V	C
		MV-I1	100	3	5 to 50	2	C	-	UL	C	V	V	C
			60	6	5 to 50	2	C	-	UL	C	V	V	C
			100	3	5 to 50	3	C	-	UL	C	V	V	C
			60	6	5 to 50	3	C	-	UL	C	V	V	C
			100	3	5 to 50	4	C	-	UL	C	V	V	C
			60	6	5 to 50	4	C	-	UL	C	V	V	C
			100	3	5 to 50	5	C	-	UL	C	V	V	C
			58	6	5 to 50	5	C	-	UL	C	V	V	C
	Archetti et al. (2012)	SV-I2	60	6	50 to 200	1	C	-	UL	C	V	V	C
		MV-I2	60	6	50 to 200	2	C	-	UL	C	V	V	C
			60	6	50 to 200	3	C	-	UL	C	V	V	C
			60	6	50 to 200	4	C	-	UL	C	V	V	C
			60	6	50 to 200	5	C	-	UL	C	V	V	C
PRP	Archetti et al. (2011)	SV-A1	480	6	14	1	C	UL	UL	C	0	V	C
		MV-A2	480	6	50	UL	C	UL	UL	C	0	V	C
		MV-A3	480	6	100	UL	C	UL	UL	C	0	V	C
	Boudia et al. (2005)	MV-B1	30	20	50	5	V	C	C	C	V	0	C
		MV-B2	30	20	100	9	V	C	C	C	V	0	C
		MV-B3	30	20	200	13	V	C	C	C	V	0	C
ARP	This paper	SV-C1	480	6	14	1	C	UL	UL	C	V	V	C
		MV-C2	480	6	50	UL	C	UL	UL	C	V	V	C
		MV-C3	480	6	100	UL	C	UL	UL	C	V	V	C
		Total	4,068										

Note. C: Constant/Capacitated, MV: Multi-vehicle, SV: Single-vehicle, UL: Unlimited, V: Varying.

instances. No instances from the MV-B3 data set were chosen since they require long computing times. From the ARP data sets, we randomly selected four instances from each class of instances, resulting in 48 instances. Therefore, we perform the parameter setting experiments on 164 instances.

The most important algorithmic parameters to set are the maximum number of iterations for the algorithm, ι^A, and the tabu search iterations coefficient, ι^V_{min} and ι^V_{max}, for the solution of the VRP_t subproblems. The rest of the parameters are the maximum number of local optimum iterations, ι^L, the maximum number of iterations without incumbent solution improvement, ι^N, the number of consecutive iterations for which the same cost update mechanism is applied, ι^U, the maximum number of $\mathcal{M}_z^{\mathcal{R}}$ subproblem iterations, $\iota^{\mathcal{R}}$, the right-hand-side of the LBI_z inequalities, r, the reduction in the aggregate fleet capacity in constraints (1.21), $1 - \lambda_t$, the gap between the solution obtained using the $\mathcal{M}_z$

subproblem and the incumbent solution, g, and the gap between the solution obtained using the $\mathcal{M}_z^{\mathcal{R}}$ subproblem and incumbent solution, $g^{\mathcal{R}}$.

We perform an extensive study on the parameter setting and arrive at the values in Table A.2. Then, we design a sensitivity analysis to make sure that the selected values are the right choice for our algorithm. Obviously, when ι^A, ι^V_{min} and ι^V_{max} increase we obtain better results (see Tables A.4, A.5 and A.6). But since the same parameter setting is used for all the problems and data sets, we have an implicit limit on the number of iterations in order to spend an acceptable computing time compared to other benchmark algorithms. Our observation indicates that the algorithm has an acceptable performance with small changes in ι^s, $\iota^{\mathcal{R}}$, g and $g^{\mathcal{R}}$ while the current setting for these four parameters helps us to reduce the necessary computing time. Also, we noticed that the best $1-\lambda_t$ varies among different IRP and PRP data sets. The last column of Table A.2 contains the ranges of the sensitivity analyses on the parameter values.

Table A.2: Parameter setting for the algorithm applied to all problems and data sets

Par	Description	Selected value	Other values for the sensitivity analysis
ι^A	Max # of algorithm iterations	200	100, 150, 250, 300
ι^V_{min}	Minimum tabu search iterations coefficient	100	50, 200
ι^V_{max}	Maximum tabu search iterations coefficient	500	400, 600
ι^L	Max # of local optimum iterations	80 (0.4*ι^A)	60 (0.3*ι^A), 100 (0.5*ι^A)
ι^N	Max # of non-improving iterations	60 (0.3*ι^A)	40 (0.2*ι^A), 80 (0.4*ι^A)
ι^s	# of iterations before $\mathcal{M}_z^{\mathcal{R}}$ subproblem can be used	5	4, 6
$\iota^{\mathcal{R}}$	Max # of $\mathcal{M}_z^{\mathcal{R}}$ subproblem iterations	10	5, 15
ι^U	# of consecutive iterations to apply each mechanism	7	5, 6, 8, 9
$1-\lambda_t$	Aggregate fleet capacity reduction amount†	$2/n$	$1/n, 3/n, 4/n$
g	Gap of $\mathcal{S}$ obtained using $\mathcal{M}_z$ subproblem and $\mathcal{S}^*$	3%	2%, 4%
$g^{\mathcal{R}}$	Gap of $\mathcal{S}$ obtained using $\mathcal{M}_z^{\mathcal{R}}$ subproblem and $\mathcal{S}^*$	0.5%	0.3%, 0.4%, 0.6%

Note. Max: Maximum, Par: Parameter, $\mathcal{S}$: Current solution, $\mathcal{S}^*$: Incumbent solution
† Up to a maximum of 25%

We used the following CPLEX setting for all problems and data sets to solve the $\mathcal{M}_y$, $\mathcal{M}_z$ and $\mathcal{M}_z^{\mathcal{R}}$ subproblems. We used CPLEX with one thread in all of our experiments. We disable all the CPLEX MIP cuts except *FlowCovers* and *Gomory*. We set the *AdvInd* parameter to zero to prevent CPLEX from spending time to recover the previous iteration's search tree with its built-in heuristic. The rest of the CPLEX settings follow the strategy of getting quality upper bounds faster rather than closing the optimality gap when solving the $\mathcal{M}_y, \mathcal{M}_z$ and $\mathcal{M}_z^{\mathcal{R}}$ subproblems. We set *Dive* to 2 (probing dive for the MIP dive strategy), *OrderType* to 1 (to use decreasing costs for the MIP priority order generation in the search tree), *CoeffReduce* to 1 (to reduce only to integral coefficients when the coefficient reduction is used by CPLEX), *DGradient* to 4 (steepest-edge pricing with unit initial norms for the dual simplex pricing algorithm) and *MIP Emphasis* to 1 (to emphasize feasibility over optimality in the search tree). These allow us to terminate CPLEX sooner and execute more iterations. We set a maximum time limit of 40 seconds for CPLEX when solving the $\mathcal{M}_y, \mathcal{M}_z$ and $\mathcal{M}_z^{\mathcal{R}}$ subproblems.

Subproblems for the PRP and IRP

In this section, we redefine the variables, objective, and constraints of our formulation to match the outbound PRP and IRP models in the literature. In PRP and IRP, the set of nodes $N_s = \{1, ..., n\}$, indexed by $i \in N_s$, represents the customers, $i = 0$ represents the plant and $N = N_s \cup \{0\}$ is the set of all nodes. Let K_i denote the storage capacity and F_{it} represent the inventory of the product (at the end of period t) at customer $i \in N_s$ and at the plant for $i = 0$. Let d_{itl} be the total demand of customer/retailer i from period t to the end of planning period l. The rest of the parameters and variables have a similar definition as in the ARP. The $\mathcal{M}_y$ subproblem for the PRP is defined as follows:

$$\min \sum_{t \in T} \left(up_t + fy_t + \sum_{i \in N} h_i F_{it} + \sigma_{0t} z_{0t} \right) \tag{A.1}$$

s.t.

$$F_{0,t-1} + p_t = \sum_{i \in N_s} q_{it} + F_{0t} \quad \forall t \in T \tag{A.2}$$

$$F_{i,t-1} + q_{it} = d_{it} + F_{it} \quad \forall i \in N_s, \forall t \in T \tag{A.3}$$

$$p_t \leq min\{C, \sum_{i \in N_s} d_{itl}\} y_t \quad \forall t \in T \tag{A.4}$$

$$q_{it} \leq min\{K_i, Q, d_{itl}\} z_{it} \quad \forall i \in N_s, \forall t \in T \tag{A.5}$$

$$\sum_{i \in N_s} q_{it} \leq Q z_{0t} \quad \forall t \in T \tag{A.6}$$

$$z_{0t} \leq m \quad \forall t \in T \tag{A.7}$$

$$F_{it} \leq K_i \ \ \forall i \in N, \forall t \in T \tag{A.8}$$

$$p_t \geq 0, y_t \in \{0,1\}, z_{0t} \in \mathbb{Z} \ \ \forall t \in T \tag{A.9}$$

$$F_{it} \geq 0 \ \ \forall i \in N \tag{A.10}$$

$$q_{it} \geq 0, z_{it} \in \{0,1\} \ \ \forall i \in N_s, \forall t \in T. \tag{A.11}$$

The objective function (A.1) minimizes the total production, setup, and inventory costs both at the plant and customers together with the vehicle dispatch cost. Constraints (A.2) and (A.3) ensure the inventory flow at the plant and at the customers, respectively. Constraints (A.4) and (A.5) force setup costs at the plant and vehicle visits to the customers, respectively. They also impose limits on production and shipment quantities. Constraints (A.6) are equivalent to constraints (1.19) for the ARP. Constraints (A.7) and (A.8) enforce the fleet size and storage limits at the plant and customers. The $\mathcal{M}_z$ subproblem for the PRP is to minimize the following objective function:

$$\min \sum_{t \in T} \left(up_t + fy_t + \sum_{i \in N} h_i F_{it} + \sum_{i \in N_s} \sigma_{it} z_{it} \right), \tag{A.12}$$

subject to constraints (A.2)-(A.5), (A.7)-(A.11) and (A.13):

$$\sum_{i \in N_s} q_{it} \leq \lambda_t m Q \ \ \forall t \in T. \tag{A.13}$$

Constraints (A.13) are the equivalent of constraints (1.21) for the PRP. Having the binary decisions y_t fixed from the solution of the $\mathcal{M}_y$ subproblem, they become constants in constraints (A.4) for the $\mathcal{M}_z$ subproblem. In the $\mathcal{M}_z$ subproblem for IRP, p_t is a parameter that makes the constraints (A.4) not applicable. The objective function of the $\mathcal{M}_z$ subproblem will then be:

$$\min \sum_{t \in T} \sum_{i \in N} (h_i F_{it} + \sigma_{it} z_{it}). \tag{A.14}$$

To comply with the replenishment process timing assumption in Archetti et al. (2007) and Archetti et al. (2011), Adulyasak et al. (2014) suggested constraints (A.15) that should be added to the $\mathcal{M}_z$ subproblem for IRP. Constraints (A.16) are equivalent to the original assumption (Archetti et al., 2007, 2011), $F_{i,t-1} + q_{it} \leq K_i$, and can be obtained by replacing the LHS from constraints (A.3). The reason for this modification is to impose them as bounds on the inventory variables rather than adding constraints to the model.

$$F_{0t} \geq p_t \quad \forall t \in T \tag{A.15}$$

$$F_{it} \leq K_i - d_{it} \quad \forall t \in T \tag{A.16}$$

Moreover, the fixed cost $\sum_{i \in N} h_i F_{i0}$ has to be added to the final solution value for the IRP instances, since Archetti et al. (2007) and Archetti et al. (2011) consider the inventory costs at the beginning of the period starting from period zero.

Upper Bound on the Number of Vehicles

We present an analysis of the upper bound on the number of necessary vehicles (Section 1.4.3). We first explain a naive procedure to merge two routes with a total load less than or equal to the vehicle capacity in the capacitated vehicle routing problem. Next, we use this in our proof of the upper bound.

Under the assumption that the triangle inequality holds, and given a solution to the capacitated vehicle routing problem, two routes with a load less than or equal to half of the vehicle capacity can be merged to obtain a new solution with a smaller or equal cost. It is sufficient to remove exactly one of the edges incident to the plant (depot) from each route and connect the resulting partial routes using exactly one new edge. This results in a solution with a smaller or equal cost, while still satisfying the vehicle capacity. Note that although this procedure gives a shorter route compared to total of the original two, it does not necessarily produce the optimal route.

Proposition A.1. *Under the assumption that the triangle inequality holds, in a feasible instance of the capacitated vehicle routing problem with the set of nodes N_s to be visited, shipments q_i, $i \in N_s$, and route capacity Q, there exists an optimal solution with a number of routes smaller than or equal to* $\max\{1, \lceil \frac{2}{Q} \sum_{i \in N_s} q_i \rceil - 1\}$.

Proof. Proof. Consider a feasible solution with $m \geq 2$ routes, and let $k = \frac{\sum_{i \in N_s} q_i}{Q}$ and Q^j denote the load in route $j \in \{1, ..., m\}$. If $m \geq \lceil 2k \rceil$, we show that there exists a better or equivalent solution with $m-1$ routes. Let j_1 and j_2 be the two routes with the smallest loads among all routes. We will prove, by contradiction, that $Q^{j_1} + Q^{j_2} \leq Q$, in which case we can merge routes j_1 and j_2, and arrive at a total number of routes equal to $m-1$. If $m-1 \geq \lceil 2k \rceil$, we repeat this route reduction procedure until we have a solution with $\lceil 2k \rceil - 1$ routes. Because this holds for any feasible solution, the number of routes in an optimal solution cannot exceed $\max\{1, \lceil 2k \rceil - 1\}$.

The proof, by contradiction, that $Q^{j_1} + Q^{j_2} \leq Q$, is as follows. Suppose that $Q^{j_1} + Q^{j_2} > Q$, then the larger load among Q^{j_1} and Q^{j_2} must be strictly larger than $Q/2$. We then have:

$$Q^{j_1} + Q^{j_2} + \sum_{j \in \{1,\ldots,m\} | j \neq j_1, j \neq j_2} Q^j > Q + (m-2)\frac{Q}{2} = m\frac{Q}{2} \geq \lceil 2k \rceil \frac{Q}{2} \geq kQ = \sum_{i \in N_s} q_i,$$

which is a contradiction. The first inequality is valid because for j_1 and j_2, $Q^{j_1} + Q^{j_2} > Q$, and each of the remaining $m-2$ routes ($j \in \{1,\ldots,m\} | j \neq j_1, j \neq j_2$) has a load greater than or equal to j_1 and j_2 by assumption. The next expression is obtained by algebraic manipulation. The second inequality is valid based on the assumption that $m \geq \lceil 2k \rceil$. The next expression is trivial because $\lceil 2k \rceil \frac{Q}{2} \geq 2k\frac{Q}{2} = kQ$. The last expression is valid based on the definition of k. This leads to the contradiction that the sum of the route loads (first term) is strictly greater than the total shipments (last term). Therefore, the sum of the two smallest loaded routes (j_1 and j_2) cannot be strictly greater than the route capacity. □

Table A.3 presents the effect of implementing this upper bound on the number of vehicles when CCJ-DH is applied on the multi-vehicle IRP and PRP instances. The results show that for the data sets with few available vehicles (MV-I1 and MV-I2) the time saving of applying this bound is negligible. On the instances with an unlimited number of vehicles, the time saving factor is about 3 (for MV-A2 data set with $n = 50$) to 4 (for MV-A3 data set with $n = 100$). The average gaps and number of BUBs are almost the same except for the MV-B3.

Further Analysis of the Algorithm

In addition to the 200 iterations that we fix for CCJ-DH for all problems and data sets as reported in the main paper, we let it run for $\iota^A = \{100, 150, 250, 300\}$ with different

Table A.3: Effect of valid upper bound on the number of vehicles on the algorithm's performance when applied to multi-vehicle IRP and PRP instances

					Gap (%)		# BUB		# NBS		Time (sec)	
Prob	Set	m	Class	Size	TB†	NB‡	TB†	NB‡	TB†	NB‡	TB†	NB‡
IRP	MV-I1	2	-	160	**2.81**	2.89	**24**	22	0	0	39	**38**
		3	-	160	2.54	**2.53**	31	**33**	**14**	13	**37**	**37**
		4	-	160	**2.89**	2.95	**27**	26	**21**	20	**37**	39
		5	-	158	**3.01**	3.09	**33**	29	**27**	25	**36**	36
	MV-I2	2	-	60	**-1.59**	-1.5	**40**	39	**40**	39	2781	**2761**
		3	-	60	**-2.55**	-2.53	**44**	**44**	**44**	**44**	**2120**	2282
		4	-	60	-4.48	**-4.49**	**50**	49	**50**	49	**2780**	2976
		5	-	60	**-4.65**	-4.41	**57**	55	**57**	55	**3232**	3233
PRP	MV-A2	UL	1	120	**-0.05**	-0.03	**101**	94	**98**	93	**201**	643
		UL	2	120	**0.02**	**0.02**	**39**	33	**39**	33	**170**	562
		UL	3	120	**-0.1**	**-0.1**	87	**90**	83	**87**	**155**	480
		UL	4	120	**-0.02**	**-0.02**	71	**77**	71	**77**	**215**	655
	MV-A3	UL	1	120	0.21	**0.18**	73	**76**	73	**76**	**1103**	5201
		UL	2	120	**0.18**	**0.18**	39	**44**	39	**44**	**961**	4441
		UL	3	120	**1.15**	1.18	**26**	**26**	**26**	**26**	**729**	3333
		UL	4	120	**0.11**	**0.11**	47	**52**	46	**52**	**1112**	4518
	MV-B1	5	-	30	0.89	**0.86**	**1**	**1**	**1**	**1**	**1825**	1980
	MV-B2	9	-	30	1.28	**1.26**	0	0	0	0	**5368**	6911
	MV-B3	13	-	30	0.22	**0.05**	12	**18**	12	**18**	**8344**	15750

Note. Other data sets include only single-vehicle instances and no tighter bound is applicable.
† Tight bound (Eq. 1.23) is applied.
‡ No bound is applied.

starting node visit costs. Tables A.4-A.7 report the average gap (%), number of BUBs, number of NBS, and computing time (seconds). Moreover, we examined the effect of employing each of the three update mechanisms separately and present in the same tables the results for $\iota^A = \{200, 250\}$ iterations. The results indicate that the mixed mechanism works better than each of the cost update mechanisms separately. The exception is on the Boudia et al. (2005) data sets for which the marginal cost update mechanism outperforms the other mechanisms. CCJ-DJ is successful to find average gaps less than 0 or in other words it outperforms the state-of-the-art algorithm (ABS-H) on the MV-I2 data set in all scenarios. On the MV-A2 data set the algorithm with 100 iterations performs almost the same or better than all the previous benchmark algorithms with different starting node visit costs. The VRP route cost update mechanism leads to substantially bigger average gaps while it still is competitive on the large-scale multi-vehicle MV-I2 instances compared to the previous state-of-the-art heuristics. Overall, different CCJ-DH scenario implementations return 1257 BUBs among which 973 are NBSs on all data sets.

Table A.4: Average gaps by different cost update mechanisms and initial node visit costs for IRP and PRP data sets (%)

					Mixed M† and T† Mechanisms															M†		T†		V†		
					$c_{0i}^{\ddagger} = c_{0i}/2$					$c_{0i}^{\ddagger} = c_{0i}$					$c_{0i}^{\ddagger} = 2c_{0i}$							$c_{0i}^{\ddagger} = c_{0i}/2$				
Prob	Set	m	Class	Size	100	150	200	250	300	100	150	200	250	300	100	150	200	250	300	200	250	200	250	200	250	OB††
IRP	SV-I1	1	-	160	1.66	1.61	1.62	1.59	1.57	1.61	1.57	1.57	1.56	1.53	1.65	1.6	1.61	1.58	1.56	3.22	2.92	2.08	2	5.3	5.16	1.39
	MV-I1	2	-	160	2.81	2.68	2.6	2.6	2.52	2.66	2.54	2.46	2.51	2.43	2.73	2.68	2.59	2.61	2.5	4.06	3.61	2.82	2.76	6.26	6.05	1.8
		3	-	160	2.54	2.51	2.4	2.35	2.24	2.44	2.44	2.36	2.3	2.19	2.49	2.45	2.42	2.33	2.21	3.73	3.62	2.77	2.68	6.97	6.49	1.67
		4	-	160	2.89	2.71	2.55	2.51	2.51	2.88	2.71	2.56	2.52	2.47	2.89	2.72	2.5	2.48	2.39	4.45	4.14	2.83	2.83	7.05	6.85	1.87
		5	-	158	3.01	2.91	2.75	2.68	2.56	2.89	2.8	2.72	2.58	2.49	2.99	2.92	2.73	2.65	2.6	4.66	4.57	3.01	3.01	7.58	7.48	2.05
	SV-I2	1	-	60	3.71	3.55	3.51	3.46	3.39	3.67	3.58	3.49	3.38	3.38	3.61	3.57	3.49	3.4	3.43	4.82	4.54	4.79	4.74	4.86	4.78	2.9
	MV-I2	2	-	60	-1.59	-1.68	-1.82	-2.05	-1.94	-1.6	-1.69	-1.74	-1.83	-1.97	-1.52	-1.78	-1.88	-1.75	-1.95	-0.61	-0.8	-0.68	-0.82	-0.11	-0.22	-2.67
		3	-	60	-2.55	-2.72	-2.86	-2.95	-3	-2.64	-2.91	-2.8	-3.04	-2.93	-2.24	-2.67	-2.91	-3	-2.99	-2.56	-2.75	-1.77	-1.83	-1.29	-1.5	-3.92
		4	-	60	-4.48	-4.57	-4.73	-4.73	-4.85	-4.45	-4.7	-4.77	-4.81	-4.78	-4.41	-4.52	-4.6	-4.74	-4.74	-3.82	-3.9	-3.69	-3.84	-3.12	-3.07	-5.63
		5	-	60	-4.65	-4.69	-4.9	-4.85	-4.86	-4.56	-4.46	-4.64	-4.7	-4.77	-4.42	-4.76	-4.84	-4.98	-5.04	-4.2	-4.26	-4.4	-4.5	-3.05	-3.18	-5.78
PRP	SV-A1	1	1	120	0.35	0.28	0.24	0.2	0.15	0.34	0.29	0.25	0.22	0.17	0.35	0.29	0.26	0.22	0.16	0.39	0.38	0.5	0.49	0.95	0.96	0.11
		1	2	120	0.06	0.04	0.03	0.03	0.03	0.06	0.04	0.04	0.03	0.03	0.06	0.04	0.04	0.03	0.03	0.07	0.06	0.08	0.08	0.16	0.15	0.01
		1	3	120	1.82	1.61	1.49	1.3	1.21	1.87	1.69	1.52	1.29	1.2	1.84	1.69	1.48	1.25	1.17	2.03	1.93	2.41	2.32	4.66	4.22	0.61
		1	4	120	0.18	0.15	0.13	0.14	0.08	0.18	0.16	0.14	0.15	0.09	0.18	0.15	0.14	0.14	0.08	0.24	0.2	0.31	0.3	0.52	0.5	0.03
	MV-A2	UL	1	120	-0.05	-0.05	-0.06	-0.05	-0.06	-0.05	-0.05	-0.05	-0.06	-0.06	-0.04	-0.04	-0.05	-0.05	-0.05	0.04	0.03	0.1	0.1	0.32	0.31	-0.09
		UL	2	120	0.02	0.02	0.01	0.01	0.01	0.02	0.02	0.02	0.01	0.01	0.03	0.02	0.02	0.01	0.01	0.02	0.02	0.06	0.06	0.11	0.11	-0.01
		UL	3	120	-0.1	-0.17	-0.2	-0.24	-0.3	-0.17	-0.22	-0.25	-0.28	-0.29	-0.11	-0.2	-0.26	-0.27	-0.28	0.27	0.19	-0.06	-0.11	1.11	1	-0.41
		UL	4	120	-0.02	-0.02	-0.03	-0.03	-0.04	-0.02	-0.02	-0.03	-0.03	-0.04	-0.01	-0.03	-0.02	-0.03	-0.03	-0.03	-0.03	0.02	0.01	0.11	0.11	-0.07
	MV-A3	UL	1	120	0.21	0.2	0.18	0.17	0.18	0.19	0.19	0.17	0.17	0.17	0.19	0.18	0.18	0.17	0.17	0.23	0.23	0.53	0.5	0.58	0.61	0.13
		UL	2	120	0.18	0.18	0.18	0.18	0.18	0.18	0.18	0.18	0.18	0.18	0.18	0.18	0.18	0.18	0.18	0.19	0.18	0.21	0.21	0.23	0.23	0.16
		UL	3	120	1.15	1.11	1.07	1.02	0.96	1.26	1.26	1.11	1.02	1.02	1.23	1.14	1.09	1.03	1.02	1.46	1.47	1.64	1.6	2.26	2.15	0.79
		UL	4	120	0.11	0.11	0.11	0.1	0.1	0.11	0.11	0.11	0.1	0.1	0.11	0.11	0.11	0.1	0.1	0.1	0.1	0.13	0.13	0.17	0.16	0.08
	MV-B1	5	-	30	0.89	0.8	0.78	0.77	0.74	0.96	0.92	0.89	0.84	0.83	1.46	1.26	1.34	1.31	1.28	0.74	0.71	1.78	1.76	2.07	2.1	0.5
	MV-B2	9	-	30	1.28	1.22	1.21	1.14	1.12	1.33	1.33	1.32	1.24	1.19	1.46	1.31	1.28	1.22	1.21	1.08	1.04	1.93	1.94	2.32	2.28	0.85
	MV-B3	13	-	30	0.22	0.21	0.18	0.16	0.17	0.27	0.22	0.2	0.19	0.2	0.26	0.19	0.17	0.18	0.15	-0.04	-0.05	1.91	1.91	2.19	2.13	-0.1

Note. The best average gap at each row is presented with the bold font.

† Transportation cost update mechanisms. M: Marginal, T: TSP route cost share, V: VRP route cost share. ‡ Initial values for node visit cost.

†† Overall best obtained solution for each instance is taken into account.

Table A.5: Number of BUBs by different cost update mechanisms and initial node visit costs for IRP and PRP data sets

					Mixed M† and T† Mechanisms															M†		T†		V†		
					$c_{0i}^{\ddagger} = c_{0i}/2$					$c_{0i}^{\ddagger} = c_{0i}$					$c_{0i}^{\ddagger} = 2c_{0i}$							$c_{0i}^{\ddagger} = c_{0i}/2$				
Prob	Set	m	Class	Size	100	150	200	250	300	100	150	200	250	300	100	150	200	250	300	200	250	200	250	200	250	OB††
IRP	SV-I1	1	-	160	30	33	31	31	31	32	33	33	30	33	30	33	32	32	33	30	32	28	28	14	16	39
	MV-I1	2	-	160	24	27	28	28	29	23	26	27	27	27	25	26	26	26	28	16	16	20	21	10	11	35
		3	-	160	31	32	31	31	34	32	32	33	32	34	32	31	31	31	36	22	25	23	24	6	7	42
		4	-	160	27	29	29	33	31	26	28	29	32	30	28	27	31	34	34	15	18	26	25	9	9	39
		5	-	158	33	34	38	38	38	33	36	36	39	39	34	33	37	38	37	24	26	30	29	11	11	48
	SV-I2	1	-	60	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	MV-I2	2	-	60	40	40	40	40	40	40	40	40	41	41	40	40	40	41	41	35	39	38	37	36	36	43
		3	-	60	44	44	45	46	48	46	46	46	49	48	44	45	45	45	46	46	46	43	43	41	40	51
		4	-	60	50	49	51	50	50	52	53	54	54	53	48	49	52	51	50	48	47	48	48	44	45	58
		5	-	60	57	57	58	57	58	54	54	55	55	56	54	55	57	56	56	52	52	57	56	49	46	60
PRP	SV-A1	1	1	120	13	12	19	20	23	14	11	19	19	21	17	13	19	21	25	12	16	3	3	1	1	39
		1	2	120	11	17	20	23	20	14	15	19	21	22	14	15	17	21	19	12	15	3	7	1	1	46
		1	3	120	3	3	5	4	3	2	3	4	4	3	2	3	4	4	3	5	5	2	2	0	0	10
		1	4	120	26	32	32	38	39	28	33	38	35	41	28	31	34	39	41	29	33	8	8	2	2	65
	MV-A2	UL	1	120	101	100	101	100	102	98	100	98	100	102	97	99	96	97	98	81	81	84	84	57	57	110
		UL	2	120	39	41	47	49	53	38	43	45	44	46	32	40	44	44	47	44	47	25	25	7	9	79
		UL	3	120	87	91	92	96	98	90	91	94	96	95	89	93	97	97	98	63	70	82	87	37	43	102
		UL	4	120	71	76	75	80	79	69	76	75	80	78	69	77	76	80	78	71	73	38	42	24	26	94
	MV-A3	UL	1	120	73	74	79	78	78	74	76	79	79	78	73	77	78	77	79	67	66	42	43	25	28	82
		UL	2	120	39	36	46	48	50	39	40	47	46	53	32	36	44	45	50	45	42	22	21	5	6	82
		UL	3	120	26	26	28	30	30	21	19	25	26	28	23	26	26	30	30	25	23	25	26	11	11	35
		UL	4	120	47	46	46	53	52	46	46	47	51	53	45	48	48	53	57	53	56	28	26	16	17	71
	MV-B1	5	-	30	1	1	2	2	3	2	2	3	3	3	0	2	2	2	3	2	3	0	0	0	0	5
	MV-B2	9	-	30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1
	MV-B3	13	-	30	12	12	12	14	15	9	11	12	11	10	10	12	14	11	12	18	19	0	0	0	0	21
Total (All Instances)				2628	888	912	955	989	1003	882	914	958	974	994	866	911	950	975	1001	815	851	675	685	406	422	1257
Total (LSMV‡† Instances)				1290	687	693	722	743	756	678	697	720	735	744	656	699	719	729	745	650	665	532	538	352	364	894

Note. The largest number of best solutions obtained at each row is presented with the bold font.

† Transportation cost update mechanisms. M: Marginal, T: TSP route cost share, V: VRP route cost share. ‡ Initial values for node visit cost.

†† Overall best obtained solution for each instance is taken into account. ‡† Large-scale multi-vehicle.

Table A.6: Number of new best solutions found by different cost update mechanisms and initial node visit costs for IRP and PRP data sets

					Mixed M† and T† Mechanisms															M†		T†		V†		
					σ_{it}‡ $= c_{0i}/2$					σ_{it}‡ $= c_{0i}$					σ_{it}‡ $= 2c_{0i}$					σ_{it}‡ $= c_{0i}/2$						
Prob	Set	m	Class	Size	100	150	200	250	300	100	150	200	250	300	100	150	200	250	300	200	250	200	250	200	250	OB††
IRP	SV-I1	1	-	160	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	MV-I1	2	-	160	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		3	-	160	14	13	13	14	14	**15**	13	**15**	**15**	14	**15**	12	13	13	**15**	11	11	10	11	3	3	19
		4	-	160	21	21	22	21	21	20	20	22	20	20	21	21	22	**23**	**23**	11	14	18	18	6	6	26
		5	-	158	27	27	31	32	31	27	29	29	**33**	32	28	26	30	32	30	20	21	26	26	10	10	39
	SV-I2	1	-	60	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	MV-I2	2	-	60	40	40	40	40	40	40	40	40	**41**	**41**	40	40	40	**41**	**41**	35	39	38	37	36	36	43
		3	-	60	44	44	45	46	48	46	46	46	**49**	48	44	45	45	45	46	46	46	43	43	41	40	51
		4	-	60	50	49	51	50	50	52	53	**54**	**54**	53	48	49	52	51	50	48	47	47	48	44	45	58
		5	-	60	57	57	**58**	57	**58**	54	54	55	55	56	54	55	57	56	56	52	52	57	56	49	46	60
PRP	SV-A1	1	1	120	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		1	2	120	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		1	3	120	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		1	4	120	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	MV-A2	UL	1	120	98	97	**99**	97	**99**	95	97	96	98	**99**	95	97	95	96	97	80	80	81	80	52	55	108
		UL	2	120	39	41	46	48	**53**	38	43	44	44	46	32	40	44	44	47	44	47	25	25	7	9	79
		UL	3	120	83	88	89	92	**95**	86	88	92	94	92	86	91	94	94	**95**	63	70	80	85	35	40	100
		UL	4	120	71	76	74	78	**79**	69	76	74	**79**	78	69	77	75	**79**	78	71	72	37	41	24	25	93
	MV-A3	UL	1	120	73	73	**79**	78	78	74	76	**79**	**79**	78	73	77	78	77	**79**	67	66	42	42	25	27	82
		UL	2	120	39	36	46	48	50	39	40	47	46	**53**	31	35	44	44	49	45	42	22	21	5	6	82
		UL	3	120	26	26	28	**30**	**30**	21	19	25	26	28	23	26	26	**30**	**30**	25	23	24	25	11	11	35
		UL	4	120	46	46	46	52	52	45	46	47	50	53	44	48	48	52	**57**	53	55	27	25	15	16	71
	MV-B1	5	-	30	1	1	2	2	**3**	2	2	**3**	**3**	**3**	0	2	2	2	**3**	2	**3**	0	0	0	0	5
	MV-B2	9	-	30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**1**	0	0	0	0	1
	MV-B3	13	-	30	12	12	12	14	15	9	11	12	11	10	10	12	14	11	12	18	**19**	0	0	0	0	21
Total (All Instances)				2628	741	747	781	799	816	732	753	780	797	804	713	753	779	790	808	691	708	577	583	363	375	973
Total (LSMV‡‡ Instances)				1290	679	686	715	732	750	670	691	714	729	738	649	694	714	722	740	649	662	523	528	344	356	889

Note. The largest number of best solutions obtained at each row is presented with the bold font.
† Transportation cost update mechanisms. M: Marginal, T: TSP route cost share, V: VRP route cost share. ‡ Initial values for node visit cost.
†† Overall best obtained solution for each instance is taken into account. ‡‡ Large-scale multi-vehicle.

Table A.7: Average running time for different cost update mechanisms and initial node visit costs for IRP and PRP data sets (seconds)

					Mixed M† and T† Mechanisms															M†		T†		V†	
					σ_{it}‡ $= c_{0i}/2$					σ_{it}‡ $= c_{0i}$					σ_{it}‡ $= 2c_{0i}$					σ_{it}‡ $= c_{0i}/2$					
Prob	Set	m	Class	Size	100	150	200	250	300	100	150	200	250	300	100	150	200	250	300	200	250	200	250	200	250
IRP	SV-I1	1	-	160	24	36	48	60	72	25	36	51	60	79	27	36	48	60	73	44	56	55	71	51	65
	MV-I1	2	-	160	39	52	68	85	99	37	60	69	88	118	36	52	68	84	98	77	95	69	92	67	77
		3	-	160	37	52	69	108	125	37	52	67	83	104	38	52	72	84	100	75	94	69	95	54	78
		4	-	160	37	53	74	84	99	38	61	77	86	102	38	54	71	94	115	78	88	68	86	45	70
		5	-	158	36	52	65	84	98	40	54	66	91	99	37	49	68	85	94	64	79	65	80	58	68
	SV-I2	1	-	60	3363	4998	6668	8295	9209	3361	5019	6831	8375	9259	3423	5061	6725	8300	9397	4394	5436	7795	9472	7617	9169
	MV-I2	2	-	60	2781	4122	5657	6914	8029	2889	4116	5351	6494	8172	3011	4718	6045	6860	8397	4617	5890	5943	7492	6473	7957
		3	-	60	2120	3513	4209	5386	6624	2088	3272	4397	5325	6923	2144	3219	4450	5478	6401	4441	6008	4159	5204	5354	6683
		4	-	60	2780	4053	5132	6236	7046	3102	4126	5289	6371	7246	2916	4199	5333	6502	7222	5918	7204	5593	6668	6880	8160
		5	-	60	3232	4319	5527	6290	7101	2983	3977	5167	6213	6952	2911	4122	5113	6211	7163	6249	7745	4733	5807	6925	8119
PRP	SV-A1	1	1	120	8	13	18	24	30	8	13	18	24	30	8	13	18	24	30	17	22	25	33	13	17
		1	2	120	8	13	18	23	29	8	13	18	23	29	8	13	18	23	29	18	22	23	30	12	15
		1	3	120	8	12	17	22	28	8	12	17	22	28	8	14	17	22	28	14	18	25	33	9	12
		1	4	120	8	13	18	24	30	8	13	18	23	29	8	13	18	23	29	15	19	25	33	15	20
	MV-A2	UL	1	120	201	300	400	497	600	201	299	400	500	598	202	301	434	503	605	522	714	365	460	301	378
		UL	2	120	170	258	344	432	524	170	258	347	433	524	169	257	355	431	662	388	525	331	422	257	323
		UL	3	120	155	233	309	385	465	157	234	313	385	467	158	236	356	394	473	353	436	288	358	176	200
		UL	4	120	215	326	434	540	661	217	329	436	525	660	216	325	470	541	654	468	669	426	656	394	448
	MV-A3	UL	1	120	1103	1612	2125	2644	3162	1084	1636	2131	2642	3174	1080	1886	2124	2637	3458	2933	3495	1720	2204	1848	1965
		UL	2	120	961	1454	1947	2432	2920	962	1461	1946	2422	2923	962	1453	1938	2431	3323	2134	2614	1791	2248	1583	1958
		UL	3	120	729	1088	1461	1802	2173	730	1093	1456	1809	2165	757	1224	1473	1846	2681	1767	2171	1485	1883	916	1147
		UL	4	120	1112	1664	2213	2767	3317	1110	1663	2214	2753	3285	1106	1661	2223	3225	3597	2293	2844	2150	2703	1985	2463
	MV-B1	5	-	30	1825	2622	3559	4446	5326	1840	2642	3546	4480	5361	1729	2494	3408	4319	5090	1347	1698	6760	8447	8602	10581
	MV-B2	9	-	30	5368	7460	9811	11773	13911	5888	8203	10876	12812	15280	5619	7925	9894	12401	14623	9488	12849	13256	16648	14130	17406
	MV-B3	13	-	30	8344	12044	15891	19329	22858	8294	11878	15747	19310	23012	8191	11705	15621	18950	22520	17955	22625	22340	27584	23869	28820

† Transportation cost update mechanisms. M: Marginal, T: TSP route cost share, V: VRP route cost share.
‡ Initial values for node visit cost.

We also present the sensitivity analysis on the relevant instances to evaluate the effect of the $\mathcal{M}_z^{\mathcal{R}}$ subproblem for the multi-vehicle instances with $m < n$. Table A.8 shows the results with and without implementing $\mathcal{M}_z^{\mathcal{R}}$ subproblem. We performed the experiments of these two tables by setting 100 iterations and $\sigma_{it} = 0.5c_{0i}$ for CCJ-DH, similar to the scenario in the sixth column of Tables A.4-A.7. Our observation is that the algorithm without using $\mathcal{M}_z^{\mathcal{R}}$ faces more infeasible VRP_t subproblems for these instances. Therefore, the $\mathcal{M}_z^{\mathcal{R}}$ subproblem implementation is crucial to obtain quality solutions. Moreover, it resulted in better average gaps and more BUBs on all data sets and classes, except for MV-I2 with $m = 3$.

Table A.8: Effect of implementing $\mathcal{M}_z^{\mathcal{R}}$ subproblem in CCJ-DH on relevant IRP and PRP instances

					Gap (%)		# BUB		# NBS		Time (sec)	
Prob	Set	m	Class	Size	$\mathcal{M}_z^{\mathcal{R}}$	NM†	$\mathcal{M}_z^{\mathcal{R}}$	NM†	$\mathcal{M}_z^{\mathcal{R}}$	NM†	$\mathcal{M}_z^{\mathcal{R}}$	NM†
IRP	MV-I1	2	-	160	**2.81**	3.62	**24**	8	0	0	39	**37**
		3	-	160	**2.54**	3.42	**31**	6	**14**	3	**37**	38
		4	-	160	**2.89**	3.42	**27**	14	**21**	11	**37**	38
		5	-	158	**3.01**	3.39	**33**	18	**27**	16	**36**	**36**
	MV-I2	2	-	60	**-1.59**	-1.56	**40**	**40**	**40**	**40**	2781	**2698**
		3	-	60	**-2.55**	-2.39	44	**45**	44	**45**	2120	**1842**
		4	-	60	**-4.48**	-4.13	**50**	44	**50**	44	2780	**1761**
		5	-	60	**-4.65**	-4.54	**57**	54	**57**	54	3232	**1917**
PRP	MV-B1	5	-	30	**0.89**	2.51	**1**	0	**1**	0	1825	**1778**
	MV-B2	9	-	30	**1.28**	2.15	0	0	0	0	5368	**4318**
	MV-B3	13	-	30	**0.22**	0.85	**12**	1	**12**	1	**8344**	9171

Note. Other data sets include only single-vehicle or unlimited multi-vehicle instances.
† Without implementing $\mathcal{M}_z^{\mathcal{R}}$ subproblem.

Appendix B – A Branch-and-Cut Algorithm for an Assembly Routing Problem

Proofs

Proposition 2.1. *Inequalities*

$$\sum_{e=t_1}^{t_2} p_e \leq I_{0k,t_1-1} + \sum_{i\in N_k} I_{ik,t_1-1} + \sum_{e=t_1}^{t_2}\sum_{i\in N_k} s_{ikt_1e}y_e \quad \forall k \in K, \forall t_1, t_2 \in T, t_1 \leq t_2 \tag{2.19}$$

are valid for the $\mathcal{M}_{ARP}$.

Proof. The inequalities for $\sum_{e=t_1}^{t_2} y_e = 0$ are trivial because $\sum_{e=t_1}^{t_2} p_e = 0$. Otherwise, let θ be the last period in which the production setup is performed, i.e., $\theta = \max_e\{t_1 \leq e \leq t_2 | y_e = 1\}$. Then,

$$\begin{aligned}
\sum_{e=t_1}^{t_2} p_e &= \sum_{e=t_1}^{\theta} p_e \\
&= \sum_{e=t_1}^{\theta} (I_{0k,e-1} - I_{0ke} + \sum_{i\in N_k} q_{ike}) \\
&= \sum_{e=t_1}^{\theta} \Big(I_{0k,e-1} - I_{0ke} + \sum_{i\in N_k} (I_{ik,e-1} - I_{ike} + s_{ike})\Big) \\
&= I_{0k,t_1-1} - I_{0k\theta} + \sum_{i\in N_k} (I_{ik,t_1-1} - I_{ik\theta} + s_{ikt_1\theta}) \\
&\leq I_{0k,t_1-1} + \sum_{i\in N_k} I_{ik,t_1-1} + \sum_{i\in N_k} s_{ikt_1\theta}
\end{aligned}$$

$$= I_{0k,t_1-1} + \sum_{i\in N_k} I_{ik,t_1-1} + \sum_{i\in N_k} s_{ikt_1\theta} y_\theta$$

$$\leq I_{0k,t_1-1} + \sum_{i\in N_k} I_{ik,t_1-1} + \sum_{e=t_1}^{t_2} \sum_{i\in N_k} s_{ikt_1e} y_e.$$

The first four equations follow from the definition of θ, constraints (2.3), constraints (2.4), and the definition of $s_{ikt_1t_2}$, respectively. The first inequality holds due to the non-negativity of inventory variables. The next equation is valid because $y_\theta = 1$. The last inequality holds because the y_e variables are nonnegative. □

Proposition 2.2. *Inequalities*

$$\sum_{e=t_1}^{t_2} q_{ike} \leq I_{ik,t_1-1} + \sum_{e=t_1}^{t_2} s_{ikt_1e} z_{ie} \quad \forall i \in N, \forall k \in K_i, \forall t_1, t_2 \in T, t_1 \leq t_2 \qquad (2.20)$$

are valid for the $\mathcal{M}_{ARP}$.

Proof. If $\sum_{e=t_1}^{t_2} z_{ie} = 0$, then the supplier i will not be visited during periods t_1 to t_2. Therefore, for these periods no shipment is possible ($\sum_{e=t_1}^{t_2} q_{ike} = 0$) and inequalities (2.20) are satisfied. Otherwise, let θ be the last period in which the supplier i will be visited, i.e., $\theta = \max_e\{t_1 \leq e \leq t_2 | z_{ie} = 1\}$. Then,

$$\sum_{e=t_1}^{t_2} q_{ike} = \sum_{e=t_1}^{\theta} q_{ike}$$

$$= \sum_{e=t_1}^{\theta} (I_{ik,e-1} - I_{ike} + s_{ike})$$

$$= I_{ik,t_1-1} - I_{ik\theta} + s_{ikt_1\theta}$$

$$\leq I_{ik,t_1-1} + s_{ikt_1\theta}$$

$$= I_{ik,t_1-1} + s_{ikt_1\theta} z_{i\theta}$$

$$\leq I_{ik,t_1-1} + \sum_{e=t_1}^{t_2} s_{ikt_1e} z_{ie}.$$

The first three equations hold due to the definition of θ, constraints (2.4), and the definition of $s_{ikt_1t_2}$, respectively. The first inequality is valid because of the non-negativity of inventory variables. The next equality is valid for the reason that $z_{i\theta} = 1$. The last inequality holds because the z_{ie} variables are nonnegative. □

Proposition 2.3. *Inequalities*

$$\sum_{e=t_1}^{t_2} \sum_{i \in N_k} q_{ike} \leq I_{00t_2} + I_{0kt_2} + \sum_{e=t_1}^{t_2} d_{et_2} \sum_{i \in N_k} z_{ie} \quad \forall k \in K, \forall t_1, t_2 \in T, t_1 \leq t_2 \tag{2.21}$$

are valid for the $\mathcal{M}_{ARP}$.

Proof. If $\sum_{e=t_1}^{t_2} \sum_{i \in N_k} z_{ie} = 0$, then no visit to the suppliers $i \in N_k$ will be made during periods t_1 to t_2 and hence no shipment of component k is possible during this period ($\sum_{e=t_1}^{t_2} \sum_{i \in N_k} q_{ike} = 0$). Then, inequalities (2.21) are satisfied because the left-hand-side (LHS) will be equal to zero and the inventory variables in the right-hand-side (RHS) are nonnegative. Otherwise, let θ be the first period in which at least one node $i \in N_k$ is visited, i.e., $\theta = \min_e\{t_1 \leq e \leq t_2 | \sum_{i \in N_k} z_{ie} \geq 1\}$. Then,

$$\begin{aligned}
\sum_{e=t_1}^{t_2} \sum_{i \in N_k} q_{ike} &= \sum_{e=\theta}^{t_2} \sum_{i \in N_k} q_{ike} \\
&= \sum_{e=\theta}^{t_2} (I_{0ke} - I_{0k,e-1} + p_e) \\
&= \sum_{e=\theta}^{t_2} \Big(I_{0ke} - I_{0k,e-1} + (I_{00e} - I_{00,e-1} + d_e) \Big) \\
&= I_{00t_2} - I_{00,\theta-1} + I_{0kt_2} - I_{0k,\theta-1} + d_{\theta t_2} \\
&\leq I_{00t_2} + I_{0kt_2} + d_{\theta t_2} \\
&\leq I_{00t_2} + I_{0kt_2} + d_{\theta t_2} \sum_{i \in N_k} z_{i\theta} \\
&\leq I_{00t_2} + I_{0kt_2} + \sum_{e=\theta}^{t_2} d_{et_2} \sum_{i \in N_k} z_{ie} \\
&= I_{00t_2} + I_{0kt_2} + \sum_{e=t_1}^{t_2} d_{et_2} \sum_{i \in N_k} z_{ie}.
\end{aligned}$$

The first four equations follow from the definition of θ, constraints (2.3), constraints (2.2), and the definition of $d_{t_1 t_2}$, respectively. The first inequality holds due to the non-negativity of inventory variables. The next inequality is valid because at least one node is visited in period θ, i.e., $\sum_{i \in N_k} z_{i\theta} \geq 1$. The last inequality is valid since the z_{ie} variables are nonnegative. The last equation holds due to the assumption that θ is the first period in which at least one node $i \in N_k$ is visited. □

Lemma B.1. *Inequalities*

$$\max\{0, Q_{it}\} \leq \sum_{e=1}^{t} \sum_{k \in K_i} b_k q_{ike} \quad \forall i \in N, t \in T$$

are valid for $\mathcal{M}_{ARP}$.

Proof. We have

$$\begin{aligned} Q_{it} &\leq \sum_{k \in K_i} b_k (s_{ik1t} + I_{ik0}) - \sum_{k \in K_i} b_k I_{ikt} \\ &= \sum_{k \in K_i} b_k \sum_{e=1}^{t} (s_{ike} + I_{ik,e-1} - I_{ike}) \\ &= \sum_{e=1}^{t} \sum_{k \in K_i} b_k q_{ike}, \end{aligned}$$

where the inequality follows from the storage capacity constraints (2.8), and the equations hold due to the definition of $s_{ikt_1t_2}$ and constraints (2.4), respectively. Because only a strictly positive Q_{it} triggers the shipment to the plant, we obtain:

$$\max\{0, Q_{it}\} \leq \textstyle\sum_{e=1}^{t} \sum_{k \in K_i} b_k q_{ike}.$$

□

Proposition 2.4. *Inequalities*

$$\left\lceil \frac{\max\left\{0, d_{1t} - I_{000}, (\sum_{k \in K} b_k I_{0k0} + \sum_{i \in N} \max\{0, Q_{it}\} - L) / \sum_{k \in K} b_k\right\}}{\min\{C, \max_{e \in \{1,\ldots,t\}}\{d_e\} + L_0\}} \right\rceil \leq \sum_{e=1}^{t} y_e \quad \forall t \in T \tag{2.22}$$

are valid for $\mathcal{M}_{ARP}$.

Proof. We first obtain two lower bounds on the cumulative production from period 1 to t.

$$\begin{aligned} \sum_{e=1}^{t} p_e &= \sum_{e=1}^{t} (d_e + I_{00e} - I_{00,e-1}) \\ &= d_{1t} + I_{00t} - I_{000} \\ &\geq d_{1t} - I_{000}. \end{aligned}$$

The first and the second equations hold because of constraints (2.2), and the definition of $d_{t_1 t_2}$, respectively. The inequality is valid due to the non-negativity of the inventory variables. Moreover,

$$\begin{aligned}
\sum_{k\in K} b_k \sum_{e=1}^{t} p_e &= \sum_{k\in K} b_k \sum_{e=1}^{t} (I_{0k,e-1} - I_{0ke} + \sum_{i\in N_k} q_{ike}) \\
&= \sum_{k\in K} b_k I_{0k0} - \sum_{k\in K} b_k I_{0kt} + \sum_{i\in N} \sum_{e=1}^{t} \sum_{k\in K_i} b_k q_{ike} \\
&\geq \sum_{k\in K} b_k I_{0k0} - L + \sum_{i\in N} \max\{0, \mathcal{Q}_{it}\}.
\end{aligned}$$

The first equation follows from constraints (2.3). The second equation is obtained by rearranging the terms. The inequality holds based on the component storage capacity at the suppliers and Lemma B.1. Next, we determine two upper bounds on the cumulative production from period 1 to t. The cumulative production amount forces a minimum number of production setups due to production capacity constraints (2.5): $\sum_{e=1}^{t} p_e \leq C \sum_{e=1}^{t} y_e$. Then, we present another expression for the minimum number of required production setups:

$$\begin{aligned}
\sum_{e=1}^{t} p_e &\leq \sum_{e=1}^{t} (d_e + I_{00e}) y_e \\
&\leq \sum_{e=1}^{t} \max_{e'\in\{1,\ldots,t\}} \{d_{e'} + I_{00e'}\} y_e \\
&= \max_{e'\in\{1,\ldots,t\}} \{d_{e'} + I_{00e'}\} \sum_{e=1}^{t} y_e \\
&\leq (\max_{e'\in\{1,\ldots,t\}} \{d_{e'}\} + L_0) \sum_{e=1}^{t} y_e.
\end{aligned}$$

The first inequality is valid since $p_t = d_t + I_{00t} - I_{00t-1} \leq d_t + I_{00t}$, and the fact that $y_t = 0$ forces $p_t = 0$. The second inequality and the equation hold trivially. The last inequality is valid because of the product storage capacity (L_0). Combining the two parts of the proof, we obtain:

$$\max\left\{0, d_{1t} - I_{000}, \left(\sum_{k\in K} b_k I_{0k0} + \sum_{i\in N} \max\{0, \mathcal{Q}_{it}\} - L\right) / \sum_{k\in K} b_k\right\} \leq \sum_{e=1}^{t} p_e \leq \min\left\{C, \max_{e\in\{1,\ldots,t\}} \{d_e\} + L_0\right\} \sum_{e=1}^{t} y_e.$$

□

Proposition 2.5. *Inequalities*

$$\left\lceil \frac{1}{Q} \max\left\{ \sum_{k\in K} b_k \max\{0, d_{1t} - I_{000} - I_{0k0}\}, \sum_{i\in N} \max\{0, Q_{it}\} \right\} \right\rceil \leq \sum_{e=1}^{t} z_{0e} \quad \forall t \in T \quad (2.23)$$

are valid for $\mathcal{M}_{ARP}$.

Proof. We obtain the first expression as follows:

$$\begin{aligned} \sum_{e=1}^{t} Qz_{0e} &\geq \sum_{e=1}^{t} \sum_{k\in K} \sum_{i\in N_k} b_k q_{ike} \\ &= \sum_{e=1}^{t} \sum_{k\in K} b_k(d_e + I_{00e} - I_{00,e-1} + I_{0ke} - I_{0k,e-1}) \\ &= \sum_{k\in K} b_k(d_{1t} + I_{00t} - I_{000} + I_{0kt} - I_{0k0}) \\ &\geq \sum_{k\in K} b_k(d_{1t} - I_{000} - I_{0k0}). \end{aligned}$$

The first inequality is valid since the LHS is the total capacity of the dispatched vehicles from period $e = 1$ to t, and the RHS is the total shipped amount over the same periods, all components and all suppliers. The first equation follows from constraints (2.3), and by replacing the p_t variables using constraints (2.2). The second equation is valid due to the definition of $d_{t_1t_2}$. The second inequality holds due to the non-negativity of inventory variables. Next, we have

$$\begin{aligned} \sum_{e=1}^{t} Qz_{0e} &\geq \sum_{e=1}^{t} \sum_{i\in N} \sum_{k\in K_i} b_k q_{ike} \\ &\geq \sum_{i\in N} \max\{0, Q_{it}\}, \end{aligned}$$

where the first inequality is valid because of the total fleet capacity, and the second inequality follows from Lemma B.1. □

Proposition 2.6. *Inequalities*

$$\left\lceil \frac{\max\{0, Q_{it}\}}{\min\left\{Q, L_i + \max_{e\in\{1,\ldots,t\}}\{\sum_{k\in K_i} b_k s_{ike}\}, \sum_{k\in K_i} b_k(I_{ik0} + s_{ik1t})\right\}} \right\rceil \leq \sum_{e=1}^{t} z_{ie} \quad \forall i \in N, \forall t \in T \quad (2.24)$$

are valid for $\mathcal{M}_{ARP}$.

Proof. Based on Lemma B.1 we know that

$$\max\{0, Q_{it}\} \leq \sum_{e=1}^{t} \sum_{k \in K_i} b_k q_{ike}.$$

Now, we present upper bounds on the cumulative shipments from node i during period 1 to t. The vehicle capacity constraints (2.10) provide the first upper bound: $\sum_{e=1}^{t} \sum_{k \in K_i} b_k q_{ike} \leq Q \sum_{e=1}^{t} z_{ie}$. Next, we have

$$\begin{aligned}
\sum_{e=1}^{t} \sum_{k \in K_i} b_k q_{ike} &\leq \sum_{e=1}^{t} (L_i + \sum_{k \in K_i} b_k s_{ike}) z_{ie} \\
&\leq \sum_{e=1}^{t} \left(L_i + \max_{e' \in \{1,\dots,t\}} \{\sum_{k \in K_i} b_k s_{ike'}\}\right) z_{ie} \\
&= \left(L_i + \max_{e' \in \{1,\dots,t\}} \{\sum_{k \in K_i} b_k s_{ike'}\}\right) \sum_{e=1}^{t} z_{ie}.
\end{aligned}$$

Where the first inequality follows from $\sum_{k \in K_i} b_k q_{ikt} \leq L_i + \sum_{k \in K_i} b_k s_{ikt}$ which is valid due to constraints (2.4) and (2.8), and the fact that $z_{it} = 0$ forces $\sum_{k \in K_i} b_k q_{ikt} = 0$. The second inequality and the equation hold trivially. Moreover, we have

$$\begin{aligned}
\sum_{e=1}^{t} \sum_{k \in K_i} b_k q_{ike} &\leq \sum_{e=1}^{t} \sum_{k \in K_i} b_k (I_{ik0} + s_{ik1e}) z_{ie} \\
&\leq \sum_{e=1}^{t} \sum_{k \in K_i} b_k \left(I_{ik0} + \max_{e' \in \{1,\dots,t\}} \{s_{ik1e'}\}\right) z_{ie} \\
&= \sum_{e=1}^{t} \sum_{k \in K_i} b_k (I_{ik0} + s_{ik1t}) z_{ie} \\
&= \sum_{k \in K_i} b_k (I_{ik0} + s_{ik1t}) \sum_{e=1}^{t} z_{ie}.
\end{aligned}$$

Where the first inequality is valid for the reason that $q_{ike} \leq I_{ik0} + s_{ik1e}$ which is valid due to constraints (2.4), the definition of $s_{ikt_1t_2}$, and the fact that $z_{it} = 0$ forces $\sum_{k \in K_i} b_k q_{ikt} = 0$. The second inequality holds trivially. The first equation follows from $\max_{e' \in \{1,\dots,t\}} \{s_{ik1e'}\} = s_{ik1t}$. The second equation holds trivially. Consequently, we obtain

$$\begin{aligned}
\max\{0, Q_{it}\} &\leq \sum_{e=1}^{t} \sum_{k \in K_i} b_k q_{ike} \\
&\leq \min\left\{Q, L_i + \max_{e \in \{1,\dots,t\}} \{\sum_{k \in K_i} b_k s_{ike}\}, \sum_{k \in K_i} b_k (I_{ik0} + s_{ik1t})\right\} \sum_{e=1}^{t} z_{ie}.
\end{aligned}$$

□

Proposition 2.7. *Inequalities*

$$\left\lceil \frac{\max\{0, d_{1t} - I_{000} - I_{0k0}\}}{\min\left\{\frac{Q}{b_k}, \max_{i\in N_k}\{I_{ik0} + s_{ik1t}\}\right\}} \right\rceil \leq \sum_{e=1}^{t} \sum_{i\in N_k} z_{ie} \quad \forall k \in K, \forall t \in T \tag{2.25}$$

are valid for $\mathcal{M}_{ARP}$.

Proof. We have

$$d_{1t} - I_{000} - I_{0k0} \leq \textstyle\sum_{e=1}^{t} \sum_{i\in N_k} q_{ike},$$

which can be obtained by replacing p_t using constraints (2.2) in constraints (2.3), and the non-negativity of the inventory variables. Next, we have

$$\textstyle\sum_{e=1}^{t} \sum_{i\in N_k} q_{ike} \leq \frac{Q}{b_k} \sum_{e=1}^{t} \sum_{i\in N_k} z_{ie},$$

which is valid due to $b_k q_{ikt} \leq Q z_{it}$. Furthermore, we have

$$\begin{aligned}
\sum_{i\in N_k} \sum_{e=1}^{t} q_{ike} &\leq \sum_{i\in N_k} (I_{ik0} + s_{ik1t}) \sum_{e=1}^{t} z_{ie} \\
&\leq \sum_{i\in N_k} \max_{i'\in N_k}\{I_{i'k0} + s_{i'k1t}\} \sum_{e=1}^{t} z_{ie} \\
&= \max_{i'\in N_k}\{I_{i'k0} + s_{i'k1t}\} \sum_{i\in N_k} \sum_{e=1}^{t} z_{ie}.
\end{aligned}$$

Where the first inequality comes from constraints (2.4), and by checking for $\sum_{e=1}^{t} z_{ie} = 0$ and $\sum_{e=1}^{t} z_{ie} \geq 1$. The second inequality and the equation are valid trivially. Finally, we obtain

$$\begin{aligned}
\max\{0, d_{1t} - I_{000} - I_{0k0}\} &\leq \sum_{e=1}^{t} \sum_{i\in N_k} q_{ike} \\
&\leq \min\left\{\frac{Q}{b_k}, \max_{i\in N_k}\{I_{ik0} + s_{ik1t}\}\right\} \sum_{e=1}^{t} \sum_{i\in N_k} z_{ie}.
\end{aligned}$$

□

Adaptation of CCJ-DH

In this section, we present the adaptation of CCJ-DH (Chitsaz et al., 2019) to the generalized version of the ARP. The algorithm decomposes the problem into three distinct subproblems. The framework of the heuristic is presented in Figure B.1.

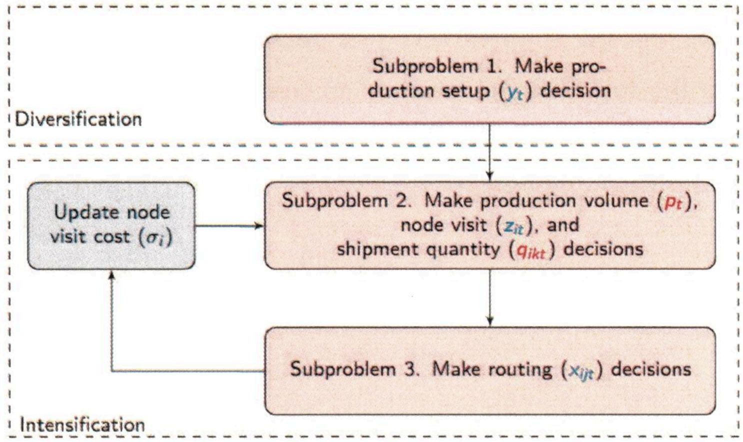


Figure B.1: CCJ-DH framework

The first subproblem returns a setup schedule. It uses an approximate transportation cost based on the number of vehicles dispatched from the plant. This results in the following objective function:

$$\min \sum_{t \in T} \left(up_t + fy_t + \sum_{k \in K^+} h_{0k} I_{0kt} + \sum_{i \in N} \sum_{k \in K_i} h_{ik} I_{ikt} + \sigma_{0t} z_{0t} \right) \tag{B.1}$$

where σ_{0t} is the cost of each vehicle dispatch. This objective function does not include any routing decision and hence constraints (2.11)-(2.12) become redundant. To impose the aggregate fleet capacity in the first subproblem, the algorithm adds the following constraints to constraints (2.3)-(2.10), and (2.13)-(2.15):

$$\sum_{i \in N} \sum_{k \in K_i} b_k q_{ikt} \leq Q z_{0t} \quad \forall t \in T. \tag{B.2}$$

After solving this subproblem using CPLEX, the algorithm fixes the setup schedule and uses it as a given parameter in the second subproblem.

The second subproblem returns node visit and shipment quantity decisions. The algorithm employs another approximation of the transportation cost in the objective function based on the cost associated with visiting each supplier (node). This results in the following objective function:

$$\min \sum_{t \in T} \left(up_t + \sum_{k \in K^+} h_{0k} I_{0kt} + \sum_{i \in N} \sum_{k \in K_i} h_{ik} I_{ikt} + \sum_{i \in N} \sigma_{it} z_{it} \right) \tag{B.3}$$

where σ_{it} represents the node visit cost estimation. Similarly as in the first subproblem, this subproblem ignores the routing decisions. To enforce the vehicle capacity and to make sure that the shipments can be packed into the available vehicles, the algorithm considers the following constraints as well as constraints (2.3)-(2.8), (2.10), and (2.14)-(2.15) in the second subproblem:

$$\sum_{i \in N} \sum_{k \in K_i} b_k q_{ikt} \leq \lambda_t m Q \quad \forall t \in T. \tag{B.4}$$

Here, $\lambda_t = 1 - \frac{2}{n}$ is a parameter. CCJ-DH solves this subproblem using CPLEX. Having the node visit and the shipment quantity decisions fixed for each time period, the algorithm solves one capacitated VRP for each period as the third subproblem. CCJ-DH uses the tabu search heuristic of Cordeau et al. (1997) to solve the VRPs.

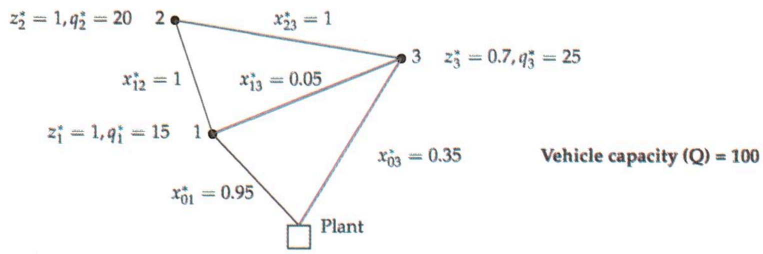
To intensify the search, CCJ-DH updates the node visit cost estimates (σ_{it}) for the next iteration. The algorithm uses two estimation mechanisms. The first mechanism is the cheapest insertion cost among all existing routes. The second mechanism splits the cost of each route (in each period) over its nodes proportional to their direct shipment cost. In this mechanism, if a node is not visited in a certain period, the algorithm considers the direct shipment cost as the estimated cost for that node. CCJ-DH switches between these two mechanisms after using each for 7 consecutive iterations.

To diversify the search, the algorithm adds a local branching type cut (Fischetti et al., 2004) to the set of constraints in the first subproblem in order to consider a new setup schedule. The stopping condition for the overall algorithm is a maximum of 200 intensification iterations. To perform a diversification, CCJ-DH considers two stopping conditions: a maximum of 80 intensification iterations, or 60 intensification iterations without incumbent solution improvement.

Examples for Fractionally Violated and Non-Violated Subtours

Figure B.2 shows an example where CVRPSEP returns a violated VRP CCC which is a non-violated ARP GFSEC in the ARP (or the IRP and the PRP). Figure B.3 shows an example for the case that a fractionally violated GFSEC or DFJ in the ARP (or the IRP and the PRP) cannot be found if the node visit variables (z_{it}) are not considered.

Figure B.2: A violated VRP CCC which is a non-violated GFSEC.



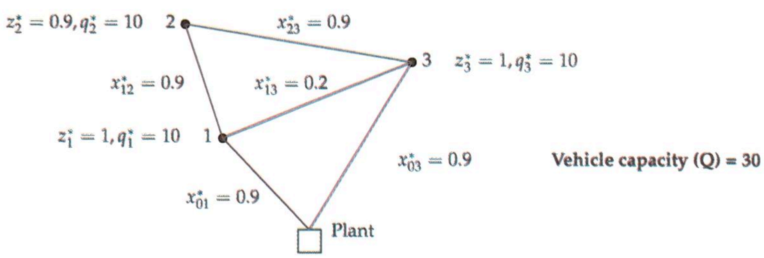
Violated VRP subtour, $S = \{1,2,3\}$: $1 + 1 + 0.05 = 2.05 > |S| - 1 = |3| - 1 = 2$

$$\textbf{LHS} = Q \sum_{(i,j)\in E(S)} x_{ij}^* = 100 * (1 + 1 + 0.05) = \textbf{205}$$

$$\textbf{RHS} = \sum_{i\in S}(Qz_i^* - \sum_{k\in K_i} b_k q_{ik}^*) = 100 * (1 + 1 + 0.7) - (15 + 20 + 25) = \textbf{210}$$

LHS $<$ **RHS** Satisfied (non-violated) fractional ARP GFSEC

Figure B.3: Violated ARP GFSEC and DFJ which is a non-violated VRP CCC and DFJ.



Non-violated VRP DFJ, $S = \{1,2,3\}$: $0.9 + 0.9 + 0.2 = 2 = |S| - 1 = |3| - 1 = 2$

Non-violated VRP CCC:

$$\textbf{LHS} = Q \sum_{(i,j)\in E(S)} x_{ij}^* = 30 * (0.9 + 0.9 + 0.2) = \textbf{60}$$

$$\textbf{RHS} = \sum_{i\in S}(Q - \sum_{k\in K_i} b_k q_{ik}^*) = 3 * (30 - 10) = \textbf{60}$$

LHS $=$ **RHS** Satisfied (non-violated) fractional VRP CCC

Violated ARP DFJ, $S = \{1,2,3\}$: $0.9 + 0.9 + 0.2 = 2 > (z_1^* + z_2^* + z_3^*) - z_1^* = (1 + 0.9 + 1) - 1 = 1.9$

LHS $>$ **RHS** Violated fractional ARP DFJ

Violated ARP GFSEC: $S = \{1,2,3\}$: $\textbf{LHS} = Q \sum_{(i,j)\in E(S)} x_{ij}^* = 30 * (0.9 + 0.9 + 0.2) = \textbf{60}$

$$\textbf{RHS} = \sum_{i\in S}(Qz_i^* - \sum_{k\in K_i} b_k q_{ik}^*) = 30 * (1 + 0.9 + 1) - (10 + 10 + 10) = \textbf{57}$$

LHS $>$ **RHS** Violated fractional ARP GFSEC

Results on the Large ARP Instances

Chitsaz et al. (2019) presented two lower bounding methods for the ARP. The first method (BC-T) is a truncated BC with a time limit of 12 hours. BC-T uses the best-bound node selection strategy. It adds inequalities (2.26) and (2.28) a priori to the model, and SECs (2.12) and (2.27) dynamically through the search using the CVRPSEP package for separation. The second method (MIP-CP) relaxes SECs (2.12) from the model and solves the resulting MIP. Then, it iteratively adds the violated SECs (2.12) as cutting planes for the resulting integral subtours and re-solves the new MIP. A time limit of five hours is set for this method.

In Table B.1, we present the performance of CCJ-DH, BC-T, and MIP-CP, and compare them with our BC. In these experiments, the BC uses all inequalities and implements algorithm $\mathcal{A}2$ to separate SECs. Two branching node selection strategies are examined: balanced between optimality and feasibility (default) or the best-bound node selection. Because BC-T is able to solve the small instances with 14 suppliers in the first set (MV-C1) to optimality in a very short time, we did not apply our BC to these instances. Columns four to six present the results for CCJ-DH: CPU, #BUB, and the average solution value as a percentage of the best lower bound obtained by the BC method (%BLB). Columns 7 to 11 show the results for BC-T: CPU, #BUB, the number of best lower bounds (#BLB), %UB, and %BUB. Columns 12 to 14 show the results for MIP-CP which only generates lower bounds: CPU, #BLB, and %BUB. Columns 15 to 19, and 20 to 24 include similar results as columns 7 to 11 for the BC of this paper with the default and with the best-bound node selection strategies, respectively.

Columns under #BUB and %UB for the BC-T and our BC methods reflect the results without considering the CCJ-DH cutoffs. The comparison of columns under %UB and %BUB for each of the BC-T and our BC methods shows the effectiveness of CCJ-DH in finding upper bounds for these large instances. Most of the BUBs for the instances with $n = 50$ and all of the BUBs for the instances with $n = 100$ are obtained by CCJ-DH. BC-T is unable to find upper bounds for the instances with $n = 100$. Therefore, it returns zero under column %UB in all four classes of these instances. Our BC with the best-bound node selection strategy is performing better than with the default node selection strategy. Moreover, it outperforms the two other methods presented in Chitsaz et al. (2019), both in terms of number of BLBs, and %BUBs.

Finally, we present more details on the performance of our BC with the default and with the best-bound node selection strategies in Table B.2. In this table we present #Node, GFS, AV^{GFS}, DFJ, and AV^{DFJ}. Although within the default node selection strategy the BC explores more nodes, the best-bound strategy returns better lower bounds. Another

Table B.1: Comparison of the BC performance with the lower bounding methods presented in Chitsaz et al. (2019)

			Chitsaz et al. (2019)											BC (This paper)									
			CCJ-DH			BC-T					MIP-CP			Default					Best-Bound				
n	Class	Size	CPU	#BUB	%BLB	CPU†	#BUB	#BLB	%UB	%BUB	CPU‡	#BLB	%BUB	CPU††	#BUB	#BLB	%UB	%BUB	CPU††	#BUB	#BLB	%UB	%BUB
50	1	120	602.8	116	99	43200	2	0	52	98.3	18000	0	97.9	3600	1	0	47.6	98.6	3600	1	120	23	99
	2	120	592.4	112	99	43200	7	1	52.1	98.5	18000	0	97.9	3600	0	1	40.6	98.6	3600	1	118	23.7	99
	3	120	467.8	119	96.2	43200	1	0	35.4	93.9	18000	0	91.7	3600	0	2	29.5	94.6	3600	0	118	10.1	96.2
	4	120	914.4	109	99.3	43200	10	1	72.6	99	18000	24	99	3600	1	0	51.3	98.9	3600	0	95	24	99.3
	Total	480	644.4	456	98.3	43200	20	2	53	97.4	18000	24	96.6	3600	2	3	42.3	97.7	3600	2	451	20.2	98.4
100	1	120	2966.6	120	97.9	43200	0	9	0	97.1	18000	4	97.3	3600	0	25	1.4	97.1	3600	0	82	3.4	97.9
	2	120	2931.6	120	97.9	43200	0	8	0	97.1	18000	2	97.3	3600	0	15	2.6	97.4	3600	0	95	2.6	97.9
	3	120	1971.3	120	91.4	43200	0	8	0	89.7	18000	1	89.2	3600	0	26	0.3	90.5	3600	0	85	0	91.3
	4	120	4212.6	120	98.6	43200	0	14	0	97.4	18000	46	98.3	3600	0	9	2.5	97.7	3600	0	51	2.6	98.5
	Total	480	3020.5	480	96.5	43200	0	39	0	95.3	18000	53	95.5	3600	0	75	1.7	95.7	3600	0	313	2.2	96.4
Total		960	1832.4	936	97.4	43200	20	41	26.5	96.4	18000	77	96.1	3600	2	78	22	96.7	3600	2	764	11.2	97.4

Size: Number of instances, † Time limit = 12 hours and maximum 30 GB memory, ‡ Time limit = 5 hours and maximum 30 GB memory, †† Time limit = one hour and maximum 25 GB memory interesting observation is that the method with the default node selection strategy applies more GFSECs and DFJs with almost the same average violation on the instances with $n = 50$. This reflects the fact that the method with the default node selection strategy explores some nodes that do not contribute much to improve the lower bound.

Table B.2: Summary of the results of the BC on the large ARP instances of Chitsaz et al. (2019) with different node selection strategies

Node Selection	n	Class	Size	%UB	%BUB	#Node	GFS	AV^{GFS}	DFJ	AV^{DFJ}
Default	50	1	120	47.6	98.6	2014.3	1625	0.21	6039	0.4
	50	2	120	40.6	98.6	1778.9	1533	0.21	5666	0.4
	50	3	120	29.5	94.6	1547	1814	0.21	5882	0.39
	50	4	120	51.3	98.9	2434.6	1069	0.22	5640	0.48
	Total		480	42.3	97.7	1944.2	1510	0.21	5806	0.42
	100	1	120	1.4	97.1	4.6	1939	0.28	3549	0.37
	100	2	120	2.6	97.4	5.3	2032	0.28	3728	0.36
	100	3	120	0.3	90.5	0.6	2263	0.25	3859	0.32
	100	4	120	2.5	97.7	35.8	1346	0.32	3429	0.48
	Total		480	1.7	95.7	11.5	1896	0.28	3641	0.38
Best-Bound	50	1	120	23	99	987.1	1160	0.22	3907	0.39
	50	2	120	23.7	99	1070.1	1146	0.22	4047	0.39
	50	3	120	10.1	96.2	653	1336	0.22	3760	0.37
	50	4	120	24	99.3	2255.2	700	0.24	3969	0.5
	Total		480	20.2	98.4	1242.1	1085	0.23	3921	0.41
	100	1	120	3.4	97.9	1.7	1921	0.28	3668	0.38
	100	2	120	2.6	97.9	1.3	2098	0.28	3730	0.37
	100	3	120	0	91.3	0.1	2140	0.26	3970	0.33
	100	4	120	2.6	98.5	22.6	1442	0.32	3664	0.48
	Total		480	2.2	96.4	6.4	1899	0.28	3757	0.39

Size: Number of instances, Time limit = 1 hour

Detailed Results on Effect of Valid Inequalities

Each type of valid inequality introduced in Section 2.3 of the main paper has a different effect on the LP relaxation value and solution time of the $\mathcal{M}_{ARP}$ model. To evaluate the effect of applying different inequality types, we performed a sensitivity analysis considering different scenarios. We consider the effect on the LP solution value when only one inequality type is added to the model. Also, we evaluate the effect when all types of valid inequalities but one are added. Furthermore, we consider the cases where no valid inequality (None), known valid inequalities (Known) from the literature (i.e., (2.18), (2.26), and (2.28)), or all valid inequalities (All) (i.e., (2.18)-(2.26), and (2.28)) are added to the model. Similar to the results presented in Table 2.4, we present the obtained lower bound as a percentage of the best upper bound found by the BC method or CCJ-

DH. Tables B.3, B.4 and B.5 present the results for each class of instances. Each column number in these tables refers to the associated valid inequality type number presented in Section 2.3 of the paper. For the first class of instances, inequalities (2.18), (2.21) and (2.24) have the greatest impact. For the second and third classes of instances, inequalities (2.18), (2.22) and (2.24) show the largest LP solution value improvements.

Table B.3: Effect of individual valid inequality types on average LP solution value as a percentage of BUB (class 1)

			Including only one type											Excluding only one type										
	Set		(l,S,WW)-type				Var Bnd				Gen Ineq			(l,S,WW)-type				Var Bnd				Gen Ineq		
$C/l/n$	Size	None	(2.18)	(2.19)	(2.20)	(2.21)	(2.22)	(2.23)	(2.24)	(2.25)	(2.26)	(2.28)	Known	(2.18)	(2.19)	(2.20)	(2.21)	(2.22)	(2.23)	(2.24)	(2.25)	(2.26)	(2.28)	All
1/4/18	5	60.4	69.6	66.3	66.1	66.1	65.5	62	67.4	60.4	60.4	60.7	69.9	82.8	86.2	84.4	84.9	84.4	86	85.2	86.6	86.6	84.2	86.6
1/4/21	5	57.2	69.9	60.8	60.9	61.5	64.6	59.8	63.1	57.3	57.3	57.6	70.3	77.6	86.1	84.4	84.9	82.5	85.3	85.3	86.3	86.3	84.1	86.3
1/4/24	5	56.5	68.5	61	61	61.9	62	58.7	63	56.5	56.5	56.8	68.9	78.6	85.6	84.2	84.8	83.2	85.6	85.3	86.3	86.3	83.6	86.3
1/4/27	5	59.1	70.1	62.4	63.4	64	65.1	60.9	65.1	59.1	59.1	59.3	70.4	78.5	85.9	84.6	85.4	83.3	85.8	85.6	86.6	86.6	84.7	86.6
1/4/30	5	62.1	76.3	65.2	65.6	66.1	68.9	63.1	68.4	62.1	62.1	62.3	76.6	80.6	90.8	89.9	90.1	87.1	90.7	89.4	91	91	88.9	91
1/4/33	5	61	73.4	64.3	65.4	65.8	67.4	62.7	67.9	61	61	61.2	73.7	80.8	89.2	88.2	88.6	86	89	88	89.7	89.7	88.2	89.7
1/4/36	5	61.2	72.3	66.7	66.2	66.9	66	62.2	67.6	61.2	61.2	61.4	72.5	82.1	87.5	85.7	86.2	85	87.4	86.9	87.9	87.9	85.9	87.9
1/4/39	5	53.9	63.7	58.2	58.4	59.2	61.9	57	59.4	53.9	54	54.4	64.2	78.4	82.4	81.3	82	79.6	82.3	82.2	83.3	83.3	80.4	83.3
1/6/15	5	67.5	79.1	71.3	70.8	72.2	71.1	70.1	72.2	67.5	67.6	67.8	79.5	85.9	92.3	91.2	90.4	91.1	91	91.3	92.4	92.4	89.8	92.4
1/6/18	5	65.8	74	67.8	70.2	72.7	68.3	68	72.4	65.8	65.8	66.1	74.2	83.8	89	87.7	86.2	87.8	87.7	87.7	89	89	87	89
1/6/21	5	56.4	72	63.4	60.7	61.8	61.7	58	62.7	56.4	56.4	56.7	72.4	79.3	86.6	85.7	85.8	85.3	86.9	86.1	87.4	87.4	85.4	87.4
1/6/24	5	60.3	74	63.9	64.8	67.3	65.5	62.4	66.1	60.3	60.4	60.6	74.3	81.4	89.9	88.4	87	87.7	89.4	89.4	90	90	88	90
1/6/27	5	63.5	76.2	67.3	67.9	69.2	67.9	64.6	69.8	63.5	63.5	63.7	76.4	82.7	90.7	89.9	89.3	89.2	91.1	89.9	91.3	91.3	89	91.3
1/6/30	5	60.5	74.3	65.6	65.6	67.4	64.4	62.5	66.5	60.5	60.5	60.9	74.7	82.7	89.6	87.9	87	89	89.1	89.2	89.8	89.8	87.1	89.8
1/6/33	5	55.9	69.2	61.3	60.8	65.8	61.1	58.8	61.9	55.9	56	56.2	69.7	82.1	86.9	86.7	85.1	85.8	87.2	86.8	88	87.8	86.2	88
1/6/36	5	54	73.6	59.8	58.8	60.1	60.7	56.8	60.9	54	54.2	54.3	74	77.7	89.7	88.1	87.3	87.6	89	88.5	89.7	89.7	87.5	89.7
1/8/12	5	69.7	79	72.1	72.9	75.6	72.4	72	74.3	69.7	69.8	70	79.3	85.8	91.6	90.9	89.1	90.4	90.6	90.8	91.7	91.7	89.9	91.7
1/8/15	5	68.9	79.1	70.6	72	74.4	72.6	70.2	74.2	69.1	69	69.3	79.5	84.4	91.2	91	89.8	89.6	91.4	89.6	91.5	91.5	89.6	91.5
1/8/18	5	64.6	78.9	68.1	67.5	71.3	68	66.4	68.7	64.7	64.7	64.9	79.3	82.4	92.2	91.4	88.7	90.3	91.8	91.4	92.2	92.1	90.2	92.2
1/8/21	5	62.7	75.3	68.2	66.7	67.4	65.7	63.7	67.4	62.7	62.7	62.8	75.5	80.6	86.9	86.6	86.7	87.4	88.2	87.7	88.4	88.3	86.9	88.4
1/8/24	5	65.4	77.5	73.1	70	70.2	68.5	67.3	70.4	65.4	65.5	65.6	77.7	86	89.8	88.3	88.7	89.9	90.3	89.9	90.4	90.3	88.2	90.4
1/8/27	5	66.6	79.7	71.3	70.5	70.9	69.5	68.2	71.9	66.6	66.7	66.9	80	84.1	90.8	89.7	89.7	90.7	91	90.1	91.2	91.2	89.4	91.2
1/8/30	5	61.3	73.8	62.8	64.6	69.4	65.2	63.7	66.9	61.4	61.4	61.8	74.5	80.8	89.5	89.1	86.9	87.7	89.2	88.4	89.7	89.6	86.8	89.7
1/8/33	5	63	74.1	69.1	66.9	68.1	66.2	64.7	67.8	63	63	63.3	74.4	82.3	86	85.1	85.1	86.6	86	86.2	86.9	86.9	84.8	86.9
1/10/9	5	67	82.7	68.2	69.2	72.5	71.2	68.3	71.2	67.3	67.1	67.3	83.1	81.8	93.3	93.2	91	92.2	93.3	92.1	93.5	93.4	92.1	93.5
1/10/12	5	67.3	78.3	68.7	70.4	74.1	71.1	68.9	71.8	67.4	67.4	67.8	78.8	84.1	91.8	91.4	89	90.1	91.9	90.9	92	91.9	89.5	92
1/10/15	5	64.5	79	67.9	67.5	68.6	67.7	66.1	69	64.6	64.6	64.8	79.4	79.6	89.9	89.7	88.8	89.6	90.5	89.8	90.7	90.5	89.1	90.7
1/10/18	5	68.2	80.6	71.8	71.9	71.8	71	69.1	73.2	68.2	68.2	68.3	80.8	82.2	90.3	89.4	89.9	90.1	90.6	90	90.8	90.7	89.4	90.8
1/10/21	5	67.3	80.5	71.2	71.1	72.5	70	68.3	72.2	67.3	67.3	67.4	80.7	83.1	91.7	90.4	89.1	90.7	91.6	91	91.7	91.6	90.3	91.7
1/10/24	5	64.2	76.7	69.1	68.2	69.4	69.3	66.2	69.6	64.2	64.3	64.4	77	83.4	89.4	88.7	88.1	89	89.4	89	89.9	89.9	88.1	89.9
1/10/27	5	64.6	74.5	67.8	68.7	70.5	66.8	67.4	69.2	64.6	64.7	64.9	74.9	81.4	87.5	86.1	85.3	87.1	86.4	87.5	87.8	87.8	86.2	87.8
1/10/30	5	62.8	74	65.9	67.7	69.6	65.5	65.4	68.3	62.8	62.8	63.1	74.4	81.6	87.8	86.7	85.7	87.6	86.8	87.6	88.2	88.2	86.1	88.2
1/12/6	5	71.2	83	73.1	74.2	74.6	74.4	73.1	75.8	71.2	71.3	71.4	83.3	84.6	93	92.2	91.8	92.6	92.8	92.1	93.1	93	91.4	93.1
1/12/9	5	63.8	75.6	67.4	68.1	70.9	66.1	66	68.7	63.8	63.8	64.1	76	82.2	88.2	87.1	86	88.1	87.5	87.7	88.5	88.5	86.8	88.5
1/12/12	5	61	78.1	63.5	64	68.2	65	62.3	66.1	61.2	61	61.3	78.4	78.2	90.7	90.4	88.8	89.4	91	90	91.1	91	89.4	91.1
1/12/15	5	66.2	82.2	69.7	69.2	69.7	70.3	67.1	70.9	66.3	66.3	66.5	82.4	81.6	92.7	92	91.8	91.5	92.9	91.7	93	92.9	91.2	93
1/12/18	5	68.6	80.4	71.8	71.8	72	72	69.7	73.6	68.7	68.6	68.8	80.7	83.5	90.9	90.5	90.6	90.8	91.5	90.3	91.6	91.6	89.5	91.6
1/12/21	5	63.9	74	65.9	67.8	70.6	68.4	64.6	68.9	64.4	64	64.4	74.5	81.8	87.2	86.7	86.7	86.1	87.8	86.5	87.9	87.7	86.1	87.9
1/12/24	5	66.2	79.3	70.4	69.5	72.3	68.6	66.8	70.2	66.2	66.2	66.4	79.5	82.1	89.5	89.5	87.6	90.2	90.5	90.1	90.6	90.5	88.8	90.6
1/12/27	5	56.8	77.1	61.7	60.6	65.5	62.8	58.5	61.9	57.1	57	57.4	77.7	79.6	90.4	90.2	88	90.2	90.9	90.3	91.1	90.8	88.9	91.1
Total	200	63	75.7	66.9	66.9	68.7	67.3	64.8	68.4	63.1	63.1	63.3	76.1	81.8	89.3	88.4	87.7	88.1	89.2	88.7	89.7	89.7	87.7	89.7

Note. $C/l/n$: Class/Number of periods/Number of suppliers, Var Bnd: Bounds on the variables, Gen Ineq: General inequalities

Table B.4: Effect of individual valid inequality types on average LP solution value as a percentage of BUB (class 2)

			Including only one type											Excluding only one type										
	Set		(l,S,WW)-type				Var Bnd				Gen Ineq			(l,S,WW)-type				Var Bnd				Gen Ineq		
C/l/n	Size	None	(2.18)	(2.19)	(2.20)	(2.21)	(2.22)	(2.23)	(2.24)	(2.25)	(2.26)	(2.28)	Known	(2.18)	(2.19)	(2.20)	(2.21)	(2.22)	(2.23)	(2.24)	(2.25)	(2.26)	(2.28)	All
2/4/18	5	71.9	81.9	71.9	74.9	72	76.6	72.7	78.5	71.9	71.9	72	82	85.9	92.8	91.8	92.7	91.2	92.4	87.8	92.8	92.7	91.7	92.8
2/4/21	5	69	76.9	69	71.5	69.2	75.6	70.7	75.1	69	69.1	69.2	77.2	85.1	89.7	88.6	89.6	86.3	89.1	85.6	89.7	89.6	88.6	89.7
2/4/24	5	64.6	78.7	64.6	67.9	65	71.7	65.7	71.5	64.6	64.7	64.8	78.9	82	91.3	90.2	91.1	89	90.8	86.4	91.3	91.2	89.7	91.3
2/4/27	5	66.7	81.3	66.7	70.1	66.9	73.7	67.7	73	66.7	66.7	66.8	81.5	83.1	92.9	91.7	92.8	90.7	92.5	88.7	92.9	92.8	91.7	92.9
2/4/30	5	68.7	80.8	68.7	72.5	68.9	74.6	69.7	75.9	68.7	68.7	68.9	80.9	85	92.6	91.3	92.6	91.2	92.4	87.9	92.6	92.6	91.4	92.6
2/4/33	5	69.4	80.6	69.4	73.4	69.6	75.1	70.3	76.2	69.4	69.4	69.5	80.7	85.3	92.3	90.7	92.2	91	92	87.8	92.3	92.2	91.1	92.3
2/4/36	5	65.6	77.5	65.6	70.4	65.7	71.1	67.6	73.3	65.6	65.7	65.8	77.8	83.7	91.7	89.9	91.7	90.1	90.7	87.1	91.7	91.7	90.2	91.7
2/4/39	5	55.2	70.3	58.2	60.9	55.9	64.9	56.7	62.6	55.2	55.2	55.4	70.6	79.2	88.3	85.2	88.4	85	87.9	84.9	88.4	88.4	86.7	88.4
2/6/15	5	72.9	82.2	72.9	77.1	73.1	76.8	74.2	77	72.9	72.9	73	82.4	85.6	92.7	90.3	92.7	91.1	92.2	90.4	92.7	92.7	91.5	92.7
2/6/18	5	63.1	77.6	63.1	68.1	63.3	68.9	61.6	68.4	63.1	63.1	63.3	77.9	79.8	90.6	87.9	90.6	88.6	90.1	87.1	90.6	90.6	88.6	90.6
2/6/21	5	73.1	79.3	73.1	77.3	73.2	76.4	74.6	78.4	73.1	73.1	73.2	79.5	86.2	90.9	88.4	90.8	89.3	90.2	87.8	90.9	90.8	89.5	90.9
2/6/24	5	72.8	84	72.8	75.8	72.8	76.5	74.5	76.2	72.8	72.8	72.9	84.2	84.1	93.2	91.5	93.2	91.6	92.1	90.8	93.2	93.2	92.2	93.2
2/6/27	5	56.7	75.8	57	64.1	57.8	62.7	57.8	63.1	56.8	56.8	56.9	76.1	75.1	89.7	86	89.7	88.4	89.3	86.9	89.7	89.5	87.9	89.7
2/6/30	5	59.8	73.3	61.5	66.2	60.6	67	61.8	66.1	59.8	59.8	60	73.7	81.8	90.2	86.4	90.2	87.8	89.8	87.6	90.3	90.2	88	90.3
2/6/33	5	59.4	76.4	59.4	66.3	60	65.4	61.1	65	59.4	59.5	59.6	76.7	77.7	90.7	87	90.6	88.8	90.1	88.5	90.7	90.4	88.6	90.7
2/6/36	5	53.8	75.4	54.1	61.5	54.2	61.3	55.4	61.8	53.8	53.8	53.9	75.6	75.5	91.8	88.1	91.7	90.1	91.3	88.1	91.8	91.6	89.9	91.8
2/8/12	5	73.7	83.8	73.7	76.8	73.8	76.6	75.2	76.8	73.7	73.7	73.9	81	83.6	92.1	90.2	92	90.8	91.2	90.1	92.1	92	91.1	92.1
2/8/15	5	71.1	83.4	71.1	74.9	71.2	75.7	72.5	74.5	71.1	71.2	71.2	83.5	83.3	92.6	90.2	92.6	91.1	91.9	90.5	92.6	92.6	91.7	92.6
2/8/18	5	76.4	82.7	76.4	80.3	76.5	79.9	77.1	80.3	76.4	76.4	76.6	82.9	87.8	92.2	89.5	92.2	90.9	91.9	89.6	92.2	92.2	90.9	92.2
2/8/21	5	63	77.7	63.1	67	63.5	72.3	65.6	68	63.1	63.2	63.3	78.2	82.6	90.2	88.6	90.2	88.2	89.1	87.9	90.2	90.1	88.6	90.2
2/8/24	5	58	73.1	58.5	64.8	59	66.9	59.7	64.3	58.1	58	58.1	73.4	78.5	88.7	85.4	88.7	85.3	88.5	86.7	88.7	88.6	87.5	88.7
2/8/27	5	52.3	70.8	53.3	60.6	53.4	62.3	54.6	60.9	52.4	52.4	52.6	71.1	78.4	90.1	86.1	90.1	87.7	89.5	87.2	90.1	89.9	87.5	90.1
2/8/30	5	60.6	79.2	60.6	66.7	61	67.1	61.8	66.4	60.6	60.6	60.7	79.4	77.6	91.9	89	91.9	90.1	91.8	89.7	91.9	91.8	90.5	91.9
2/8/33	5	63.8	79.3	63.9	69.1	64.1	71.3	65.3	68.4	63.8	63.8	63.9	79.6	81.8	91.9	88.7	91.9	90.2	91.1	89.6	91.9	91.9	90.1	91.9
2/10/9	5	69.6	79.9	69.6	75	69.7	72.3	70.8	73.6	69.6	69.6	69.8	80.1	81.7	90.6	87.2	90.6	89.5	89.9	88.9	90.6	90.6	89.2	90.6
2/10/12	5	62	73.7	62	67.6	62.6	70.9	63.7	68.8	62	62.1	62.3	74.1	82.5	88.3	86.2	88.3	85.8	88	85.5	88.3	88.2	86.1	88.3
2/10/15	5	60.8	77	62.2	67.7	61.5	69.5	61.6	67	60.8	60.8	61	77.2	81.1	89.9	86.6	90.1	88.9	90	88	90.1	89.9	88.2	90.1
2/10/18	5	51.8	70	52.6	61.7	53.1	59.8	52.8	60.7	51.9	51.9	52.1	70.4	76.4	90.6	85.6	90.6	87.6	90.5	87.8	90.6	90.5	87.3	90.6
2/10/21	5	65	79	65	71.3	65.3	71.6	66.2	70.1	65	65.1	65.1	79.2	81.7	90.8	87	90.8	89.6	90.4	89.4	90.8	90.7	89.6	90.8
2/10/24	5	59	74.1	60.2	68.1	59.4	65.1	60.7	65.9	59	59.1	59.3	74.5	79.8	90.8	84.9	90.8	90	90.4	89.1	90.8	90.7	88.4	90.8
2/10/27	5	62.2	77.3	62.2	68	62.4	66.5	64	67.1	62.2	62.2	62.4	77.6	77.2	89.8	86.6	89.8	88.2	88.7	87.8	89.8	89.8	88.3	89.8
2/10/30	5	52.6	65.6	56.4	56.7	54.3	60.5	55.6	58.3	52.7	52.9	53	66.2	73	81	80.4	82.5	79.6	81.9	80.6	82.5	82.4	81.1	82.5
2/12/6	5	70.5	79.5	70.5	75.7	70.6	73.3	72.4	74.3	70.5	70.6	70.8	79.7	82.4	89.3	86.2	89.3	88.7	88.1	87.8	89.3	89.3	88.2	89.3
2/12/9	5	68.7	77.5	69.5	74.8	69	73.5	70.5	73.1	68.7	68.7	68.9	77.7	84.1	89.8	85.9	89.8	88.8	88.8	88.2	89.8	89.8	88.4	89.8
2/12/12	5	65.2	76	65.7	73	65.8	69.9	65.9	71.6	65.2	65.2	65.3	76.1	81.8	89.1	85.2	89.2	88.6	89.1	87.4	89.3	89.2	87.7	89.3
2/12/15	5	55.3	73.8	58.1	64.4	56.5	61.2	56.6	62.4	55.4	55.4	55.6	74	77.3	90.6	85.3	90.6	89.5	90.5	88.5	90.6	90.3	87.4	90.6
2/12/18	5	52.4	71.6	52.8	54.9	53.2	68.5	56.2	57.6	52.8	52	52.8	72.4	77.7	85.4	85	85.4	81.3	84.9	83.2	85.4	85.1	84.7	85.4
2/12/21	5	52.5	62	53.7	58	54.5	64.4	55.5	60.5	52.9	52.8	52.9	62.5	78.8	82.9	80.6	82.8	78.4	82.4	79.1	82.9	82.6	80.4	82.9
2/12/24	5	56.5	73.5	56.6	65.1	57.8	63.7	58.2	64	56.6	56.6	56.7	73.6	76.2	88.2	84.6	88.2	86.5	88.1	86.6	88.2	88	86.9	88.2
2/12/27	5	54.6	72.9	56.9	64.5	55.9	63.2	55.6	62.4	54.7	54.6	54.7	73.1	77.5	89.5	84.2	89.6	88.8	89.5	88.4	89.6	89.4	87.6	89.6
Total	200	63	76.6	63.6	68.6	63.6	69.6	64.6	69	63.1	63.1	63.2	76.9	81	90.2	87.4	90.2	88.3	89.7	87.5	90.2	90.1	88.6	90.2

Note. $C/l/n$: Class/Number of periods/Number of suppliers, Var Bnd: Bounds on the variables, Gen Ineq: General inequalities

Table B.5: Effect of individual valid inequality types on average LP solution value as a percentage of BUB (class 3)

	Set		Including only one type												Excluding only one type										
			(l,S,WW)-type				Var Bnd				Gen Ineq			(l,S,WW)-type				Var Bnd				Gen Ineq		
C/l/n	Size	None	(2.18)	(2.19)	(2.20)	(2.21)	(2.22)	(2.23)	(2.24)	(2.25)	(2.26)	(2.28)	Known	(2.18)	(2.19)	(2.20)	(2.21)	(2.22)	(2.23)	(2.24)	(2.25)	(2.26)	(2.28)	All
3/4/18	5	68.1	70.4	68.1	68.8	68.1	74.3	69.9	82.1	68.1	68.3	68.3	70.9	91.9	92.5	92.4	92.5	88	92	77.9	92.5	92.5	90.2	92.5
3/4/21	5	66.5	68.4	66.5	66.8	66.6	74.6	68.4	78.2	66.6	66.7	66.9	68.9	89.8	90.6	90.5	90.6	83.5	89.9	78	90.6	90.5	88.1	90.6
3/4/24	5	64.7	68.1	64.7	65.8	64.7	76.4	66.1	77.3	64.7	64.9	65	68.5	91.5	92.9	92.7	92.9	83	92.5	81.1	92.9	92.8	91.6	92.9
3/4/27	5	65.3	67.6	65.3	66.1	65.3	78.2	66.2	78.4	65.3	65.4	65.5	68	94.3	94.3	94.2	94.3	83.6	94.2	80.6	94.3	94.1	92.2	94.3
3/4/30	5	67	70.5	67	67.2	67	77.4	68.6	79.4	67	67.1	67.3	71	92.6	93.9	93.9	93.9	85.5	93.1	81.2	93.9	93.9	92.1	93.9
3/4/33	5	64.6	68.3	64.6	65.2	64.6	73.9	66.3	78.5	64.6	64.8	64.9	68.9	91.6	92.9	92.8	92.9	85.9	92.5	78.4	92.9	92.9	89.9	92.9
3/4/36	5	61.5	66.8	61.5	62.1	61.7	71.6	65.6	75.9	61.7	62	62.2	67.8	91.5	92.3	92.1	92.3	86.6	90.3	78.1	92.3	92.3	90.2	92.3
3/4/39	5	46.1	53.5	46.1	48.2	46.2	62.2	48	66.7	46.2	46.2	46.4	53.9	87.1	88.7	88.4	88.7	78.1	88	69.6	88.7	88.5	85.7	88.7
3/6/15	5	70.4	73.5	70.4	71.2	70.5	76.8	72.2	81.3	70.4	70.6	70.8	74	91	92	91.9	92	87.7	91.4	81.2	92	91.9	90.2	92
3/6/18	5	69.3	72.9	69.3	70.4	69.3	75.4	70.3	79.5	69.3	69.4	69.5	73.2	89	89.9	89.6	89.9	86.4	89.5	79.6	89.9	89.7	87.2	89.9
3/6/21	5	63.6	69	63.6	65.5	63.7	70.8	65.6	74.2	63.7	63.8	63.9	69.6	86.9	88.2	87.7	88.2	85.1	87.5	77.8	88.2	87.7	84.9	88.2
3/6/24	5	65.9	68.3	65.9	67.5	66	72.9	67.1	75.9	66	66	66.3	68.8	88.1	88.4	88.2	88.4	83.4	88.1	77.8	88.4	88.2	84.3	88.4
3/6/27	5	67.3	71.6	67.3	68	67.4	76.4	68.9	78.1	67.4	67.4	67.5	71.9	90.4	91	90.7	91	85.5	90.5	80.1	91	91	88.7	91
3/6/30	5	60.9	67	60.9	62.3	61	74.6	62	71.8	61	61	61.1	67.3	89.3	90.5	90.2	90.5	81.7	90.5	79.2	90.5	90	87.9	90.5
3/6/33	5	65.5	68	65.5	66.7	65.5	72.6	69.2	73.2	65.5	65.9	66.1	69	86.4	87.1	86.6	87.1	81.6	85.4	79.9	87.1	86.5	84.6	87.1
3/6/36	5	60.3	69.5	60.3	61.9	60.4	73.2	63.1	69.2	60.4	60.5	60.9	70.2	86.9	89.3	88.7	89.3	83.1	88.4	81.3	89.3	89.1	86.5	89.3
3/8/12	5	73.4	74.2	73.4	74.9	73.5	78.5	77.4	81.1	73.5	73.7	73.8	74.9	90.7	91	91	91	86.4	89.4	84.6	91	91	89.6	91
3/8/15	5	65.8	72.3	65.8	67.1	65.8	75.6	67.1	74.7	65.8	65.8	65.9	72.7	87	89.3	89.2	89.3	83.7	89	81.1	89.3	89.2	87.7	89.3
3/8/18	5	71.5	75.9	71.5	73.2	71.6	76.6	73.4	79.1	71.6	71.6	71.8	76.3	87.6	89.8	89.3	89.8	86.9	89.5	83.2	89.8	89.7	87.6	89.8
3/8/21	5	67.7	70.7	67.7	68.8	67.8	75.1	69.9	74.8	67.8	67.8	68	71.1	86.5	87.9	87.6	87.9	82	87.3	80.8	87.9	87.7	85.5	87.9
3/8/24	5	63.5	67.6	63.5	65.3	63.5	70.2	64.9	73.2	63.5	63.6	63.9	68.1	84.1	85.3	85.1	85.3	81.5	85.1	76	85.3	85.3	82.1	85.3
3/8/27	5	71.5	74.3	71.5	72	71.5	77	73.9	79.1	71.5	71.6	71.7	74.7	88.3	89.3	89	89.3	85.6	88.7	81.7	89.3	89.2	87.2	89.3
3/8/30	5	70.6	74.4	70.6	71.4	70.6	75.8	71.6	78.2	70.6	70.6	70.8	74.8	86.3	88	87.8	88	84.9	87.8	80.2	88	87.9	86.1	88
3/8/33	5	65.4	73	65.4	66.5	65.5	73.4	66.5	73.2	65.4	65.5	65.6	73.3	84.2	87.4	87.2	87.4	83.8	87.3	79.7	87.4	87	85.6	87.4
3/10/9	5	66	71.9	66	67.8	66.2	74.2	71.5	72.2	66.1	66.5	66.5	73	85.7	88.8	88.3	88.8	83.5	86.5	85.4	88.8	88.8	87.7	88.8
3/10/12	5	64.2	69.9	64.2	66.9	64.3	70.9	66.8	72.4	64.3	64.4	64.7	70.6	83.6	85.8	85	85.8	82.8	84.7	80.3	85.8	85.3	83.8	85.8
3/10/15	5	67.3	73.4	67.3	69.2	67.4	73.3	69.4	75	67.4	67.4	67.6	73.8	84.5	87.4	87	87.4	84.7	86.9	81.2	87.4	87.3	85.4	87.4
3/10/18	5	63	67.5	63	64.7	63.1	68.9	65.4	71.4	63	63.1	63.2	67.9	82	84	83.5	84	80.6	83.7	76.2	84	83.6	81.7	84
3/10/21	5	65.7	67.2	65.7	67.6	65.8	70.8	68.5	73.9	65.7	65.9	66	67.7	84.9	85.6	85.2	85.6	81.2	85.2	77.9	85.6	85.6	82.3	85.6
3/10/24	5	65.8	69.9	65.8	67.5	65.9	72.1	67.6	73.8	65.8	65.9	66.1	70.3	84.2	86.1	85.7	86.1	82	86.1	79	86.1	86	83.8	86.1
3/10/27	5	67.7	71.8	67.7	69.7	67.8	73.7	68.3	76.4	67.8	67.8	67.9	72.1	85.3	87.1	86.8	87.1	83.4	87.1	79.2	87.1	86.9	84.9	87.1
3/10/30	5	66.3	72.1	66.3	67.7	66.4	74.2	68.1	73.4	66.3	66.4	66.5	72.5	85	86.9	86.4	86.9	82.9	86.7	80.1	86.9	86.8	85	86.9
3/12/6	5	70.4	74	70.4	72.8	70.5	74.4	72.3	78.6	70.4	70.5	70.6	74.3	86.1	88.2	88	88.2	85.7	87.7	81.8	88.2	88	86.6	88.2
3/12/9	5	69.5	73.8	69.5	71	69.5	75	70.7	76	69.5	69.5	69.7	74.1	85.3	87.6	87.4	87.6	84	87.2	81.5	87.6	87.4	85.1	87.6
3/12/12	5	67.6	71.7	67.6	70.1	67.9	72.5	70.4	74.4	67.7	67.9	67.9	72.2	83.8	85.8	85.3	85.8	83	85.3	81	85.8	85.7	84.2	85.8
3/12/15	5	68.7	71.3	68.7	70.3	68.8	73.3	70.8	74.8	68.7	68.8	69	71.8	83.2	84.5	84.2	84.5	81.3	83.9	79.4	84.5	84.4	82.3	84.5
3/12/18	5	65.7	70.1	65.7	67.6	65.8	73.7	68.2	71.7	65.8	65.8	66.2	70.8	84.5	86.2	86	86.2	80.9	85.9	81.4	86.2	85.7	83.2	86.2
3/12/21	5	65.2	70.1	65.2	67.6	65.3	70.5	66.9	73.5	65.2	65.3	65.4	70.4	83.5	85.8	85.4	85.8	83.1	85.7	79	85.8	85.4	82.9	85.8
3/12/24	5	66.3	71.9	66.3	68.4	66.3	72	69.4	74.4	66.3	66.5	66.5	72.4	84.3	87.4	87.3	87.4	84.2	86.1	80.9	87.4	87.3	85.9	87.4
3/12/27	5	60.1	68.8	60.1	62.9	60.7	70.2	62.5	68.2	60.2	60.3	60.8	69.7	83.2	86.9	86.5	86.9	81.8	86.3	80.9	86.9	86.7	84.5	86.9
Total	200	65.9	70.3	65.9	67.4	66	73.6	68	75.3	65.9	66.1	66.2	70.8	87.1	88.7	88.4	88.7	83.7	88.1	79.9	88.7	88.5	86.4	88.7

Note. $C/l/n$: Class/Number of periods/Number of suppliers, Var Bnd: Bounds on the variables, Gen Ineq: General inequalities

Appendix C – Multi-Product Production Routing Under Decoupled Planning Periods

Proofs

Theorem 3.1. *$\mathcal{R}^B_{MP-PRP}$ and $\mathcal{R}^S_{MP-PRP}$ are valid reformulations for $\mathcal{M}^B_{MP-PRP}$ and $\mathcal{M}^S_{MP-PRP}$, respectively.*

Proof. First we show that for every feasible solution of the $\mathcal{M}^B_{MP-PRP}$ model, there exists a feasible solution to the $\mathcal{R}^B_{MP-PRP}$ model with the same solution value. Suppose that $\bar{y}$, $\bar{p}$, $\bar{I}$, $\bar{q}$, $\bar{z}$ and $\bar{x}$ satisfy the system of (3.1)-(3.18) (feasible in $\mathcal{M}^B_{MP-PRP}$).

- For every $\tau \in \mathcal{T}^\pi$ and for every $k \in \mathcal{K}$, we let $\bar{\mathbf{y}}_{kt} = \bar{y}_{k\tau}$ and $\bar{\mathbf{p}}_{kt} = \bar{p}_{k\tau}$ where $t = \pi(\tau - 1) + 1$. Constraints (3.37) fix the rest of the $\bar{\mathbf{y}}$ and $\bar{\mathbf{p}}$ variables to zero.
- For every $\omega \in \mathcal{T}^\rho$ and for every $i \in \mathcal{N}$, we let $\bar{\mathbf{z}}_{it} = \bar{z}_{i\omega}$ where $t = \omega\rho$. Constraints (3.38) fix the rest of the $\bar{\mathbf{z}}$ variables to zero.
- For every $\omega \in \mathcal{T}^\rho$, for every $i \in \mathcal{N}$ and for every $k \in \mathcal{K}$, we let $\bar{\mathbf{q}}_{ikt} = \bar{q}_{ik\omega}$ where $t = \omega\rho$. Constraints (3.38) fix the rest of the $\bar{\mathbf{q}}$ variables to zero.
- For every $\omega \in \mathcal{T}^\rho$, we let $\bar{\mathbf{z}}_{0t} = \bar{z}_{0\omega}$ where $t = \omega\rho$. Constraints (3.39) fix the rest of the $\bar{\mathbf{z}}_{0t}$ variables to zero.
- For every $\omega \in \mathcal{T}^\rho$ and for every $(i,j) \in \mathcal{E}$, we let $\bar{\mathbf{x}}_{ijt} = \bar{x}_{ij\omega}$ where $t = \omega\rho$. Constraints (3.35) and (3.38)-(3.39) force the rest of the $\bar{\mathbf{x}}$ variables to zero.

- For every $\omega \in \mathcal{T}^{\rho}$, for every $i \in \mathcal{N}$ and for every $k \in \mathcal{K}$, we let $\bar{\mathbf{I}}_{ikt} = \bar{I}_{ik\omega}$ where $t = \omega\rho$. For the rest of the micro periods ($t \in \mathcal{T}, t \bmod \rho \neq 0$), we let $\bar{\mathbf{I}}_{ikt} = \bar{I}_{ik,\lfloor \frac{t}{\rho} \rfloor}$.
- The inventory variables (and hence the solutions) at the plant level, $\bar{I}_{0kt}$, are defined on the micro periods and are the same in both formulations.

One observes that the solution $\bar{\mathbf{y}}$, $\bar{\mathbf{p}}$, $\bar{\mathbf{I}}$, $\bar{\mathbf{q}}$, $\bar{\mathbf{z}}$ satisfies the system of constraints (3.2), (3.13), (3.28)-(3.44) and hence is feasible in $\mathcal{R}^{B}_{MP-PRP}$. Similarly, we can show that for every feasible solution in $\mathcal{R}^{B}_{MP-PRP}$ there exists a feasible solution in $\mathcal{M}^{B}_{MP-PRP}$. Thus, $\mathcal{R}^{B}_{MP-PRP}$ is a valid reformulation of $\mathcal{M}^{B}_{MP-PRP}$. In the same way, we can show $\mathcal{R}^{S}_{MP-PRP}$ is a valid reformulation of $\mathcal{M}^{S}_{MP-PRP}$. □

Proposition 3.1.

$$\sum_{e=t_1}^{t_2} \mathbf{p}_{ke} \leq I_{0kt_2} + \sum_{i \in \mathcal{N}} \mathbf{I}_{ikt_2} + \sum_{e=t_1}^{t_2} \Big(\sum_{i \in \mathcal{N}} \mathbf{d}_{iket_2} \Big) \mathbf{y}_{ke} \forall k \in \mathcal{K}, \forall t_1, t_2 \in \mathcal{T}, t_1 \leq t_2 \tag{3.51}$$

are valid for $\mathcal{R}^{B}_{MP-PRP}$, $\mathcal{R}^{S}_{MP-PRP}$.

Proof. If $\sum_{e=t_1}^{t_2} \mathbf{y}_{ke} = 0$, then no setup will be done during periods t_1 to t_2 and hence no production of product $k \in \mathcal{K}$ is possible during these periods ($\sum_{e=t_1}^{t_2} \mathbf{p}_{ke} = 0$). Then, inequalities (3.51) are satisfied because the left-hand-side (LHS) will be equal to zero and the inventory variables in the right-hand-side (RHS) are nonnegative. Otherwise, let θ be the first period in which the production setup for product $k \in \mathcal{K}$ is performed, i.e., $\theta = \min_e \{t_1 \leq e \leq t_2 | \mathbf{y}_{ke} = 1\}$. Then,

$$\begin{aligned}
\sum_{e=t_1}^{t_2} \mathbf{p}_{ke} &= \sum_{e=\theta}^{t_2} \mathbf{p}_{ke} \\
&= \sum_{e=\theta}^{t_2} (I_{0ke} - I_{0k,e-1} + \sum_{i \in \mathcal{N}} \mathbf{q}_{ike}) \\
&= \sum_{e=\theta}^{t_2} \Big(I_{0ke} - I_{0k,e-1} + \sum_{i \in \mathcal{N}} (\mathbf{I}_{ike} - \mathbf{I}_{ik,e-1} + \mathbf{d}_{ike}) \Big) \\
&= I_{0kt_2} - I_{0k,\theta-1} + \sum_{i \in \mathcal{N}} (\mathbf{I}_{ikt_2} - \mathbf{I}_{ik,\theta-1} + \mathbf{d}_{ik\theta t_2}) \\
&\leq I_{0kt_2} + \sum_{i \in \mathcal{N}} (\mathbf{I}_{ikt_2} + \mathbf{d}_{ik\theta t_2}) \\
&= I_{0kt_2} + \sum_{i \in \mathcal{N}} \mathbf{I}_{ikt_2} + \sum_{i \in \mathcal{N}} \mathbf{d}_{ik\theta t_2} \mathbf{y}_{k\theta}
\end{aligned}$$

$$\leq I_{0kt_2} + \sum_{i\in\mathcal{N}} \mathbf{I}_{ikt_2} + \sum_{e=\theta}^{t_2} \big(\sum_{i\in\mathcal{N}} \mathbf{d}_{iket_2}\big)\mathbf{y}_{ke}$$

$$= I_{0kt_2} + \sum_{i\in\mathcal{N}} \mathbf{I}_{ikt_2} + \sum_{e=t_1}^{t_2} \big(\sum_{i\in\mathcal{N}} \mathbf{d}_{iket_2}\big)\mathbf{y}_{ke}.$$

The first four equations follow from the definition of θ, constraints (3.28), constraints (3.29), and the definition of $\mathbf{d}_{ikt_1t_2}$, respectively. The first inequality holds due to the non-negativity of inventory variables. The next equation is valid because $\mathbf{y}_{k\theta} = 1$. The last inequality is valid since the $\mathbf{y}_{ke}$ variables are nonnegative. The last equation holds due to the assumption that there is no setup from period t_1 to θ. □

Proposition 3.2. *Inequalities*

$$\left\lceil \frac{\max\left\{0, \sum_{i\in\mathcal{N}} \max\{0, \mathbf{d}_{ik1t} - \mathbf{I}_{ik0}\} - I_{0k0}\right\}}{C/\theta_k} \right\rceil \leq \sum_{e=1}^{t} \mathbf{y}_{ke} \quad \forall k \in \mathcal{K}, \forall t \in \mathcal{T} \tag{3.52}$$

are valid for $\mathcal{R}^B_{MP-PRP}$ and $\mathcal{R}^S_{MP-PRP}$.

Proof. First we show:

$$\sum_{e=1}^{t} \mathbf{p}_{ke} = \sum_{e=1}^{t} \big(\sum_{i\in\mathcal{N}} \mathbf{q}_{ike} + I_{0ke} - I_{0k,e-1}\big)$$

$$= \sum_{e=1}^{t} \Big(\sum_{i\in\mathcal{N}} (\mathbf{d}_{ike} + \mathbf{I}_{ike} - \mathbf{I}_{ik,e-1}) + I_{0ke} - I_{0k,e-1}\Big)$$

$$= \sum_{i\in\mathcal{N}} (\mathbf{d}_{ik1t} + \mathbf{I}_{ikt} - \mathbf{I}_{ik0}) + I_{0kt} - I_{0k0}$$

$$\geq \sum_{i\in\mathcal{N}} (\mathbf{d}_{ik1t} - \mathbf{I}_{ik0}) - I_{0k0}.$$

The first two equation are obtained based on constraints (3.28) and (3.29), respectively. The third equation holds due to the definition of $\mathbf{d}_{ikt_1t_2}$. The first inequality follows from the non-negativity of inventory variables. We can write

$$\textstyle\sum_{e=1}^{t} \mathbf{p}_{ke} \geq \max\left\{0, \sum_{i\in\mathcal{N}} \max\{0, \mathbf{d}_{ik1t} - \mathbf{I}_{ik0}\} - I_{0k0}\right\},$$

because only a strictly positive product shortage triggers the production at the plant. Finally, the validity of the proposition comes from the fact that:

$$\max\left\{0, \sum_{i\in\mathcal{N}} \max\{0, \mathbf{d}_{ik1t} - \mathbf{I}_{ik0}\} - I_{0k0}\right\} \leq \sum_{e=1}^{t} \mathbf{p}_{ke}$$

$$\leq C/\theta_k \sum_{e=1}^{t} \mathbf{y}_{ke}.$$

□

Proposition 3.3. *Inequalities*

$$\sum_{e=t_1}^{t_2} \mathbf{q}_{ike} \leq \mathbf{I}_{ik,t_2} + \sum_{e=t_1}^{t_2} \mathbf{d}_{iket_2}\mathbf{z}_{ie} \quad \forall i \in \mathcal{N}, \forall k \in \mathcal{K}, \forall t_1, t_2 \in \mathcal{T}, t_1 \leq t_2 \tag{3.53}$$

are valid for $\mathcal{R}^{B}_{MP-PRP}$, $\mathcal{R}^{S}_{MP-PRP}$.

Proof. If $\sum_{e=t_1}^{t_2} \mathbf{z}_{ie} = 0$, then customer $i \in \mathcal{N}$ will not be visited during periods t_1 to t_2. This results in no shipment of product $k \in \mathcal{K}$ to that customer during the associated periods. Then, inequalities (3.53) are satisfied because the inventory variables in the RHS are nonnegative. Otherwise, let θ be the first period in which customer $i \in \mathcal{N}$ is visited, i.e., $\theta = \min_e\{t_1 \leq e \leq t_2 | \mathbf{z}_{ie} = 1\}$. Then,

$$\begin{aligned}
\sum_{e=t_1}^{t_2} \mathbf{q}_{ike} &= \sum_{e=\theta}^{t_2} \mathbf{q}_{ike} \\
&= \sum_{e=\theta}^{t_2} (\mathbf{I}_{ike} - \mathbf{I}_{ik,e-1} + \mathbf{d}_{ike}) \\
&= \mathbf{I}_{ikt_2} - \mathbf{I}_{ik,\theta-1} + \mathbf{d}_{ik\theta t_2} \\
&\leq \mathbf{I}_{ikt_2} + \mathbf{d}_{ik\theta t_2} \\
&= \mathbf{I}_{ikt_2} + \mathbf{d}_{ik\theta t_2}\mathbf{z}_{i\theta} \\
&\leq \mathbf{I}_{ikt_2} + \sum_{e=\theta}^{t_2} \mathbf{d}_{iket_2}\mathbf{z}_{ie} \\
&= \mathbf{I}_{ikt_2} + \sum_{e=t_1}^{t_2} \mathbf{d}_{iket_2}\mathbf{z}_{ie}.
\end{aligned}$$

The first three equations hold because of the definition of θ, constraints (3.10) for periods θ to t_1, and the definition of $\mathbf{d}_{ikt_1t_2}$. The first inequality is valid due to the non-negativity of the inventory variables. The fourth equation follows from $\mathbf{z}_{i\theta} = 1$. The last inequality and equation are valid because the $\mathbf{y}_{ke}$ variables are nonnegative. □

Proposition 3.4. *Inequalities*

$$\left\lceil \frac{1}{Q} \sum_{i \in \mathcal{N}} \sum_{k \in \mathcal{K}} b_k \max\{0, \mathbf{d}_{ik1t} - \mathbf{I}_{ik0}\} \right\rceil \leq \sum_{e=1}^{t} \mathbf{z}_{0e} \quad \forall t \in \mathcal{T} \tag{3.54}$$

are valid for $\mathcal{R}^B_{MP-PRP}$ and $\mathcal{R}^S_{MP-PRP}$.

Proof. We have

$$\begin{aligned}\sum_{e=1}^{t} Q\mathbf{z}_{0e} &\geq \sum_{e=1}^{t}\sum_{i\in\mathcal{N}}\sum_{k\in\mathcal{K}} b_k\mathbf{q}_{ike}\\ &= \sum_{e=1}^{t}\sum_{i\in\mathcal{N}}\sum_{k\in\mathcal{K}} b_k(\mathbf{d}_{ike}+\mathbf{I}_{ike}-\mathbf{I}_{ik,e-1})\\ &= \sum_{i\in\mathcal{N}}\sum_{k\in\mathcal{K}} b_k(\mathbf{d}_{ik1e}+\mathbf{I}_{ike}-\mathbf{I}_{ik0})\\ &\geq \sum_{i\in\mathcal{N}}\sum_{k\in\mathcal{K}} b_k(\mathbf{d}_{ik1e}-\mathbf{I}_{ik0}).\end{aligned}$$

The first inequality is valid since the LHS is the total fleet capacity for period $e = 1$ to t, and the RHS is the total shipment for the same periods. The first equation follows from constraints (3.29). The second equation is valid due to the definition of $\mathbf{d}_{ikt_1t_2}$. The second inequality holds due to the non-negativity of inventory variables. The proposition is valid because only strictly positive demand shortages necessitate vehicles' dispatch. □

Proposition 3.5. *Inequalities*

$$\left\lceil \frac{\sum_{k\in\mathcal{K}} b_k \max\{0, \mathbf{d}_{ik1t}-\mathbf{I}_{ik0}\}}{\min\left\{Q, L_i+\max_{1\leq\theta\leq t}\{\sum_{k\in\mathcal{K}} b_k\mathbf{d}_{ik\theta}\}\right\}}\right\rceil \leq \sum_{e=1}^{t}\mathbf{z}_{ie} \quad \forall i\in\mathcal{N}, \forall t\in\mathcal{T} \tag{3.55}$$

are valid for $\mathcal{R}^B_{MP-PRP}$ and $\mathcal{R}^S_{MP-PRP}$.

Proof. Similar to the proof presented in Proposition 3.4 we have

$$\textstyle\sum_{k\in\mathcal{K}} b_k(\mathbf{d}_{ik1t}-\mathbf{I}_{ik0}) \leq \sum_{e=1}^{t}\sum_{k\in\mathcal{K}} b_k\mathbf{q}_{ike}.$$

Thus,

$$\textstyle\sum_{k\in\mathcal{K}} b_k \max\{0, \mathbf{d}_{ik1t}-\mathbf{I}_{ik0}\} \leq \sum_{e=1}^{t}\sum_{k\in\mathcal{K}} b_k\mathbf{q}_{ike},$$

is valid for the reason that only strictly positive product shortage volumes force shipments. The vehicle capacity constraints (3.33) provide the first upper bound:

$$\textstyle\sum_{e=1}^{t}\sum_{k\in\mathcal{K}} b_k\mathbf{q}_{ike} \leq Q\sum_{e=1}^{t}\mathbf{z}_{ie}.$$

Next, we have

$$\begin{aligned}\sum_{k\in\mathcal{K}} b_k \mathbf{q}_{ikt} &= \sum_{k\in\mathcal{K}} b_k(\mathbf{d}_{ikt} + \mathbf{I}_{ikt} - \mathbf{I}_{ik,t-1}) \\ &\leq \sum_{k\in\mathcal{K}} b_k(\mathbf{d}_{ikt} + \mathbf{I}_{ikt}) \\ &\leq \sum_{k\in\mathcal{K}} b_k \mathbf{d}_{ikt} + L_i,\end{aligned}$$

which gives

$$\textstyle\sum_{k\in\mathcal{K}} b_k \mathbf{q}_{ikt} \leq \left(\sum_{k\in\mathcal{K}} b_k \mathbf{d}_{ikt} + L_i\right)\mathbf{z}_{it}.$$

Therefore, we deduce

$$\textstyle\sum_{k\in\mathcal{K}} b_k \max\{0, \mathbf{d}_{ik1t} - \mathbf{I}_{ik0}\} \leq \sum_{e=1}^{t} \sum_{k\in\mathcal{K}} b_k \mathbf{q}_{ike} \leq \min\left\{Q, L_i + \max_{1\leq\theta\leq t}\{\sum_{k\in\mathcal{K}} b_k \mathbf{d}_{ik\theta}\}\right\} \sum_{e=1}^{t} \mathbf{z}_{ie}.$$

□

Impact of Inequalities on Lower Bound Improvement

Table C.1 reports the improvement of the lower bounds obtained by incorporating the valid inequalities in the small- and big-bucket models. On the small-bucket instances, applying the valid inequalities results in an average increase of the lower bounds by 70.5%, 38.7%, and 25.6%, respectively for $\rho = 1$, $\rho = 2$ and $\rho = 3$. On the big-bucket instances, the lower bound improvements obtained by the addition of the valid inequalities are 48.8%, 32.8%, and 22.7%, respectively for $\pi = 1$, $\pi = 2$ and $\pi = 3$. Notice that in this table, for the cases where the inequalities improve the lower bound more than twice, the percentage increase reported is more than 100%. Overall, the larger (more periods, products, and nodes) and the harder to solve (smaller ρ and π) the instances are, the bigger the improvement is.

Table C.1: Lower bound improvement with valid inequalities

			Small-Bucket LSP									Big-Bucket LSP								
			$\rho = 1$			$\rho = 2$			$\rho = 3$			$\pi = 1$			$\pi = 2$			$\pi = 3$		
l	k	n	None	All	(%)	None	All	(%)	None	All	(%)	None	All	(%)	None	All	(%)	None	All	(%)
12	4	5	32209	32277	0.2	32722	32722	0.0	33124	33124	0.0	32149	32231	0.3	32296	32296	0.0	32796	32797	0.0
		10	37338	37826	1.3	38464	38663	0.5	38576	38743	0.4	37270	38021	2.0	38165	38841	1.8	40690	41902	3.0
		15	35620	46818	31.4	46793	47836	2.2	47675	47998	0.7	45810	46788	2.1	46775	47339	1.2	47581	48216	1.3
		20	36045	51379	42.5	48405	52614	8.7	52233	52553	0.6	46319	51293	10.7	49448	52076	5.3	50711	54006	6.5
		25	37750	56009	48.4	45767	58209	27.2	57714	58240	0.9	45843	56432	23.1	52459	57442	9.5	54083	58588	8.3
		30	38823	58977	51.9	48663	63216	29.9	56972	63068	10.7	45756	60708	32.7	56186	61856	10.1	58221	64360	10.5
		35	36530	65812	80.2	54449	71021	30.4	61081	70838	16.0	48263	68569	42.1	52506	70189	33.7	56035	72779	29.9
	6	5	41381	41807	1.0	41868	41869	0.0	42431	42431	0.0	41549	41551	0.0	41725	41726	0.0	41633	41634	0.0
		10	33196	47535	43.2	42981	47923	11.5	48248	48491	0.5	46726	47330	1.3	46612	47484	1.9	47389	47435	0.1
		15	30768	54604	77.5	43703	55253	26.4	51466	56256	9.3	44733	54216	21.2	51539	54458	5.7	53907	54722	1.5
		20	30843	58067	88.3	43318	59456	37.3	55455	60735	9.5	47355	57941	22.4	51515	58387	13.3	57946	58790	1.5
		25	27061	61644	127.8	42123	64643	53.5	57162	66127	15.7	35044	62377	78.0	42987	62716	45.9	55226	63144	14.3
		30	24347	64207	163.7	43812	69387	58.4	58190	71632	23.1	35242	66655	89.1	42974	67181	56.3	52642	67172	27.6
		35	22774	65167	186.1	44497	71968	61.7	57718	75366	30.6	32224	69960	117.1	41089	70541	71.7	44369	70382	58.6
	8	5	42840	51861	21.1	47042	52465	11.5	52703	52911	0.4	51640	51646	0.0	50137	51646	3.0	51718	51720	0.0
		10	37550	61433	63.6	49309	62554	26.9	49285	62761	27.3	37317	60607	62.4	43597	60667	39.2	55322	60799	9.9
		15	33033	65826	99.3	49858	68103	36.6	52410	68933	31.5	42816	65689	53.4	50349	65761	30.6	59367	66560	12.1
		20	35314	73324	107.6	54670	77393	41.6	59535	79663	33.8	40900	73662	80.1	49006	74018	51.0	55803	74858	34.1
		25	30298	73816	143.6	50749	78962	55.6	59812	81722	36.6	38864	74784	92.4	46606	75041	61.0	51947	76288	46.9
		30	33969	80129	135.9	49220	85443	73.6	62520	89341	42.9	37900	80278	111.8	43849	80730	84.1	53250	81233	52.6
		35	26186	80766	208.4	51421	89173	73.4	66248	93864	41.7	35488	83457	135.2	44488	83776	88.3	52561	84461	60.7
18	4	5	39762	41065	3.3	41912	42162	0.6	43152	43201	0.1	40764	44644	9.5	44757	45002	0.5	46421	46750	0.7
		10	35596	45740	28.5	44683	46895	5.0	46987	47177	0.4	43648	46648	6.9	46533	47592	2.3	48684	49143	0.9
		15	39802	57618	44.8	50978	59761	17.2	59067	60456	2.4	51158	60741	18.7	59330	62126	4.7	62987	64340	2.1
		20	49660	67317	35.6	58505	74722	27.7	72111	74928	3.9	59455	74832	25.9	66375	76946	15.9	74651	79524	6.5
		25	47834	73942	54.6	61869	81651	32.0	73752	82170	11.4	58963	81305	37.9	69300	83406	20.4	79007	86216	9.1
		30	49062	82540	68.2	69313	89963	29.8	75892	94457	24.5	62571	94454	51.0	72992	97335	33.4	79337	100585	26.8
	6	5	34801	47730	37.2	44269	48296	9.1	47045	48300	2.7	43733	47348	8.3	47474	47534	0.1	48132	48133	0.0
		10	37572	59059	57.2	44256	59903	35.4	49879	60562	21.4	53397	58827	10.2	57935	59162	2.1	59077	60428	2.3
		15	37114	64768	74.5	52130	69758	33.8	56146	70331	25.3	51033	67416	32.1	63308	67732	7.0	68518	70757	3.3
		20	36140	71067	96.6	51488	78906	53.3	59969	80467	34.2	50530	75849	50.1	59250	76265	28.7	71374	79473	11.3
		25	38148	77909	104.2	55098	85132	54.5	64799	90294	39.3	46074	84758	84.0	51369	85181	65.8	58356	89193	52.8
		30	31380	82567	163.1	54260	83962	54.7	67555	95073	40.7	44227	89723	102.9	54882	90230	64.4	59948	94703	58.0
	8	5	37779	60800	60.9	45791	61484	34.3	52629	62055	17.9	53963	59791	10.8	59814	59823	0.0	60239	60329	0.2
		10	36584	71601	95.7	49792	74072	48.8	53239	75366	41.6	51253	71851	40.2	63055	71997	14.2	69282	72735	5.0
		15	37505	75001	100.0	49870	82299	65.0	58549	85056	45.3	46867	81133	73.1	60809	81250	33.6	70824	81722	15.4
		20	40095	84329	110.3	60068	93408	55.5	66786	95822	43.5	53448	90199	68.8	60627	90448	49.2	74627	90514	21.3
		25	44748	92675	107.1	60036	99228	65.3	73512	106397	44.7	51905	100244	93.1	55547	100348	80.7	64873	100563	55.0
		30	34879	100873	189.2	61152	105629	72.7	81373	116921	43.7	46385	110800	138.9	56943	110893	94.7	67506	111051	64.5
24	4	5	36759	42541	15.7	43126	43587	1.1	43972	43972	0.0	43961	45648	3.8	45847	46200	0.8	48018	48345	0.7
		10	50151	62422	24.5	54811	63598	16.0	61985	64155	3.5	63563	70675	11.2	70741	72049	1.8	74116	74980	1.2
		15	51188	68504	33.8	61052	78158	28.0	71569	79344	10.9	61418	81087	32.0	69306	83543	20.5	78508	85904	9.4
		20	49617	78196	57.6	66930	86937	29.9	77359	91181	17.9	67335	96631	43.5	73930	98913	33.8	83805	101629	21.3
		25	56931	89466	57.1	76430	94160	23.2	93332	104852	12.3	70695	112880	59.7	81229	115793	42.6	92936	118924	28.0
	6	5	41723	55935	34.1	48982	56743	15.8	54927	57256	4.2	50097	56736	13.3	55861	57344	2.7	56765	58253	2.6
		10	38617	65411	69.4	45974	70576	53.5	51752	71797	38.7	53410	69747	30.6	66603	70258	5.5	70141	73279	4.5
		15	43692	77271	76.9	56361	85754	52.2	61809	89464	44.7	58591	88695	51.4	65734	89648	36.4	77048	94911	23.2
		20	47217	85307	80.7	60577	88328	45.8	70340	97616	38.8	56619	98396	73.8	62872	99074	57.6	71175	104064	46.2
		25	46044	92487	100.9	62432	97536	56.2	78391	110354	40.8	59196	115140	94.5	65476	115886	77.0	73358	122441	66.9
	8	5	38655	68186	76.4	46545	69136	48.5	51114	70006	37.0	58883	67577	14.8	64180	67725	5.5	62049	68559	10.5
		10	39921	75831	90.0	49937	84312	68.8	56734	86259	52.0	56690	83155	46.7	66275	83691	26.3	78877	85354	8.2
		15	42789	87070	103.5	57925	98548	70.1	65219	100268	53.7	57544	99720	73.3	68433	100247	46.5	82500	102014	23.7
		20	50100	102655	104.9	67126	103666	54.4	78414	113330	44.5	63257	115110	82.0	70139	116232	65.7	82650	119220	44.2
		25	58504	104333	78.3	70501	116094	64.7	88498	128494	45.2	67318	132210	96.4	72592	133255	83.6	81444	136383	67.5
30	4	5	41914	51320	22.4	49256	52661	6.9	52745	53506	1.4	49829	55906	12.2	55029	57370	4.3	58277	59437	2.0
		10	50881	64490	26.7	59218	73973	24.9	70425	75481	7.2	71829	84310	17.4	81387	86160	5.9	86628	88413	2.1
		15	54462	77646	42.6	67796	86645	27.8	78062	91194	16.8	69166	98065	41.8	74267	100129	34.8	81083	102269	26.1
		20	60842	89133	46.5	78042	92806	18.9	87915	104617	19.0	76932	119510	55.3	85681	121507	41.8	96420	124022	28.6
	6	5	35596	59195	66.3	44430	60855	37.0	51872	62278	20.1	48684	61609	26.5	59139	61885	4.6	62722	63888	1.9
		10	42848	71091	65.9	53919	81988	52.1	58050	84645	45.8	63260	88507	39.9	72299	89646	24.0	83197	94600	13.7
		15	44575	81966	83.9	59043	86568	46.6	66065	95959	45.2	58460	105409	80.3	65654	106106	61.6	76228	110982	45.6
		20	57842	94307	63.0	69322	99637	43.7	82484	113483	37.6	71824	131624	83.3	76242	132635	74.0	86641	137613	58.8
	8	5	44742	76233	70.4	52823	80236	51.9	60955	80973	32.8	60149	79002	31.3	72118	79472	10.2	75038	80878	7.8
		10	44031	83196	88.9	57220	92055	60.9	62042	95017	53.1	63638	100983	58.7	75878	103136	35.9	87215	106309	21.9
		15	49287	96033	94.8	68020	102163	50.2	77332	113785	47.1	67947	129109	90.0	81801	132076	61.5	93833	136203	45.2
		20	60258	101687	68.8	72111	116843	62.0	88653	124326	40.2	71759	146924	104.7	75811	149248	96.9	89133	152974	71.6
Average			40781	69512	70.5	53412	74086	38.7	61591	77385	25.6	52131	77562	48.8	59078	78434	32.8	65558	80452	22.7