HEC MONTRÉAL
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Essays on Credit Risk and Callable Bonds Valuation

par
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RÉSUMÉ

Cette thèse comporte trois essais sur l'évaluation des obligations avec risque de défaut et des obligations rachetables.

Le premier essai traite de l'impact de l'hétérogénéité d'un groupe de firmes sur l'écart de crédit moyen du groupe. L'essai propose un modèle de groupe hétérogène de firmes dans lequel les fondamentaux financiers des firmes présentent des distributions transversales et des dynamiques temporelles. Le modèle est construit de façon à ce qu'à chaque instant le prix de chaque obligation soit donné par le modèle de Merton. Utilisant une approche par simulation, l'essai montre que la forme des distributions transversales ainsi que les paramètres gouvernant les dynamiques temporelles des fondamentaux ont un effet sur l'écart de crédit moyen du groupe et sur le biais auquel l'on expose si l'on venait à ignorer l'hétérogénéité du groupe.

Le deuxième essai a pour but d'estimer la part de l'écart de défaut dans l'écart total des obligations corporatives. Pour ce faire, l'essai propose une extension de l'approche standard d'estimation du modèle de Merton à partir des prix d'actions pour y incorporer l'information sur les taux de défaut historiques tout en prenant en compte l'hétérogénéité des firmes. Il est démontré que l'approche proposée permet une estimation plus précise de la part de l'écart de défaut dans l'écart total. L'approche est ensuite appliquée sur un échantillon de près de 3 millions d'observations journalières de près de 1286 firmes nord-américaines. L'un des principaux résultats est que l'écart de défaut moyen représente 40% de l'écart total moyen des obligations BBB de maturité 4 ans, soit près des deux tiers de la part obtenue en ignorant l'information sur les taux de défaut historiques et près de deux fois la part obtenue en ignorant l'hétérogénéité.

Le troisième essai propose une approche de tarification des obligations rachetables dans un modèle de taux multi-facteurs. Le problème de tarification est posé sous la forme d'un programme dynamique. Afin de solutionner le problème, l'essai propose un algorithme récursif basé sur la distribution jointe des facteurs sous la mesure forward, sur les séries de Chebyshev tronquées et sur la méthode d'intégration par Clenshaw-Curtis. Une preuve de convergence de l'algorithme pour un nombre quelconque de facteurs gaussiens est proposée. L'approche est illustrée à l'aide du modèle de Vasicek à deux facteurs, un modèle choisi pour sa parcimonie. L'effet des différents paramètres du modèle tel que la corrélation entre les deux facteurs de taux sur la valeur de l'option de rachat est analysé. L'avantage de l'approche proposée par rapport aux méthodes alternatives est la stabilité et la précision numérique qu'elle offre dans un temps de calcul raisonnable.

Mots clés: risque de crédit, taux de défaut historiques, estimation, hétérogénéité, écart de défaut, écart des obligations corporatives, obligations rachetables, modèle de taux multifacteur, tarification.

ABSTRACT

This thesis contains three essays on the evaluation of credit risky bonds and callable bonds.

The first essay deals with the effect of the heterogeneity of a group of firms on the average credit spread of the group. The essay proposes a model of a heterogeneous group of firms whose financial fundamentals have cross-sectional distributions and time-series dynamics. The model is built in such a way that at each point in time, the spread of each firm is given by the Merton model. Using a simulation approach, it is shown that the shape of the cross-sectional distributions and the parameters governing the time-series dynamics of the fundamentals have an effect on the average credit spread of the group and on the bias to which one is exposed if heterogeneity is ignored.

The second essay aims at estimating the share of the default spread in the total corporate bond spread. The essay proposes an extension of the standard estimation approach of the Merton model from equity prices in order to incorporate historical default rates in the estimation while taking into account the heterogeneity of firms. It is shown that the proposed approach yields a more accurate estimation of the share of the default spread in the total spread than the standard approach does. The proposed approach is then applied on a sample of nearly 3 millions daily observations of 1,286 North American companies. One of the main results is that the average default spread represents 40% of the total average spread of BBB bonds with a maturity of 4 years. This share is about two-thirds of the share obtained if historical default rates are not included in the estimation and twice the share obtained if heterogeneity is ignored.

The third essay proposes a pricing approach for callable bonds in multifactor interest rate models. The pricing problem is posed in the form of a dynamic program. To solve the problem, the essay proposes a recursive algorithm based on the joint distribution of the factors under the forward measure, on truncated Chebyshev series and on Clenshaw-Curtis integration. A proof of convergence of the algorithm for any number of Gaussian factors is proposed. The approach is illustrated under the two-factor Vasicek model which is chosen for its parsimony. The impact of various parameters of the model such

as the correlation between the two interest rate factors on the embedded call price is analyzed. The advantage of the proposed algorithm compared to alternative methods is the stability and the numerical accuracy that it offers in a reasonable computing time.

Keywords: credit risk, historical default rates, estimation, heterogeneity, default spread, corporate bonds spreads, callable bond, multifactor interest model, pricing.

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LIST OF ABBREVIATIONS

2FV	Two-Factor Vasicek
ATA	At The Average
ATM	At The Median
CIR	Cox Ingersoll Ross
CS	Cross-Sectional
CV	Coefficient of Variation
DP	Default Probability
HBF	Heterogeneity Bias Free
HH	Huang and Huang
RAM	Random Access Memory
RMSE	Root Mean Square Error
TS	Time-Series

NOTATION

n	number of firms
τ	horizon of fundamentals time-series
t	time
μ_{it}	expected rate of return on firm i asset value at time t
σ_{it}	expected volatility of the return on firm i asset value at time t
V_{it}	asset value of firm i at time s
dz_{it}	standard Brownian increment of the asset value of firm i at time s
Q	risk neutral measure
dz_{it}^{Q}	Q standard Brownian increment of the asset value of firm i at time s
r_t	riskless rate at time t
T	debt maturity
B_{it}	debt principal of firm i at time t
ρ_i	recovery rate of firm i
θ_{it}	set of fundamentals of firm i at time t:
$\mathbb{E}_t^Q[.]$	risk neutral expectation operation conditional on time t information
$N(.)$	cumulative normal distribution function
$E(\theta_{it})$	equity price of firm i at time t
$F(\theta_{it})$	bond price of firm i at time t
$\Phi(.)$	individual bond default rate function
$\Psi(.)$	individual bond spread function
$\overline{\theta}$	vector of average fundamentals of the group of firms
θ_{HH}	vector of HH fundamentals of the group of firms
θ_{median}	vector of median fundamentals of the group of firms
M	number of simulation paths or rounds
W_t^{λ}	standard Brownian motion for the mean-reverting leverage process
W_t^{σ}	standard Brownian motion for the stochastic volatility process
W_t^{r}	standard Brownian motion for the stochastic interest rate process

κ_λ	speed of mean reversion of leverage
θ_λ	target leverage
σ_λ	leverage volatility
κ_σ	speed of mean reversion of volatility
θ_σ	long run average volatility
σ_σ	volatility of the volatility
$\eta_{\sigma v}$	correlation between asset value and asset volatility
κ_r	speed of mean reversion of the riskless rate
θ_r	long run average riskless rate
σ_r	volatility of riskless rate
η_{rv}	correlation between asset value and asset volatility
R^2	coefficient of determination
σ	asset volatility
λ	leverage
σ_E	equity volatility
σ_D	debt volatility
σ_{ED}	is the correlation between bond returns and equity returns
E	equity value
V	asset value
B	debt principal
r	riskless rate
ϕ_{it}	default rate of firm i at time t
y_{itT}	firm specific leverage adjustment factor
$\Phi_T(.)$	Merton model default rate function at the maturity T
ϕ_{lT}	average default rate of a group l of firms over the period of interest
y_{lT}^G	group specific leverage adjustment factor

$\widehat{\sigma}$	estimated asset volatility
$\widehat{\lambda}_1$	estimated first stage leverage
$\widehat{\lambda}_2$	estimated second stage leverage
V_0	initial asset value
P	total debt principal in the He-Xiong model
C	total coupon rate in the He-Xiong model
ε	liquidity spread in the He-Xiong model
V_B	default boundary in the He-Xiong model
α	fraction of asset value recovered at default in the He-Xiong model
π	tax rate in the He-Xiong model
ξ	intensity of liquidity shocks in the He-Xiong model
$\widehat{V}$	estimated asset value in the first stage
ζ	asset payout rate in the He-Xiong model
$\overline{\rho}^p$	average recovery rate of group p
ρ_{pl}	average recovery rate of firms common to group p and group l
α_l	economic sector specific adjustment factor to recovery rates
L	set of economic sectors
Π	set of rating categories
Ω^p	set of all observations falling in group p
q	total number of coupons in the callable bond
t_k	k^{th} coupon date except t_0 which is the pricing date
t_{q^*}	first coupon date on which the bond is callable
c	callable bond coupon rate
Δ	coupon period
K_k	callable bond strike price at coupon date t_k
δ	notification period

x_t	vector of interest rate factors at time t
x_{it}	value of interest rate factor i at time t
$r(.)$	riskless rate as a function of the vector of risk factors x_t
$B_t(.)$	value function at time t of a zero coupon bond maturing at time T
$C_t(.)$	callable bond value function
$H_t(.)$	holding value function of the callable bond
$\mathbb{E}^{Q}_{t,x}[.]$	risk neutral expectation operator conditional on $x_t = x$
Q^{Δ}	forward measure using $B_{t_{k-1}-\delta}(x, t_k - \delta)$ as numeraire
Q^{d_k}	forward measure using $B_{t_k}\left(x, t_{q^*} - \delta\right)$ as numeraire
Q^T	forward measure using $B_u(x, T)$ as numeraire
$\mathbb{E}^{Q^{\Delta}}_{t,x}[.]$	expectation operator under the measure Q^{Δ} conditional on $x_t = x$
$\mathbb{E}^{Q^{d_k}}_{t,x}[.]$	expectation operator under the measure Q^{d_k} conditional on $x_t = x$
$\mathbb{E}^{T}_{u,x}[.]$	expectation operator under the measure Q^T conditional on $x_u = x$
$Var^{T}_{u,x}[.]$	Variance operator under the measure Q^T conditional on $x_u = x$
a	vector of speeds of mean reversion of interest rate factors
a_i	speed of mean reversion of interest rate factor i
$\bar{x}$	vector of long run mean of interest rate factors
$\bar{x}_i$	long run mean of interest rate factor i
ν	vector of the volatilities of interest rate factors
ν_i	volatility of interest rate factor i
N	total number of interest rate factors
$x_{i\min}$	lower bound of the truncation interval for interest rate factor i
$x_{i\max}$	upper bound of the truncation interval for interest rate factor i
m	degree of truncation of the Chebyshev series.
κ	parameter controlling the width of the truncation interval of the interest rate factors
I_l	the multidimensional Chebyshev polynomial of order l
$b(l)$	multidimensional Chebyshev series coefficients of order l
W_t	Q multidimensional Brownian motion for the interest rate factors processes
W_{it}	Q standard Brownian motion for interest rate factor i process
W^T_t	Q^T multidimensional Brownian motion for the interest rate factors processes
W^T_{it}	Q^T standard Brownian motion for interest rate factor i process

To my beloved fiancée Sandrine,

To my mother Regine,

To my father Jean,

To my sister Lucrèce,

To my brother John-Vital,

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CHAPTER 1

INTRODUCTION

A bond is a financial security which entitles its holder to: (a) an amount called *face value* or *principal* to be paid at the expiry date of the security which is called *maturity* and (b) intermediary payments called *coupons* due at regular time intervals with the last payment due at maturity. If the coupons are all equal to zero then the bond is called a *zero coupon bond*. Coupons are defined as the product of a *coupon rate* times the principal. The coupon rate is usually fixed but can be variable.

Bonds often include several options such as call options, put options, currency options, convertibility options, caps and floors. A call option allows the issuer of the bond, to buy back the bond before maturity at a predetermined price. A put option allows the bondholder to sell back the bond to its issuer before maturity at a predetermined price. A currency option gives the issuer the right, but not the obligation, to pay coupons and principal in its currency of choice at predetermined exchange rates. A convertibility option allows the bond holder to convert the bond into equity thereby exchanging future coupons and principal payments for dividends. A cap is an option to pay a predetermined fixed rate instead of a floating rate whenever the floating rate exceeds the fixed rate. A floor is an option to receive a predetermined fixed rate instead of a floating rate whenever the floating rate drops below the fixed rate.

The value of a bond depends on several risk factors including interest rate risk, default risk, liquidity risk, macroeconomic risks and embedded options risks. Interest rate risk is the risk that interest rates rise or fall in the markets. While rising rates reduce the bond value, decreasing rates imply lower coupon rates on new fixed rate issues and on floating rate notes. Default risk is the risk that the issuer of the bond misses the promised payments. Liquidity risk is the risk that the demand for the bond is so weak that one might have to offer a premium on the price in order to attract buyers. Macroeconomic risks include inflation risk which erodes the currency value and the risk of an economic downturn that generates losses in financial markets and raises investors' risk aversion. Embedded options add additional risks in the bond such as the risk of an early call or put, the risk of conversion to equity and currency risks.

Each risk source is so complex to analyze that there is a specialized literature for each of them. This thesis deals separately with two risk factors: default risk and call risk.

In default risk analysis, it is important to distinguish between corporate bonds and sovereign bonds. Default risk for corporate bonds is mainly driven by the financial health of the specific bond issuer. This financial health is generally measured by several indicators including: leverage, interest coverage ratio, earnings stability, return on assets, working capital to assets and reserves to assets among others. On the other hand, the default risk of a sovereign entity is mainly driven by the economic shape of the country and its political climate. In this thesis, our default risk analysis will focus on corporate bonds.

Bonds are generally valued by discounting their future cash flows at the required rate of return. This rate of return is determined as the rate of return on riskless securities plus an appropriate *spread* that compensates the bondholder for the risks incurred. The riskless rate is generally assumed to be the rate of return on the US government T-bills in such a way that the difficulty in bonds valuation is to determine the appropriate spread.

In the case of default risk, the appropriate spread is generally called *credit spread*. It is clear that the credit spread should compensate the bond holder at least for the expected loss due to default. For instance, if the probability of default is 1%, and the loss given default is 60% of the bond value then the appropriate spread should be at least equal to 60 basis points (bps). Actually, the observed credit spreads on the market are much higher than the expected default loss because investors require a risk premium on top of the expected loss. This risk premium is the investor's profit. In other words, investors value bonds as if the probability of default risk were higher than what it actually is. The default probability that justifies the observed spreads is commonly termed the *risk neutral* default probability. The actual default probability is commonly termed *physical* or *objective* default probability. Thus, finding the credit spread consists of finding the risk neutral default probability and the loss given default. Alternatively, it consists of finding the physical default probability, the risk premium and the loss given default.

Several models have been developed in the literature for the valuation of corporate bonds in the presence of default risk. They may be classified in five categories: regression models, structural models, reduced form models, hybrid models and macro-structural models. Regression models build and estimate the probability of default of a firm or its credit spread as a function of the financial ratios of the firm, the characteristics of the bond and the overall market conditions. Structural models build a mathematical probabilistic model for the future evolution of the firm assets or cash flows, define the boundary value of firm assets or cash flows at which default occurs, and solve the model for the physical default probabilities, the risk neutral default probabilities and credit spreads. Reduced form models assume that default is an exogenous event that occurs randomly at a given rate and do not endogenize the link between this default rate and the financial health of the firm. Hybrid models are regression models where the probability of default predicted from a structural model is used as an additional explanatory variable. Macro-structural models are structural models where asset values and risk premia are endogenized and related to macro-economic fundamentals.

The first essay of this thesis studies a problem related to structural models. The problem arises when one is not interested in an individual bond credit spread but rather in the average credit spread of a group of bonds. While structural models have widely been studied in the case of individual bonds, the rigorous study of their predictions at a group level have not drawn the attention of researchers until recently. Paradoxically, it is the average spread at a group level that is of interest in many situations. In fact, it was generally assumed that the prediction of these models at a group level would be similar to the models' prediction for a typical firm of the group. This assumption was generally justified by a homogeneity assumption of the groups considered. Recent empirical evidence has shown however that the groups in question are actually highly heterogeneous. The goal of the essay is to study how different forms of heterogeneity affect average spreads at the group level.

In parallel with the default risk valuation literature, a literature that focuses on corporate bond spreads decomposition has been developed. This literature attempts to identify the different risk factors that are priced in corporate bond spreads and to quantify the proportion of the total spread attributable to each risk factor. The literature has uncovered a dilemma that is called the *credit spread puzzle*. The puzzle is that default risk does not appear to be as important as expected in corporate bond spreads. Traditionally, it was thought that the main difference between a US government bond and a corporate bond is that the corporate bond embeds a higher default risk than the US government bond. So it was thought that the corporate bond spreads were mainly related to default risk. For this reason, corporate bond spreads were and are still called credit spreads in the literature. However, some researchers have recently found that default risk may represent less than 20% of corporate bond spreads. This finding has raised two important questions. First of all, is this figure of 20% accurate? Second, if it is accurate then what explains the rest of the spreads?

The second essay of this thesis attempts to answer the above first question of the credit spread puzzle literature. The essay proposes a simple approach based on structural models which aims at accurately capturing the proportion of corporate bond spreads attributable to default risk. The approach fully accounts for the heterogeneity problem discussed in the first essay. The approach is then applied on a large sample of firms and the share of default risk in the spreads is estimated. The implications of our finding for the spread decomposition literature, for corporate bond pricing, for risk management and firms capital structure are discussed.

Bonds with embedded call options are often named *callable bonds*. Callable bonds differ from ordinary bonds in four ways. First, the maturity of a callable bond is uncertain. The maturity is decided by the bond issuer when he calls the bond. It is therefore difficult to assess the length of the period during which the bondholder is exposed to the bond risks. Second, callable bonds cash flows are uncertain. If the bond is called then future coupon payments are lost. Third, callable bonds are likely to be called when interest rates are low in the market compared to the bond coupon rate. As a result, there is a reinvestment risk in callable bonds. This reinvestment risk is the risk that when the bond is called, the bondholder is not likely to find similar bonds with the same returns in which to invest his money. Fourth, callable bonds offer limited capital appreciation opportunities. These bonds will always trade around their call price because there is the permanent risk that the bond is bought back by its issuer at the call price. Fifth, callable bonds have different sensitivities to market risk factors than ordinary bonds. This is because the embedded call options also have their own sensitivities to market risk factors.

Because of their complexity, no closed-form formula is available for the pricing of callable bonds. The literature has therefore focused on developing efficient numerical algorithms for the pricing of these bonds. The questions studied in this literature are the questions of numerical instability, convergence, computing speed, and computer memory requirements. Numerical instability is the problem that a small change in an input of the model may cause a large change in the computed price. The question of convergence arises because numerical algorithms often involve a discretization of the state space and/or the time space. Convergence is the desirable property that as the discretization becomes finer and finer the computed price becomes virtually constant. The question of computing speed is natural in the sense that faster algorithms are always preferable. Computing speed is especially relevant in the context of high frequency trading, where a millisecond is worth several millions. Finally, numerical algorithms often requires the storage of large amounts of data in the RAM (Random Access Memory) of the computer. The computer memory problem arises when the algorithm requires a larger RAM than what is available.

Although there is a wide empirical evidence that interest rates are explained by at least two stochastic factors, efficient pricing algorithms for callable bonds are built on one-factor interest rate models. This is because it is not easy to design multifactor interest rate algorithms that are numerically stable, convergent and have reasonable computing power requirements.

The third essay of this thesis overcomes this difficulty. It proposes a numerical valuation methods of callable bonds under a multifactor interest rate model. The essay demonstrates that the algorithm has all the desirable properties of an efficient numerical algorithm. The effect of the different model parameters including the correlation between the factors on the prices are discussed.

The rest of this thesis is divided in 4 chapters. Chapter 2 presents our first essay on structural models and average credit spreads. Chapter 3 presents our second essay on the share of default risk in corporate bond spreads. Chapter 4 presents our third essay on the pricing of callable bonds. Finally, chapter 5 concludes.

CHAPTER 2

STRUCTURAL MODELS AND AVERAGE SPREADS: THE ROLE OF HETEROGENEITY

Structural models of default are models that predict corporate bond spreads from firm fundamentals such as leverage, asset volatility, debt maturity, recovery rate, payout rate, tax rate and riskless rate among other fundamentals. Famous structural models include the Merton [50] model with a zero coupon debt, the Black and Cox [5] model with bond covenants and debt subordination, the Leland and Toft [43] model with taxes, endogenous default and optimal capital structure, the Longstaff and Schwartz [45] model with stochastic interest rate, the Collin-Dufresne and Goldstein [14] model with dynamic leverage and more recently the He and Xiong [32] model with liquidity risk. These models have wide applications in several areas of finance including bonds valuation, bond ratings, capital structure analysis, financial distress resolution, bond covenants analysis, agency problems analysis and financial systems regulations [58].

On the empirical side however, these models have not had much success. Jones et al. [38] have shown that the pricing error from their extension of the Merton [50] model to callable bonds are comparable to those of a naive model which assumes that debt is riskless. Eom et al. [26] tested several structural models including the Merton [50] model, the Geske [31] model, the Longstaff and Schwartz [45] model, the Leland and Toft [43] model and the Collin-Dufresne and Goldstein [14] model and they found that they all underestimate short term spreads and high quality bonds spreads while they overestimate long term spreads and junk bonds spreads.

The problem with the empirical studies of Jones et al. [38] and Eom et al. [26] is that their implementations of the models are not calibrated to observed default losses so that their findings may be due to under-estimations or over-estimations of expected default loss. Huang and Huang [33] have fixed this problem. They implemented several of the above models as well as models with jumps and time varying risk premia and they found that once these models are forced to match historical default experience, they yield similar spreads and they all underestimate bond spreads, especially at the short end of the term structure and for high quality bonds. The findings of Huang and Huang [33] were so striking that they gave rise to a new family of structural models with macroeconomic risk where there exists some states of the world where default rates are very high and at the same time Sharpe ratios are very high [3, 9, 11].

A common feature of empirical tests of credit risk models is that they use various clusterings of the bonds in their samples in order to see whether the model performs better for a given group of bonds than for another. Various authors such as Elton et al. [25], Eom et al. [26], Jones et al. [38], Longstaff et al. [47] use clustering by rating categories as assigned by credit rating agencies to analyze how their models perform for safe bonds versus junk bonds. Given the well known concerns about the reliability of ratings, many authors use their own clustering of firms. For instance, Elkamhi et al. [24] grouped bonds by a measure of distance to default implied by the Merton model. Longstaff et al. [47] used a two-dimensional grouping by rating category and by economic sector. The problem with this clustering approach is that it is often assumed that the obtained groups are homogenous. Many works including the seminal paper of Huang and Huang [33] and the papers of He and Xiong [32], Leland [41], Leland and Toft [43], Longstaff et al. [47] make that homogeneity assumption. Yet there is an overwhelming empirical evidence that these groups are highly heterogeneous both in the cross-section and in the time-series [3, 11, 18, 20, 21, 24, 25, 55].

In this essay, we study how the heterogeneity of a group of bonds in either the cross-section or the time-series affects the average credit spreads predicted by the Merton model at the group level. In our model, cross-section heterogeneity arises because firms in the same group at a given period in time do not necessarily have the same fundamentals. Time-series heterogeneity arises from the fluctuation of these fundamentals over time while the firms remain in the group. At each point in time, individual bonds are valued using the Merton model, but the inputs of the Merton model model are revised over time as the fundamentals of the issuing firm change. Thus, our methodology mimics the common practice of valuing securities using a given model and frequently calibrating the inputs of the model to the market.

A key advantage of our modeling approach is that it allows us to detect the impact, if any, of dynamic leverage strategies, stochastic volatility and stochastic interest rates on average spreads, even though these time-series variations in firm fundamentals are not priced in individual bond values. Our approach differs from that taken in dynamic leverage models such as that of Collin-Dufresne and Goldstein [14], in stochastic interest rate models such as that of Longstaff and Schwartz [45] and in stochastic volatility models such as that of Zhang et al. [62]. In these models, the time-series variation in the fundamentals are priced in individual securities while it is not case in our approach. Besides, the goal of these models is to examine how much compensation bondholders require ex-ante for the anticipated fluctuation of firm fundamentals whereas one of the goals of this paper is to examine how future revisions of the inputs of the Merton model affect the average spreads predicted by the model. The second advantage of our approach is that we can detect the impact of the distributions of the heterogeneous fundamentals in the cross-section on the measured average spreads. This allows us to study how the shape of these distributions affects the average spreads.

We quantify the biases due to ignoring heterogeneity. Ignoring heterogeneity generally consists of choosing a representative firm for the group and of taking the average spread of the group to be the spread of that representative firm as in Huang and Huang [33]. We consider three choices of representative firms which are the average firm, the median firm and the firm whose fundamentals are set to the average fundamentals except asset volatility which is calibrated to average default rates of the group. We study the biases due to each of these ways of ignoring heterogeneity and we study how the moments of the distributions in the case of cross-sectional heterogeneity and the parameters governing the dynamics of the time-series heterogeneity affect the size and sign of these biases. We also study the impact of the non-heterogeneous fundamentals on the sign and size of the biases.

There are few other papers that discuss the role of heterogeneity in pricing corporate bonds. These include Bhamra et al. [3], Chen et al. [11], Cremers et al. [16], David [20], Feldhutter and Schaefer [29]. This paper distinguishes itself from all these other contributions in that it is the sole to jointly: (a) use the Merton model for individual bond spreads, (b) study the impact of different distribution shapes in the cross-section on the average spreads and on the heterogeneity biases, (c) model future revisions in the Merton model inputs such as asset volatility, leverage and interest rate as stochastic processes and study the impact of the characteristics of these processes on the credit spreads, (d) simultaneously study several ways of ignoring heterogeneity and to compare the biases due to each of them and (e) study the impact of non-heterogeneous parameters on the heterogeneity biases.

The rest of this essay is organized as follows. Section 2.1 describes the framework of the study. Section 2.2 studies cross-sectional heterogeneity. Section 2.3 studies time-series heterogeneity. Finally section 2.5 summarizes the findings and concludes.

2.1 The setup

Consider a group of firms $i = 1, 2, ..., n$ followed over a discrete time period $t = 0, 1, ..., \tau$.

Assume that the market is frictionless i.e, there are no taxes, no indivisibility problem, no transaction costs, no restrictions on short-selling, securities are perfectly liquid and there exists a market where one can lend and borrow at the same riskless rate. Further assume that the Modigliani-Miller theorem holds in such a way that the firm value is independent of the firm capital structure and that trading takes place in continuous time.

Assume that a time s, the future evolution of each firm asset value is governed by a Gauss-Wiener process:

$$dV_{it} = \mu_{is} V_{it} dt + \sigma_{is} V_{it} dz_{it}, \text{ for all } t > s, \tag{2.1}$$

where μ_{is} is the expected rate of return on firm i asset value at time s, σ_{is} is the expected volatility of the return on firm i asset value at time s, dz_{it} is a standard Gauss-Wiener process under the physical probability measure, and dz_{it} is independent of dz_{jt} for $j \neq i$. In the risk neutral world, at time s, future asset values are determined by:

$$dV_{it} = r_s V_{it} dt + \sigma_{is} V_{it} dz^{Q}_{it}, \text{ for all } t > s, \tag{2.2}$$

where r_s is the expected level of the riskless rate of the horizon $[s, \tau]$ and dz_{it}^Q is a standard Gauss-Wiener process under the risk neutral probability measure, and dz_{it}^Q is independent of dz_{jt}^Q for $j \neq i$.

The above asset value dynamics are myopic in the sense that at a given period in time, asset values are projected as if the riskless rate, the expected asset returns and the asset volatilities will remain constant at their current values in the future. However, it is recognized ex-ante that at subsequent periods in time, if riskless rates, expected asset returns or asset volatilities change, their new values will be used to project the asset values.

Our choice of these myopic asset value dynamics is motivated by the fact that these dynamics are consistent with how stochastic models are used in practice. An example is the case of option pricing within the Black-Scholes model. In this model, stock volatility is assumed to be constant. At each point in time, investors perform the so-called *calibration to market* by observing implied volatilities from the option market and use them as the volatility in the Black-Scholes model. By using the implied volatility in the Black-Scholes setup, investors are implicitly making the assumption that the volatility will remain constant at the observed level. Yet, the next day when the market is updated, it is the new implied volatilities that will be used as inputs to the model. Another example is the case of interest rate models with deterministic shifts such as the extended Vasicek model and the extended CIR models that are designed to be frequently calibrated to the market. As the market fluctuates, the models are re-calibrated and their inputs are revised. However, ex-ante, the change in inputs is ignored in the projection of interest rate dynamics. These myopic dynamics are often justified by the argument that it suffices to frequently calibrate the model to market in order to obtain accurate prices [34]. Asset dynamics in structural models are also frequently calibrated to market as described in Bharath and Shumway [4], Crosbie and Bohn [17], Elkamhi et al. [24]. Thus our asset dynamics may be viewed as a faithful representation of the practice of frequent calibration to market.

We further assume that each firm continuously issues new debt of maturity T and uses the proceeds to retire the existing debt at no profit or loss in such a way that at date

t, each firm has only two types of securities: equity and a zero-coupon debt of principal B_{it} with a constant maturity T. A similar assumption of permanent refinancing is also made by Leland and Toft [43]. This assumption of permanent refinancing is not binding for our study. It is made for simplicity reasons only. The role of this assumption is to ensure that the group always contains bonds with the same maturity T. An alternative formulation which achieves the same goal and does not require the assumption of permanent refinancing would be to assume that it is not the same group of firms that is followed over time and that at each time a new group of firms with debt principal B_{it} and debt maturity T is formed. The rest of our analysis would be still valid under this alternative formulation.

The firms do not pay dividends nor do they do share repurchases. At maturity, in the event of default by firm i, bondholders of firm i receive the minimum between the asset value and a fraction ρ_i of the face value of the debt.

Let θ_{it} be the vector of fundamentals [1] of firm i at time t:

$$\theta_{it} = (B_{it}, V_{it}, \lambda_{it}, \sigma_{it}, \mu_{it}, \rho_i, r_t, T), \tag{2.3}$$

where:

$$\lambda_{it} = \frac{B_{it}}{V_{it}} \text{ is the leverage ratio.}$$

Since at each point in time, firm fundamentals except asset value are expected to remain constant at their current level, individual securities values at given point in time are given by the Merton [49] model with the sole difference that one must account for the recovery rate as in Eom et al. [26]. Thus, the equity price of firm i at time t is given by:

1. By fundamentals, we mean all the parameters required to value the firm liabilities and to assess the firm physical default rate. It is for this reason that, we include the riskless rate r_t and the asset drift μ_{it} in the vector of all firm fundamentals.

$$
\begin{aligned}
E\left(\theta_{it}\right) &= e^{-r_t T}\mathbb{E}_t^Q\left[\left(V_{it+T}-B_{it}\right)\times 1_{\left\{V_{it+T}>B_{it}\right\}}\right], \\
&= V_{it}N\left(d_{1it}\left(\lambda_{it}\right)\right)-B_{it}e^{-r_t T}N\left(d_{2it}\left(\lambda_{it}\right)\right),
\end{aligned}
$$

where:

$$
\begin{aligned}
d_{1it}\left(x\right) &= \frac{-\ln\left(x\right)+\left(r_t+\frac{\sigma_{it}^2}{2}\right)\left(T\right)}{\sigma_{it}\sqrt{T}}, \\
d_{2it}\left(x\right) &= \frac{-\ln\left(x\right)+\left(r_t-\frac{\sigma_{it}^2}{2}\right)\left(T\right)}{\sigma_{it}\sqrt{T}},
\end{aligned}
\tag{2.4}
$$

$N\left(.\right)$ is the cumulative normal distribution function,

the bond price of firm i at time t is given by:

$$
\begin{aligned}
F\left(\theta_{it}\right) &= e^{-r_t T}\mathbb{E}_t^Q\left[B_{it}\times 1_{\left\{V_{it+T}>B_{it}\right\}}+\min\left(\rho_i B_{it},V_{it+T}\right)\times 1_{\left\{V_{it+T}<B_{it}\right\}}\right], \\
&= B_{it}e^{-r_t T}N\left(d_{2it}\left(\lambda_{it}\right)\right)+V_{it}N\left(-d_{1it}\left(\rho_i\lambda_{it}\right)\right) \\
&+ \rho_i B_{it}e^{-r_t T}\left(N\left(d_{2it}\left(\rho_i\lambda_{it}\right)\right)-N\left(d_{2it}\left(\lambda_{it}\right)\right)\right),
\end{aligned}
\tag{2.5}
$$

the physical default rate of firm i at time t is given by:

$$
\Phi\left(\theta_{it}\right) = N\left(-d_{3it}\left(\lambda_{it}\right)\right), \tag{2.6}
$$

where:

$$
d_{3it}\left(x\right) = \frac{-\ln\left(x\right)+\left(\mu_{it}-\frac{\sigma_{it}^2}{2}\right)\left(T\right)}{\sigma_{it}\sqrt{T}}, \tag{2.7}
$$

and the spread of firm i at time t is given by:

$$
\Psi\left(\theta_{it}\right) = -\frac{1}{T}\ln\left(\frac{F\left(\theta_{it}\right)}{B_{it}}\right)-r_t. \tag{2.8}
$$

Our interest is to measure the average spread of the group of firms over the considered time horizon.

Let $\overline{\theta}$ be the vector of the average fundamentals of the group over the considered period and θ_{median} be the vector of the median fundamentals of the group over the considered period. For readability reasons, we will refer to the firm whose vector of fundamentals coincides with $\overline{\theta}$ as the *average firm* and we will refer to the firm whose vector of fundamentals coincides with θ_{median} as the *median firm*. The average firm and the median firm need not be physical firms [2].

Define θ_{HH} as the vector of parameters where all parameters except asset volatility are equal to their average values in the group over the considered period, and asset volatility is determined in such a way that the average default rate of the group of firms over the period is equal to the default rate of the firm with parameter θ_{HH}, i.e.

$$\Phi(\theta_{HH}) = \frac{1}{n(\tau+1)} \sum_{i=1}^{n} \sum_{t=0}^{\tau} \Phi(\theta_{it}). \tag{2.9}$$

For readability reasons, we will often refer to the firm whose vector of fundamentals coincides with θ_{HH} as the *HH firm* and we will refer to the volatility of the HH firm as the *HH volatility* [3].

Now define the following four estimators of average spread: the ATA estimator, ATM estimator, the HH estimator and the HBF estimator respectively.

Definition The ATA (read At-The-Average) estimator of the average spread is $\Psi\left(\overline{\theta}\right)$, the spread of the average firm.

Definition The ATM (read At-The-Median) estimator of the average spread is $\Psi(\theta_{median})$, the spread of the median firm.

2. The notions of average firm and median firm are abstract statistical concepts. In practice, there may be no firm in the group whose vector of fundamentals coincides with either $\overline{\theta}$ or θ_{median}. However, it is always possible to find statistically $\overline{\theta}$ or θ_{median}.

3. The notion of HH firm is also an abstract mathematical concept. In practice, there may be no firm in the group whose vector of fundamentals coincides with θ_{HH}. However, it is always possible to calculate θ_{HH}. This is because the default rate function is strictly increasing in asset volatility, can be as close to 0 as one wishes for low enough asset volatility and can be as close to 1 as one wishes for high enough asset volatility.

Definition The HH (read Huang and Huang) estimator of the average spread is $\Psi(\theta_{HH})$, the spread of the median firm.

Definition The HBF (read Heterogeneity-Bias-Free) estimator of the average spread is the average of the spreads of the individual firms in the group over the considered time period. The HBF estimator is equal to:

$$\overline{\Psi}_T = \frac{1}{n(\tau+1)} \sum_{i=1}^{n} \sum_{t=0}^{\tau} \Psi_T(\theta_{it}). \tag{2.10}$$

.

These four estimators are conceptual abstractions of how one might attempt to measure the average spread of a group of firms. For instance, a very common grouping of firms in the corporate bond pricing literature is the grouping by rating categories (see for instance Crouhy et al. [18], Elkamhi et al. [24], Elton et al. [25], He and Xiong [32], Huang and Huang [33], Jones et al. [38], Leland and Toft [43], Longstaff and Schwartz [45]). He and Xiong [32], Leland and Toft [43] use the ATA estimator in their models. Crouhy et al. [18] argued that the ATA estimator may lead to biased measures because the distribution of fundamentals inside rating categories is skewed. They suggested to use the ATM estimator instead. Huang and Huang [33] used the HH estimator with the argument that it is essential to calibrate structural models to historical default rates. Elton et al. [25], Jones et al. [38] used the HBF estimator in their studies.

If firms are observed at one period only, i.e. $\tau = 0$ and the considered group is homogeneous, i.e, $\theta_{i0} = \theta$ for $i = 1,...,n$ then all the four estimators are exactly the same. Intuitively, this is simply because in this case, $\overline{\theta} = \theta_{median} = \theta_{HH} = \theta$ and the average of individual spreads is the same as individual spreads.

In principle, none of the four estimators is better or more appropriate than others in the absolute sense. The appropriateness of an estimator depends on its intended use.

If the goal is to get a measure of what the spread of a typical firm of the group would be, then the ATM estimator would be appropriate because of potential asymmetries in the distribution of parameters inside the groups [18].

If the goal is to compare the model average spread to an historical average spread, then the appropriate estimator is naturally the same as the one used for the computation of the historical average spread. For instance, if the historical average has been calculated using the mean of observed individual spreads in the group, then the equivalent model average should be calculated using the mean of model individual spreads, that is using the HBF estimator. If the spread of the firm with the average fundamentals in the group is observable and the historical average has been taken to be that spread, then the equivalent model average should be calculated in a similar fashion, that is by using the ATA estimator.

If calibration to default rates is an important consideration for the intended application, for instance in an analysis of corporate bond risk premia, then the HH estimator would be more appropriate than the ATA and the ATM estimators. In fact, Huang and Huang [33] have shown that calibration to default rates is essential in order to avoid an overestimation or underestimation of expected default loss component of the spreads.

A context where the HBF estimator is the appropriate estimator is the following. Assume that all individual bonds in the group are held in equal weights by an investor. In this case, the spread earned on the portfolio is given by the average spreads of the individual spreads in the group, that is, the HBF estimator. This implies that investment decisions in this portfolio of bonds would be based on the HBF estimator instead of ATA, ATM or HH estimators.

In the rest of this paper we assume that the HBF estimator is the appropriate estimator. This assumption is justified by two reasons. First, the most common intended use of a model average spread in the literature of performance tests of structural models is to compare it with an historical average spread calculated using the mean of observed individual spreads in the group. Second, the HBF estimator has the intuitive interpretation of a portfolio average spread. Throughout the rest of this essay, the term *average spreads* refer to the HBF average spreads unless otherwise mentioned.

In the rest of the paper we will also be interested in the consequences of using one of the ATA, ATM or HH estimators instead of HBF estimator. This often happens in the structural models testing literature, perhaps because it is much easier and simpler to use the ATA, ATM or HH estimators than the HBF estimator. Using the HBF estimator entails estimating the fundamentals for all individual firms while average or median fundamentals are generally available in the literature or could be estimated by economic arguments. Also average historical default probabilities per such groups as rating categories or economic sectors are made available to the public by rating agencies. Thus, using the ATA, ATM or HH estimators saves estimation time.

As explained earlier, using the ATA, ATM or HH estimators does not pose any problem if the group of firms is homogeneous and is observed at one period of time only. However, commonly used grouping of firms such as rating categories or economic sectors are highly heterogeneous and are observed over long time periods [3, 24, 25, 45]. This implies that using the ATA, ATM or HH estimators exposes the econometrician to potential biases. We shall refer to these biases as heterogeneity biases. We shall also interpret the ATA, ATM or HH estimators as estimators that assume homogeneity or that ignore heterogeneity.

There are three possible biases: the ATA bias, the ATM bias and the HH bias. These are the ratios over the HBF estimator of the difference between the HBF estimator and the ATA estimator, the difference between the HBF estimator and the ATM estimator and the difference between the HBF estimator and the HH estimator respectively.

In the next sections, we adopt a simulation approach to study the impact of heterogeneity on the HBF average spreads and on the heterogeneity biases. This approach allows us to abstract from sample bias issues and make inferences at the population level. Besides, it allows us to study the effect of heterogeneity in each fundamental separately and to vary the distribution shape of one fundamental while keeping the other fundamentals constant. The ability to study different distribution shapes is important because distribution shapes vary widely from groups to groups. For instance, the distribution of leverage for AA firms is less dispersed than that for BBB firms [55]. Furthermore, we can generate different asset value paths from Equation 2.1 and use them to evaluate theoretically the effect of time-series heterogeneity.

2.2 Cross-sectional heterogeneity bias

Here we assume that $\tau = 0$ so that the group is made of different firms observed at the initial time period. The firms in the group are heterogeneous in either leverage λ_{i0} or volatility σ_{i0} but not both. All fundamentals except the heterogeneous one are the same. To simplify notation, when a fundamental is non-heterogeneous we will not index it by a subscript. So λ, σ, ρ, r, and μ refer to the non-heterogeneous values of leverage, asset volatility, recovery rate, riskless rate, and expected asset return respectively.

2.2.1 The simulation procedure

For each heterogeneous fundamental (leverage or volatility), we consider various distribution shapes and compute the HBF spread and the ATA, ATM and HH biases for each shape. The simulation procedure is as follows.

For each heterogeneous fundamental, and for each distribution shape, we generate an heterogeneous sample of n firms by randomly drawing n values of the fundamental from the distribution. We then compute the ATA, ATM, HH and HBF estimators for this sample of firms and take the relevant differences and ratios to compute the biases. Next, we repeat this procedure M times in order to obtain the distributions of the biases of the ATA, ATM and HH estimators. We then measure the average and the standard errors of the biases of each estimator. For each estimator, the standard error is calculated as the standard deviation of the sample of the M biases divided by the square root of M. The biases and their standard errors are expressed as a percentage of the HBF average spreads. Along with the biases, we also report the average of the HBF average spreads and its corresponding standard error (calculated as the standard deviation of the sample of the M HBF estimates divided by the square root of M). In the simulations, we choose $n = 1000$, about the number of publicly traded and rated firms in the US and $M = 10000$, a number large enough so that standard errors are small enough for statistical significance.

2.2.2 Comparing the HBF average spread and the heterogeneity biases for different distribution shapes

In all simulations, we fix $\mu = 0.13$, which is at the midpoint between the values of 0.12 in [42] and of 14.66 in [33], $r = 0.05$, as in [29], $\rho = 0.4$, which is well inside the range of recovery rates in [57] and $T = 4$, which is close to the average maturity of BBB firms in [42]. When leverage is the heterogeneous parameter we fix $\sigma = 0.3$, which is close to the median asset volatility in [24], and, when volatility is the heterogeneous parameter, we fix $\lambda = 0.4$, which is close to the average leverage in [24].

We consider 8 distribution shapes of leverage and volatility, mimicking observed features of these fundamentals inside rating categories or economic sectors, including uniform distributions, normal distributions and skewed distributions. To calibrate the distributions, we back out daily leverage ratios and asset volatilities from equity prices of all North American firms except financial firms common to the databases COMPUSTAT and CRSP over the period 1984-2013. The sample has about 2.8 Millions firm-days observations. The characteristics of the distributions are chosen to fit observed ranges or moments of the distributions of leverage and asset volatility in selected rating categories. The negative skewness distributions have been observed in selected group of firms of the economic sector and rating categories. Tables 2.I and 2.II summarize the selected distributions for leverage and volatility respectively.

Results for the cross-sectional leverage heterogeneity study are shown in table 2.III and those for the cross-sectional volatility heterogeneity study are shown in table 2.IV. The tables show the HBF average spread in bps in column 5 and the ATA, ATM and HH biases as a percentage of the HBF average spread in columns 2, 3 and 4 respectively. The tables also report standard errors in parenthesis. On can see that the obtained standard errors are small enough to ensure statistical significance. The results show that the characteristics of the cross-section distribution of leverage and volatility have an impact on the measured average spread and on the sign an size of the heterogeneity biases.

First consider leverage heterogeneity.

The two highest average spreads are obtained with the Uniform-HighRange distribu-

Table 2.I: Selected distribution for the study of cross-sectional leverage heterogeneity

Distribution	Matched empirical feature	Characteristics
Uniform-LowRange	AAA range	range=[0.1,0.6446]
Uniform-HighRange	BBB range	range=[0.13,10.4469]
Normal-LowStd	BBB average	mean=0.4, sd=0.05
Normal-HighStd	BBB average	mean=0.4, sd=0.1
Pearson-PosSkew1	AAA first 4 moments	mean=0.0876, sd=0.0849, skewness=1.8, kurtosis=6.81
Pearson-PosSkew2	BBB first 4 moments	mean=0.4, sd=0.1, skewness=4.88, kurtosis=95.13
Pearson-NegSkew1	Utility-BBB first 4 moments	mean=1.8011, sd=0.4727, skewness=-0.4710, kurtosis= 2.9398
Pearson-NegSkew2	Communications-AA first 4 moments	mean=0.1175, sd=0.0394, skewness=-0.6538, kurtosis=2.6743

This table describes the considered distributions for the study of cross-sectional heterogeneity in leverage. To calibrate the distributions, we back out daily leverage ratios from equity prices of all North American firms (except financial firms) common to the databases COMPUSTAT and CRSP over the period 1984-2013 . The sample has about 2.8 Millions firm-days observations. The characteristics of the distributions are chosen to fit observed ranges or moments of the distributions of leverages in selected groups.

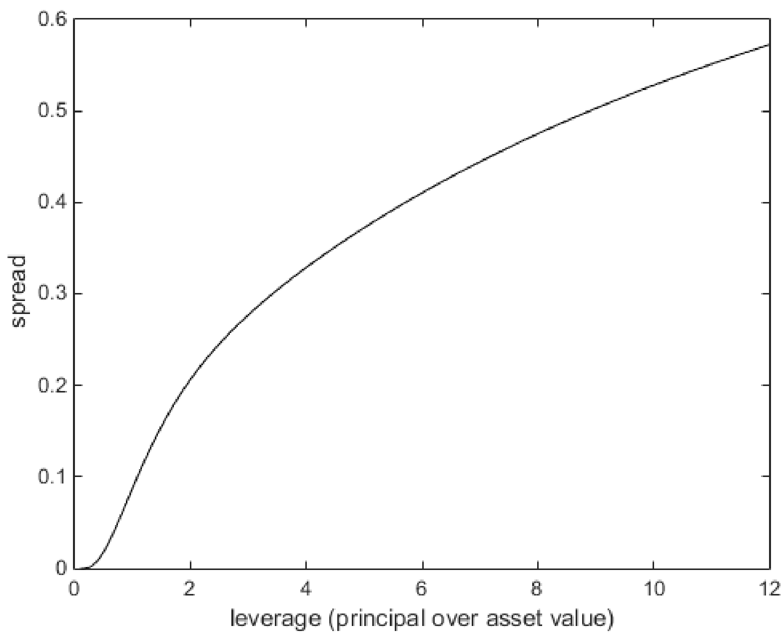
Table 2.II: Selected distribution for the study of cross-sectional volatility heterogeneity

Distribution	Matched empirical feature	Characteristics
Uniform-LowRange	AAA range	range=[0.1384,0.5852]
Uniform-HighRange	B range	range=[0.1182,4.1037]
Normal-LowStd	AA average	mean=0.2539, sd=0.03
Normal-HighStd	AA first 2 moments	mean=0.2539, sd=0.0698
Pearson-PosSkew1	AA first 4 moments	mean=0.2539, sd=0.0698, skewness=0.89, kurtosis=4.73
Pearson-PosSkew2	BBB first 4 moments	mean=0.3132, sd=0.1943, skewness=9.31, kurtosis=136.01
Pearson-NegSkew1	Technology-AAA first 4 moments	mean=0.2795, sd=0.0492, skewness=-0.3566, kurtosis= 2.474
Pearson-NegSkew2	Capital goods-AAA first 4 moments	mean=0.2642, sd=0.0252, skewness=-0.7708, kurtosis=1.7745

This table describes the considered distributions for the study of cross-sectional heterogeneity in volatility. To calibrate the distributions, we back out daily asset volatilities from equity prices of all North American firms (except financial firms) common to the databases COMPUSTAT and CRSP over the period 1984-2013. The sample has about 2.8 Millions firm-days observations. The characteristics of the distributions are chosen to fit observed ranges or moments of the distributions of asset volatility in selected groups.

tion and the Pearson-NegSkew1 distribution. These two distributions have exceptionally high average leverages of 528% and 180%, justifying their corresponding high average spreads of 3492.38 bps and 1819.77 bps respectively. For these two distributions, the sign of the heterogeneity biases are negative indicating that ignoring heterogeneity leads to an overestimation of the average spreads. This negative sign contrasts with the convexity bias argument of David [20] and Feldhutter and Schaefer [29] that the heterogeneity bias is positive due to the convexity of the spread function. Actually, the spread function is indeed convex for low leverage values but it becomes concave for high leverage values as shown in figure 2.1. It is this concavity that justifies the negative sign that we obtain here.

Figure 2.1: The spread predicted by the Merton model as a function of leverage



This graph shows that for a realistic range of leverage as observed in our sample of BBB firms, the spread function is convex for low leverage values but become concave for high leverage values. Parameter values are $\mu = 0.13$, $\sigma = 0.3$, $\rho = 0.4$, $T = 4$, $r = 0.05$.

The two lowest average spreads are obtained with the Pearson-NegSkew2 and Pearson-PosSkew1 distributions. These two distributions have very low average leverages of 11.75% and 8.76% respectively justifying their corresponding low average spreads of

0.62 bps and 3.73 bps respectively. Interestingly, the Pearson-NegSkew2 has a higher average than the Pearson-PosSkew1 distribution but the first has a lower spread than the second. This shows that the higher moments of the distributions play an important role in explaining the obtained average spreads. Indeed, the Pearson-PosSkew1 distribution has a positive third moment which means that it is asymmetric to the right and admits few very high leverages that pull the average spread up. In contrast, the Pearson-NegSkew1 has a negative third moment which implies that it is asymmetric to the left and admits few very low leverages that pull the average spread down. Regarding the heterogeneity biases, we see that they exceed 40% of the HBF average spread. This strengthens the point that it is important to account for heterogeneity.

The cases of the normal distributions are interesting in that the average is kept constant while the standard deviation is doubled for the Normal-HighStd distribution versus the Normal-LowStd distribution. This allows us to isolate the impact of higher dispersion on the measured average spread and on the size of the bias. We find that higher dispersion leads to larger heterogeneity biases. This finding is expected as higher dispersion implies larger heterogeneity and, as a result, we expect the effect of heterogeneity to be stronger. We all also find that higher dispersion leads to larger average spreads. This finding is due to the fact that the spread function increases more rapidly for leverage values above the average than for leverage values below the average.

We now turn to cross-sectional volatility heterogeneity.

The two largest average spreads are obtained for the Uniform-HighRange and Uniform-LowRange distributions. These distributions have the highest average volatilities of 270% and 36% respectively. The exceptionally high average volatility of the Uniform-HighRange distribution justifies its corresponding exceptionally high average spread of 8818.59 bps. Ignoring heterogeneity in these cases biases the measured average spreads downward by more than 17% except for the HH bias in the Uniform-LowRange distribution case, where the bias is negative (HH is higher than HBF).

Similarly to the case of leverage heterogeneity, we observe that for the case of the two normal distributions, the distribution with the larger dispersion shows a higher average spread and larger heterogeneity biases. The justification is the same as the one for leverage heterogeneity.

We also observe that the ATA bias is positive for all the considered cases. This is because the spread function is convex in volatility over the considered range of parameters (see Figure 2.2). The ATM bias is also positive for all cases but the for the Pearson-NegSkew2 distribution. The negative sign of the ATM bias in this case cannot be justified by a concavity argument since the spread function is convex. In this case, the negative sign occurs because, for this particular distribution, the median volatility is well above the average volatility and above the volatility that equalizes the individual spread function to the average spread. This happens because of the negative skew of the distribution. It is also because of this negative skew that the ATM bias is lower than the ATA bias for the Pearson-NegSkew1 distribution. It is for a similar reason that the ATM bias is higher than the ATA bias for the positive skew distributions since in this case the median volatility is lower than the average volatility. Consistently with this argument, the ATA and ATM biases are similar for the symmetric uniform and normal distributions. Notice that this relation between the ATA bias and the ATM bias also holds in the case of leverage heterogeneity.

Finally we observe that the HH bias is often negative. This negative sign is also explained by the relative position of the average volatility and the HH-volatility, i.e. the volatility such that default probability at this volatility is equal to the average default probability. A negative sign of the HH bias implies that the HH-volatility is well above the average volatility and above the volatility that equalizes individual spreads with the average spread.

Overall, we find that ignoring cross-sectional heterogeneity in leverage or volatility may severely bias the measured average spreads. The sign and the magnitude of this bias depends on the characteristics of the distribution. Indeed our results illustrate that the first three moments are important to understand the sign and size of the biases. These moments are also important in understanding the variation in average spreads from one distribution to the other. Another critical determinant of the sign and magnitude of the bias is the shape of the spread function.

Table 2.III: Cross-sectional leverage heterogeneity bias (std. err. in parenthesis)

Distribution	Heterogeneity bias			HBF average spreads (bps)
	ATA bias	ATM bias	HH bias	
Uniform-LowRange	35.42% (0.013%)	35.36% (0.038%)	-0.85% (0.002%)	109.64 (0.035)
Uniform-HighRange	-9.91% (0.006%)	-9.90% (0.011%)	-1097.60% (0.226%)	3491.75 (0.464)
Normal-LowStd	4.33% (0.002%)	4.34% (0.01%)	-0.59% (0.001%)	94.85 (0.013)
Normal-HighStd	14.83% (0.007%)	14.84% (0.019%)	-1.40% (0.001%)	106.58 (0.025)
Pearson-PosSkew1	99.12% (0.002%)	99.96% (0.001%)	100.00% (0.001%)	3.73 (0.006)
Pearson-PosSkew2	11.48% (0.009%)	32.85% (0.021%)	-3.32% (0.005%)	102.61 (0.035)
Pearson-NegSkew1	-3.34% (0.002%)	-5.68% (0.007%)	-7.27% (0.014%)	1819.92 (0.158)
Pearson-NegSkew2	61.33% (0.017%)	44.54% (0.037%)	100.00% (0.001%)	0.62 (0.001)

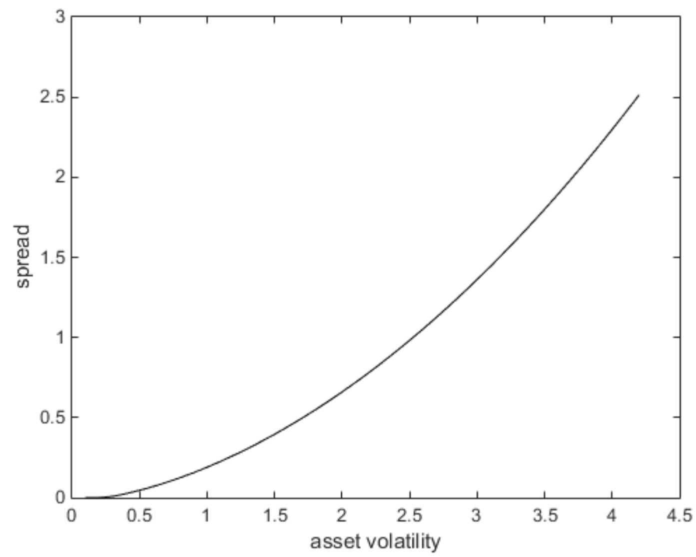
This table shows computed ATA, ATM and HH biases (columns 2 to 4) as a percentage of the HBF average spread (column 5), in the case of cross-sectional leverage heterogeneity. The ATA (At-The-Average) estimator is the spread at the average firm. The ATM (At-The-Median) estimator is the spread at the median firm. The HH (Huang and Huang) estimator is the spread for the firm whose default probability equals the average default probability of the heterogeneous group. The HBF (Heterogeneity-Bias-Free) estimator is the average of the individual spreads. In each simulation, 1000 leverages are randomly drawn from the selected distribution, the average spread is estimated using the HBF, ATA, ATM and HH estimators respectively and the biases are calculated. The simulation is repeated 10000 times to calculate the average biases and the standard errors of the estimations. In the simulation the other firm parameters are held constant: $\sigma = 0.3$, $\mu = 0.13$, $r = 0.05$, $\rho = 0.4$, $T = 4$.

Table 2.IV: Cross-sectional volatility heterogeneity bias (std. err. in parenthesis)

Distribution	Heterogeneity bias			HBF average spreads (bps)
	ATA bias	ATM bias	HH bias	
Uniform-LowRange	20.32% (0.01%)	20.30% (0.036%)	-9.49% (0.006%)	233.93 (0.066)
Uniform-HighRange	17.56% (0.007%)	17.52% (0.028%)	63.65% (0.009%)	8817.25 (2.266)
Normal-HighStd	10.67% (0.005%)	10.69% (0.016%)	-9.20% (0.004%)	43.88 (0.009)
Normal-LowStd	36.18% (0.015%)	36.21% (0.029%)	-23.26% (0.009%)	61.50 (0.022)
Pearson-PosSkew1	35.73% (0.013%)	47.74% (0.027%)	-27.75% (0.011%)	61.06 (0.026)
Pearson-PosSkew2	35.32% (0.051%)	66.83% (0.049%)	13.43% (0.089%)	170.80 (0.263)
Pearson-NegSkew1	14.84% (0.008%)	8.43% (0.021%)	-10.85% (0.005%)	76.67 (0.017)
Pearson-NegSkew2	6.39% (0.003%)	-31.51% (0.012%)	-5.07% (0.003%)	52.23 (0.007)

This table shows computed ATA, ATM and HH biases (columns 2 to 4) as a proportion of the HBF average spread (column 5) in the case of cross-sectional volatility heterogeneity. The ATA (At-The-Average) estimator is the spread at the average firm. The ATM (At-The-Median) estimator is the spread at the median firm. The HH (Huang and Huang) estimator is the spread for the firm whose default probability equals the average default probability of the heterogeneous group. The HBF (Heterogeneity-Bias-Free) estimator is the average of the individual spreads. In each simulation, 1000 asset volatilities are randomly drawn from the selected distribution, the average spread is estimated using the HBF, ATA, ATM and HH estimators respectively and the biases are calculated. The simulation is repeated 10000 times to calculate the average biases and the standard errors of the estimations. In the simulation the other firm parameters are held constant: $\lambda = 0.4$, $\mu = 0.13$, $r = 0.05$, $\rho = 0.4$, $T = 4$.

Figure 2.2: The spread predicted by the Merton model as a function of volatility



This graph shows that for a very wide range of volatility, the spread function is convex. The other model parameters are as follows $\mu = 0.13$, $\lambda = 0.4$, $\rho = 0.4$, $T = 4$, $r = 0.05$.

2.2.3 Effect of other fundamentals on average spreads and heterogeneity biases

In the previous subsection, apart from the heterogeneous fundamentals, the other fundamentals were kept fixed. In this section, we study the impact of the level of these other fundamentals on the average spreads and the heterogeneity biases. For space reasons, we specialize on the case of the Pearson-SkewPos2 distributions for the two cases of leverage and volatility heterogeneity. These distributions also have the advantage that their first four moments have been calibrated to the observed moments of leverage and volatility for BBB firms.

To evaluate the effect of the non-heterogeneous fundamentals, we repeat the simulation procedure of the previous subsection for different levels of each non-heterogeneous fundamentals. The default values of the non-heterogeneous fundamentals are as in the previous subsection. Results for the cross-section heterogeneity in leverage are shown in Figure 2.3 and those for the cross-section heterogeneity in volatility are shown in Figure 2.4.

2.2.3.1 Effect of debt maturity

Figures 2.3 and 2.4 show that the term structure of average spreads has a hump shape. Average spread is increasing for short maturities and declining for long maturities. As explained by Merton [50], longer maturities represent longer investment periods and pose greater risk to the investor in that sense, but, at the same time, they delay the possibility of a default. For leverage values lower than 1, the first effect dominates in the short run and the second effect dominates in the long run.

We observe on these figures that the ATA and ATM biases steadily decline with maturity. Two effects are at play here. First, the average spread tends to increase with maturity for short maturities so that the denominators of the biases increase. Second, maturity affects the shape of the spread function. As shown in Figure 2.5, the concavity of the spread function for high leverage values increases with maturity. This reduces the convexity effect. Also, as shown in Figure 2.5, the convexity of the spread as a function of volatility is stronger for shorter maturities than for longer maturities. Thus, the second effect reduces the numerator of the biases. These two effects explain the decline in the ATA and ATM biases.

In contrast with the other two, the HH bias tends to increases with maturity, although there is a slight decline at very short maturities in the case of leverage heterogeneity. The difference with the two other biases is that maturity also affects θ_{HH} through the average default probabilities in such a way the overall effect is not easily predictable.

2.2.3.2 Effect of recovery rate

As one may have expected, average spreads decrease with recovery rates since the consequences of default become less severe with higher recovery rates. This decline puts an upward pressure on the magnitude of the heterogeneity biases. However, recovery rates also affect the shape of the spread function, so that the effect of recovery rates on the heterogeneity biases is non-trivial.

In the case of cross-sectional volatility heterogeneity, the convexity of the spread function becomes more pronounced with higher recovery rates as shown in Figure 2.5.

This reinforces the convexity effect and puts additional upward pressure on the volatility heterogeneity biases. The resulting effect of these two upward pressures is an increase in the biases.

In the case of cross-sectional leverage heterogeneity, for low recovery rates, the concavity of the spread function becomes more pronounced as the recovery rate increases. This reinforces the concavity effect, putting downward pressures on the biases, but the opposite effect is observed for high recovery rates. The combination of these pressures justifies the slight declining trend of the leverage heterogeneity biases for low recovery rates and their upward trend for high recovery rates.

2.2.3.3 Effect of riskless rate

Higher riskless rates are associated with lower spreads, as the drift in the risk neutral world is then higher. This first effect tends to increase the magnitude of the heterogeneity biases. This implies a downward pressure for the HH bias, which is negative, and an upward pressure for the other biases since they are positive. Additionally, the concavity of the spread function for high leverages becomes less pronounced in the case of the leverage heterogeneity, as shown in Figure 2.5, and the convexity of the spread function becomes more pronounced in the case of the volatility heterogeneity, as shown in Figure 2.6. This second effect tends to increase the biases. Since the two effects operate in the upward direction for the ATA and ATM biases, they both increase. In the case of HH bias, the second effect appears to dominate the first and the HH bias also increases.

2.2.3.4 Effect of volatility

In the case of cross-sectional heterogeneity in leverage, average spreads increase with the asset volatility as expected. This first effect puts a downward pressure on the magnitude of the heterogeneity biases. This results in an upward pressure for the HH bias, which is negative, and a downward pressure for the other biases, which are positive. At the same time, the concavity of the spread function for high leverage values becomes less pronounced, as shown in Figure 2.5. This second effect puts an upward pressure on the biases. The result of these two effects is an increase in the HH bias and a decrease in the ATA and ATM biases. This suggests that for the ATA and ATM biases, the downward pressure from the increase in spreads dominates the upward pressure from the shape of the spread function.

2.2.3.5 Effect of leverage

In the case of cross-sectional heterogeneity in volatility, average spreads increase with leverage, as expected. This first effect puts a downward pressure on the heterogeneity biases. Additionally, the convexity of the spread function becomes less pronounced, as shown in Figure 2.6. This second effect puts an additional downward pressure on the biases. As a result of these two downward effects, the biases decline.

Figure 2.3: Cross-sectional heterogeneity in leverage: Effect of non-heterogeneous fundamentals on average spreads and heterogeneity biases

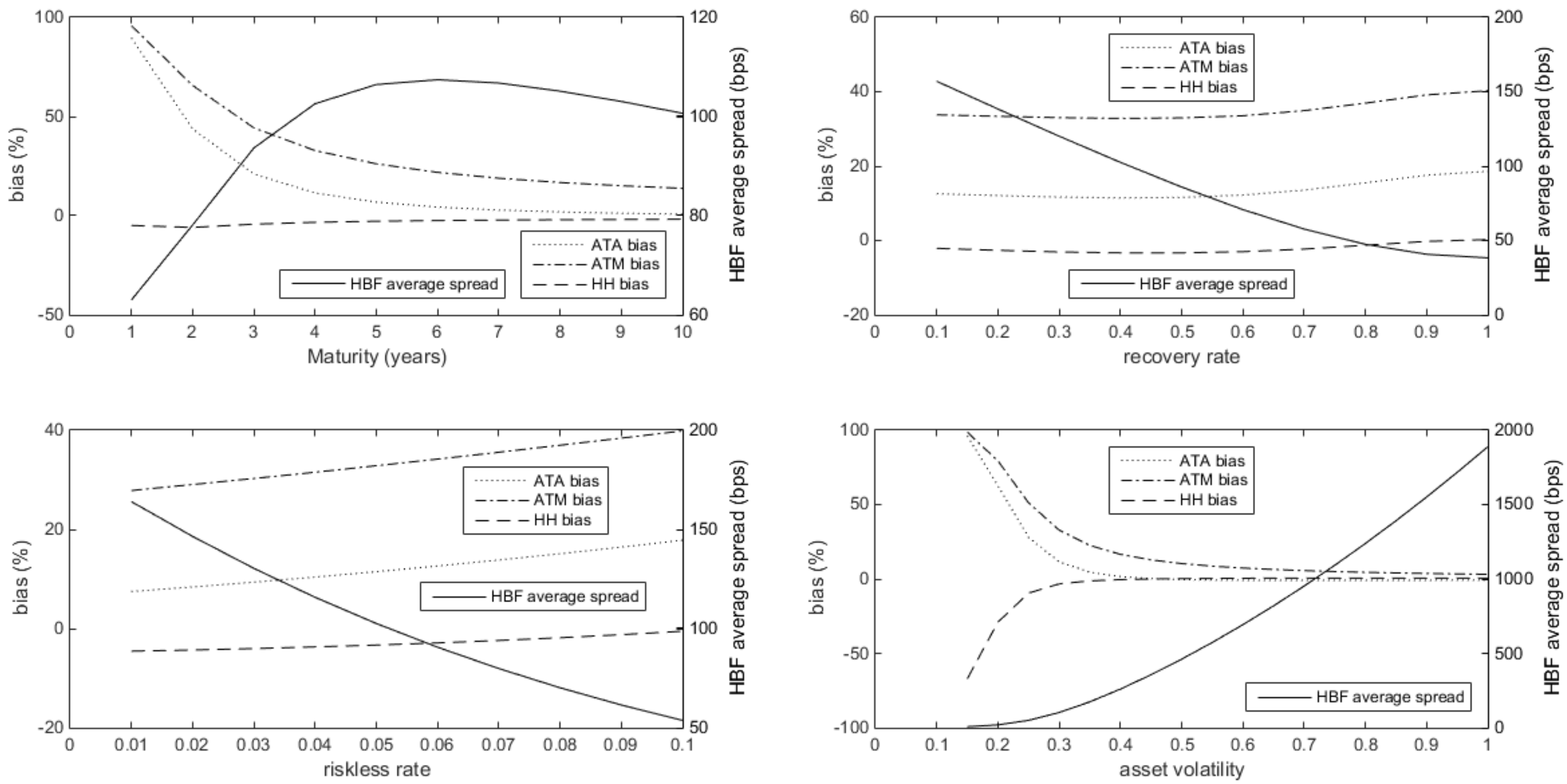


Figure 2.4: Cross-sectional heterogeneity in volatility: Effect of non-heterogeneous fundamentals on average spreads and heterogeneity biases

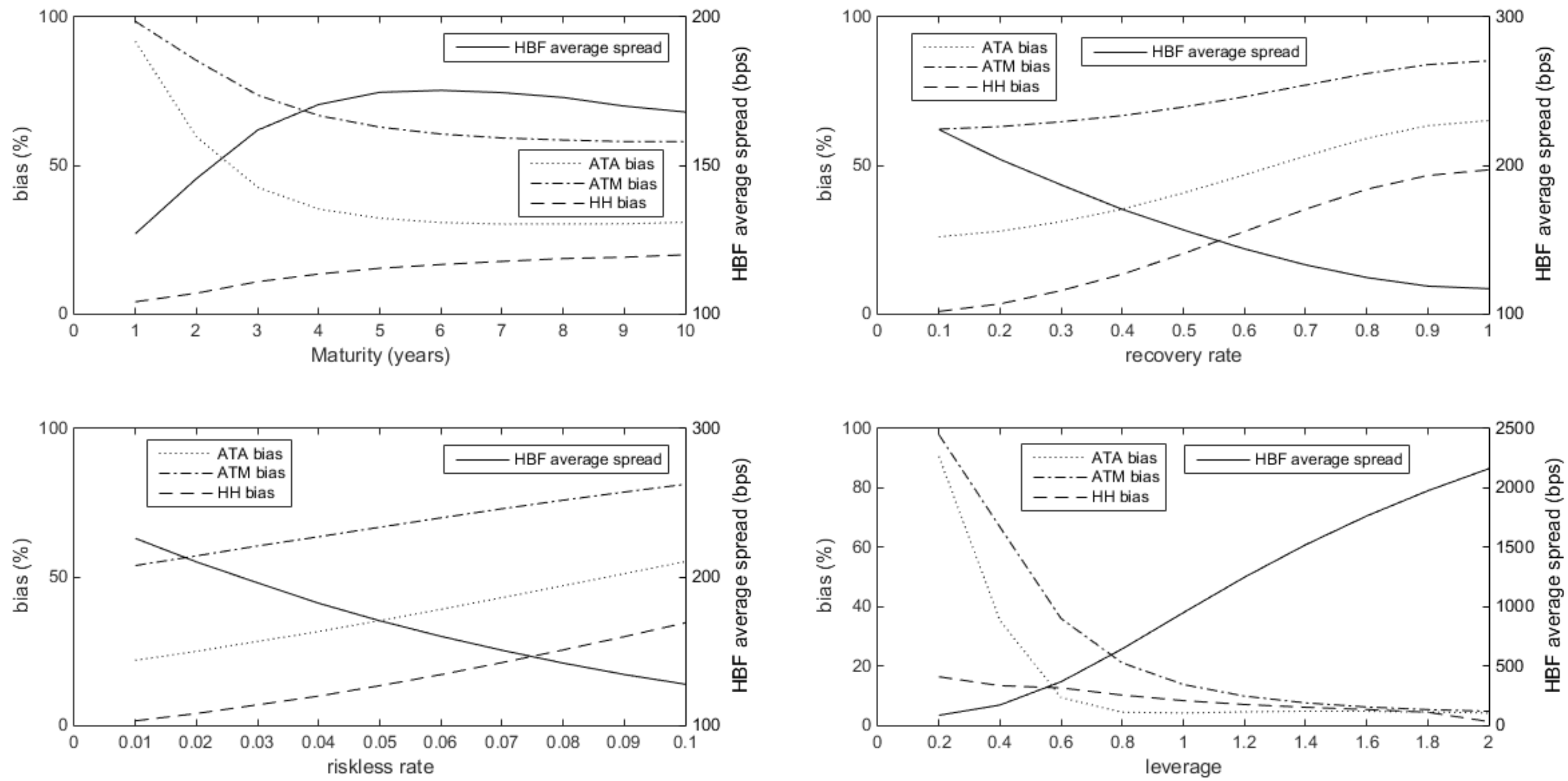


Figure 2.5: Spread as a function of leverage: Effect of other firm fundamentals

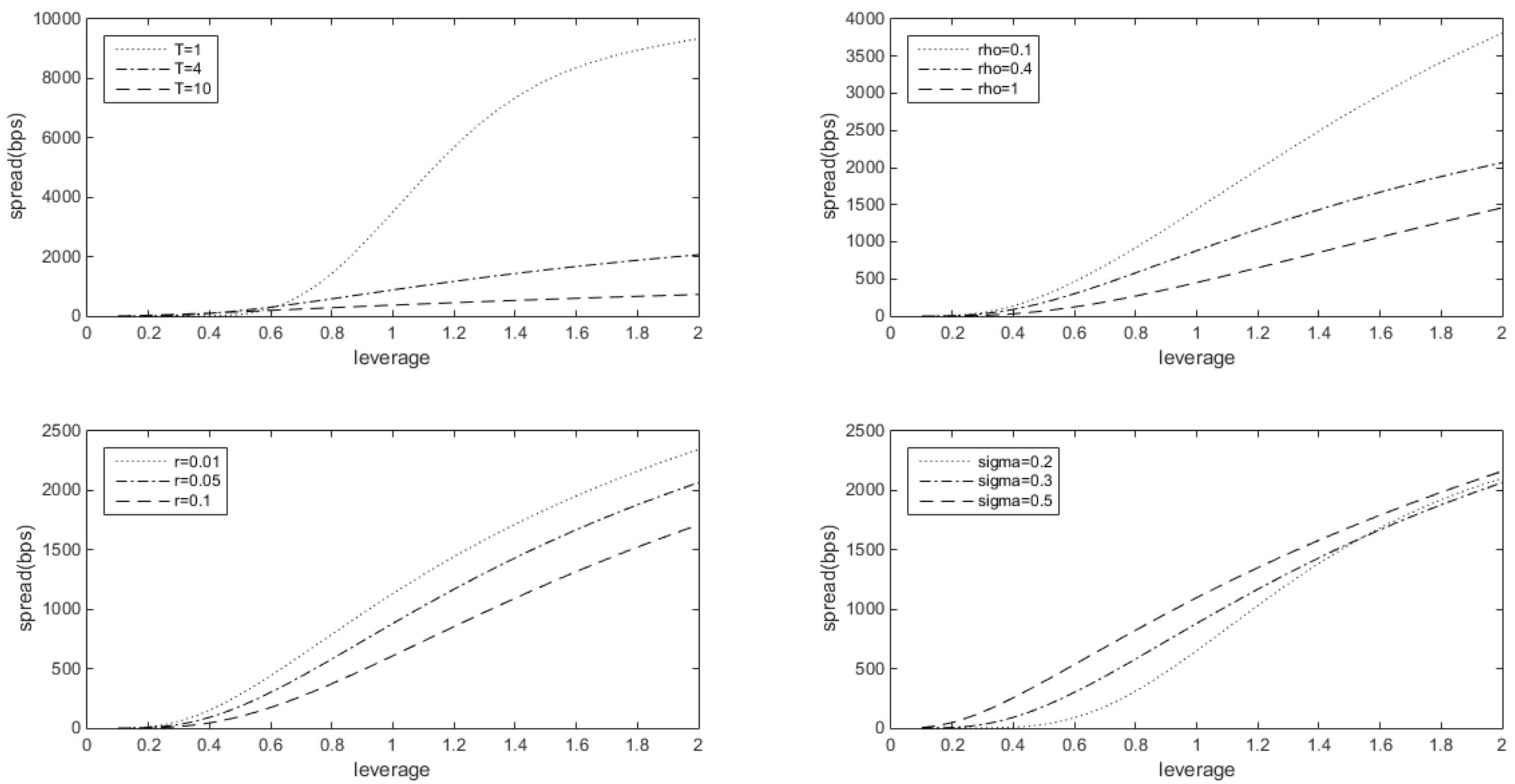
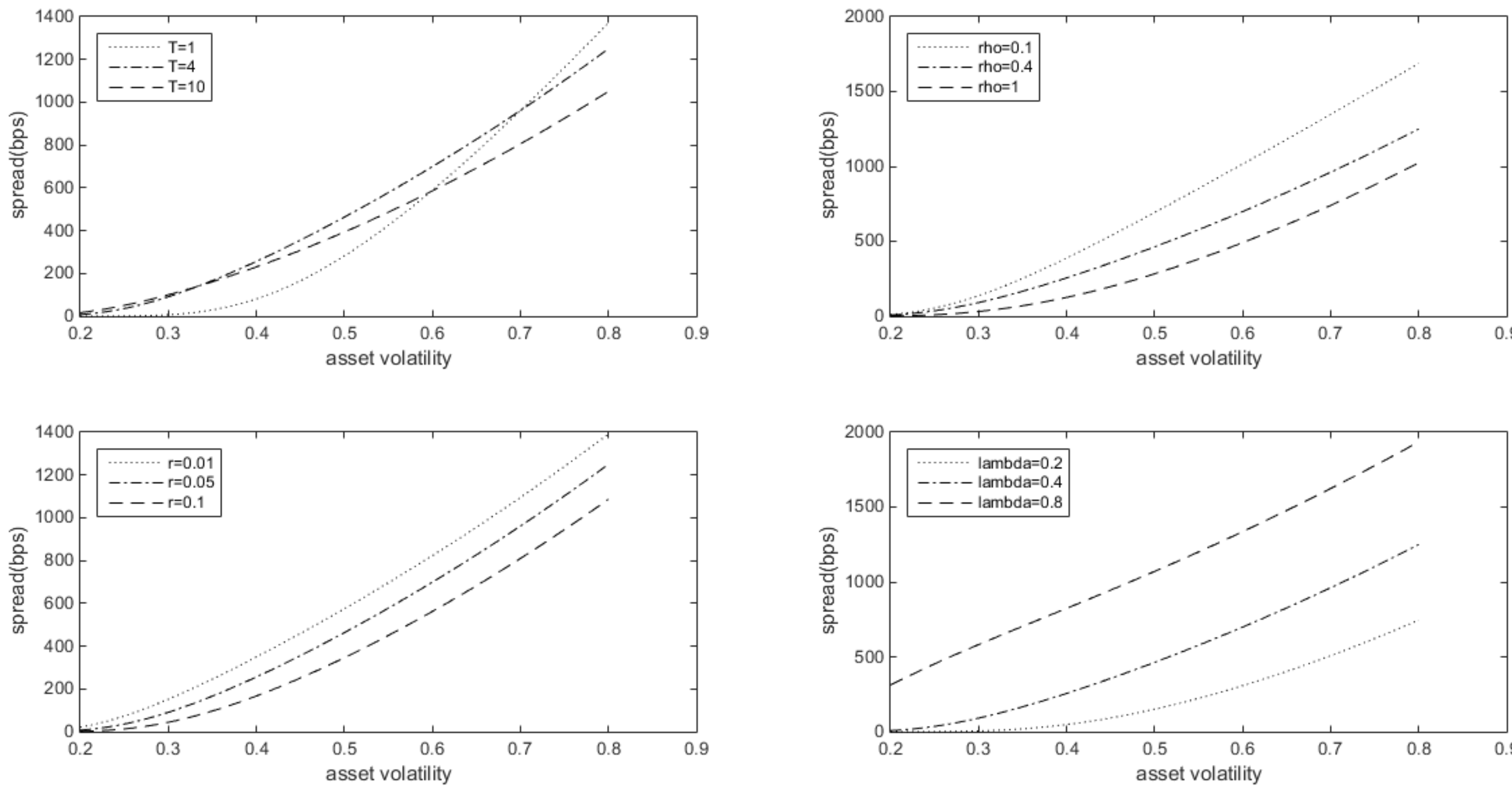


Figure 2.6: Spread as a function of volatility: Effect of other firm fundamentals



2.3 Time-series heterogeneity bias

In this section, we consider the case where $n = 1$, and we study the impact of future variations in leverage, asset volatility and riskless rate on the average spread of the firm over the horizon $[0, \tau]$. We remind the reader that even though leverage, asset volatility and riskless rate vary over time, our model of average spread is different from existent dynamic leverage models, stochastic volatility models and stochastic interest rate models. In these models, future variations in debt principal, volatility or interest rates are explicitly priced in individual bond values at each point in time. In contrast, in our model, individual bond values at each point in time assume, in a myopic way, that debt principal, volatility and riskless rate will remain constant at their current values. But then at a subsequent point in time, when debt principal, volatility or riskless rate change, the bond value is updated. Therefore, in our model, at no point in time is stochastic debt principal, volatility or riskless rate priced in an individual bond value, but the fluctuations in these fundamentals in time will materialize in the average spread over the considered time period.

The procedure to calculate average spreads and heterogeneity biases is as follows. We simulate M paths of monthly asset values from the dynamics of the heterogeneous fundamentals. For each path, we calculate the HBF average spread, and the corresponding ATA, ATM and HH biases as a proportion of the HBF average spread. We then calculate the overall HBF average spread and the average ATA, ATM and HH biases over the M paths. We also calculate the standard error of the estimations as the standard deviation of each path result divided by $\sqrt{M}$.

In this section, to simplify notation, a fundamental that is non-heterogeneous is not indexed by a subscript. So λ, σ, ρ, r, and μ refer to the non-heterogeneous values of leverage, asset volatility, recovery rate, riskless rate, and expected asset return respectively.

The default value of the firm fundamentals are: expected asset return $\mu = 0.13$, asset volatility $\sigma = 0.3$, debt maturity $T = 4$, recovery rate $\rho = 0.4$, debt principal $B = 40$, initial asset value $V_0 = 100$, riskless rate $r = 0.05$, horizon $\tau = 10$. We choose $M =$

20000.

2.3.1 Time-series heterogeneity in leverage with constant debt principal

In this subsection, we consider the case where debt principal is constant; accordingly, the time-series variation in leverage comes from the time-series variation in asset value, as described by Equation 2.1. Figure 2.7 plots the computed average spreads and heterogeneity biases as a function of expected asset return, asset volatility, maturity and recovery rate. Figure 2.8 plots the computed average spreads and heterogeneity biases as a function of the horizon τ. 95% confidence bands are provided when standard errors of the estimations are large. The findings are as follows.

Average spreads decrease with expected asset returns. This is because, higher expected asset returns imply lower expected future leverages which in turn imply lower expected future spreads. The lower expected future spreads pull down the average spreads. Regarding the heterogeneity biases, for expected asset returns lower than 15%, the HH bias is less than 3% of the average spreads while the ATA and ATM biases may be as large as 40%. The ATA and ATM biases increase monotonically with expected asset return simply reflecting the decrease in average spreads. For low expected asset returns, the HH bias declines with the expected asset return and for high expected asset returns, the HH bias increases with the expected asset return. The different behavior of the HH bias compared to the other biases is due to the fact that the expected asset return matters for default probabilities.

Average spreads increase with asset volatility. Here, in addition to the traditional channel of higher default risk, there is also an heterogeneity channel. Indeed, higher asset volatility implies more variability in future leverages, so that the convexity effect is amplified and average spreads are higher. Regarding the heterogeneity biases, they decrease with the asset volatility despite the stronger convexity effect because of the additional increase in average spreads from the higher default risk channel. The biases are larger than 50% for asset volatilities around 20%.

The observed term structure of average spreads is declining. This contrasts with the hump shaped term structure of the Merton model without heterogeneity. The declining effect is explained by the effect of maturity on the shape of the spread function. For low leverages, higher maturity weakens the convexity of the function so that the effect of heterogeneity is less important and average spreads are lower. Of course, the traditional effects of maturity on spreads in absence of heterogeneity are present but the convexity effect appears to be dominate these traditional effects. Regarding the heterogeneity biases, they are more severe at the short end of the term structure than at the long term. This is also explained by the effect of maturity on the spread function convexity.

Average spreads decline with recovery rates, as expected. The time-series heterogeneity biases are not very sensitive to recovery rates for low recovery rates. In fact, for low recovery rates, the decline in average spreads appears to be compensated by the weaker convexity effect. For high recovery rates, the decline in average spreads is amplified by the the stronger convexity effect and the heterogeneity biases increase.

Average spreads decline with the horizon τ over which the firm is followed . This effect is explained by the fact that leverage is expected to decline over time. As the horizon gets further, leverage is expected to decline further and future spreads are expected to be lower. This pulls down the average spreads. The heterogeneity biases increase, as expected, with the horizon. Indeed, the longer the horizon, the larger future variation and the stronger the convexity effects.

Figure 2.7: Log normal leverage process: Effect of other firm fundamentals on average spreads and heterogeneity biases (1)

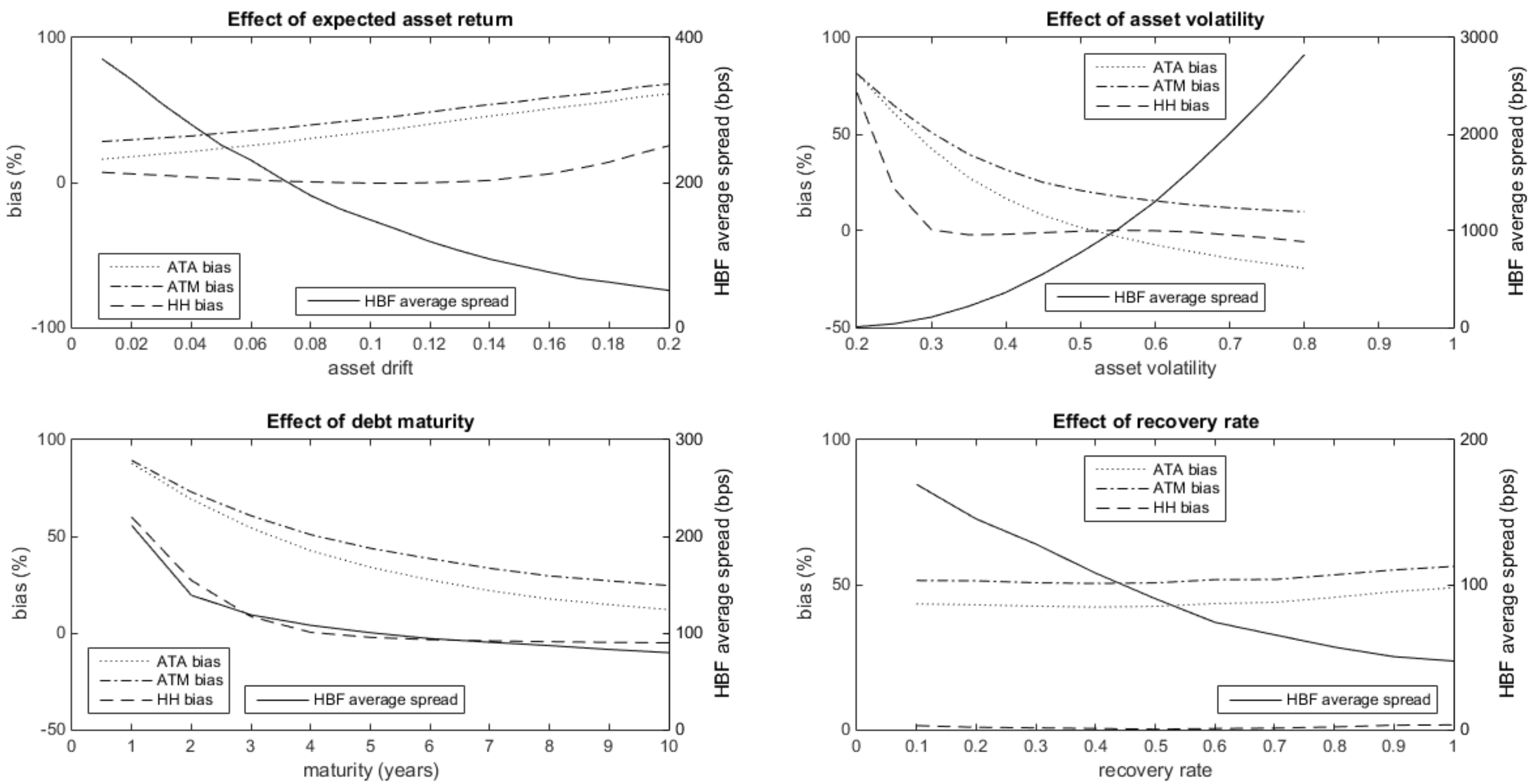
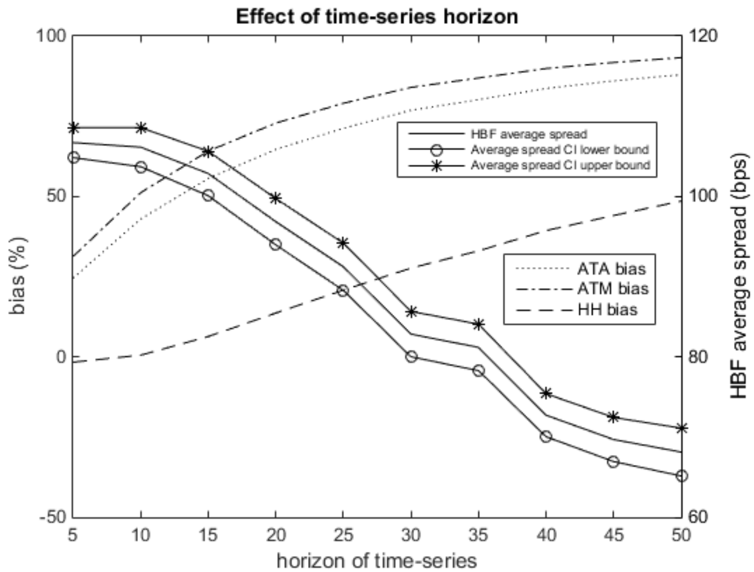


Figure 2.8: Log normal leverage process: Effect of other firm fundamentals on average spreads and heterogeneity biases (2)



2.3.2 Time-series heterogeneity in leverage: Mean reverting leverage policy

The lognormal leverage model of the previous subsection implies that leverage is expected to decline steadily over time. However, it is known that firms have a leverage level that optimizes the combined effect of tax advantages and financial distress costs of debt [24, 43]. It is therefore reasonable to assume that when leverage exceeds the optimal level, firms would tend to reduce it in order to reduce the financial distress costs of debt, while when leverage is below the optimal level, firms would tend to increase it in order to increase the tax benefits of debt. As a result, leverage would be mean reverting as in the dynamic leverage model of Collin-Dufresne and Goldstein [14]. To study the impact of such a leverage policy on average spreads, we model debt principal as the product of asset value, given by Equation 2.1, and leverage, which is assumed to follow a square-root-type process according to the following equation:

$$d\lambda_t = \kappa_\lambda(\theta_\lambda - \lambda_t) + \sigma_\lambda \sqrt{\lambda_t} dW_t^\lambda, \tag{2.11}$$

where κ_λ is the parameter governing the speed of mean reversion of leverage, θ_λ is the target leverage, and σ_λ is the parameter governing the volatility of leverage.

The default values for the parameters of the leverage dynamics are: $\kappa_\lambda = 0.4$, $\theta_\lambda = 0.4$, $\sigma_\lambda = 0.3$ and $\lambda_0 = 0.4$.

Figure 2.9 plots the computed average spreads and heterogeneity biases in function of κ_λ, θ_λ, σ_λ, and maturity T. Figure 2.10 plots the computed average spreads and heterogeneity biases in function of the recovery rate ρ and the horizon τ. When standard errors of the simulations are large, we also plot the corresponding 95% confidence bands. The findings are as follows.

Average spreads decrease with the speed of mean reversion of the leverage to its target level. In fact, as the speed of mean reversion increases, leverage can drift less freely away from its target level so that time-series variability in leverage is reduced. This in turns reduces the heterogeneity effect on the average spreads and average spreads go down. This heterogeneity effect is measured by the heterogeneity biases, which all decline with the speed of mean reversion.

As the target leverage increases, the average spread also increases, reflecting the higher credit risk associated with higher leverage. However, the heterogeneity biases decline with the target leverage. This decline is the result of two effects. First, the target leverage has no effect on the variability of leverage so that in absolute terms the heterogeneity biases are constant; however, as a proportion of average spreads, which are increasing in the target leverage, they tend to decline. Second, the concavity of the spread function is more pronounced for higher leverages. This second effect also reduces the heterogeneity biases.

Average spreads increase with the volatility of leverage since the risk of being insolvent at maturity increases. This first effect tends to reduce the heterogeneity biases. At the same time, higher leverage volatility imply greater variability in leverage, so that the convexity effect is stronger. This second effect tends to increase the heterogeneity biases. We observe that the ATA and the ATM biases increase with the volatility of leverage. This implies that the second effect dominates the first for these biases. The two effects appear to neutralize each other for the HH bias which remains quasi-constant and close to 0%.

The term structure of average spreads in this model is the same as with the log-normal leverage model for the same reasons. The impact of recovery rates on the average spreads is also similar to the case of the log-normal leverage model.

The impact of time horizon τ on the average in this model is the opposite of that in the log-normal leverage model. In this model, as the time horizon increases, there are more episodes where leverage drifts away from the target level and is then pulled back to this target level. As a result there is more heterogeneity in the time-series and average spreads are increased. Consistently with this explanation, the ATA and ATM biases increase with the time horizon.

Figure 2.9: Mean reverting leverage: Effect of other firm fundamentals on spreads and heterogeneity biases (1)

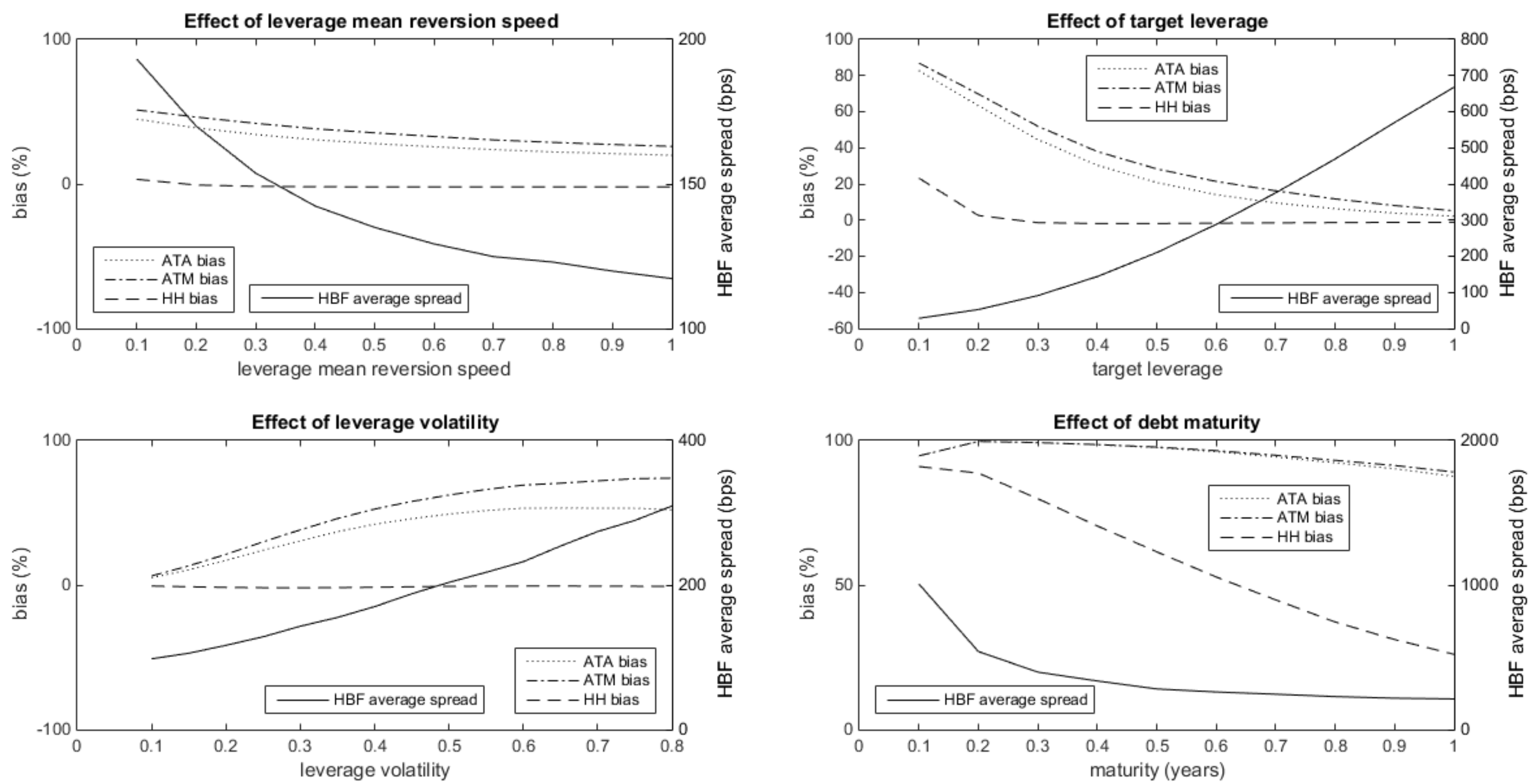
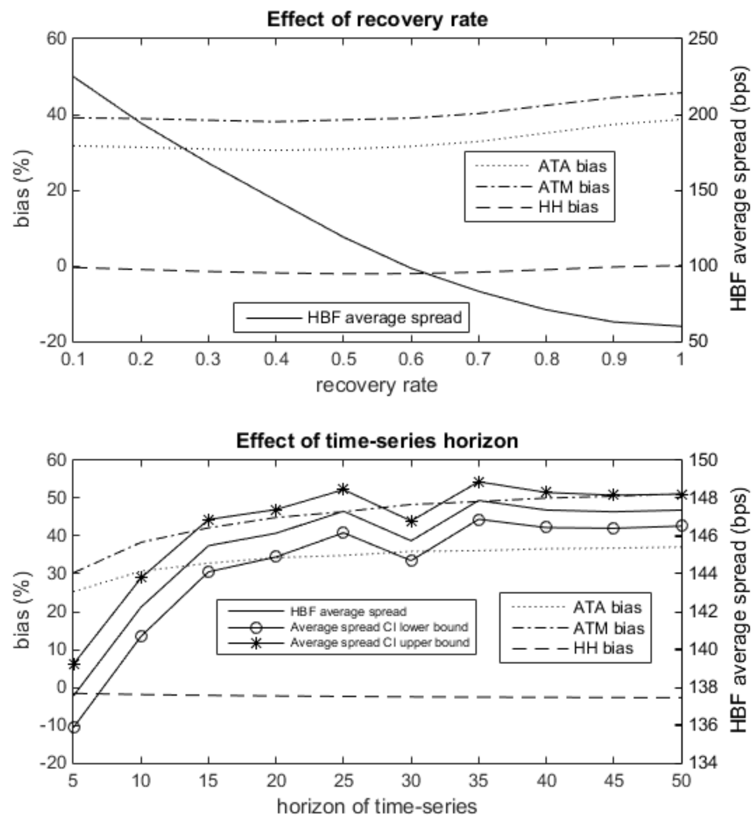


Figure 2.10: Mean reverting leverage: Effect of other firm fundamentals on spreads and heterogeneity biases (2)



2.3.3 Time-series heterogeneity in volatility

In this subsection, we assume that asset volatility follows a square-root-type process given by the following equation:

$$d\sigma_t = \kappa_\sigma(\theta_\sigma - \sigma_t) + \sigma_\sigma\sqrt{\sigma_t}dW_t^\sigma, \tag{2.12}$$

where κ_σ is the parameter governing the speed of mean reversion of volatility, θ_σ is the long run average volatility and σ_σ is the parameter governing the volatility of the volatility. The correlation between asset value and asset volatility is $\eta_{\sigma v}$.

The default values for the parameters of the volatility dynamics are: $\kappa_\sigma = 0.4$, $\theta_\sigma = 0.3$, $\sigma_\sigma = 0.1$, $\eta_{\sigma v} = 0$, $\sigma_0 = 0.3$.

Figure 2.11 plots the computed average spreads and heterogeneity biases in function of the volatility dynamics parameters κ_σ, θ_σ, σ_σ and $\eta_{\sigma v}$. Figure 2.12 plots the computed average spreads and heterogeneity biases in function of maturity T, recovery rate ρ, and horizon τ. When standard errors of the simulations are large, we also plot the corresponding 95% confidence bands. The findings are as follows.

The speed of mean reversion of volatility to its long term average value is negatively related to average spreads. An increase in speed of mean reversion reduces the variability of volatility around its average, so that there is less heterogeneity in volatility in the time series. The reduced heterogeneity pulls down the average spreads. The heterogeneity biases do not appear to be very sensitive to the speed of mean reversion but they also decline with speed of mean reversion.

Average spreads increase with the long term average volatility, as expected. Since the long term average volatility does not affect the variability of volatility, the ATM and ATA heterogeneity biases are constant in absolute terms and decline as a proportion of average spreads. The HH bias declines for low levels of the long term average volatility and then increases. In fact, recall that the HH bias is different from the other biases in that it depends on the long term average volatility. Higher long term average volatility increases the average default probability and this implies a higher HH volatility, which increase the HH average spread. An increase in the HH bias reflects a higher increase in the HBF average spread than in the HH average spread.

As the volatility of volatility increases, average spreads increase. This is justified by the greater variability in volatility. For the same reason, the heterogeneity biases increase, except for the HH bias, which remains close to 0%.

Average spreads decrease with the correlation between asset value and asset volatility. To understand this effect, recall that higher volatility has a positive effect on default risk, while higher asset value has a negative effect on default risk. With high negative correlation, volatility is high when asset value is low in such a way that the asset value effect exacerbates the volatility effect. As a result, default risk is very high and spreads are also very high. With high positive correlation, volatility is high when asset value is high in such a way that the asset value effect dilutes the volatility effect. As a result, default risk is lower and spreads are also lower. Between these two extremes, as the correlation increases, the asset value effect has more dilution power on the volatility effect and spreads decrease. For similar reasons, the heterogeneity biases also decline with the correlation.

The model predicts a declining term structure of average spreads. The reason is that higher maturity yields weakens the convexity of the spread as a function of volatility. For the same reason, the heterogeneity biases decline with maturity.

Average spreads decline with recovery rates, as expected. The heterogeneity biases are almost insensitive to the recovery rate for the same reason as in the case of time-series heterogeneity in leverage.

As the time horizon τ increases, the average spread first increases, then reaches a peak and declines. This effect results from two opposing forces. The first is due to asset value, which on average drifts upward, reduces leverage and pulls down the average spread. The second is due to the variability in volatility that increases with the horizon, and that generates higher average spreads through a stronger convexity effect. In the short run, the volatility effect appears to be stronger than the asset value effect thereby increasing the average spreads. In the long run, the asset value effect appears to be dominant. Regarding the heterogeneity biases, they all increase, reflecting the higher variability in volatility.

Figure 2.11: Time-series heterogeneity in volatility: Effect of other firm fundamentals on spreads and heterogeneity biases (1)

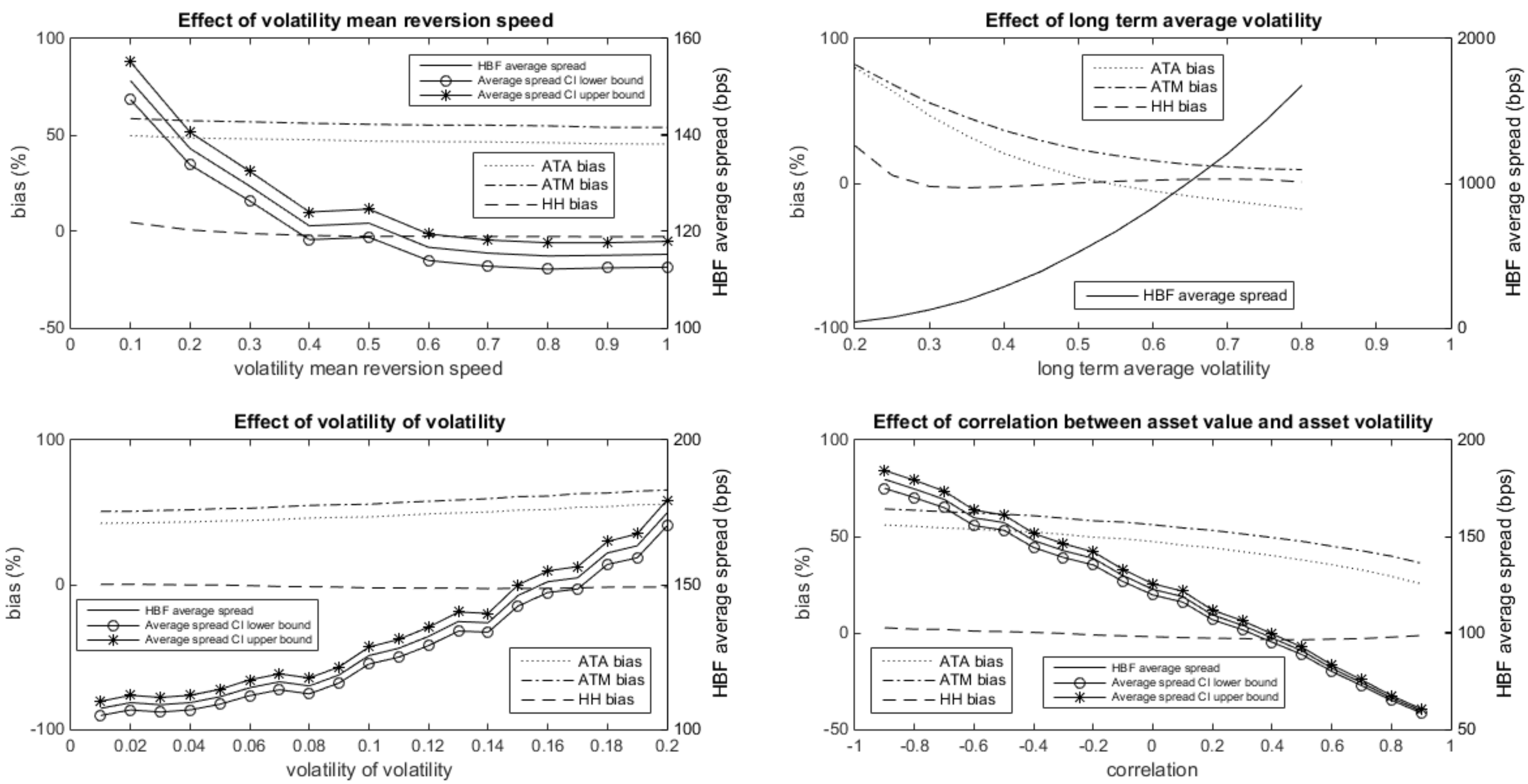
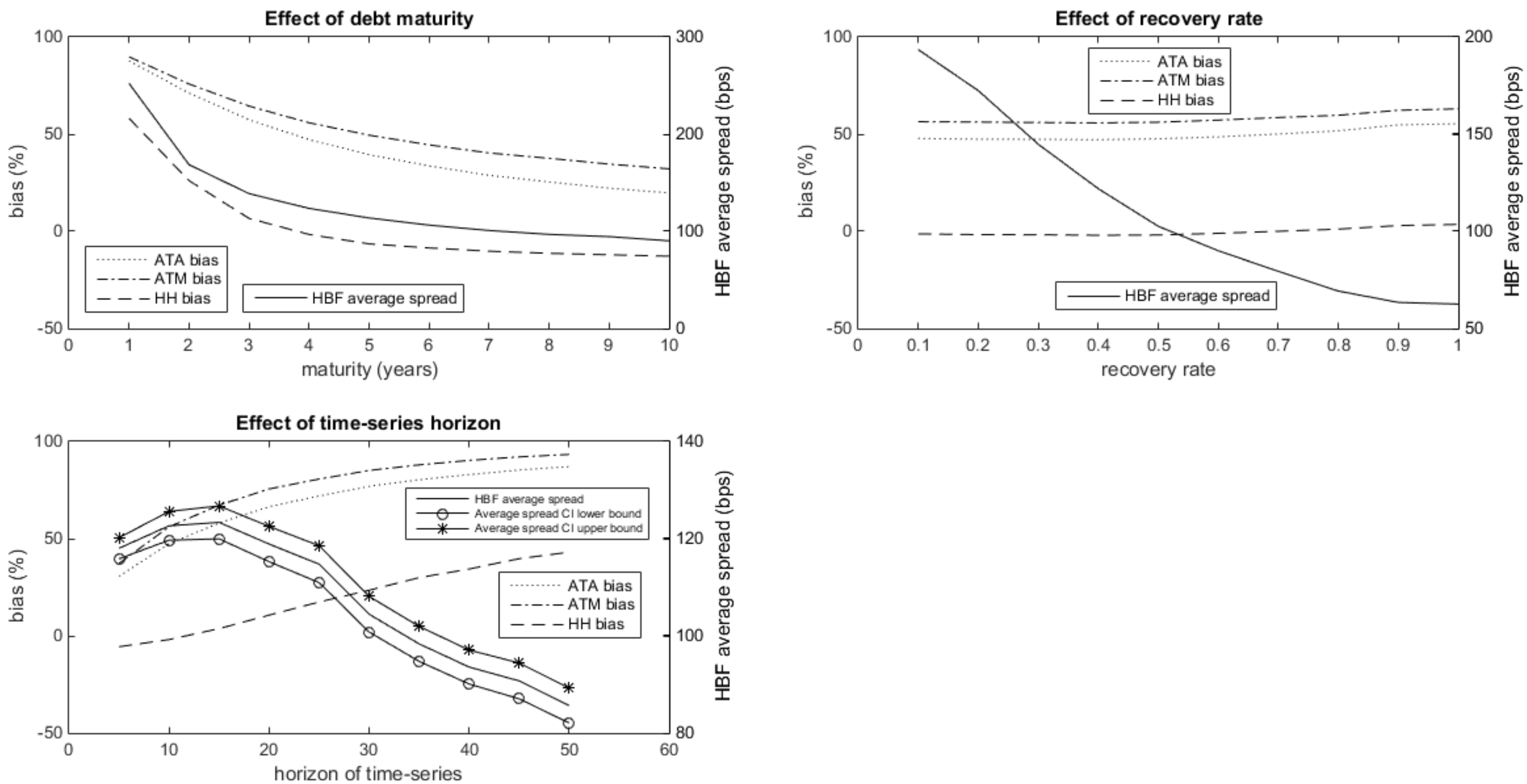


Figure 2.12: Time-series heterogeneity in volatility: Effect of other firm fundamentals on spreads and heterogeneity biases (2)



2.3.4 Time-series variation in riskless rate

In this subsection, we assume that the riskless rate follows a square-root-type process given by the following equation:

$$dr_t = \kappa_r(\theta_r - r_t) + \sigma_r\sqrt{r_t}dW_t^r, \tag{2.13}$$

where κ_r is the parameter governing the speed of mean reversion of the riskless rate, θ_r is the long run average of the riskless rate and σ_r is the parameter governing the volatility of the riskless rate. The correlation between asset value and riskless rate is η_{rv}.

The default values for the parameters of the interest rate dynamics are: $\kappa_r = 0.4$, $\theta_r = 0.05$, $\sigma_r = 0.1$, $\eta_{rv} = 0$, $r_0 = 0.05$.

Figure 2.13 plots the computed average spreads and heterogeneity biases in function of the interest rate dynamics parameters κ_r, θ_r, σ_r and η_{rv}. Figure 2.14 plots the computed average spreads and heterogeneity biases in function of maturity T, recovery rate ρ, and horizon τ. When standard errors of the simulations are large, we also plot the corresponding 95% confidence bands. The findings are as follows.

For $M = 20000$, the effect of the speed of mean reversion on the average spreads is somewhat noisy, as shown by the corresponding confidence bands. Overall, we can detect a weak downward trend for low speeds and no trend for high speeds. This would be consistent with the argument that higher mean reversion reduces time-series heterogeneity and therefore the convexity effect on the average spreads. This effect is weak however, as shown by the heterogeneity biases which are almost insensitive to the speed of mean reversion of the interest rate.

Average spreads decline with the long term average riskless rate, as expected. In fact, a long term average risk free rate implies higher expected asset returns in the risk neutral world and reduces default risk. This first effect tends to increase the heterogeneity biases. Additionally, the convexity of the spread function is stronger for higher riskless rates. This second effect also tends to increase the heterogeneity effects. As a result, the heterogeneity biases increase.

The effect of interest rate volatility is also noisy, although there is an upward trend.

This trend is explained by the higher variability in interest rates associated with higher volatility. This higher volatility commands stronger heterogeneity effect which tends to increase the spreads.

Average spreads increase with the correlation between riskless rate and asset value. To understand this, recall that the riskiness of bonds in the risk neutral world increases with lower asset value and lower riskless rate. A higher correlation between asset value and riskless rate implies that asset value is more likely to be low when the riskless rate is low, so that the riskiness of the bond is greater. This commands higher spreads. The same correlation effect has been found by Longstaff and Schwartz [45] in their model of stochastic interest rate. It is interesting that we detect the same effect using the Merton model simply by allowing for the possibility of future revisions of the riskless rate used as input in the Merton model. For similar reasons, heterogeneity biases increase with the correlation.

Average spreads decline with maturity, reflecting the fact that the convexity of the spread function is less pronounced for higher maturities than it is for shorter maturities. For the same reason, the heterogeneity biases also decline with maturity.

Here again, we observe that average spreads decline with recovery rates, as expected, and that recovery rates do not have a sizable impact on the heterogeneity biases.

Finally, we observe that average spreads increase with the time horizon for short time horizons and decrease with time horizon for long time horizons. As in the case of time-series heterogeneity in volatility, this happens because a longer horizon implies a greater room for variability in short rates and asset value but, at the same time, it implies lower average leverages. The first effect appears to dominate in the short run while the second takes the lead in the long run.

Figure 2.13: Time-series heterogeneity in riskless rate: Effect of other firm fundamentals (1)

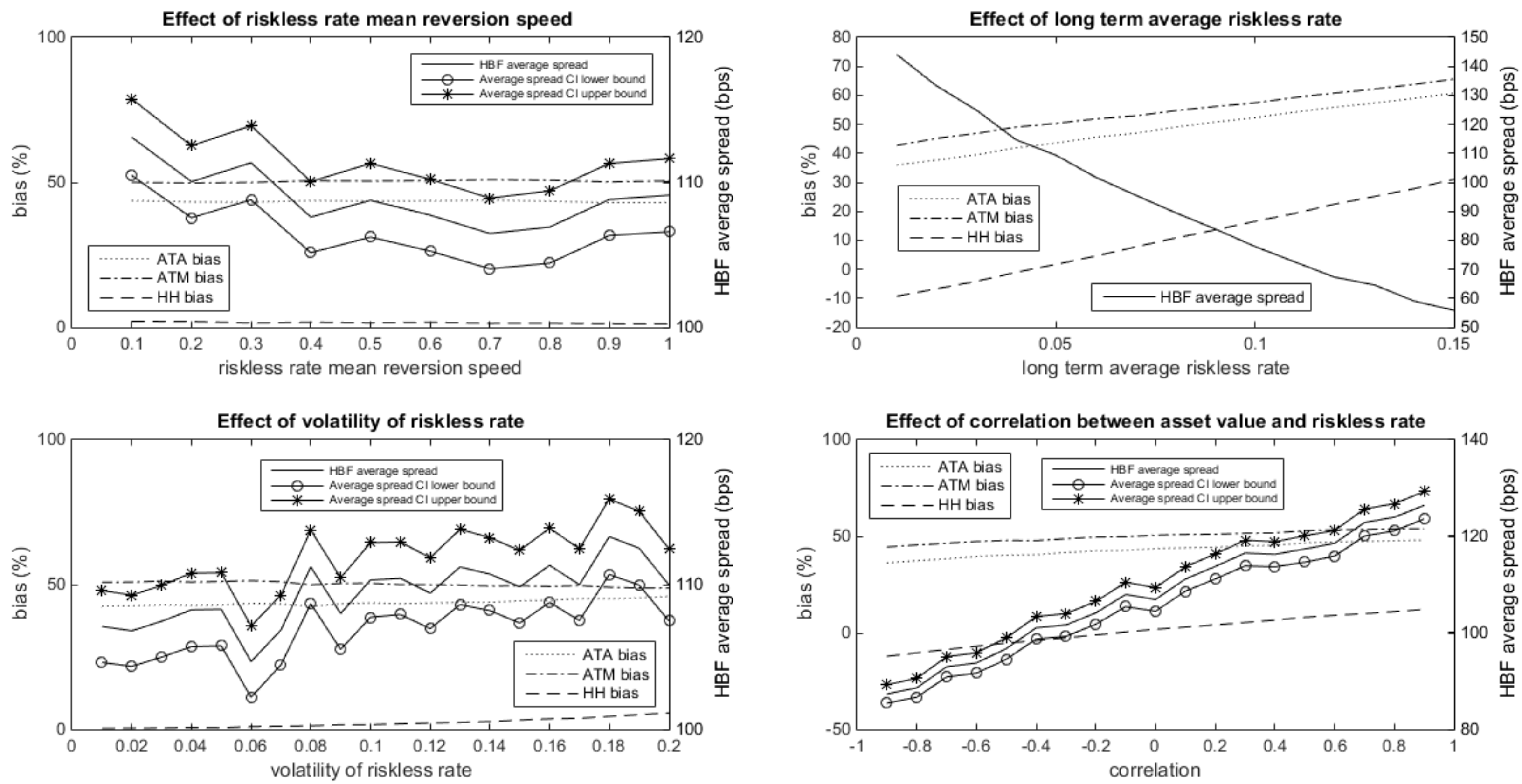
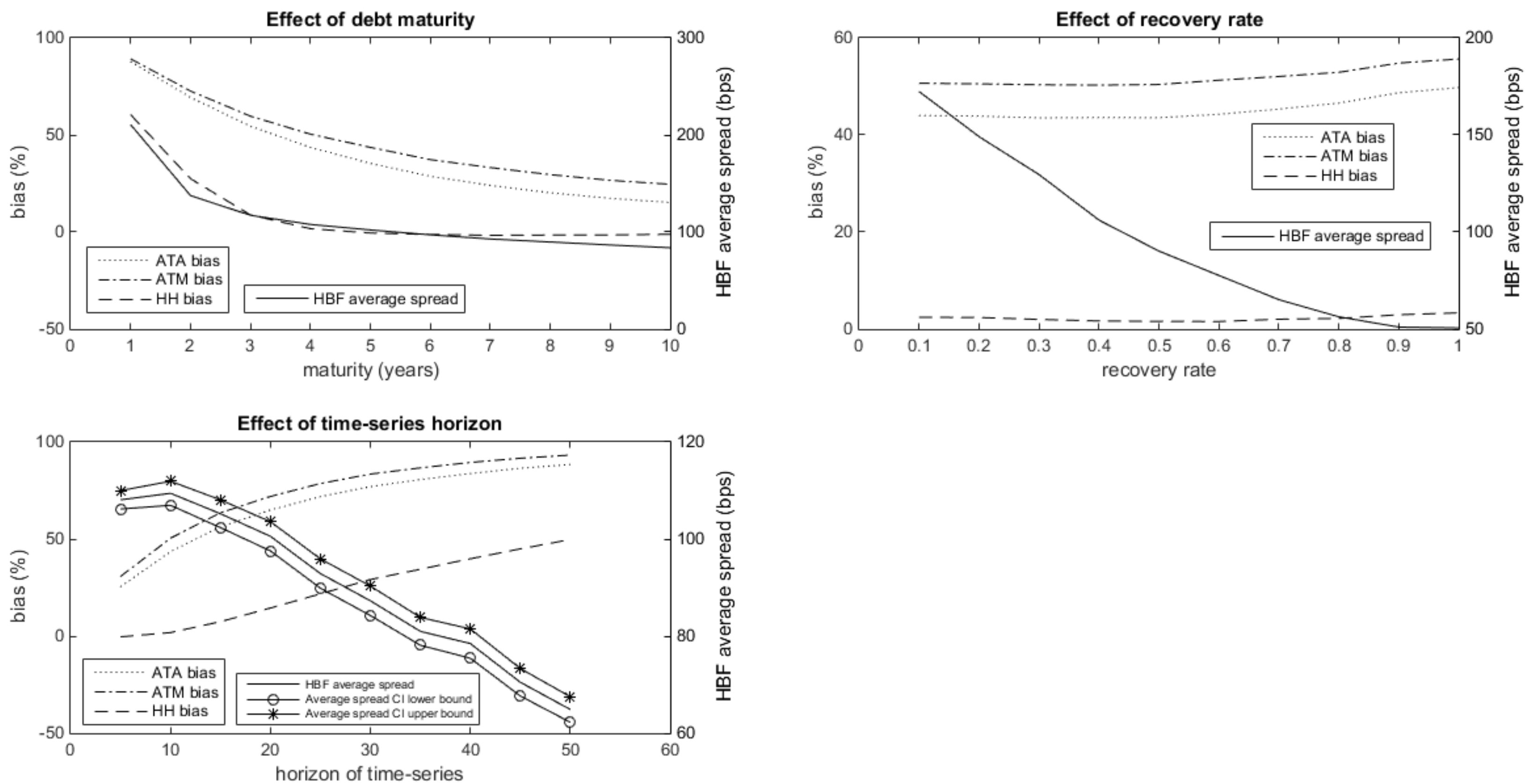


Figure 2.14: Time-series heterogeneity in riskless rate: Effect of other firm fundamentals (2)



2.4 Regression analysis

In this section, we often use the abbreviations TS for (time-series) and CS for (cross-sectional), to indicate the dimension of heterogeneity that is being considered.

The theoretical findings of the previous sections give rise to the following testable hypotheses:

- **H1**: The cross-sectional variations in the TS-average spread of firms, as predicted by the Merton model, is partly explained by the cross-sectional variation in the higher order moments (than the mean) of the TS-distributions of firm fundamentals.
- **H2**: The time-series variations in the CS-average spread of a group of firms, as predicted by the Merton model, is partly explained by the change in the higher order moments (than the mean) of the CS-distributions of firm fundamentals.

2.4.1 Data

These hypotheses can be tested using our panel data on asset volatilities and leverages. This dataset can be used to measure the moments of cross-sectional and time-series distributions of asset volatilities and leverages and test the effect of these moments on average spreads. As stated earlier, the calibration of our theoretical distributions of section 2.2 was based on a large dataset of asset volatilities and leverages. In summary, this dataset has been constructed by backing out daily asset volatilities and asset values from equity prices and equity volatilities using data from CRSP and COMPUSTAT [4]. The estimation methodology is the same rolling horizon approach as in Bharath and Shumway [4] except that we use a rolling window of 4 years of observations. Leverage is measured as the total debt principal divided by the estimated asset value. The sample consists of daily data on 1286 firms for a total of 2735873 firm-days from December 26, 1984 to December 31, 2013. The sample also include information on rating categories and economic sectors.

Using this dataset, we computed the predicted spread by the Merton model for each

4. Detailed information on the construction of this database will be given in section 3.4

firm, at each day. We used a horizon $T = 4$, a recovery rate $\rho = 0.4$ and a riskless rate $r = 0.05$. Based on the same dataset, we also measured for each firm, the first 4 moments of the distributions of the time-series of its leverage and asset volatility. Firms with less than 300 observations were dropped in this estimation. We then selected three groups of firms: the group of all firms, the group of BBB firms and the group of firms in the technology sector. We measured for each group of firm, for each month, the first 4 moments of the distributions of leverage and asset volatility of firms in this group. Months with less than 30 observations were dropped.

2.4.2 Summary statistics

Summary statistics of the average spreads and the moments of the distributions of asset volatility and leverage are displayed in Table 2.V.

Consider first the cross-sectional variation in TS heterogeneity. First, as Table 2.V shows, there is considerable cross-sectional variation in the TS-average spread of firms. In fact, the standard deviation of the TS-average spread is more than 5 times the median of 244 bps. This means that while some firms have enjoyed a low TS-average spread over our sample period, others have paid extremely high spreads on average. Second, we observe that the first moments of the TS-distributions of leverage and asset volatility are also widely variable across firms. The standard deviation of the TS-average leverage and of the TS-average asset volatility are 65% and 25% respectively while their medians are 43% and 34% respectively. Third, there is also substantial variations in the higher order moments of the TS-distribution of leverage and asset volatility. For instance, while many firms had a very stable leverage, others had a highly variable leverage. This variability is measured by the standard deviation of the TS-standard-deviation of leverages which is 39%. The median of the TS-standard-deviation of leverage is 14%.

Consider now the time-series variation in CS heterogeneity. CS-average spreads have fluctuated considerably over our sample period. CS-Average spreads were high in some months and low in some other months. For instance, for the group of BBB firms, the median of the CS-average spread was 180 bps. However, the predicted BBB average spread reached 532 bps in March 2001, date at which the US officially entered into recession following the burst of the dot-com bubble and 409 bps in March 2009 following the recent financial crisis. CS-average leverages and CS-average asset volatilities have also been very variable. For instance, in the technology sector, the median of the CS-average asset volatility was 41% with a standard deviation of 11%. Finally, the higher order moments of the CS-distributions of leverage and asset volatility have also widely fluctuated. For instance for the group of all firms, the CS-skewness of asset volatility had a median of about 5.79 and a standard deviation of 1.98.

2.4.3 Regressions

In order to test the hypothesis H1, we run the following cross-sectionnal and firm-level regression:

$$\begin{aligned} mean\_spread_i &= \beta_0 + \beta_1 \times mean\_leverage_i + \beta_2 \times mean\_asset\_vol_i \\ &+ \beta_3 \times std\_leverage_i + \beta_4 \times std\_asset\_vol_i + \beta_5 \times skew\_leverage_i \\ &+ \beta_6 \times skew\_asset\_vol_i + \beta_7 \times kurt\_leverage_i \\ &+ \beta_8 \times kurt\_asset\_vol_i + Z_i, \end{aligned} \tag{2.14}$$

where Z_i is the error term.

In order to test the hypothesis H2, we run the following time-series regression for the selected groups of firms:

$$\begin{aligned} mean\_spread_t &= \beta_0 + \beta_1 \times mean\_leverage_t + \beta_2 \times mean\_asset\_vol_t \\ &+ \beta_3 \times std\_leverage_t + \beta_4 \times std\_asset\_vol_t + \beta_5 \times skew\_leverage_t \\ &+ \beta_6 \times skew\_asset\_vol_t + \beta_7 \times kurt\_leverage_t \\ &+ \beta_8 \times kurt\_asset\_vol_t + Z_t, \end{aligned} \tag{2.15}$$

where Z_t is the error term.

The results of the regressions are displayed in Table 2.VI. The table shows that many of the coefficients associated with higher order moments (than the mean) are significant.

This is clear evidence that the variation in average spreads is partly related to the variation in higher order moments (than the mean) of the distributions of heterogeneous parameters.

To get a measure of how much explanatory power the higher order moments have, Table 2.VI also displays the percentage of the R^2 that is attributable to each moment. For H1 (time-series heterogeneity) we can see that as much as a quarter of the overall R^2 is attributable to the higher order moments. For H2 (cross-section heterogeneity), the higher order moments claim an even larger portion of the R^2. Indeed, the portion of the R^2 attributable to the higher order moments in the H2 regressions are 35%, 43% and 50.3% for the group of all firms, the group of BBB firms, and the group of technology firms respectively. Finally, we can see that it is mainly the second order moments that claim most of the explanatory power of the higher order moments although the importance of skewness and kurtosis is not negligible.

Table 2.V: Summary statistics

Variable	TS		CS: All firms		CS: BBB firms		CS: Tech firms	
	median	std	median	std	median	std	median	std
mean_spread (bps)	244	1367	419	139	180	137	322	251
mean_leverage	0.43	0.65	0.51	0.10	0.42	0.11	0.29	0.09
std_leverage	0.14	0.39	0.57	0.16	0.25	0.12	0.24	0.11
skew_leverage	0.65	1.37	3.51	4.03	1.13	1.64	1.29	0.68
kurt_leverage	2.90	38.33	23.53	109.41	4.34	34.12	4.39	3.33
mean_asset_vol	0.34	0.25	0.32	0.05	0.29	0.04	0.41	0.11
std_asset_vol	0.05	0.10	0.20	0.04	0.14	0.08	0.19	0.11
skew_asset_vol	0.12	0.93	5.79	1.98	3.62	3.40	1.74	1.16
kurt_asset_vol	2.06	3.65	54.28	32.51	22.65	43.71	7.45	5.31

This table gives summary statistics of cross-sectional variation of time-series (TS) heterogeneity (columns 2 and 3) and the time-series variation of cross-sectional (CS) heterogeneity for the group of all firms (columns 4 and 5), for the group of BBB firms (columns 6 and 7) and for the group of Technology firms (columns 8 and 9). Heterogeneity is measured by the first four moments of the distributions (TS or CS) and leverage and asset volatility as shown in column 1. The table also display the predicted average spread (TS or CS) from the Merton model.

Table 2.VI: Regression results

	Dependent variable: mean_spread			
	H1	H2: All firms	H2: BBB firms	H2: Technology sector
mean_leverage	0.1186***	0.0975***	0.0906***	0.1588***
	(0.0038, 21.22)	(0.0021, 36.35)	(0.0022, 38.17)	(0.0081, 13.38)
mean_asset_vol	0.4327***	0.1115***	0.0924***	0.1181***
	(0.0074, 53.02)	(0.0024, 28.71)	(0.0042, 19.28)	(0.0045, 36.33)
std_leverage	-0.0771***	0.0003	0.0037	-0.0317***
	(0.0068, 12.23)	(0.0016, 15.11)	(0.0041, 13.79)	(0.0066, 13.36)
std_asset_vol	-0.0493***	0.0837***	0.0713***	0.1162***
	(0.018, 11.93)	(0.0041, 8.14)	(0.0046, 15.15)	(0.0062, 27.29)
skew_leverage	-0.0006	-0.0008***	-0.0015***	0.0068***
	(0.0018, 0.19)	(0.0001, 3.68)	(0.0004, 3.71)	(0.0014, 0.55)
skew_asset_vol	0.0086***	-0.0011***	-0.0004	-0.0041***
	(0.0019, 1.08)	(0.0002, 3.13)	(0.0003, 5.08)	(0.0012, 5.1)
kurt_leverage	0.0001	0.0001***	0.0001***	-0.0013***
	(0.0001, 0.08)	(0, 2.18)	(0, 1.56)	(0.0003, 0.62)
kurt_asset_vol	-0.0005	0.0001***	0.0001	0.0006**
	(0.0005, 0.25)	(0, 2.69)	(0, 3.25)	(0.0002, 3.38)
observations	1124	290	290	290
Total R2	0.89801	0.99528	0.97239	0.9785

This table tests the hypothesis that higher order moments (than the mean) of the distribution of heterogeneous fundamentals impact predicted average spreads by the Merton Model. H1 considers time-series heterogeneity (column 2), while H2 considers cross-sectional heterogeneity for either all firms (Column 3), BBB firms (Column 4) or technology firms (Column 5). Standard errors and Shapleys' percentage of R2 are in parenthesis in this order. (*** Significant at 1% level, ** significant at 5% level, and *Significant at 10% level)

2.5 Summary and concluding remarks

In this essay, we have introduced a model of a heterogeneous group of bonds whose individual values are given by the Merton model. We have shown that the average spread of the group depends on the cross-sectional and time-series heterogeneity in the group. We have also shown that the characteristics of the heterogeneity, i.e., the fundamentals that are heterogeneous, the distributions of these heterogeneous fundamentals in the cross-section and their dynamics in the time-series, are key determinants of the average spreads. We have also established that the biases arising from ignoring heterogeneity depend on the level of the non-heterogeneous fundamentals through their impacts on the shape of the spread function.

The most striking findings are the following. Higher order moments (than the mean) of the distribution of the heterogeneous fundamentals are important in explaining the variation in the size of the average spreads. Expected asset returns matter for average spreads, even though they do no matter for individual spreads. The correlation between changes in riskless rate and asset returns has a positive impact on average spreads, even though the stochastic nature of interest rates is not priced in individual bond spreads. The correlation between asset returns and changes in volatility has a negative impact on average spreads, even though the Merton model ignores stochastic volatility.

These findings imply that stochastic volatility, stochastic interest rates and dynamic leverages must be accounted for in empirical tests of the Merton model at a time-series average level and that cross-sectional heterogeneity must be accounted for in empirical tests of the Merton model at a group level.

We have studied the consequences of ignoring heterogeneity on the value of various estimators for the average spread. We have found that these various estimators may be severely biased, with biases that can reach 100% of the average spreads. The biases may be negative or positive, depending on the moments of the distributions and on the shape of the spread function.

Finally, we have found that, in general, the heterogeneity biases are larger when spreads are low than when they are high. Also, these biases are larger at the short end of the term structure than for the long term. These findings suggest that, by accounting for cross-sectional heterogeneity and for revisions in structural models inputs in the measurement of model-generated average spreads, one may be able to elucidate, at least partly, the credit spread puzzle which is the finding that structural models predict spreads that are too low compared to observed average spreads. Indeed, the heterogeneity biases are more important exactly where it has been found, while ignoring heterogeneity, that the gap between observed and predicted spreads is larger.

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CHAPTER 3

CAPTURING THE DEFAULT COMPONENT OF CORPORATE BOND SPREADS: A STRUCTURAL MODEL APPROACH

How much spread do bond investors charge for the default risk that they incur? Is the observed level of average spreads entirely due to expected default losses and default risk premia? These question, crucial for any fixed income market participant, do not have a simple answer. Many papers have attempted to answer them from different angles, often arriving to different answers [3, 11, 15, 20, 25, 29, 33, 47]. Some of these papers use a regression approach in order to measure the portion of spreads variability that is explained by default variables. Others use reduced form models or structural models to measure the default component of observed spreads.

In this essay, we attempt to capture the default component of the spreads using the structural modeling framework. This framework has the advantage to avoid endogeneity issues that affect regressions. In addition, unlike the other approaches, the structural approach derives endogenously the functional form of the relation that links default spreads and default probabilities to firm fundamentals.

Three main issues must be dealt with when using the structural modeling approach to measure the default component of corporate bond spreads.

The first and most obvious issue is the selection of a structural model. Several structural models are available, involving different assumptions and different levels of complexity. A non-exhaustive list includes the Merton [50] zero-coupon-bond model, the Leland and Toft [43] model with endogenous default, the Longstaff and Schwartz [45] model with stochastic interest rate, the Mella-Barral and Perraudin [48] model with strategic debt service, the Collin-Dufresne and Goldstein [14] with dynamic leverage, the Zhou [63] model with asset value jumps, and, more recently, the He and Xiong [32] model with rollover risk. Clearly, a trade-off between realistic assumptions and complexity must be made in the choice of a structural model.

The second issue is the estimation of the model inputs, the main constraint being data availability. Key model inputs, such as asset value and asset volatility, are not observable. In addition, while equity is traded frequently, not all debt is traded, so that the market value of total debt is not observable. Moreover, the model must be calibrated to historical default and recovery rates in order to ensure that it does not overstate, or understate, expected default losses [33]. This poses additional difficulties, as historical default and recovery rates are not observable for individual bonds. Clearly, the more complex is the model, the more difficult is its estimation.

The third issue, which is the most recent and perhaps the least well understood, is the heterogeneity of rating categories. The traditional way of assessing the performance of a structural model is to compare the model average spread to observed average spreads, rating-wise. It is generally assumed in this exercise that rating categories are homogenous in such a way that the model average spread for a given rating category is assumed to be the model spread for the typical firm in that rating category. However, the rating-through-the-cycle policy of rating agencies and the small number of rating categories make rating categories heterogeneous [3, 20, 24, 43]. Recent empirical studies have argued that ignoring this heterogeneity may lead to important biases in the measured default spread [3, 20, 29], while others found the opposite result [11, 33]. One fact is certain, heterogeneity makes the calibration to historical default and recovery rates more difficult because parameters must be estimated for individual bonds, but historical default and recovery data are only available at the categorical level.

In this essay, we propose a simple two-stage approach to capture the size of default risk in the spreads. Our approach uses a calibrated version of the Merton model with a simple extension to introduce loss given default, and also fully accounts for heterogeneity.

We choose the Merton model because of its simplicity and parsimony. In fact, we have no reason to believe that more complicated structural models, where the sole source of risk is the asset value diffusion, would generate significantly different default spreads than our calibrated version of the Merton model. It is well recognized that the main reason why more complicated structural models perform better than the Merton model is that they are able to better replicate the observed term structure of default rates [14, 42, 43] and to adequately model bankruptcy costs [1, 48].

In this essay, we calibrate the Merton model to the observed term structure of default rates and we extend the Merton model by introducing an exogenous recovery rate parameter that we calibrate to observed recovery rates by rating category and economic sector. Our calibrated Merton model is therefore likely to yield similar default spreads as the more complicated asset diffusion structural models. In addition, models with additional sources of risk, such as jump models, stochastic volatility models and macroeconomic models, may certainly lead to more accurate measures of the default spread. However, adequate estimation and calibration of a structural model in a way that accounts for heterogeneity but does not overestimate or underestimate expected default losses is a prickly issue. Using a more complex model might unnecessarily divert our efforts and the attention of the reader from the more important issues of heterogeneity and calibration to observed expected losses. In this context, we believe that the Merton model is a good starting point that provides a useful benchmark for future works based on more complex models.

The first stage of our approach is the same as the standard estimation approach for the Merton model, which consists of inverting the model in order to estimate the firm fundamentals from equity prices and equity volatility. This standard approach is used by several authors, such as Bharath and Shumway [4], Crosbie and Bohn [17], Elkamhi et al. [24]. The second stage consists of adjusting individual first-stage leverages in order to ensure that the model default rate corresponds to the historical default rate. The adjustment may be done at the individual bond level if one has access to proxy data for historical individual default rates. Otherwise, it can be done at the rating category level, as historical default rate data by category is made public by rating agencies. What we call "implied leverages" are the adjusted leverages, implied from historical default data.

We show that the second stage equations have unique solutions and that numerical solutions to these equations can be easily found. We then compare the robustness our two-stage approach to that of the standard approach by performing two simulation experiments. In the first experiment, it is assumed that the assumptions of the Merton model hold, but the actual capital structure is not perfectly known to the econometrician.

In the second experiment, it is assumed that the world is perfectly described by the He and Xiong [32] model, which may be the most recent and elaborate among structural models. In this model, there are market liquidity frictions, taxes, endogenous default decision, coupons and debt rollover, while all of these assumptions are ignored in the Merton model. In these two situations, we find that our two-stage approach based on the Merton model gives a more accurate estimation of the actual default spread than the standard estimation approach. The second experiment is of high importance as it shows that, despite the imperfections of the Merton model, our two-stage estimation approach can capture the default component of the spreads. On the other hand, the standard estimation approach leads in that case to the biased conclusion that the entire spread is due to default risk, while, in reality, part of the spread is explained by liquidity.

We then conduct an empirical study that applies our two-stage estimation approach to measure the share of default risk in the spreads. Our empirical analysis is based on a sample of daily data on 1286 firms for a total of 2735873 firm-days from December 26, 1984 to December 31, 2013. To our knowledge, this is the longest data-set (in terms of period covered) ever used for this type of study. Contrarily to Feldhutter and Schaefer [29], we use a 32 year long historical default rate data because it reflects the normal level of default rates over the last 90 years [1]. We also perform several robustness checks regarding the key constants of our study, such as the level of historical default rates, the heterogeneity of recovery rates, the calibration procedure, the period of study and the considered structural model. Our findings confirm that rating categories are highly heterogeneous and that heterogeneity plays an important role in adequately measuring the default spread. We find that the heterogeneity bias free default sizes are about 1%, 8%, 11% and 40% of AAA, AA, A, BBB bond spreads.

These measures of the size of the default spread have important implications for the understanding of bond markets. They imply that investment grade bond investors are more than compensated for the default risk due to asset diffusion that they incur. This does not exclude the possibility that investment grade bond investors require compen-

1. In fact when one excludes the World War II period and its aftermath from the 90 years default data, one arrives at similar default rates to the default rates based on the 32 year default data [40]

sation for other sources of default risk, such as jumps or stochastic volatility or time-varying macro-economic conditions, or that they require compensation for other sources of risk, such as liquidity risk. Our findings therefore support the development of structural models that generate higher default risk premia, such as models with jump risk, stochastic volatility risk, or counter-cyclical default risk premia [3, 11, 13, 20, 62, 63].

Our results are also consistent with empirical studies establishing the presence of tax premia, market liquidity premia and individual bond liquidity premia in the spreads [10, 15, 25, 47]. They support the development of structural models where non-default factors such as liquidity risk and macroeconomic risks are explicitly considered. From a capital structure perspective, our findings support the argument of He and Xiong [32] that the low leverage puzzle may be explained by liquidity risk, a non-default factor.

The rest of this essay is organized as follows. Section 3.1 briefly reviews existing estimation methods. Section 3.2 presents our approach. Section 3.3 presents the performance tests of our approach. Section 3.4 presents the data that we use for our empirical analysis. Section 3.5 presents the results. Section 3.6 presents the robustness checks. Section 3.7 discuss the results and section 3.8 concludes.

3.1 An overview of existing estimation approaches

In order to estimate the share of default risk in the spreads with the structural modeling approach, one has first to predict the default spread using the selected structural model, and second to divide the predicted default spread by the observed spread. The second step is straightforward once the first is completed and the first step is also straightforward if one knows the inputs of the selected model. Therefore, the main challenge is how to estimate these inputs. The key inputs to estimate are asset volatility and asset value. The literature on the estimation of these two inputs is the object of this section.

Recall that in the Merton model, the equity price is given by:

$$\begin{aligned} E &= VN(d_1(\lambda)) - Be^{-rT}N(d_2(\lambda)). \qquad (3.1) \\ \text{where :} & \\ d_1(x) &= \frac{-\ln(x) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \\ d_2(x) &= \frac{-\ln(x) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \\ \text{and } \lambda &= \frac{B}{V} \text{ is the initial leverage ratio,} \end{aligned}$$

B is debt principal, T is the debt maturity, r is the riskless rate, V is the asset value and σ is the asset volatility.

The bond price if one introduces a recovery rate ρ in the Merton model as in Eom et al. [26] is given by:

$$F = Be^{-rT}N(d_2(\lambda)) + V_0N(-d_1(\rho\lambda)) + \rho Be^{-rT}(N(d_2(\rho\lambda)) - N(d_2(\lambda))). \quad (3.2)$$

The objective default rate at the horizon T is given by:

$$\Phi_T(\lambda, \sigma, \mu) = N\left(-\frac{-\ln(\lambda) + \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right). \quad (3.3)$$

The spread at the horizon T is given by:

$$\Psi_T(\lambda, \sigma, \rho, r) = -\frac{1}{T}\ln\left(\frac{F}{B}\right) - r \quad (3.4)$$

Regarding asset value, the simplest and common approach is to proxy asset value by the book value of debt plus market value of equity. The first drawback to this solution is that market value of equity is observed daily, while book value of debt is at best observed quarterly. The second drawback is the assumption that market value of debt is equal to book value of debt. A variant of this approach, suggested by Jones et al. [38], is to assume that asset value is equal to market value of equity plus market value of traded debt plus book value of non-traded debt scaled by the ratio of book to market of traded debt. Implicit to this solution is the assumption that the book to market ratio of traded debt and non-traded debt are similar. However, there is no reason to believe that this assumption holds. Moreover, to implement this solution, one must be able to disentangle book value of traded debt from book value of non traded debt. Such disentangled data is not readily available. A quite complex variant of this approach has been suggested by Davydenko [21]. It includes tax benefit of debt and bankruptcy costs in the estimation of asset value and uses estimates of the market price of every type of liability to calculate market value of debt. This approach is difficult to use in practice because of data limitations.

Regarding asset volatility, it is possible to estimate it from asset values as the volatility of daily asset returns. However, if asset value is calculated using one of the above approaches, the assumptions made about the market value of debt also affect the estimated asset volatility, mainly because book value of debt is less volatile than market value of debt. Alternatively, one may estimate equity return volatility using time-series of equity returns, and debt return volatility using available data on publicly traded debt. Then asset volatility may be estimated as a weighted average of the equity return volatility and the debt return volatility. The drawback of this approach is that the weights are often chosen arbitrarily. Schaefer and Strebulaev [55] propose an extended version of the weighted average approach where it is possible to take into consideration the correlation between bond returns and equity returns. Schaefer and Strebulaev [55] assume that asset volatility is given by:

$$\sigma^2 = (1-\lambda)^2 \sigma_E^2 + \lambda^2 \sigma_D^2 + 2\lambda (1-\lambda) \sigma_{ED}, \tag{3.5}$$

where σ is the asset volatility, σ_E is equity return volatility, σ_D is bond return volatility and σ_{ED} is the correlation between bond returns and equity returns. Schaefer and Strebulaev [55] assume that λ is book leverage. The main drawback of this approach is that it implicitly assumes that leverage is constant. However, there is an overwhelming evidence that leverage varies over time [2]. The other drawback is that it is subject to frequent bond price data availability.

There is another estimation approach, described by Crosbie and Bohn [17], which allows to estimate directly both asset value and asset volatility. According to Duan et al. [23], the approach has the advantage of being equivalent to a maximum likelihood estimation of asset volatility and asset value. In addition, the approach is simple and based only on equity prices, which are observable. As a result, this approach has established itself as the standard estimation approach of asset value and volatility. It is based on the Merton [50] model, and consists of solving a system of two equations: one equation for the equity value E given by

$$E = VN(d_1) - Be^{-rT}N(d_2), \tag{3.6}$$

and the other equation for the volatility of equity returns σ_E given by

$$\sigma_E = \sigma E_V \frac{V}{E} = \sigma N(d_1)\frac{V}{E}, \tag{3.7}$$

where V is the asset value, T is the debt maturity, B is the debt principal and r is the riskless rate.

If the equity value E, the equity volatility σ_E, the horizon T, the debt principal B and the riskless rate r are known, then one can solve simultaneously and numerically equations (3.6) and (3.7) for the asset volatility σ and the asset value V.

According to Crosbie and Bohn [17], the asset volatility obtained from the simultaneous solution of equations (3.6) and (3.7) fluctuates too much with leverage. For that reason, they suggest an alternative iterative procedure that is better described by Bharath and Shumway [4]. The procedure is the following: One first guesses a value of σ. Equation (3.6) is then used to find the corresponding V for every day of the previous four years. The resulting asset returns are used to estimate a new asset volatility σ which becomes the input of the next iteration. Bharath and Shumway [4] suggest to use $\sigma_E \frac{E}{E+B}$

2. Even if debt principal is constant, asset value fluctuates over time.

as a starting value for σ.

The problem with the above approaches is that they completely ignore the information on the level of physical default probabilities. As a result, they do not guarantee that the level of default rate implied by the estimated parameters be equal to the observed level of default rates. However, having such a guarantee is crucial for the goal of this study, which is to measure the share of default risk in the spreads. Indeed, default spreads are composed of two parts: the expected loss [3] from default and the risk premia. Since the expected loss component is the physical probability of default times the loss given default, this component could be made arbitrarily as large as one wishes by increasing the probability of default. This implies that it is possible to generate arbitrarily large default spreads by counter-factually using high physical probability of default. To ensure that the measured spreads are not driven by counter-factually large default probabilities arising from the estimation procedure, it is important to control the level of physical default probability implied by the estimation procedure.

Feldhutter and Schaefer [29] suggest an extension of the Schaefer and Strebulaev [55] procedure that incorporates historical default probabilities. Leverage is calculated as the ratio of book value of debt over book value of debt plus equity value. Asset volatility is obtained by first calculating the quantity $(1-\lambda)\,\sigma_E$, which is interpreted as a lower boundary for the asset volatility. This lower boundary is then multiplied by an adjustment ratio estimated by Schaefer and Strebulaev [55]. The adjustment ratio is the ratio of the asset volatility calculated using equation 3.5 to the lower boundary $(1-\lambda)\,\sigma_E$. Finally, Feldhutter and Schaefer [29] use a variant of the Merton model where the default boundary is different from the debt principal. They estimate the default boundary as the default boundary that minimizes a measure of the distance between the model default rates and historical default rates simultaneously across all maturities and ratings.

There are several problems with the estimation procedure of Feldhutter and Schaefer [29]. First, their computation of leverage assumes that market value of debt is equal to book value of debt, an assumption which does not hold empirically. Second, it shares the

3. Expectation here is taken under the physical measure.

problem of constant leverage assumption of Schaefer and Strebulaev [55] in the calculation of asset volatilities. Third, the approach assumes that the default boundary is the same for all the firms. However, there is strong evidence that there is huge heterogeneity in default boundaries across firms, even if one focuses solely on firms with the same rating (see Davydenko [21]). Fourth, very surprisingly, Feldhutter and Schaefer [29] find that the default boundary that minimizes the distance between the model default rates and historical default rates is exactly the debt principal. This implies that including historical default rates in the estimations do not change their estimates, and is therefore redundant in their approach. Finally, it is known that the Merton model, when calibrated to one particular maturity, cannot fit the whole term structure. This implies that the fitting error in the minimization problem of Feldhutter and Schaefer [29] is very likely large because they use only one variable (the default boundary) to fit simultaneously the term structures of all rating categories.

3.2 Our estimation approach

Here, we propose a new approach for the estimation of the share of default risk in corporate bond spreads. Our approach builds on the standard estimation approach described above but incorporates historical default rates in the estimation.

The approach consists of two stages. The first stage is identical to the standard estimation procedure, which consists of inverting the Merton model from equity prices and equity volatility as described in the previous section. The second stage consists of a maturity-specific adjustment of the leverage obtained in the first stage so that the model default rate is equal to the observed default rate at each horizon. The adjustment can be done at the individual firm level or at a group level depending on default data availability.

Definition *(Individual level adjustment)* Suppose that the default rate for firm i at time t, which we denote ϕ_{itT}, is observable. Then the individual adjustment consists of solving the following equation for y_{itT}

$$\Phi_T\left(\lambda_{it}e^{-y_{itT}}, \sigma_{it}, \mu_{it}\right) = \phi_{itT}, \tag{3.8}$$

where $\Phi_T(.)$ is the Merton model default rate function at the maturity T and λ_{it}, σ_{it} and μ_{it} are respectively the estimated leverage, asset volatility and asset drift from the first stage for firm i at time t.

Definition *(Categorical level adjustment)* Suppose that the average default rate of a group l of n firms over a certain period τ, which we denote ϕ_{lT}^{G}, is observable. Then the categorical adjustment consists of solving the following equation for y_{lT}^{G},

$$\frac{1}{n\tau}\sum_{t=1}^{\tau}\sum_{i=1}^{n}\Phi_T\left(\lambda_{it}e^{-y_{lT}^{G}},\sigma_{it},\mu_{it}\right)=\phi_{lT}^{G}, \tag{3.9}$$

where λ_{it}, σ_{it} and μ_{it} are the estimated fundamentals from the first stage for firm i at time t.

Proposition 3.2.1. *There exists a unique solution to the individual adjustment equation 3.8.*

Proof. The default rate function $\Phi_T(\lambda,\sigma,\mu)$ is continuous, strictly increasing in leverage λ over the interval $]0,+\infty[$. The function e^{-y} is continuous and strictly decreasing in y over $\Re$. This implies that the function $g(y)=\Phi_T(\lambda e^{-y},\sigma,\mu)$ is continuous and strictly decreasing in y over $\Re$.

Furthermore, $\forall\lambda>0, \Phi_T(\lambda,\sigma,\mu)\in\,]0,1[$ and has a limit of 0 for $\lambda=0$ and a limit of 1 as λ goes to $+\infty$. This implies that $\forall y\in\Re, g(y)\in\,]0,1[$ and has a limit of 0 as y goes to $-\infty$ and a limit of 1 as y goes to $+\infty$.

As a result, $g(y)$ is a bijection from $\Re$ to $]0,1[$. As a consequence for any $\phi\in\,]0,1[$, there exists a unique $y\in\Re$ such that $g(y)=\phi$. $\square$

Proposition 3.2.2. *There exists a unique solution to the categorical adjustment equation 3.9.*

Proof. Following the proof of 3.2.1, it is clear that the function

$$h(y)=\frac{1}{n\tau}\sum_{t=1}^{\tau}\sum_{i=1}^{n}\Phi_T\left(\lambda_{it}e^{-y},\sigma_{it},\mu_{it}\right), \tag{3.10}$$

is continuous and strictly decreasing in y over $\Re$ as the sum of functions that are continuous and strictly decreasing in y over $\Re$. Furthermore, $\forall\, y \in \Re$, the function $g_{it}(y) = \Phi_T(\lambda_{it} e^{-y}, \sigma_{it}, \mu_{it})$ takes values in $]0,1[$, has a limit of 0 as y goes to $-\infty$ and a limit of 1 as y goes to $+\infty$. It follows that the same holds for the function $h(y)$.

As a result, $h(y)$ is a bijection from $\Re$ to $]0,1[$. As a consequence, for any $\phi \in\,]0,1[$, there exists a unique $y \in \Re$ such that $h(y) = \phi$. $\square$

Definition *(Implied leverage)* The default probability implied leverage or simply the implied leverage is the leverage that makes the model default probability consistent with the observed default probability. For an individual adjustment by Equation 3.8, the leverage adjustment is firm-specific, time-specific and maturity-specific and the implied leverage for firm i at time t is equal to $\lambda_{it} e^{-y_{itT}}$. For a categorical adjustment by Equation 3.9, the leverage adjustment is category-specific and maturity-specific, and the implied leverage for firm i at time t in the category l is equal to $\lambda_{it} e^{-y^G_{lT}}$.

The implied leverage defined above is a fictitious leverage value that has nothing to do with actual leverage or book leverage. It is an artifice that is designed to mitigate the impact of the potential imperfections of the Merton model and the estimation procedure on the measured default spread. Because of that, the implied leverage figure taken alone does not have any economic interpretation. However, comparing the first-stage leverage with the implied leverage is interesting from an economics perspective. If the implied leverage is lower (resp. higher) than the first-stage leverage then it means that the estimated parameters of the first stage imply higher (resp. lower) default rates than observed default rates.

The implied leverage is opaque to the cause of the potential overestimation or underestimation of the default rate at the first stage. Whether the cause is biased estimates of asset volatility, leverage and/or asset drift in the first stage or incorrect assumptions of the model about for instance, the capital structure, the absence of tax, the perfect market assumptions or the Miller-Modigliani theorem, the use of the implied leverage instead of the first stage leverage ensures that any estimation bias or incorrect assumption has a relatively weak impact on the predicted default rate by the model. In short, implied leverages are a tool to mitigate not the causes, but the consequences of biased estimators or wrong model assumptions.

The first advantage of our two-stage procedure is that, thanks to Propositions 3.2.1 and 3.2.2, it ensures that our estimated parameters fit exactly the observed term structure of default rates. Second, solving Equations 3.8 or 3.9 poses no numerical difficulty because they consist of finding the unique root of a very smooth and strictly increasing function [4]. Third, it can be operationalized at the rating category or economic sector levels where historical default rate data are publicly available. Fourth, asset value and volatility are estimated exactly as in the estimation procedure of Bharath and Shumway [4], Crosbie and Bohn [17] which is the standard. So our approach retains some of the advantages of the standard procedure: It is easy to operationalize; There is no need of observing market value of book debt; It is based on a structural model [5] that links firm fundamentals to equity prices in such a way that the estimated asset value and volatility capture very well the observed dynamics of equity prices. Fifth, and most importantly, it ensures that the default spread is not severely over or under estimated. This last advantage is demonstrated in the next subsection.

However, these advantages come at a cost. The estimated leverage and volatilities no more satisfy the system of two equations derived from the Merton model. The perfect fit of observed equity values is lost and it is no more possible to back the estimations by a maximum likelihood argument. This cost is discussed further in the empirical section.

3.3 Performance tests

The main problem in estimating structural models is that there are often frictions between some of the model assumptions and the reality.

One such friction, in the case of the Merton model, is the assumption that the firm has a unique outstanding zero coupon debt. In reality, firms have several outstanding bonds

4. Under weaker conditions, the Newton-Raphson method guarantees a quadratic convergence for a suitable choice of initial point.

5. The fact that it is based on the simple Merton model does not seem to be a limitation. Elkamhi et al. [24] find that the estimated parameters from the Merton model are consistent with the more advanced structural model of Leland and Toft [43].

with different maturities. Simplifying assumptions on the actual capital structure must therefore be made in order to use the Merton model. These simplifying assumptions may severely bias the results of the model inversion. In this section, we consider the most common capital structure simplifying assumption and we compare the biases resulting from using the standard inversion approach to ours.

Another friction is the assumption of perfect liquidity of securities in the Merton model. In reality, as demonstrated by the recent financial crisis, the liquidity component of corporate bonds spreads can be substantial. To our knowledge, potential biases resulting from inverting the Merton model in a world with liquidity frictions have not yet been studied. In this section, we fill this gap in the literature. We consider a world in which there is liquidity risk and we compare the biases resulting from using the standard inversion approach to ours.

3.3.1 Estimation accuracy in presence of capital structure frictions

Firms generally have much more complex capital structures than postulated in the Merton model. In order to use the Merton model, a common simplifying assumption about the capital structure is that the total debt principal B of the Merton model is equal to the short term debt plus half the long term debt [4, 17, 24, 61]. Assumptions about debt maturity T vary widely. Elkamhi et al. [24] use an average maturity of about 4 years, Bharath and Shumway [4], Crosbie and Bohn [17] use a maturity of one year. Here we study how accurate the measured spread resulting from the standard estimation procedure and our estimation procedure are with respect to these simplifying capital structure assumptions.

In order to keep things simple, we assume that all the assumptions of the Merton model hold, so that the actual spread, default rate, equity value and equity volatility are given by Equations 3.4, 3.3, 3.6 and 3.7 respectively, but the econometrician does not have this information.

The actual firm fundamentals are $\sigma = 0.28$, $r = 0.06$, $\mu = 0.12$, $T = 10$, $V = 100$, $B = 50$, $\rho = 0.4$. This implies that the actual spread at the 10 year horizon, the default rate at the 10 year horizon, the equity value and the equity volatility are respectively 99 basis points, 4.5%, 73.9454 and 0.3679.

The econometrician does not know the asset volatility, the asset value and the debt maturity but he knows that the debt is long term [6]. He knows the riskless rate r, the expected asset return μ, the recovery rate ρ and the debt principal B. He also observes the equity value, the equity volatility, the default probability and the spread.

Assume that the econometrician makes the assumption that $T = 4$ and that it can be reasonably assumed that half of the debt would be due in 4 years.

In the standard approach, he would then solve Equations 3.6 and 3.7 simultaneously for σ, and V using $T = 4$ and a debt principal of 25. He would find that $\widehat{\sigma} = 0.2911$ and that $\widehat{V} = 93.5838$. This implies an estimated leverage of 26.71%. Based on these estimates of leverage and asset volatility, his measured 10 year spread would be 32 basis points only, which is 68% less than the actual spread.

In our two-stage approach, the first stage is identical to the standard approach, so he would estimate that the asset volatility is $\widehat{\sigma} = 0.2911$ and that the first stage leverage $\widehat{\lambda}_1$ is 26.71%. He would then solve the individual adjustment equation 3.8 and find an implied leverage $\widehat{\lambda}_2$ of 45.64%, which is much closer to the actual leverage. His measured 10 year spread would now be 96 basis points which is only 4% less than the actual spread. The improvement in the estimation accuracy is huge.

Notice that using the implied leverage guarantees a model default rate equal to the actual default rate of 4.5%, while using the first stage leverage implies a model default rate of 1.14%. Thus the standard approach drastically underestimates the default probability in this example. This justifies why using implied leverage leads to a more accurate spread.

We now repeat the above example for several values of debt principal $B = 10$, 20, ..., 100, several debt maturities $T = 2$, 3, ..., 20, and several values of asset volatility $\sigma = 0.2$, 0.22 ,..., 0.5. The risk free rate is kept at 6% (same as in Elkamhi et al. [24]), asset drift is fixed at 12% (same as in Leland and Toft [43]), recovery rate is fixed at 0.4, and initial asset value is fixed at 100. The combinations of parameters for which no

6. Here we are referring to the accounting classification of debt maturities. Any debt due in more than a year is classified as long term. This is the classification adopted in the COMPUSTAT database.

numerical solution is found for the system of two equations in the standard approach or for our calibration equation are discarded. In the end, we have a total of 1775 different combination of parameters. Table 3.I displays a summary of the results. For interested readers, Table 3.II displays detailed results for selected combination of parameters. The results show that either on an absolute basis or a relative basis, despite the capital structure frictions, our two-stage estimation procedure gives a quite accurate estimation of the actual spread compared to the standard estimation approach.

Next, we study the performance of our approach at the group level. In this case, we relax the assumption that the individual default rates are observable and make the weaker assumption that only the average default rate of a group of firms is observable. Our two-stage approach is still applicable by using the group calibration equation 3.9. We consider four distribution shapes for each heterogeneous parameter in the group: uniform, normal, positive skew and negative skew. The heterogeneous parameters are leverage and volatility. The shapes considered are the shapes Uniform-LowRange, Normal-HighStd, Pearson-PosSkew2 and Pearson-NegSkew2 of Tables 2.I and 2.II . Each time, we draw 1000 firms randomly from the distribution of interest. The other parameters are as in the above example. We then calculate the actual default spread and the corresponding estimates from the standard approach and our two-stage approach. The simulation is then repeated 10000 times and overall averages and standard errors are computed. Table 3.III summarizes the results. The results show that even with the group level adjustment, our two-stage approach still gives a more accurate estimation of the default spread compared to the standard approach if there are capital structure frictions.

Table 3.I: Performance of our two-stage approach and the standard approach in presence of capital structure frictions: Summary.

	mean		median		max		RMSE	
	Standard approach	Two-stage approach	Standard approach	Two-stage approach	Standard approach	Two-stage approach	Standard approach	Two-stage approach
Absolute error (bps)	131	11	121	7	435	103	151	17
Relative error	56%	5%	58%	3%	89%	27%	58%	7%

This table reports the summary statistics of the difference between the estimated and actual spreads using the standard approach and our approach in presence of capital structure frictions. It is assumed that all the assumptions of the Merton model hold but the econometrician does not have this information. Equity price, equity volatility, default rate and debt principal are assumed to be observable but debt maturity T, asset value and asset volatility are unknown to the econometrician. It is known however that the debt is a long term debt. The econometrician assumes that debt maturity is 4 year and that the portion of debt principal due is half the total debt. The standard approach consists of inverting the equity price and volatility equations for asset value and volatility. Our two-stage approach consists of a first stage identical to the standard approach and a second stage where leverage is adjusted to fit the actual default rate. The obtained leverage and volatility from each approach are then used to predict spread at horizon T. The study is repeated for several values of debt principal $B = 10, 20, 30, ..., 100$, several debt maturities $T = 2, 3, 4, ..., 20$, and several values of asset volatility $\sigma = 0.2, 0.22, 0.24, ..., 0.5$. The other parameters are $\mu = 0.12$, $r = 0.06$, $\rho = 0.4$, $V_0 = 100$.

Table 3.II: Performance of our two-stage approach and the standard approach in presence of capital structure frictions: Selected cases.

Actual parameters			Spreads(basis points)			% difference from actual spread		Default rate		Estimated parameters			
σ	B	T	Actual	Standard approach	Two-stage approach	Standard approach	Two-stage approach	Actual	Standard approach	V	σ	Standard approach leverage	Implied leverage
0.24	50	10	58	14	54	-75%	-7%	0.0172	0.0032	92.90	0.25	0.27	0.44
0.24	70	2	336	417	267	24%	-21%	0.0561	0.0906	64.47	0.38	0.54	0.48
0.24	70	4	249	162	206	-35%	-17%	0.0664	0.0504	73.58	0.31	0.48	0.52
0.24	70	20	60	9	71	-86%	19%	0.0211	0.0008	107.90	0.22	0.32	0.96
0.28	30	4	16	3	15	-80%	-10%	0.0032	0.0006	88.24	0.32	0.17	0.24
0.28	50	2	96	62	81	-36%	-16%	0.0154	0.0116	75.11	0.37	0.33	0.35
0.28	50	4	128	63	111	-51%	-13%	0.0348	0.0183	80.83	0.34	0.31	0.37
0.28	50	10	99	32	96	-68%	-4%	0.0450	0.0114	93.58	0.29	0.27	0.46
0.28	70	2	496	620	416	25%	-16%	0.0953	0.1446	64.10	0.44	0.55	0.47
0.28	70	4	349	241	306	-31%	-12%	0.1124	0.0863	74.27	0.36	0.47	0.52
0.28	70	10	177	62	174	-65%	-2%	0.0942	0.0251	92.62	0.29	0.38	0.67
0.28	70	20	96	19	111	-80%	15%	0.0576	0.0045	109.01	0.25	0.32	1.02
0.36	30	4	76	28	71	-63%	-7%	0.0239	0.0086	88.43	0.40	0.17	0.24
0.36	30	10	102	40	101	-61%	-2%	0.0615	0.0204	96.31	0.37	0.16	0.28
0.36	30	20	91	34	95	-63%	5%	0.0759	0.0196	103.90	0.34	0.14	0.38
0.36	50	2	277	226	249	-18%	-10%	0.0572	0.0519	74.82	0.48	0.33	0.35
0.36	50	4	280	169	260	-40%	-7%	0.1022	0.0640	81.68	0.43	0.31	0.38
0.36	50	10	203	86	202	-58%	0%	0.1370	0.0501	95.50	0.36	0.26	0.49
0.36	50	20	142	47	153	-67%	8%	0.1322	0.0285	107.53	0.33	0.23	0.74
0.36	70	2	807	1010	731	25%	-9%	0.1795	0.2465	63.69	0.55	0.55	0.46
0.36	70	4	544	404	512	-26%	-6%	0.2113	0.1655	75.96	0.44	0.46	0.54
0.36	70	10	298	126	301	-58%	1%	0.2124	0.0771	95.75	0.35	0.37	0.73

This table compares the spread measured by our two-stage approach and that measured by the standard approach to the actual spread. It is assumed that all the assumptions of the Merton model hold but the econometrician does not have this information. Equity price, equity volatility, default rate and debt principal are assumed to be observable but debt maturity T, asset value and asset volatility are unknown to the econometrician. It is known however that the debt is a long term debt. The econometrician assumes that debt maturity is 4 year and that the portion of debt principal due is half the total debt. The standard approach consists of inverting the equity price and volatility equations for asset value and volatility. Our two-stage approach consists of a first stage identical to the standard approach and a second stage where leverage is adjusted to fit the actual default rate. The obtained leverage and volatility from each approach are then used to predict spread at horizon T. The study is repeated for several values of debt principal $B = 10, 20, 30, ..., 100$, several debt maturities $T = 2, 3, 4, ..., 20$, and several values of asset volatility $\sigma = 0.2, 0.22, 0.24, ..., 0.5$. The other parameters are $\mu = 0.12$, $r = 0.06$, $\rho = 0.4$, $V_0 = 100$.

Table 3.III: Performance of our two-stage approach and the standard approach in presence of capital structure frictions and heterogeneity: Summary.

Heterogeneous parameter and distribution	average spreads (bps)			average DPs		
	Actual	Standard	Our approach	Actual	Standard	Our approach
Leverage, Uniform-LowRange	76.95	39.91	65.83	2.15%	1.24%	2.15%
	(2.6427)	(1.6276)	(2.2992)	(0.0008)	(0.0005)	(0.0008)
Leverage, Normal-HighStd	71.29	32.32	60.87	1.88%	0.93%	1.88%
	(1.877)	(1.1596)	(1.6159)	(0.0006)	(0.0004)	(0.0006)
Leverage, Pearson-PosSkew2	63.37	27.69	53.77	1.64%	0.80%	1.64%
	(2.057)	(1.4935)	(1.7501)	(0.0007)	(0.0005)	(0.0007)
Leverage, Pearson-NegSkew2	0.18	0.01	0.20	0.00%	0.00%	0.00%
	(0.0069)	(0.0003)	(0.0083)	(0)	(0)	(0)
Volatility, Uniform-LowRange	218.48	132.82	211.58	8.72%	5.49%	8.72%
	(6.2944)	(4.3081)	(6.2418)	(0.0028)	(0.0019)	(0.0028)
Volatility, Normal-HighStd	54.04	23.13	48.75	1.58%	0.73%	1.58%
	(1.9371)	(1.0499)	(1.8344)	(0.0007)	(0.0004)	(0.0007)
Volatility, Pearson-PosSkew2	159.48	108.02	149.85	3.92%	2.69%	3.92%
	(25.6518)	(23.8246)	(24.8525)	(0.0037)	(0.0034)	(0.0037)
Volatility, Pearson-NegSkew2	44.37	14.95	38.26	1.02%	0.36%	1.02%
	(0.6288)	(0.2562)	(0.562)	(0.0002)	(0.0001)	(0.0002)

This table compares for selected heterogeneous groups of firms, the average spreads and default probabilities measured by the standard approach and by our approach to the actual average spreads and default probabilities in the presence of capital structure frictions. Each group is heterogeneous in either leverage or volatility and the distribution of the heterogeneous parameter is given in Column 1. The distributions are described in tables 2.I and 2.II. For each selected distribution, we perform 10000 rounds of simulations, each round consisting of simulating 1000 firms from the distribution and calculating the relevant quantities. The figures displayed are the overall averages and standard errors (in parenthesis) over the 10000 rounds. It is assumed that all the assumptions of the Merton model hold but the econometrician does not have this information. Equity price, equity volatility, default rate and debt principal are assumed to be observable but debt maturity T, asset value and asset volatility are unknown to the econometrician. It is known however that the debt is a long term debt. The econometrician assumes that debt maturity is 4 year and that the portion of debt principal due is half the total debt. The standard approach consists of inverting the equity price and volatility equations for asset value and volatility. Our two-stage approach consists of a first stage identical to the standard approach and a second stage where leverage is adjusted to fit the actual average default rate of the group. The actual values of the firm fundamentals are: riskless rate $r = 0.06$, asset drift $\mu = 0.12$, horizon $T = 10$, recovery rate $\rho = 0.4$, asset volatility $\sigma = 0.28$ (for leverage heterogeneity only) and leverage $\lambda = 0.5$ (for volatility heterogeneity only).

3.3.2 Robustness to the presence of liquidity risk and other frictions

We now compare the performance of our two-stage approach to that of the standard approach when there are other frictions, namely, liquidity risk, asset payout, coupons, taxes and endogenous default decision.

In order to introduce these frictions, we assume that the world is perfectly described by the He and Xiong [32] model. In this model, the asset value follows a log normal diffusion process with a proportional asset payout rate of ζ. Default may happen at any moment during the life of the debt contract. It is assumed that default happens when the asset value falls bellow a given value called the default boundary and denoted by V_B. This default boundary is endogenously determined by equity holders as the asset value at which equity becomes worthless. When default occurs, only a portion α of the asset value is recovered. There is a tax rate π and there are liquidity frictions in the market. Liquidity shocks occur as Poisson events with intensity ξ and, in the event of a liquidity shock, a fraction ε of the bond value is lost. The firm has a stationary capital structure with total debt principal P and total annual coupon payment C. At each instant in the interval of time $[t, t+dt]$, a portion $\frac{1}{T}dt$ of the total debt matures and is replaced by new issues of maturity T, annual coupon $\frac{C}{T}$ and principal $\frac{P}{T}$. He and Xiong [32] derived closed form solutions for the spread on new issues and equity price. Because the formulas are lengthy we do not report them here. The default component of the spreads is defined as the total spread minus the liquidity spread $\xi\varepsilon$.

As an illustration, we consider the base case parameters suggested by He and Xiong [32], $\zeta = 0.02$, $\alpha = 0.6$, $\xi = 0.01$, $\varepsilon = 1$, $C = 6.39$, $P = 61.68$, $r = 0.08$, $\pi = 0.27$, $V = 100$. In addition, we assume that the asset volatility is $\sigma = 0.28$, the asset drift is $\mu = 0.14$ and the maturity of new issues is $T = 4$. Using the closed form solutions of He and Xiong [32], we find that the actual spread on the new issues is 356 basis points and that the equity price is $E = 41.8781$. With a liquidity spread of 100 basis points, the actual share of default in the spreads is 72%. To find equity volatility we first numerically compute the first derivative of equity value with respect to asset value E_V. We find that $E_V = 1.0834$. Using the following relationship between equity volatility and asset volatility,

$$\sigma_E = \sigma E_V \frac{V}{E}, \tag{3.11}$$

we find that equity volatility is equal to 0.7244. The formula for the physical default probability of Leland [42], Leland and Toft [43] remains valid in this context except that one must use the default boundary of He and Xiong [32]. We find that the He and Xiong [32] default boundary V_B is equal to 51.6769. This implies a 4 year default probability of 0.1097.

Consider now an econometrician who ignores that the world is described by the He and Xiong [32] model and uses the Merton model to measure the spreads. We assume that he observes the equity value, the equity volatility, the default rate, the debt principal and its maturity, the coupon, the default boundary and the asset drift but he does not observe the asset volatility and the asset value. Since there is no coupon in the Merton model, it must be accounted for in the debt principal of the Merton model: $B = (P/T + C) = 21.81$. The recovery rate as a fraction of debt principal is equal to $\frac{\alpha V_B}{P} = 0.3351$.

Suppose that the econometrician estimates asset volatility and asset value by the standard approach. He would invert Equations 3.6 and 3.7 and find that $\widehat{V} = 55.6032$ and $\widehat{\sigma} = 0.5728$. This implies a leverage of 39.22%. The estimated parameters imply a default spread of 582 basis points which is 226 basis points more than the actual total spread, and 326 basis points more than the actual default spread.

Now suppose instead that the econometrician uses our two-stage approach. Because the first stage is identical to the standard approach, he would estimate a first stage leverage $\widehat{\lambda}_1$ of 39.22% and an asset volatility of $\widehat{\sigma} = 0.5728$. In the second stage, he would calculate the implied leverage by solving Equation 3.8. He would find an implied leverage of 22.25%. The corresponding default spread is 277 basis points which is only 21 basis points more than the actual default spread.

In summary, our two stage approach gives an estimated share of default risk in the spread of 78%, which is very close to the actual default share of 72%, while the standard approach puts the estimated default share at 163%. Interestingly, the estimate from the standard approach gives the inaccurate impression that there is no liquidity spread.

To ensure that this good performance of our two-stage approach is not specific to the above particular parametrization of the He-Xiong Model, we repeat the study for several maturities $T = 1,2,3,...,20$, asset volatilities $\sigma = 0.2,0.22,0.24,...,0.5$, total debt principal $P = 10,20,30,...,100$, coupon rates $\frac{C}{P} = 0.1,0.15$ and liquidity spread $\varepsilon = 25,50,100$ basis points. The other parameters of the model are as in the illustration. We discard the combinations of parameters for which there were numerical difficulties either in our approach or the standard approach or for which the actual size of default spread was less than 1 basis points. In total we are left we 7460 different combinations of parameters. Table 3.IV summarizes the results. The table shows that in both absolute and relative terms, our approach gives much lower estimation errors than the standard approach. For interested readers, Table 3.V provides detailed results for selected combinations of parameters.

We next study the performance of our approach at the group level. We relax the assumption that the individual default rates are observable and make the weaker assumption that only the average default rate of a group of firms is observable. In this case, our two-stage approach is still applicable by using the group calibration equation 3.9. It is assumed that only leverage is heterogeneous. We consider four distribution shapes for leverage heterogeneity: uniform, normal, positive skew and negative skew. The shapes considered are the shapes Uniform-LowRange, Normal-HighStd, Pearson-PosSkew2 and Pearson-NegSkew2 of Tables 2.I and 2.II . Each time we draw 1000 leverages randomly from the distribution of interest. The other parameters are as in the above example. We then calculate the actual default spread and the corresponding estimates from the standard approach and our two-stage approach. The simulation is then repeated 10000 times and overall averages and standard errors are computed. Table 3.VI summarizes the results. These results show that even with the group level adjustment, our two-stage approach still gives a more accurate estimation of the default spread compared to the standard approach if there are liquidity and other frictions.

Table 3.IV: Performance of our two-stage approach and the standard approach in presence of liquidity and other frictions: Summary.

	mean		median		max		RMSE	
	Standard approach	Two-stage approach	Standard approach	Two-stage approach	Standard approach	Two-stage approach	Standard approach	Two-stage approach
Absolute error (bps)	147	36	40	26	2724	372	287	53
Relative error	41%	14%	28%	13%	286%	50%	57%	17%

This table reports the summary statistics of the difference between the estimated and actual shares of default risk using the standard approach and our approach in presence of liquidity and other frictions. In this exercise, we use the He-Xiong model to simulate a world with liquidity risk, taxes, endogenous default, payout rates and complex capital structure. We then attempt to capture the share of default risk in the spreads by the Merton model using the standard approach and our two-stage approach as described in the text. In the He-Xiong model σ is the asset volatility, ε is the liquidity cost as a fraction of bond price, T is the maturity of new issues, C is the total annual coupon payment and P is the total debt principal. The exercise in performed for several maturities $T = 1,2,3,...,20$, asset volatilities $\sigma = 0.2, 0.22, 0.24, ..., 0.5$, total debt principal $P = 10,20,30,...,100$, coupon rates $\frac{C}{P} = 0.1, 0.15$ and liquidity spread $\varepsilon = 25,50,100$ basis points. The other parameter choices are: asset drift $\mu = 0.14$, initial asset value $V_0 = 100$, tax rate $\pi = 0.27$, asset payout rate $\zeta = 0.02$, liquidity shock intensity $\xi = 1$ and risk free rate $r = 0.08$.

Table 3.V: Performances of our two-stage approach and the standard approach in presence of liquidity and other frictions: Selected cases

He-Xiong model parameters						Share of default risk in the spreads			Default rates	
σ	ε	T	C	P	Actual total spread	Actual	Standard	Our approach	Actual	Standard to actual
0.24	0.005	4	6	60	177	72%	113%	73%	0.0433	1.62
0.24	0.005	4	9	60	195	74%	111%	74%	0.0494	1.54
0.24	0.005	10	6	60	147	66%	105%	57%	0.0504	2.00
0.24	0.005	10	9	60	196	75%	142%	57%	0.0797	2.64
0.24	0.01	4	6	60	238	58%	106%	59%	0.0497	1.87
0.24	0.01	4	9	60	260	61%	106%	62%	0.0571	1.80
0.24	0.01	10	4	40	126	21%	10%	21%	0.0101	0.40
0.24	0.01	10	6	60	201	50%	90%	44%	0.0555	2.25
0.24	0.01	10	9	60	255	61%	130%	47%	0.0880	2.90
0.32	0.005	4	4	40	131	62%	50%	74%	0.0313	0.65
0.32	0.005	4	6	60	366	86%	185%	97%	0.1408	1.88
0.32	0.005	4	9	60	413	88%	186%	95%	0.1593	1.91
0.32	0.005	10	4	40	146	66%	56%	74%	0.0667	0.73
0.32	0.005	10	6	60	277	82%	154%	82%	0.1723	1.85
0.32	0.005	10	6	40	176	72%	75%	76%	0.0915	0.98
0.32	0.01	4	4	40	185	46%	41%	55%	0.0338	0.73
0.32	0.01	4	6	60	426	77%	177%	87%	0.1497	1.98
0.32	0.01	4	9	60	477	79%	181%	87%	0.1698	2.00
0.32	0.01	10	4	40	197	49%	46%	56%	0.0698	0.79
0.32	0.01	10	6	40	229	56%	63%	60%	0.0959	1.06

This table reports for selected cases, the difference between the estimated and actual shares of default risk using the standard approach and the actual approach in presence of liquidity and other frictions. In this exercise, we use the He-Xiong model to simulate a world with liquidity risk, taxes, endogenous default, payout rates and complex capital structure. We then attempt to capture the share of default risk in the spreads by the Merton model using the standard approach and our two-stage approach as described in the text. In the He-Xiong model σ is the asset volatility, ε is the liquidity cost as a fraction of bond price, T is the maturity of new issues, C is the total annual coupon payment and P is the total debt principal. The values of these parameters are indicated in the first 5 columns of the table. The other parameter choices are: asset drift $\mu = 0.14$, initial asset value $V_0 = 100$, tax rate $\pi = 0.27$, asset payout rate $\zeta = 0.02$, liquidity shock intensity $\xi = 1$ and risk free rate $r = 0.08$.

Table 3.VI: Performance of our two-stage approach and the standard approach in presence of liquidity and other frictions and heterogeneity in leverage: Summary.

Distribution	Average spreads, actual	Default component			average DPs		
		Actual	Standard	Our approach	Actual	Standard	Our approach
Uniform-LowRange	170.98 (2.6923)	41% (0.0092)	70% (0.0247)	43.87% (0.0101)	2.83% (0.0012)	4.52% (0.0024)	2.83% (0.0012)
Normal-HighStd	161.03 (1.8852)	37.89% (0.0073)	48.59% (0.0212)	40.40% (0.0077)	2.29% (0.0008)	2.77% (0.0017)	2.29% (0.0008)
Pearson-PosSkew2	154.28 (0)	35.18% (0)	46.54% (0)	37.01% (0)	2.02% (0)	2.53% (0)	2.03% (0)
Pearson-NegSkew2	100.08 (0.0154)	0.08% (0.0002)	0.00% (0)	0.03% (0)	0.00% (0)	0.00% (0)	0.00% (0)

This table compares, for selected heterogeneous groups of firms, the average spreads and default probabilities measured by the standard approach and by our approach to the actual average spreads in the presence of liquidity and other frictions. Each group is heterogeneous in leverage and the distribution of leverage in the group is given in Column 1. The distributions are described in tables 2.I and 2.II. For each selection distribution, we perform 10000 rounds of simulations, each round consisting of simulating 1000 firms from the distribution and calculating the relevant quantities. The figures displayed are the overall averages and standard errors (in parenthesis) over the 10000 rounds. In this exercise, we use the He-Xiong model to simulate a world with liquidity risk, taxes, endogenous default, payout rates and complex capital structure. We then attempt to capture the share of default risk in the spreads by the Merton model using the standard approach and our two-stage approach as described in the text. Our two-stage approach consists of a first stage identical to the standard approach and a second stage where leverage is adjusted to fit the average default rate of the group using equation 3.9. The parameters of the He-Xiong model are: asset volatility $\sigma = 0.28$, liquidity cost $\varepsilon = 1$, maturity of new issues $T = 4$, total annual coupon payment $C = 6.39$, total debt principal $P = 61.68$, asset drift $\mu = 0.14$, initial asset value $V_0 = 100$, tax rate $\pi = 0.27$, asset payout rate $\zeta = 0.02$, liquidity shock intensity $\xi = 1$, risk free rate $r = 0.08$.

3.4 Data

3.4.1 Firm raw data and risk free rate

Our sample covers all North American firms common to the databases COMPUSTAT and CRSP for which we have available data, excluding financial firms. To construct the database, we retrieve from COMPUSTAT quarterly data on debt in current liabilities (COMPUSTAT variable name: dlcq), long term debt (COMPUSTAT variable name: dlttq) and economic sector (COMPUSTAT variable name: spcseccd). Regarding credit ratings, we used S&P credit rating on domestic long term debt issuer (COMPUSTAT label: splticrm). The rating data is available at monthly frequency. We also retrieve from CRSP daily data on share closing price (CRSP variable name PRC), daily number of outstanding shares (CRSP variable name: SHROUT), share code (CRSP variable name: SHRCD), and ex-dividend daily returns (CRSP variable name: RETX). For the risk-free rate r, we use the monthly data on one year Treasury Constant Maturity Rate obtained from the Board of Governors of the Federal Reserve system [7]. After first round cleaning (keeping common shares only, merging the data-set and keeping observations with no missing data and strictly positive debt principal), the data-set consists of 1536 firms from January 2, 1981 to December 31, 2013 for a total of 4794915 firm-days.

3.4.2 Historical default rates

We use historical default rate data over the period 1981 to 2013 (32 years) from the table 24 of Standard and Poors [57]. This data is reprinted in Table 3.VII of this essay. Feldhutter and Schaefer [29] argued that there is too much statistical uncertainty on default rates based on 28 years default data, and that a longer default data (90 years) should be used in order to reduce that uncertainty. However, caution must be made in using 90 year historical data because this means going back to the 1920s up until the current decade. This also means including in the data, the special spells of the great depression period going from mid-1929 through mid-1939, the World War II period and the post-World War II period up until 1970. As the exhibits 9 and 10 of Keenan et al.

7. Available at http://research.stlouisfed.org/fred/data/irates/gs1 (H.15 Release).

[40] show, the default rates in these special spells were abnormally high and when these abnormal periods are discarded, default rates from the 1920s to the 1980s are similar to those from the 1980s to the 2010s. This implies that the default rates data that we use reflects very well the normal level of default rates over the last 90 years.

Table 3.VII: Historical default rates (%)

Rating	Maturity in years			
	1	2	4	10
AAA	0	0.03	0.24	0.74
AA	0.02	0.07	0.24	0.84
A	0.07	0.17	0.43	1.59
BBB	0.21	0.6	1.53	4.33
BB	0.8	2.46	6.29	14.39
B	4.11	9.27	16.99	26.97
Below B	26.87	36.05	44.27	51.35

Source: Standard and Poors [57]

3.4.3 Recovery rates

Average recovery rates by rating categories for senior unsecured bonds over the period 1982-2010 are retrieved from Exhibit 22 of Moody's [51]. Average recovery rates by economic sector are retrieved from Table 4 of Standard and Poors [56]. Tables 3.VIII and 3.IX show the recovery rates by rating and by economic sector respectively. In this study, we would like to have some heterogeneity in recovery rates, at least across economic sectors. Unfortunately, we do not have data on average recovery rates by economic sectors inside a given rating category, and for that reason we suggest the following procedure to generate them. Moody's annual reports provide average recovery rates by economic sector $\overline{\rho}^p, p \in \Pi$ and average recovery rates by rating $\overline{\rho}^l, l \in L$. Π and L are the sets of economic sectors and rating categories respectively. Based on this information, we generate average recovery rates stratified by rating and economic sector simultaneously ρ_{pl}.

We assume that the recovery rate of a firm with a given rating in a given economic sector is equal to the average recovery rate inside that economic sector scaled by an adjustment factor that depends of the firm rating, that is, ρ_{pl} has the following form:

$$\rho_{pl} = \alpha_l \overline{\rho}^p.$$

Accordingly, two firms with the same rating but belonging to different economic sectors will have different recovery rates.

To identify α_l, we require that the average recovery rate inside a given rating category in our sample matches the historical average reported by the rating agencies.

Let Ω^p be the set of all observations in our sample falling in sector $p, p \in \Pi$, and let Ω^l be the set of all observations in our sample falling in rating category $l, l \in L$. The identification condition is:

$$\sum_{p \in \Pi} \frac{card\left(\Omega^p \cap \Omega^l\right)}{card\left(\Omega^l\right)} \alpha_l \overline{\rho}^p = \overline{\rho}^l, l \in L.$$

This implies that:

$$\alpha_l = \frac{\overline{\rho}^l}{\sum_{p \in \Pi} \frac{card\left(\Omega^p \cap \Omega^l\right)}{card\left(\Omega^l\right)} \overline{\rho}^p}, l \in L.$$

This procedure allows us to assign different recovery rates to firms inside a given rating category based on their economic sector, while ensuring that the average recovery rate for the rating category exactly matches its historical counterpart. The obtained recovery rates are shown in Table 3.X. The procedure appears to do a good job at generating heterogeneity inside rating categories.

Table 3.VIII: Average recovery rates by rating categories and maturity

Rating	number of years prior to default			
	1	2	4	10
AAA	0.3724	0.4015	0.5043	0.3880
AA	0.3724	0.4015	0.5043	0.3880
A	0.3177	0.4756	0.3990	0.4182
BBB	0.4147	0.4302	0.4457	0.4269
BB	0.4711	0.4461	0.4081	0.4080
B	0.3790	0.3606	0.3806	0.4135
Below B	0.3550	0.3481	0.3533	0.3496

Note: Due to extremely small sample sizes in the AAA category, we assume the same recovery rates as AA firms. Due to data unavailability at the 10 year horizon, we use the 5 year horizon recovery rate as a proxy. Source: Moody's [51]

Table 3.IX: Average recovery rates by economic sectors

Sector	recovery rate
Basic Materials	0.353
Capital goods	0.353
Communication services	0.391
Consumer Staples	0.336
Consumer Cyclicals	0.336
Energy	0.461
Health Care	0.347
Technology	0.328
Transportation	0.379
Utilities	0.641

Source: Standard and Poors [56]

Table 3.X: Heterogeneous recovery rates across economic sectors

sector	AAA	AA	A	BBB	BB	B	Below B	AAA	AA	A	BBB	BB	B	Below B
				T=1							T=2			
Transportation	39.45%	36.63%	30.62%	38.66%	48.79%	39.89%	38.02%	42.53%	39.49%	45.83%	40.11%	46.20%	37.96%	37.29%
Utilities		61.95%	51.78%	65.39%	82.52%	67.47%	64.31%		66.79%	77.52%	67.83%	78.14%	64.19%	63.06%
Health care	36.12%	33.53%	28.03%	35.40%	44.67%	36.52%	34.81%	38.94%	36.15%	41.96%	36.72%	42.30%	34.75%	34.14%
Capital Goods	36.74%	34.11%	28.52%	36.01%	45.44%	37.16%	35.42%	39.61%	36.78%	42.69%	37.36%	43.03%	35.35%	34.73%
Energy	47.98%	44.55%	37.24%	47.03%	59.35%	48.52%	46.25%	51.73%	48.03%	55.75%	48.79%	56.20%	46.17%	45.35%
Technology	34.14%	31.70%	26.50%	33.46%	42.22%	34.52%	32.91%	36.81%	34.17%	39.67%	34.71%	39.98%	32.85%	32.27%
Basic Materials		34.11%	28.52%	36.01%	45.44%	37.16%	35.42%		36.78%	42.69%	37.36%	43.03%	35.35%	34.73%
Communication services	40.69%	37.79%	31.59%	39.89%	50.33%	41.16%	39.23%	43.87%	40.74%	47.28%	41.38%	47.66%	39.16%	38.47%
Consumer cyclicals	34.97%	32.47%	27.14%	34.28%	43.25%	35.37%	33.71%	37.70%	35.01%	40.63%	35.56%	40.96%	33.65%	33.06%
Consumer Staples	34.97%	32.47%	27.14%	34.28%	43.25%	35.37%	33.71%	37.70%	35.01%	40.63%	35.56%	40.96%	33.65%	33.06%
				T=4							T=10			
Transportation	53.42%	49.60%	38.45%	41.55%	42.26%	40.06%	37.84%	41.10%	38.16%	40.30%	39.80%	42.25%	43.52%	37.45%
Utilities		83.89%	65.03%	70.28%	71.48%	67.75%	64.00%		64.54%	68.16%	67.31%	71.46%	73.61%	63.33%
Health care	48.91%	45.41%	35.20%	38.04%	38.70%	36.68%	34.65%	37.63%	34.94%	36.90%	36.44%	38.69%	39.85%	34.28%
Capital Goods	49.75%	46.20%	35.81%	38.70%	39.37%	37.31%	35.25%	38.28%	35.54%	37.54%	37.07%	39.36%	40.54%	34.88%
Energy	64.97%	60.33%	46.77%	50.54%	51.41%	48.73%	46.03%	49.99%	46.42%	49.02%	48.41%	51.40%	52.94%	45.55%
Technology	46.23%	42.92%	33.28%	35.96%	36.58%	34.67%	32.75%	35.57%	33.03%	34.88%	34.44%	36.57%	37.67%	32.41%
Basic Materials		46.20%	35.81%	38.70%	39.37%	37.31%	35.25%		35.54%	37.54%	37.07%	39.36%	40.54%	34.88%
Communication services	55.11%	51.17%	39.67%	42.87%	43.60%	41.33%	39.04%	42.40%	39.37%	41.58%	41.06%	43.59%	44.90%	38.63%
Consumer cyclicals	47.36%	43.97%	34.09%	36.84%	37.47%	35.52%	33.55%	36.44%	33.83%	35.73%	35.28%	37.46%	38.59%	33.20%
Consumer Staples	47.36%	43.97%	34.09%	36.84%	37.47%	35.52%	33.55%	36.44%	33.83%	35.73%	35.28%	37.46%	38.59%	33.20%

3.4.4 Historical spreads

For historical spreads, we rely on the estimates of Huang and Huang [33]. These historical estimates are based on the period 1973-1993 for 10 year investment grade bonds and on the period 1985-1998 for 4 year investment grade bonds. There are many reasons for using these estimates. First, we could have used data on actual transaction prices from Trace database but these prices are available only from 2002. Using TRACE would mean throwing away almost two thirds of the period that we cover. Second, historical average spreads in our study serve only as benchmark for comparison purposes. Since many studies rely on the Huang and Huang [33] estimates, they seem to be a good choice of benchmark. Third, there is little disagreement between different measures of historical average spreads over different time periods (see Table 4 of Bhamra et al. [3]).

3.5 Results and discussions

3.5.1 Standard approach estimates

Using the raw data-set, we exclude financial firms and observations for which the standard estimation procedure of Crosbie and Bohn [17] and Bharath and Shumway [4] do not converge. Finally, to ensure that our analysis is not driven by extreme values and outliers, we winsorize the data by eliminating the first 0.5% quantile and the last 99.5% quantile. The final sample includes daily data on 1286 firms for a total of 2735873 firm-days from December 26, 1984 to December 31, 2013. Table 3.XI shows the number of firms and observations in the final sample by rating and economic sector.

Table 3.XI: Number of firms and observations by rating category and economic sector

Sector	AAA	AA	A	BBB	BB	B	Below B	Total
Panel A: Number of firms by rating category and economic sector								
Transportation	1	2	6	20	22	25	13	48
Utilities	0	14	62	81	27	13	4	120
Health care	5	10	23	29	43	31	2	94
Capital Goods	1	10	39	75	100	46	14	180
Energy	1	3	15	41	45	30	4	96
Technology	2	9	23	41	53	45	11	101
Basic Materials	0	6	39	69	63	54	20	148
Communication services	1	5	12	7	16	21	9	43
Consumer cyclicals	1	11	53	113	169	159	55	290
Consumer Staples	2	17	50	63	66	60	27	166
Total	14	87	322	539	604	484	159	1286
Panel B: Number of firms and observations by economic sector								
Transportation	1037	1771	10401	41343	25708	23531	4797	108588
Utilities	0	19219	87189	153839	22403	8959	475	292084
Health care	15180	17432	44691	39793	35817	20733	95	173741
Capital Goods	720	21014	90153	133967	121065	42107	2449	411475
Energy	2236	5361	35049	67866	60227	22651	384	193774
Technology	3561	15598	47182	48530	48090	30866	1961	195788
Basic Materials	0	14663	88234	134157	75762	51307	3772	367895
Communication services	2193	1650	15957	9722	6798	18722	2336	57378
Consumer cyclicals	436	16891	104274	154378	180342	105702	11927	573950
Consumer Staples	1945	35792	95620	100187	64604	56560	6492	361200
Total	27308	149391	618750	883782	640816	381138	34688	2735873

Note about the last column of Panel A: The total exclude duplicates due to changing firm ratings.

Table 3.XII provides summary statistics of asset volatilities, asset drifts and leverage ratios estimated by the standard approach by rating category. As one can see, there is substantial heterogeneity inside rating categories. Consider the BBB category for instance. Asset volatilities vary from 11.47% to 363% with an interquartile range of 22.64% to 36.16%. Asset drifts vary from -28.19% to 220.93% with an interquartile range of 3.41% to 19.50%. Leverage ranges from 0.07% to 581.87% with an interquartile range of 11.03% to 30.67%. Coefficients of variations are large, suggesting that there is a sizable variation around the means. The interquartile ranges are important, indicating that the observed large dispersions are not driven by extreme values. We also observe substantial positive skewness and kurtosis in leverage, asset drift and asset volatility. This shows that required risk premia can be substantially large for some firms at some times, and that leverage ratios and asset volatilities can be extremely high relative to the average. The largest skewnesses and kurtoses are observed for BBB rated firms. Globally, heterogeneity is less important in high quality rating categories than in low quality rating categories.

Table 3.XII: Summary statistics of estimated firm parameters from the standard approach

Rating	mean	sd	CV	range	p99-p1	skewness	kurtosis	min	p1	p25	p50	p75	p99	max
Panel A: Asset Volatility σ_V (in percent per annum)														
AAA	25.08	6.67	0.27	45.31	35.42	1.56	7.93	13.22	13.84	20.61	24.19	28.32	49.26	58.52
AA	25.39	6.98	0.27	57.41	34.87	0.89	4.73	11.47	12.13	20.39	24.42	29.53	47.00	68.88
A	27.73	10.92	0.39	111.10	54.16	3.15	22.17	11.47	12.51	21.08	26.00	31.69	66.67	122.58
BBB	31.32	19.43	0.62	351.73	73.38	9.31	136.01	11.47	12.47	22.64	28.68	36.16	85.84	363.21
BB	40.31	19.21	0.48	313.75	101.05	4.26	39.05	11.48	15.76	29.36	36.83	46.37	116.81	325.23
B	49.71	28.87	0.58	398.54	196.34	4.69	38.33	11.82	16.67	34.63	44.18	56.23	213.01	410.37
Below B	65.34	53.99	0.83	391.93	286.66	3.60	17.63	18.49	19.43	40.00	52.33	67.70	306.09	410.43
Panel B: Asset drift μ_V (in percent per annum)														
AAA	14.24	12.62	0.89	77.35	60.79	0.46	3.21	-17.59	-12.00	5.16	13.11	21.92	48.78	59.77
AA	12.51	11.30	0.90	101.28	54.41	0.79	4.09	-20.63	-9.17	5.04	10.67	18.56	45.24	80.65
A	12.61	12.87	1.02	137.66	66.96	1.60	9.29	-28.16	-11.23	4.75	10.90	18.42	55.73	109.50
BBB	13.03	17.10	1.31	249.11	83.83	3.03	25.65	-28.19	-15.74	3.41	10.48	19.50	68.09	220.93
BB	17.54	20.97	1.20	249.07	105.98	1.43	9.40	-28.19	-20.95	3.65	15.05	28.07	85.03	220.88
B	13.95	24.44	1.75	246.23	115.82	1.82	10.52	-28.19	-24.56	-2.25	9.61	25.40	91.26	218.04
Below B	8.10	34.56	4.27	221.78	184.43	2.48	10.13	-28.19	-27.18	-12.09	-1.29	14.03	157.25	193.58
Panel C: Leverage (in percent per annum)														
AAA	6.07	5.67	93.26	50.26	29.14	2.05	9.75	0.07	0.09	2.05	4.64	8.35	29.23	50.33
AA	10.54	8.87	84.17	82.58	40.62	1.82	7.57	0.20	0.68	4.49	7.95	13.63	41.29	82.78
A	15.81	12.00	75.92	156.24	53.57	1.43	5.77	0.05	0.62	7.11	12.31	21.71	54.20	156.29
BBB	22.60	17.33	76.67	581.80	73.02	5.19	98.42	0.07	1.81	11.03	18.91	30.67	74.82	581.87
BB	32.07	26.72	83.33	589.93	127.10	2.88	24.48	0.10	2.21	14.38	24.62	42.38	129.31	590.02
B	57.81	46.56	80.53	1302.29	217.76	3.09	25.34	0.89	4.40	26.73	46.65	76.30	222.16	1303.18

This table shows summary statistics and quantiles of firm fundamentals across S&P credit ratings. CV stands for Coefficient of Variation and is defined as the ratio of the standard deviation to the mean. Range is the maximum minus the minimum. Panels A, B and B display statistics for asset volatility, asset drift and leverage respectively. The first column presents the rating categories considered. μ_V, the drift of asset returns (in percent per annum), is calculated for each firm and year as the mean of daily asset returns multiplied by 252. σ_V, the instantaneous volatility of asset returns (in percent per annum), is calculated for each firm as the standard deviation of the daily asset return multiplied by the square root of 252. Leverage is short term debt plus half long term debt divided by asset value.

Table 3.XII also shows that distributions are highly overlapping. We often think of the median asset volatility of a given rating category to be exclusively typical to firms inside that category. However, the overlapping distributions contradict this common assertion. For instance, the median asset volatility for BBB rating category is 28.68% in our sample. It is true that this asset volatility is typical to BBB firms but it is not typical to BBB firms only. This value of 28.68% is close to the center of the distribution of AA and A rating categories also. In other words, a significant proportion of AA firms operate “riskier” businesses than typical BBB firms. And a significant proportion of BBB rated businesses are “safer” than AA rated firms.

The large dispersions, skewnesses, kurtoses and the highly overlapping distributions suggest that the assumption of homogeneity can hardly be justified.

Panel A of Table 3.XIII shows the share of default spread in the yield spreads using estimated parameters from the standard approach and the HBF estimator. Thanks to our large sample size, standard errors are very small so that they can be ignored. The table indicates that credit spreads estimated from the standard approach represent only a tiny fraction of AAA, AA and A bond spreads. For instance for AA bonds, of the 65 basis points observed spreads at the 4 year horizon, only 4 basis points are attributable to credit risk. The measured share of default spread increases when bond quality decreases. For BBB bonds, which form the bulk of investment grade bonds, the estimated share of default risk in the spreads is 58% at the 4 year horizon and 55% at the 10 year horizon. For junk bonds, the standard approach estimates of the default spreads come very close to the total observed average spreads. For BB bonds, the estimated share of default risk in the spreads is 85% at the 4 year horizon and 77% at the 10 year horizon. For B bonds, it seems that investors are not charging enough spreads for the default risk that they incur. The estimated default spread for B bonds from the standard approach is 81% and 28% higher than observed average B spreads at the 4 year and 10 year horizons respectively.

The problem with the estimated shares of default spreads from the standard approach is the risk that the low shares for very high quality bonds may be due to an underestimation of expected default losses by the model. Similarly there is a risk that the higher shares for BBB bonds and junk bonds be due to an overestimation of the expected default losses by the model. To check this, we calculate the average physical probability of default implied by the estimated parameters of the first stage. Table 3.XIV compares this average default rate to the historical average. We can see that the standard approach estimates of default rates are quite different from historical estimates. The default rates implied by the standard approach estimates represent only 20% and 60% of historical average default rates of AAA and AA bonds at the 4 year horizon. This implies that the standard approach understates the share of default risk in the spreads for AAA, and AA bonds at the 4 year horizon. Apart from the AAA and AA bonds at the 4 year horizon, the default rates estimated from the standard approach are much higher than their historical counterpart. For instance, for BBB bonds, the default rates estimated from the standard approach are more than double the corresponding historical default rates. This implies that the standard approach overstates the share of default risk in the spreads for A rated bonds and lower quality bonds.

Table 3.XIII: Estimated share of default risk in the spreads

Rating	Spread (basis points)		Historical spread (basis points)		Default size		Default sizes of Huang and Huang [33]	
	T=4	T= 10	T=4	T= 10	T=4	T= 10	T=4	T= 10
Panel A: Spreads estimated from the standard approach								
AAA	0	1	55	63	0%	2%	2%	16%
AA	4	7	65	91	5%	8%	9%	16%
A	16	27	96	123	17%	22%	10%	19%
BBB	92	106	158	194	58%	55%	20%	29%
BB	272	247	320	320	85%	77%	54%	60%
B	849	607	470	470	181%	129%	95%	83%
Panel B: Spreads from our two-stage approach								
AAA	1	1	55	63	1%	2%	2%	16%
AA	5	5	65	91	8%	5%	9%	16%
A	11	14	96	123	11%	11%	10%	19%
BBB	62	66	158	194	39%	34%	20%	29%
BB	159	185	320	320	50%	58%	54%	60%
B	447	376	470	470	95%	80%	95%	83%

Table 3.XIV: Comparison of physical default rates estimated from the standard approach and historical default rates

Rating	Model DP from standard approach		Historical DP		Ratio	
	T=4	T= 10	T=4	T= 10	T=4	T= 10
AAA	0.05%	0.83%	0.24%	0.74%	0.2	1.1
AA	0.14%	1.19%	0.24%	0.84%	0.6	1.4
A	0.75%	3.33%	0.43%	1.59%	1.7	2.1
BBB	3.46%	9.21%	1.53%	4.33%	2.3	2.1
BB	11.10%	18.47%	6.29%	14.39%	1.8	1.3
B	31.12%	40.11%	16.99%	26.97%	1.8	1.5

3.5.2 Our two-stage approach estimates

As shown in section 3.2, with our two-stage estimation approach, the risk of default spread underestimation or overestimation is minimized by the requirement in the second stage that at each maturity and for every rating category, the model average default rate exactly matches its historical counterpart.

3.5.2.1 Implied leverages

Recall that the first stage of our approach is the same as the standard approach. Here, since our historical default data is at the rating category level, the second stage consists of solving Equation 3.9 for each maturity and rating category. This second stage gives us the implied leverages from historical default rates.

We emphasize that the obtained implied leverages should not be interpreted as being more accurate estimations of actual leverages than the leverages of the first stage. They should be regarded simply as an econometric artifice to stem the repercussions of potential biases in the estimation on the obtained share of default risk and should be interpreted relatively to the first stage leverages. If implied leverages are greater (resp. lower) than first stage leverages, it means that the standard estimation approach underestimates (resp. overestimates) historical default rates.

We therefore expect the implied leverages to be higher than the estimated leverages from the standard approach for all rating categories and maturities except for the AAA and AA categories at the 4 year horizon where we have found that the standard approach underestimates the historical default rates. Implied leverages are shown in Table 3.XV. As expected, the table shows that the implied leverages are smaller than the standard approach leverages except for the AAA category at the 4 year horizon. For instance, for the BBB category, the estimated leverage from the standard approach is 22.6% and the implied leverages are 14.14% and 10.43% at the 4 and 10 year horizons respectively.

Table 3.XV: Summary statistics of implied leverages from historical default rates and our two-stage estimation approach

Rating	mean	sd	CV	range	p99-p1	skewness	kurtosis	min	p1	p25	p50	p75	p99	max
Panel A: Leverage at 4 year maturity														
AAA	8.99	8.39	0.93	74.40	43.13	2.05	9.75	0.11	0.13	3.04	6.87	12.36	43.27	74.51
AA	12.36	10.40	0.84	96.80	47.61	1.82	7.57	0.24	0.79	5.26	9.32	15.98	48.40	97.04
A	12.95	9.83	0.76	128.03	43.90	1.43	5.77	0.04	0.51	5.83	10.09	17.79	44.41	128.08
BBB	14.14	10.84	0.77	364.04	45.69	5.19	98.42	0.04	1.13	6.90	11.83	19.19	46.82	364.08
BB	20.75	17.29	0.83	381.69	82.24	2.88	24.48	0.06	1.43	9.30	15.93	27.42	83.67	381.75
B	28.69	23.10	0.81	646.25	108.06	3.09	25.34	0.44	2.18	13.26	23.15	37.86	110.24	646.70
Below B	59.80	71.60	1.20	1030.90	382.33	4.56	35.52	1.16	3.11	22.28	38.99	71.49	385.45	1032.00
Panel B: Leverage at 10 year maturity														
AAA	5.69	5.30	0.93	47.06	27.28	2.05	9.75	0.07	0.08	1.92	4.35	7.81	27.37	47.13
AA	8.45	7.11	0.84	66.17	32.55	1.82	7.57	0.16	0.54	3.59	6.37	10.92	33.09	66.34
A	9.27	7.04	0.76	91.62	31.41	1.43	5.77	0.03	0.37	4.17	7.22	12.73	31.78	91.65
BBB	10.43	7.99	0.77	268.40	33.68	5.19	98.42	0.03	0.83	5.09	8.72	14.15	34.52	268.44
BB	21.65	18.04	0.83	398.23	85.80	2.88	24.48	0.07	1.49	9.71	16.62	28.61	87.29	398.29
B	22.63	18.22	0.81	509.65	85.22	3.09	25.34	0.35	1.72	10.46	18.26	29.86	86.94	510.00
Below B	31.43	37.63	1.20	541.75	200.92	4.56	35.52	0.61	1.64	11.71	20.49	37.57	202.56	542.36

This table shows summary statistics and quantiles of implied leverage ratios across S&P credit ratings at different maturities. CV stands for Coefficient of Variation and is defined as the ratio of the standard deviation to the mean. Range is the maximum minus the minimum.

3.5.2.2 Equity fit

The standard estimation approach ensures that equity prices are exactly matched, at the cost of an imprecision on predicted default rates. On the other hand, by requiring that estimated default rates match observed default rates, we loose the ability to match equity prices exactly. Clearly a choice has to be made on whether fitting equity prices is more important than matching default probabilities.

There are two reasons why we choose to match default probabilities at the cost of loosing the perfect fit of equity prices. First, our goal is to measure the share of default risk in the spreads. To achieve this goal, it is necessary to ensure that expected default losses are not overestimated or underestimated, as argued by Bhamra et al. [3], Chen et al. [11], Huang and Huang [33]. The other reason is that the same factors that affect bond prices, such as default risk, macroeconomic risk, liquidity risk and taxes, are also present in equity prices [32, 61]. This indicates that if a model is calibrated to capture only the default component of spreads, it would very likely also capture only the default component of equity prices. This implies that it is normal that the estimated parameters do not fit exactly equity prices, if the goal is to capture default components of the spread, which is the case here.

Nevertheless, it is important to look at the implications of our estimation approach for the fitting of equity prices. We therefore attempt to answer the following two questions: How close are the equity prices, estimated from our two-stage approach, to actual equity prices? How much of the variation in actual equity prices is explained by the variation in the equity prices estimated from our two-stage approach?

We answer the first question by computing the ratio of the difference between individual estimated equity prices and observed equity prices to observed equity prices and taking the squared root of the average squared ratio of the obtained relative differences. This is equivalent to using a RMSE measure on a relative basis instead of an absolute basis, in order to avoid that large companies bias the RMSE measure. The further this measure is from 0, the worst the estimation fit. We find that the deviation of estimated equity prices from observed equity prices is about 3% for the standard estimation approach, while it is and about 108% for our two-stage approach. This finding clearly demonstrates that the requirement to fit historical default rates has a negative impact on our ability to match equity price levels. As we argued earlier, an alternative reading of these results is that there are other risk factors than default risk that are important in explaining the level of equity prices.

To answer the second question, we compute the overall correlation between observed equity prices and estimated equity prices. This is equivalent to regressing observed equity prices on estimated equity prices. We find a correlation of 99.97% for the standard estimation approach and a correlation of 95.37% for our two-stage approach. This finding shows that the estimated equity prices from both the standard approach and our two-stage approach mimic very well the dynamics of observed equity prices. This implies that our two-stage approach is able to capture the observed variation in equity prices, even if it cannot capture their level.

3.5.2.3 Default spread size

Panel B of Table 3.XIII shows estimated default spreads using our two-stage approach. A general observation is that estimated default spreads from the two-stage approach are smaller than those from the standard approach, except for the AAA and AA categories where an economically insignificant increase of 1 basis point is observed. The difference between our two-stage estimated approach and the standard approach is striking for BBB and junk bonds. For BBB bonds for instance, the estimated default spread from the two-stage approach is 62 basis points at the 4 year horizon and 66 basis points at the 10 year horizon. This represents 39% and 34% of observed BBB spreads at the 4 year and 10 year horizons respectively. The reduction from the standard approach estimates is 30 and 40 basis points at the 4 year and 10 year horizons respectively. For B rated bonds the estimated default spread is reduced by 402 basis points and 231 basis points at the 4 year and 10 year horizons respectively. This brings the estimated share of default risk in the spreads for B rated bonds from 181% (129%) down to 95% (80%) at the 4 (10) year horizon.

3.6 Robustness checks

We now study the robustness of the estimated shares of default risk in the spreads by our two-stage approach with respect to some key assumptions of our work. We consider the following cases:

- Case 1: Average historical default probability is increased by 50%.
- Case 2: Heterogeneity in recovery rates is increased by attributing truncated normally distributed random recovery rates to firms inside rating categories while still matching average historical recovery rates.
- Case 3: The model default rate for each individual bond is calibrated to match the historical default rate for the corresponding rating category and horizon.
- Case 4: Leverages are fixed to our sample average and asset volatilities are adjusted to match historical default rates.
- Case 5: Leverages are fixed to the same average as Huang and Huang [33] and asset volatilities are adjusted to match historical default rates.
- Case 6: The sample period is restricted to 1985-1998, the same as Huang and Huang [33].
- Case 7: An advanced structural model: Leland and Toft [43] model instead of the Merton model

For clarity and space parsimony, we only present for each case the estimated share of default risk in the spreads by our two-stage approach. The results are shown in Table 3.XVI. The base case scenario is the one used in the earlier part of our study.

3.6.1 Sensitivity to historical default rates

A key parameter of our analysis is the average historical default rate. This parameter is not measured with certainty. Keenan et al. [40] provide some estimates of the volatility of one year default rates and ten year default rates by rating categories. At the one year horizon, the volatility of default rates ranges from 0% for AAA bonds to 0.28% for BBB bonds and 4.99% for B rated bonds. At the 10 year horizon, the volatility of default rates ranges from 1.93% for AAA bonds to 10.23% for BBB bonds and 15.94% for B rated

Table 3.XVI: Robustness checks

rating	base case	case 1	case 2	case 3	case 4	case 5	case 6	case 7
				Maturity = 4 years				
AAA	1%	2%	1%	0%	4%	4%	0%	1%
AA	8%	11%	9%	5%	8%	6%	3%	7%
A	11%	15%	11%	11%	9%	3%	11%	8%
BBB	39%	47%	39%	38%	22%	3%	56%	22%
BB	50%	72%	51%	109%	34%	16%	120%	39%
B	95%	143%	98%	140%	55%	55%	164%	73%
				Maturity = 10 years				
AAA	2%	3%	2%	7%	2%	0%	0%	1%
AA	5%	8%	6%	19%	5%	1%	3%	5%
A	12%	16%	12%	37%	6%	0%	11%	9%
BBB	34%	43%	34%	74%	12%	0%	42%	14%
BB	58%	95%	59%	151%	48%	30%	94%	44%
B	80%	131%	82%	136%	50%	43%	114%	53%

This table shows results of our robustness checks. The base case default size is the ratio of the default spread as predicted by our Merton model with bankruptcy costs to the observed spreads. The data and estimation approaches are as described in the corresponding sections of this work. Case 1: Average default probability is increased by 50%. Case 2: Heterogeneity in recovery rates is increased by attributing truncated normally distributed random recovery rates to firms inside rating category while matching average historical recovery rates. Case 3: The model default rate for each bond is calibrated to match the historical default rate for the corresponding rating category and horizon. Case 4: Leverages are fixed to the sample average and asset volatilities are adjusted to match historical default rates. Case 5: Leverages are fixed to the same average as Huang and Huang [33] and asset volatilities are adjusted to match historical default rates. Case 6: The sample period is restricted to 1985-1998, the same as Huang and Huang [33]. Case 7: Default sizes based on the Leland and Toft [43] model.

bonds. Overall, the volatility of default rates tends to decrease with credit quality and to increase with time to maturity. The authors examined cohorts of bonds from 1920 to 1998, excluding the 1950-1965 period, which makes roughly 63 cohorts. Assuming independence of the cohorts, the volatility of each sample average default rate could be taken to the volatility of the sample divided by the square root of the sample size. This yields an estimate of coefficients of variation of average default rates no greater than 17% for each rating category. These estimates show that the uncertainty about average default rates can be substantial. The true coefficient of variation could be even larger than 17% due to correlation between the cohorts. It is therefore possible that the gap between observed spreads and our estimated default spread be due to true default rates that are larger than their historical averages. To control for this possibility, we consider a worst scenario case, increasing the average historical default probabilities by 50%, and measure the bias-free default sizes. The results are shown in the column labeled "Case 1" of Table 3.XVI. As expected, the obtained default sizes are larger, but the increase is not substantial.

3.6.2 Recovery rates

We used a highly stylized approach to assign different recovery rates to firms inside the same rating category according to their economic sectors, while ensuring that at each maturity the average recovery rate matches empirical observations. One might argue that this approach does not generate enough heterogeneity of recovery rates inside rating categories, so that the effect of heterogeneity in recovery rates is underestimated.

As an attempt to deal with this issue, we consider an approach that generates larger heterogeneity in recovery rates than in our base case. We assume that for each rating category and maturity, recovery rates follow a normal distribution with mean equal to the historical average for that rating category and maturity. We arbitrarily consider a coefficient of variation of 50%. Recovery rates for individual firms are drawn from this distribution but are truncated at a maximum of 99% and a minimum of 1%. This procedure leads to much more heterogeneity in recovery rates than in our base case. Indeed, even inside the same economic sector, for the same rating and horizon, firms have different recovery rates. In addition, the assumed range of dispersion of recovery rates inside rating categories is now 98% while it was roughly 30% in our base case.

The new default sizes are displayed in the column labeled "Case 2" of Table 3.XVI. One can observe that there is no marked difference from our base case findings.

3.6.3 Alternative calibration approaches

The second stage of our estimation procedure consists of adjusting the leverage ratios in our analysis so that the model average default rates for each rating category match their corresponding historical counterpart. There may be three concerns with this approach. First we impose a uniform adjustment to leverage ratios for all firms with a given rating, while one may have preferred firm-specific adjustments. Second, we adjust leverage ratios, while one could have chosen to adjust asset volatilities. Third, the physical total leverage ratios in our model are lower than those of Huang and Huang [33], which may bias our estimates downward.

To address these concerns, we consider three alternative calibration procedures. The first alternative is to force the model default rate of each individual firms to match the historical average default rate of the rating category to which the firm belongs, thereby allowing firm-specific adjustments. The second alternative is to calibrate the asset volatility instead of leverage, in such a way that leverage is kept at our sample average. The third alternative is to consider the same average leverage ratios as Huang and Huang [33] and calibrate asset volatilities to match historical default rates. The results are shown respectively in columns labeled "Case 3", "Case 4" and "Case 5" of Table 3.XVI. The results speak for themselves. None of these alternative procedures have substantially increased the explained default size for investment grade bonds. In most cases, the model default size is even lower than in the base case.

3.6.4 Period of study

Our model estimates of default spreads are based on the period 1984 to 2013. However, we compare these models spreads with historical spreads estimated by Huang and

Huang [33]. These historical estimates are based on the period 1973-1993 for 10 year investment grade bonds and on the period 1985-1998 for 4 year investment grade bonds. Despite the fact that there is little disagreement between different measures of historical average spreads (see Table 4 of Bhamra et al. [3]) over different time periods, it could be the case that this difference in time frames bias our measured default sizes downward. To account for this possibility, we reconducted our study over the period 1985-1998, to be consistent for the time frame for 4 year investment grade bonds. The results are shown in the column labeled "Case 6" of Table 3.XVI. The new default sizes are smaller for high quality bonds and are larger for lower quality bonds than in the base case, but there is no marked change for investment grade bonds in general.

3.6.5 Is it specific to the Merton Model?

To investigate whether our findings would change if we would consider more elaborated structural models, we consider here the Leland and Toft [43] model, which is one of the most successful extensions of the Merton Model. Some particularities of this model relatively to the Merton Model are the following. Default occurs endogenously once the firm value falls below an optimally determined level. There are corporate taxes. The firm continuously pays a coupon and replaces maturing debt with new debt at par. The model also accounts for payout rates and tax benefits of debt, and generates heterogeneity in recovery rates. Following Elkamhi et al. [24], we choose the coupon rate to be equal to the risk free rate times the debt principal. Following Leland [42] we choose the tax rate $\pi = 15\%$, the asset payout rate $\zeta = 6\%$, and the bankruptcy recovery parameter $\alpha = 70\%$. Asset value and volatility are the same as in the Merton model, so the first stage of the estimation does not change. The second stage of the estimation is also very similar, with the difference that the model-implied physical default probabilities are from the Leland and Toft [43]. Average spreads based on this advanced structural model are given in the column labeled "Case 7" of Table 3.XVI. We can see that the share of default risk in the spreads is lower than in our base case, but that the difference is not substantial.

3.7 Further discussions

3.7.1 Importance of heterogeneity in understanding the spreads

An advantage of the approach proposed in this essay is that it allows to account for firms heterogeneity when aggregating at the group level. In fact, the rating level average spreads in the above results are obtained by averaging individual spreads in both the cross-section and the time series. As a consequence, our results reported above are free of any heterogeneity bias.

Now assume that each rating level average spread is measured as the model spread for the firm whose fundamentals are the average fundamentals observed in the rating category of interest, which would produce an heterogeneity bias. This specific bias was called the ATA bias in Chapter 2. Table 3.XVII reports the ATA bias in absolute terms. Consistently with our results of Chapter 2, we expect this bias to be positive. This is because the distributions of firm parameters in rating categories are positively skewed, so that the convexity region of the spread function outweighs its concavity region. In Panel A of Table 3.XVII, we assume that recovery rates are homogeneous, while we relax this assumption in Panel B by introducing heterogeneity in recovery rates across economic sectors. These two panels allow us to study separately the impact of heterogeneity in recovery rates.

As expected, in both panels, the bias is positive. In Panel A, it ranges from 0 basis points for one-year and 2-year AAA-rated bonds to 362 basis points for 1-year B-rated bonds. The bias is larger for junk bonds than for investment grade bonds. The bias is typically around 5 basis points for high quality bonds and typically above 100 basis points for junk bonds. For investment grade bonds, the bias is larger at long maturities than at short maturities. For low quality bonds, the bias is larger at short maturities than at long maturities. For BBB-rated bonds the bias is no larger than 59 basis points. Comparing the results of Panel A with those of Panel B, we find that heterogeneity in recovery rates has no impact on the size of the bias. Indeed we see that the difference between the results reported in Panels A and B is at most 1 basis point.

Now assume that the rating level average spread is measured as the model spread for the firm whose parameters are fixed at the categorical average except the asset volatility which is calibrated to match historical default rates. This would produce a different heterogeneity bias that we called the HH bias in Chapter 2. Panels A and B of Table 3.XVIII display the HH bias in absolute terms without and with heterogeneous recovery rates respectively.

Based on our results of Chapter 2, we expect that these biases could be positive or negative, and this is what we observe in Table 3.XVIII. The biases range from -154 to 33 basis points. As we argued in Chapter 2, the reason of the negative bias is that the calibrated asset volatilities in the HH estimator may be so large that it causes an overestimation of the average model spread. This intuition is confirmed empirically here in Table 3.XIX. Also, consistently with our theoretical findings in Chapter 2, the HH biases are much smaller in absolute value than the ATA biases. By comparing Panels A and B, we see that in this case also heterogeneity in recovery rates does not have an important role.

Another way of gauging the role of heterogeneity is to compare our measured shares of default spread in the total spread to the measures of Huang and Huang [33]. Indeed the main difference in our study is that we account for heterogeneity while they do not. This comparison is done in Panel B of Table 3.XIII. The default spread sizes of Huang and Huang [33] are reported in the last column. At the 4 year horizons, our default spread sizes are pretty much similar except for the BBB rating category [8]. In fact, our measured default spread sizes are twice as large as those of Huang and Huang [33] for BBB bonds. At the 10 year horizon, our measured default sizes are lower than those of Huang and Huang [33] for high quality AAA, AA and A bonds, but are higher for BBB bonds. These results are important because they show that heterogeneity can have a significant impact on the perceived importance of default risk in the spreads.

8. This may be because the BBB category is the most heterogeneous one in our sample.

Table 3.XVII: Convexity bias

	Maturity (Years)			
Rating	1	2	4	10
Panel A: Homogeneous recovery				
AAA	0	0	1	1
AA	4	6	5	5
A	6	7	10	12
BBB	34	52	59	56
BB	75	109	93	46
B	362	266	132	98
Panel B: Heterogeneous recovery				
AAA	0	0	1	1
AA	4	6	5	5
A	7	7	11	12
BBB	34	52	60	57
BB	77	110	94	47
B	359	265	132	99

Table 3.XVIII: Heterogeneity bias

	Maturity (Years)			
Rating	1	2	4	10
Panel A: Homogeneous recovery				
AAA	0	-2	-9	-24
AA	2	1	-4	-19
A	-2	-3	-9	-28
BBB	16	19	5	-29
BB	5	-30	-94	-154
B	33	-22	-55	-61
Panel B: Heterogeneous recovery				
AAA	0	-2	-9	-24
AA	2	1	-4	-19
A	-1	-2	-9	-28
BBB	16	19	6	-29
BB	6	-28	-93	-154
B	30	-23	-55	-61

Table 3.XIX: Typical firm: Calibrated and uncalibrated asset volatilities

	Calibrated asset volatility to historical default rates(%)				
	Maturity (years)				
Rating	1	2	4	10	Average asset volatility(%)
AAA	25.08	51.35	45.47	43.43	25.08
AA	59.05	46.55	40.21	39.11	25.39
A	56.88	46.87	41.83	41.14	27.73
BBB	79.48	55.23	47.05	46.13	31.32
BB	74.51	60.52	54.72	54.12	40.31
B	74.59	62.96	58.63	58.95	49.71

3.7.2 Is the credit spread puzzle a myth?

Our findings contrast with those of Feldhutter and Schaefer [29], who find that the default spread explains all the spread after correcting for the heterogeneity bias. Their finding implies that the credit spread puzzle is due to a statistical error, while our findings imply the opposite. Although we use the same model to capture the default spread, our estimation methodologies and the data that we use are different, as explained in Sections 3.1 and 3.4.2. This explains the difference in our findings.

First, while our calibration approach consists of solving an equation that we demonstrate to have a unique solution, their calibration approach is a minimization problem and their obtained minimal error may still be large. In fact, since it is known that the Merton model when calibrated to one particular maturity cannot fit the whole term structure, their calibration approach is likely to result in large fitting errors. In contrast, our approach ensures that the term structure of default rate for every rating category is exactly fitted. In other other words, our approach ensures that expected default losses are not over- or underestimated, contrarily to their approach. Their larger spreads might therefore reflect overestimated default losses.

Second, while we use 32 years of default rate data, they use 90 years of default data. Because the 90 years default rates are higher than the 32 years default rates, they estimate higher default losses than we do. This may justify their higher spreads. However, their 90 years default data does not reflect the normal levels of average default rates, as they include the abnormally high default rate periods of the great depression and World War II periods. Thus the higher spreads of Feldhutter and Schaefer [29] may not reflect usual market expectations about default losses.

It is worth mentioning that it is hard to reconcile the findings of Feldhutter and Schaefer [29] with the extensive literature providing empirical evidence that jump risk, volatility risk and liquidity risk are priced in the spreads [10, 15, 16, 25, 47, 62] and that risk premia is time-varying and counter-cyclical [3, 11, 13, 20]. In contrast, our findings are well aligned with this literature. This point is discussed further in the next subsection.

3.7.3 Implications for pricing

Our finding that the default spread, as measured by the Merton model with bankruptcy costs, is low relatively to the observed level of spreads has important pricing implications. On one hand, it may imply that spreads are actually explained by default risk but the model that we use to capture default risk underestimates the price of default risk. This is plausible because in the Merton model, default risk premia arise solely from asset risk augmented with the option-like nature of default. The model ignores that volatility is time-varying and that investors want to be compensated for volatility risk. The model also ignores jumps and the price of jump risk. Moreover, the model ignores that risk premia is time-varying and counter-cyclical. Clearly, incorporating these additional prices of risk in the price of default risk or making the price of default risk counter-cyclical may be helpful in finding out whether default risk solely justifies corporate bond spreads. There exist models with one or several of these features, such as those of Zhou [63], Zhang et al. [62], Cremers et al. [16], Bhamra et al. [3], Chen et al. [11, 13], David [20]. To the best of our knowledge, though the default spreads based on these models are higher than those based on the Merton model, none of these models is able to entirely explain the spreads.

The other possibility is that spreads are explained by additional factors to default risk, such as individual bond liquidity, market liquidity, tax differentials and asymmetric information. This possibility is consistent with the empirical findings of Collin-Dufresne et al. [15], Elton et al. [25], Longstaff et al. [47] and CHEN et al. [10]. There could also be interaction effects between default risk and liquidity as the works of Ericsson and Renault [27], He and Xiong [32] have shown.

Whether spreads are entirely explained by default risk or not, our study makes it clear that there is more in the spreads than just expected default losses and the default risk premia arising from asset risk. This is important for corporate finance because it implies that firms must pay attention to more than expected default losses and asset diffusion risk premia when issuing new debt. By doing so, they may be able to better anticipate their financing costs or obtain lower financing rates by adequately timing their issues. This is also important for bond market investors because it provides guidance on bond valuation models. Bond values predicted from models of default losses and asset risk premia only are likely to be wrong ex-post as they ignore other sources of default risk and the additional factors to default risk that the market cares about.

3.7.4 Implications for capital structure

One puzzling fact in corporate finance is the low leverage of firms [24]. The puzzle is that, if one balances tax benefits of debt with bankruptcy costs, most firms would seem to have an advantage in using more leverage than they do. According to He and Xiong [32], one possible answer to the low leverage puzzle is liquidity risk. By using more debt, firms expose themselves to more liquidity shocks, higher spreads, and higher rollover losses when the time comes to refinance the debt. This possibility increases bankruptcy risk and costs. With high enough expected liquidity risk and losses, the apparent advantage of higher leverage may disappear, thereby explaining the low leverage puzzle. Our findings suggest that this low-leverage puzzle may indeed be explained by the non-default components of the spreads.

3.8 Conclusion

In this study, we have attempted to capture the share of default risk in corporate bond spreads. Towards this end, we have proposed a new calibration approach of the Merton model. The approach builds on the standard calibration approach, which consists of inverting the Merton model in order to estimate asset volatility and asset value from equity volatility and equity price. The novelty in the approach is that it incorporates information on observed default probabilities in the estimation.

The proposed approach has the advantages of jointly offering the possibility of: (a) accounting for the heterogeneity of firms in the estimation of average spreads, (b) calibrating firm fundamentals to historical default rates while preserving their heterogeneity, (c) using individual firm historical default rates or group level historical default rates depending on data availability, (d) using equity prices and equity volatilities in the estimation, (e) using the Merton model which is parsimonious and very tractable.

Using simulation studies, we have shown that the proposed approach leads to much smaller estimation errors on the share of default risk in the spreads than the standard approach does.

We have applied the approach on a large sample of about 3 Millions firm days observations from 1984 to 2013. We have estimated the share of default risk in the spreads for BBB firms to be about 40% at the 4 year horizon. This share is lower for higher quality bonds and higher for lower quality bonds. The estimated shares were subjected to a battery of robustness checks regarding the period of study, the uncertainty on historical default rates, the use of more elaborated structural models than the Merton model and other assumptions in the calibration approach.

Our findings imply that heterogeneity plays an important role in the estimated shares of default risk. Indeed, our share of default risk in BBB spreads is twice as large as that obtained when heterogeneity is ignored.

Our findings also suggest that the non-default component of the spreads is large. This has important implications for bonds valuation and capital structure analysis.

We have shown that using the Leland-Toft model instead of the Merton model does not change significantly our findings. Future research may try to answer the question of whether using models with jump risk, stochastic volatility or time-varying risk premia instead of the Merton model would lead to significantly larger share of default risk in the spreads.

Another possible research avenue consists of using Moody's EDF default data as a proxy for historical individual default rates in order to perform the calibration to historical default rates at the individual level instead of at the rating category level.

Finally, our empirical analysis focused on the share of default risk in the first moments of spreads. In future works, one may also be interested in studying how much of the second moment of default spreads is explained by default risk, using our two-stage approach.

CHAPTER 4

PRICING CALLABLE BONDS UNDER THE TWO-FACTOR VASICEK MODEL

4.1 Introduction

Many corporate and government debt instruments include call or put provisions. The call provision allows the bond issuer to buy back the bond at a pre-determined price if she wishes to. The put provision gives the bond holder the right but not the obligation to sell back the bond to its issuer at a pre-determined price. Callable and putable bonds are very popular as they can be used as an effective way to manage interest rate risk [53], they can help mitigate credit risk problems [13], they offer attractable yields and they provide an easy way of trading interest rate options [28].

Just like a plain vanilla bond, a callable or putable bond is characterized by its maturity t_q and its coupon rate c that is due every Δ fraction of year (for instance Δ is equal to 0.5 for a semi-annual coupon). The coupon dates are denoted by $t_k, k = 1,2,...,q$, and the pricing date is denoted by t_0. The face value of the bond can be normalized to 1$ without loss of generality. There is an initial protection period during which the embedded option cannot be exercised. After that protection period, the call or put option can be exercised at any coupon date. We denote by t_{q^*}, the first coupon date on which the embedded option is exercisable. The option strike varies with time to maturity: It is generally above the face value for earlier exercise dates, and declines over time to reach compulsorily the face value at maturity. We denote by K_k the strike price at the coupon date t_k, $k = 1,2,...,q$. The bond holder (issuer) must announce in advance her intention to put (call) in order to exercise the option at the coming exercise date. Thus, the decision to exercise at date t must be taken at date $t-\delta$, where δ is the notification period.

From a pricing perspective, callable and putable bonds may be viewed as the combination of two securities. In fact, a long position in a callable bond is equivalent to a long position in a plain vanilla bond plus a short position in the embedded call option, while a long position in a putable bond is equivalent to a long position in a plain vanilla bond plus a long position in the embedded put option. Since the valuation of a plain vanilla bond is straightforward, the most difficult component to value in a callable or a putable bond is the embedded option. This embedded option can be viewed as a Bermudan option where the underlying asset is the plain vanilla coupon bond. In the rest of the paper, we focus on callable bonds without loss of generality.

No closed-form solution exists for the valuation of callable bonds. Traditional approaches for the valuation of callable bonds are binomial trees [39], trinomial trees [35–37] and finite differences [6]. These approaches are very popular, probably because they were the firsts to be developed, however they have several drawbacks. They generally give numerically unstable prices and they may yield negative prices [8, 22]. Numerical instability is a serious problem for users of these methods, as a slight change in some parameters of the valuation problem may lead to substantial changes in the price. This poses difficulties for the computation of Greeks. It also raises questions about the validity of the computed price because numerical instability implies too much uncertainty on the calculated price. In fact, parameters are generally estimated with some degree of error and it is common practice to consider a price range over the range of plausible values of the parameters instead of a single price. With numerical instability, a slight error of estimation would translate into an undesirably wide price range.

Developing stable numerical algorithms for the computation of callable bond values has attracted the attention of many researchers including Ben-Ameur et al. [2], Buttler and Waldvogel [8], D'Halluin et al. [22]. Buttler and Waldvogel [8] propose a stable dynamic program algorithm based on Green's function, but they acknowledged that their approach is limited to special one-factor interest rate models with constant parameters, namely, the Vasicek and the CIR models, for which analytical expressions of the Green functions are available. Also for the Vasicek and CIR models, D'Halluin et al. [22] propose a stable finite difference algorithm using advanced methods, such as the Crank-Nicholson scheme and flux limiters. Ben-Ameur et al. [2] develop a stable dynamic program algorithm using piecewise linear interpolation, which they apply to the Vasicek model, the CIR model, the extended Vasicek model and the extended CIR model.

It is well established in the literature that the term structure of interest rates is driven by several factors [19, 44]. However, developing stable pricing algorithms for the pricing of Bermudan type interest rate derivatives in general and callable bonds in particular under multiple factor interest rate models has been a largely unexplored field. Under multiple factor models, Bermudan options, including the embedded option in callable bonds, may be valued using simulation techniques but simulations have the disadvantage of yielding estimation error relatively too large for statistical inference, for reasonable number of paths [59].

This essay proposes a dynamic programming algorithm for the valuation of callable bonds under the two-factor Vasicek model. This model has the advantage to have multiple factors, to be analytically tractable and to allow correlation between the factors. Dai and Singleton [19] find that factor correlations play a very important role in explaining interest rate dynamics. The proposed method is based on the Clenshaw-Curtis integration scheme and Chebyshev series expansion. Although the method is applied to the two-factor Vasicek model in this essay, it is applicable to any multi-factor model, as long as the distribution of factors under the forward measure is known in closed form or may be efficiently calculated using numerical methods.

The rest of this essay is organized as follows. Section 4.2 presents the valuation model and derives the distribution of the factors under the forward measure. Section 4.3 presents our numerical solution method for the valuation problem. Section 4.4 presents numerical illustrations of our approach and section 4.5 concludes.

4.2 Valuation Model

4.2.1 The dynamic program

Let,

X_t be the vector of the underlying interest rate factors process under the risk-neutral measure,

x denote a vector of the underlying interest rate factors values,

$r(x)$ be the risk-free short rate when the values of the underlying factors are given by x,

$B_t(x,T)$ be the value at x and time t of a zero coupon bond maturing at time T,

$C_t(x)$ be the callable bond value at time t when the values of the underlying factors are given by x,

$H_t(x)$ be the holding value (ex-coupon) of the callable bond i.e the value at x and t of the callable bond if its not called until the next call date, excluding the upcoming coupon.

At maturity, the callable bond is worth its face value plus the coupon:

$$C_{t_q}(x) = 1 + c.$$

It immediately follows that the value of the callable bond at the last decision date $t_q - \delta$ is given by:

$$C_{t_q-\delta}(x) = (1+c)\,B_{t_q-\delta}\left(x, t_q\right). \tag{4.1}$$

Now, assume that for a given value of $k = q^* + 1, \ldots, q$ the callable bond value at time $t_k - \delta$ is known.

One period earlier, at time $t_{k-1} - \delta$, the callable bond value is the minimum between its holding value and the discounted call strike,

$$C_{t_{k-1}-\delta}(x) = \min\left(H_{t_{k-1}-\delta}(x), K_{k-1}B_{t_{k-1}-\delta}(x, t_{k-1})\right) + cB_{t_{k-1}-\delta}(x, t_{k-1}). \tag{4.2}$$

The holding value is the conditional expected value under the risk neutral measure of the discounted callable bond value at the next call date,

$$H_{t_{k-1}-\delta}(x) = \mathbb{E}^{Q}_{t_{k-1}-\delta,x}\left[\exp\left(-\int_{t_{k-1}-\delta}^{t_k-\delta} r(X_s)\,ds\right) C_{t_k-\delta}(X_{t_k-\delta})\right],$$

where $\mathbb{E}^{Q}_{t,x}[.]$ denotes the expectation operator under the risk neutral measure Q, conditional on the information that the values of the underlying factors at time t are given by the vector x.

The holding value can also be evaluated under the forward measure Q^{Δ}. Under this measure, $B_{t_{k-1}-\delta}\left(x,t_k-\delta\right)$ is used as a numeraire. This implies using the change of numeraire results of Geman et al. [30], that is:

$$H_{t_{k-1}-\delta}\left(x\right)=B_{t_{k-1}-\delta}\left(x,t_k-\delta\right)\mathbb{E}_{t_{k-1}-\delta,x}^{Q^{\Delta}}\left[C_{t_k-\delta}(X_{t_k-\delta})\right]. \tag{4.3}$$

At the initial pricing date, the value of the callable bond is simply the sum of the discounted values of the coupon payments to be received during the protection period and the expected discounted value of the callable bond at the first decision date,

$$C_{t_0}\left(x\right)=c\sum_{k=1}^{q^*-1}B_{t_0}\left(x,t_k\right)+\mathbb{E}_{t_0,x}^{Q}\left[\exp\left(-\int_{t_0}^{t_{q^*}-\delta}r\left(X_s\right)ds\right)C_{t_{q^*}-\delta}(X_{t_{q^*}-\delta})\right],$$

Let $d_0=t_{q^*}-\delta-t_0$. Using $B_{t_0}\left(x,t_{q^*}-\delta\right)$ as a numeraire, the initial callable bond price for is given by

$$C_{t_0}\left(x\right)=c\sum_{k=1}^{q^*-1}B_{t_0}\left(x,t_k\right)+B_{t_0}\left(x,t_{q^*}-\delta\right)\mathbb{E}_{t_0,x}^{Q^{d_0}}\left[C_{t_{q^*}-\delta}(X_{t_{q^*}-\delta})\right]. \tag{4.4}$$

4.2.2 Interest rate model

4.2.2.1 The Vasicek (VAS) model

The Vasicek model was proposed by Vasicek [60]. In this model, it is assumed that the short rate is governed by a single stochastic factor. The single factor follows a mean reverting affine diffusion process given by:

$$\begin{aligned} dX_t &= a\left(\bar{x}-X_t\right)dt+\nu dW_t, \qquad (4.5)\\ r(x) &= x, \end{aligned}$$

where a is the parameter controlling the speed of mean reversion, $\bar{x}$ is the long run average and ν is the volatility parameter.

Bond prices under the Vasicek model are given by

$$B_t^{VAS}(x, t+\tau_1) = \exp\left(-\bar{x}\tau_1 - \frac{1-e^{-a\tau_1}}{a}(x-\bar{x}) - \frac{\nu^2}{4a^3}\left(-3+2a\tau_1+4e^{-a\tau_1}-e^{-2a\tau_1}\right)\right). \tag{4.6}$$

Advantages and drawbacks of the Vasicek model have been listed by Brigo and Mercurio [7]. One the positive side, the model is simple, analytically tractable and captures the observed mean reversion of interest rates. On the negative side, short rates may be negative, the volatility is constant and the model cannot generate term structures observed on the market. There are many other one-factor interest rate models that improve the Vasicek model in many respects, such as the CIR model, the Dothan model, the Extended Vasicek model, the Extended CIR model, the Black-Karasinski model and the Exponential Vasicek model (see Brigo and Mercurio [7] for a review of these models). However, all one factor models have the undesirable feature that rates of all maturities are perfectly correlated and are constrained to move in the same direction as the short rate [52]. Moreover, there is an imposing literature that shows that the term structure of interest rates is driven by at least two factors [19, 44].

4.2.2.2 The two-factor Vasicek (2FV) Model

In this model, the interest rate is the sum of two factors X_{1t}, X_{2t}:

$$\begin{aligned} dX_t &= a(\bar{x}-X_t)\,dt + \nu dW_t, \\ r(x) &= 1'x, \end{aligned} \tag{4.7}$$

where

$$
\begin{aligned}
X_t &= \begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix}, a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \\
\bar{x} &= \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}, dW_t = \begin{pmatrix} dW_{1t} \\ dW_{2t} \end{pmatrix} \\
v &= \begin{pmatrix} v_1 & 0 \\ v_2\eta & v_2\sqrt{1-\eta^2} \end{pmatrix}, 1' = \begin{pmatrix} 1 & 1 \end{pmatrix},
\end{aligned}
$$

a_1, a_2 are the two mean reversion parameters, v_1, v_2 are the volatility parameters, $\bar{x}_1$, $\bar{x}_2$ are the unconditional average of each factor, η is the correlation parameter between the two factors, and W_{1t} and W_{2t} are two independent Brownian motions.

This two-factor model has the advantage of generating a wide variety of term structures including those observed on the market [7]. It is consistent with the empirical evidence that multiple factors are required to explain the short rate dynamics. Importantly, it can capture the correlation between the two-factors. This correlation plays an important role in explaining the short rate dynamics [19]. Moreover, the model is still analytically tractable, contrarily to the two-factor CIR model where analytical tractability is lost when there is correlation between the factors.

The two-factor Vasicek model may still generate negative rates, but some authors have argued that negative rates are not a very important problem. For instance, Longstaff and Schwartz [45] argue that this type of model impacts prices mainly through the expected value of the short rate, which is positive. In addition, even if the short rate may be theoretically negative, the probability of this happening is very close to zero for reasonable values of the parameters [7].

Bond prices under this model are given by:

$$B_t^{2FV}(x,t+\tau_1)=\exp\left(\begin{array}{c} -\bar{x}_1\tau_1-\frac{1-e^{-a_1\tau_1}}{a_1}(x_1-\bar{x}_1)-\frac{\nu_1^2}{4a_1^3}\left(-3+2a_1\tau_1+4e^{-a_1\tau_1}-e^{-2a_1\tau_1}\right)\\ -\bar{x}_2\tau_1-\frac{1-e^{-a_2\tau_1}}{a_2}(x_2-\bar{x}_2)-\frac{\nu_2^2}{4a_2^3}\left(-3+2a_2\tau_1+4e^{-a_2\tau_1}-e^{-2a_2\tau_1}\right)\\ -\frac{\eta\nu_1\nu_2}{a_1a_2}\left(\tau_1-\frac{1-e^{-a_1\tau_1}}{a_1}-\frac{1-e^{-a_2\tau_1}}{a_2}+\frac{1-e^{-(a_1+a_2)\tau_1}}{a_1+a_2}\right)\end{array}\right). \tag{4.8}$$

The distribution of the factors under the forward measure is given by the following proposition.

Proposition 4.2.1. *Under the forward measure Q^T, conditional on the value at time u of the factors, the two factors of the 2FG model follow a bivariate normal distribution with mean and covariance matrix given by:*

$$\begin{aligned}
\mathbb{E}_{u,x_1}^T[X_{1t}] &= x_1e^{-a_1(t-u)}+\bar{x}_1\left(1-e^{-a_1(t-u)}\right)\\
&\quad-\frac{\nu_1^2}{a_1^2}\left(\left(1-e^{-a_1(t-u)}\right)-\frac{e^{-a_1(T-t)}}{2}\left(1-e^{-2a_1(t-u)}\right)\right)\\
&\quad-\frac{\eta\nu_1\nu_2}{a_2}\left(\frac{1}{a_1}\left(1-e^{-a_1(t-u)}\right)-\frac{e^{-a_2(T-t)}}{a_1+a_2}\left(1-e^{-(a_1+a_2)(t-u)}\right)\right),\\
\mathbb{E}_{u,x_2}^T[X_{2t}] &= x_2e^{-a_2(t-u)}+\bar{x}_2\left(1-e^{-a_2(t-u)}\right)\\
&\quad-\frac{\nu_2^2}{a_2^2}\left(\left(1-e^{-a_2(t-u)}\right)-\frac{e^{-a_2(T-t)}}{2}\left(1-e^{-2a_2(t-u)}\right)\right)\\
&\quad-\frac{\eta\nu_1\nu_2}{a_1}\left(\frac{1}{a_2}\left(1-e^{-a_2(t-u)}\right)-\frac{e^{-a_1(T-t)}}{a_1+a_2}\left(1-e^{-(a_1+a_2)(t-u)}\right)\right),\\
Var_{u,x}^T[X_t] &= \begin{pmatrix} \frac{\nu_1^2}{2a_1}\left(1-e^{-2a_1(t-u)}\right) & \frac{\eta\nu_1\nu_2}{a_1+a_2}\left(1-e^{-(a_1+a_2)(t-u)}\right)\\ \frac{\eta\nu_1\nu_2}{a_1+a_2}\left(1-e^{-(a_1+a_2)(t-u)}\right) & \frac{\nu_2^2}{2a_2}\left(1-e^{-2a_2(t-u)}\right)\end{pmatrix}.
\end{aligned}$$

Proof. It is well known that the change of measure from the risk neutral measure to the forward measure involves a predictable process γ_t and a new multidimensional Brownian motion W_t^T such that :

$$dW_t = dW_t^T + \gamma_t dt.$$

Chen and Huang [12] have shown that :

$$\gamma_t = -\nu' \frac{1}{B_t(x,T)} \frac{\partial B_t(x,T)}{\partial x}.$$

Applying this result of Chen and Huang [12] to the two-factor Vasicek case, it is straightforward to find that :

$$\begin{aligned}
\gamma_t &= -\begin{pmatrix} \nu_1 & \nu_2\eta \\ 0 & \nu_2\sqrt{1-\eta^2} \end{pmatrix} \begin{pmatrix} \frac{1-e^{-a_1(T-t)}}{a_1} \\ \frac{1-e^{-a_2(T-t)}}{a_2} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{\nu_1}{a_1}\left(1-e^{-a_1(T-t)}\right) - \nu_2\eta\frac{1-e^{-a_2(T-t)}}{a_2} \\ -\frac{\nu_2\sqrt{1-\eta^2}}{a_2}\left(1-e^{-a_2(T-t)}\right) \end{pmatrix}.
\end{aligned}$$

We now proceed to find the dynamics of the two-factors under the forward measure.

$$\begin{aligned}
\begin{pmatrix} dX_{1t} \\ dX_{2t} \end{pmatrix} &= \begin{pmatrix} a_1(\bar{x}_1 - X_{1t}) \\ a_2(\bar{x}_2 - X_{2t}) \end{pmatrix} dt \\
&\quad + \begin{pmatrix} \nu_1 & 0 \\ \nu_2\eta & \nu_2\sqrt{1-\eta^2} \end{pmatrix} \times \\
&\quad \begin{pmatrix} dW_{1t}^T - \frac{\nu_1}{a_1}\left(1-e^{-a_1(T-t)}\right)dt - \nu_2\eta\frac{1-e^{-a_2(T-t)}}{a_2}dt \\ dW_{2t}^T - \frac{\nu_2\sqrt{1-\eta^2}}{a_2}\left(1-e^{-a_2(T-t)}\right)dt \end{pmatrix}.
\end{aligned}$$

Focusing on the first factor, we find that :

$$\begin{aligned}
dX_{1t} &= a_1(\bar{x}_1 - X_{1t})dt - \frac{\nu_1^2}{a_1}\left(1-e^{-a_1(T-t)}\right)dt \\
&\quad - \frac{\nu_1\nu_2\eta}{a_2}\left(1-e^{-a_2(T-t)}\right)dt + \nu_1 dW_{1t}^T.
\end{aligned}$$

It is useful to make the change of variable $Z_{1t} = e^{a_1 t} X_{1t}$.

Taking the differential of Z_{1t} we find that :

$$
\begin{aligned}
dZ_{1t} &= e^{a_1 t} dX_{1t} + a_1 e^{a_1 t} X_{1t} dt, \\
&= e^{at} \left(a_1 \left(\bar{x}_1 - X_{1t} \right) dt - \frac{\nu_1^2}{a_1} \left(1 - e^{-a_1 (T-t)} \right) dt \right) + \\
&\quad e^{at} \left(- \frac{\nu_1 \nu_2 \eta}{a_2} \left(1 - e^{-a_2 (T-t)} \right) dt + \nu_1 dW_{1t}^T \right) + a_1 e^{a_1 t} X_{1t} dt, \\
&= a_1 \bar{x}_1 e^{a_1 t} dt - \frac{\nu_1^2}{a_1} \left(e^{a_1 t} - e^{-a_1 T} e^{2 a_1 t} \right) dt \\
&\quad - \frac{\nu_1 \nu_2 \eta}{a_2} \left(e^{a_1 t} - e^{-a_2 T} e^{(a_1 + a_2) t} \right) dt + \nu_1 e^{a_1 t} dW_{1t}^T .
\end{aligned}
$$

Integrating both sides of the above equation, we find that :

$$
\begin{aligned}
Z_{1t} - Z_{1u} &= \bar{x}_1 \left(e^{a_1 t} - e^{a_1 u} \right) - \frac{\nu_1^2}{a_1^2} \left(\left(e^{a_1 t} - e^{a_1 u} \right) - \frac{e^{-a_1 T}}{2} \left(e^{2 a_1 t} - e^{2 a_1 u} \right) \right) \\
&\quad - \frac{\nu_1 \nu_2 \eta}{a_2} \left(\frac{1}{a_1} \left(e^{a_1 t} - e^{a_1 u} \right) - \frac{e^{-a_2 T}}{a_1 + a_2} \left(e^{(a_1 + a_2) t} - e^{(a_1 + a_2) u} \right) \right) \\
&\quad + \nu_1 \int_u^t e^{a_1 s} dW_{1s}^T, \\
&= \bar{x}_1 e^{a_1 t} \left(1 - e^{-a_1 (t-u)} \right) - \frac{\nu_1^2 e^{a_1 t}}{a_1^2} \left(\left(1 - e^{-a_1 (t-u)} \right) - \frac{e^{-a_1 (T-t)}}{2} \left(1 - e^{-2 a_1 (t-u)} \right) \right) \\
&\quad - \frac{\nu_1 \nu_2 \eta e^{a_1 t}}{a_2} \left(\frac{1}{a_1} \left(1 - e^{-a_1 (t-u)} \right) - \frac{e^{-a_2 (T-t)} e^{a_2 t}}{a_1 + a_2} \left(1 - e^{-(a_1 + a_2)(t-u)} \right) \right) \\
&\quad + \nu_1 \int_u^t e^{a_1 s} dW_{1s}^T .
\end{aligned}
$$

Multiplying both sides of the above equation by $e^{-a_1 t}$, we find that the closed form expression of X_{1t} is

$$
\begin{aligned}
X_{1t} &= X_{1u}e^{-a_1(t-u)} + \bar{x}_1\left(1-e^{-a_1(t-u)}\right) \\
&\quad -\frac{\nu_1^2}{a_1^2}\left(\left(1-e^{-a_1(t-u)}\right)-\frac{e^{-a_1(T-t)}}{2}\left(1-e^{-2a_1(t-u)}\right)\right) \\
&\quad -\frac{\nu_1\nu_2\eta}{a_2}\left(\frac{1}{a_1}\left(1-e^{-a_1(t-u)}\right)-\frac{e^{-a_2(T-t)}}{a_1+a_2}\left(1-e^{-(a_1+a_2)(t-u)}\right)\right) \\
&\quad +\nu_1 e^{-a_1 t}\int_u^t e^{a_1 s}dW_{1s}^T.
\end{aligned} \tag{4.9}
$$

Now consider the second factor. We have that :

$$
\begin{aligned}
dX_{2t} &= a_2\left(\bar{x}_2 - X_{2t}\right)dt + \nu_2\eta\left(dW_{1t}^T - \frac{\nu_1}{a_1}\left(1-e^{-a_1(T-t)}\right)dt - \nu_2\eta\frac{1-e^{-a_2(T-t)}}{a_2}dt\right) \\
&\quad +\nu_2\sqrt{1-\eta^2}\left(dW_{2t}^T - \frac{\nu_2\sqrt{1-\eta^2}}{a_2}\left(1-e^{-a_2(T-t)}\right)dt\right), \\
&= \left(a_2\left(\bar{x}_2 - X_{2t}\right) - \frac{\nu_1\nu_2\eta}{a_1}\left(1-e^{-a_1(T-t)}\right) - \frac{\nu_2^2}{a_2}\left(1-e^{-a_2(T-t)}\right)\right)dt \\
&\quad +\nu_2\eta dW_{1t}^T + \nu_2\sqrt{1-\eta^2}dW_{2t}^T.
\end{aligned}
$$

It is useful to make the change of variable $Z_{2t} = e^{a_2 t}X_{2t}$.

Taking the differential of Z_{2t}, we find that :

$$
\begin{aligned}
dZ_{2t} &= e^{a_2 t} dX_{2t} + a_2 e^{a_2 t} X_{2t} dt \\
&= e^{a_2 t} \left(a_2 \left(\bar{x}_2 - X_{2t} \right) dt - \frac{\nu_1 \nu_2 \eta}{a_1} \left(1 - e^{-a_1 (T-t)} \right) dt \right) \\
&\quad + e^{a_2 t} \left(- \frac{\nu_2^2}{a_2} \left(1 - e^{-a_2 (T-t)} \right) dt + \nu_2 \eta dW_{1t}^T + \nu_2 \sqrt{1-\eta^2} dW_{2t}^T \right) \\
&\quad + a_2 e^{a_2 t} x_{2t} dt, \\
&= \left(a_2 \bar{x}_2 e^{a_2 t} - \frac{\nu_1 \nu_2 \eta}{a_1} \left(e^{a_2 t} - e^{-a_1 T} e^{(a_1 + a_2) t} \right) - \frac{\nu_2^2}{a_2} \left(e^{a_2 t} - e^{-a_2 T} e^{2 a_2 t} \right) \right) dt \\
&\quad + \nu_2 \eta e^{a_2 t} dW_{1t}^T + \nu_2 \sqrt{1-\eta^2} e^{a_2 t} dW_{2t}^T .
\end{aligned}
$$

Integrating both sides of the above equation, we find that :

$$
\begin{aligned}
Z_{2t} - Z_{2u} &= a_2 \bar{x}_2 \left(e^{a_2 t} - e^{a_2 u} \right) \\
&\quad - \frac{\nu_1 \nu_2 \eta}{a_1} \left(\frac{1}{a_2} \left(e^{a_2 t} - e^{a_2 u} \right) - \frac{e^{-a_2 T}}{a_1 + a_2} \left(e^{(a_1 + a_2) t} - e^{(a_1 + a_2) u} \right) \right) \\
&\quad - \frac{\nu_2^2}{a_2^2} \left(\left(e^{a_2 t} - e^{a_2 u} \right) - \frac{e^{-a_2 T}}{2} \left(e^{2 a_2 t} - e^{2 a_2 u} \right) \right) \\
&\quad + \nu_2 \eta \int_u^t e^{a_2 s} dW_{1s}^T + \nu_2 \sqrt{1-\eta^2} \int_u^t e^{a_2 s} dW_{2s}^T \\
&= a_2 \bar{x}_2 e^{a_2 t} \left(1 - e^{-a_2 (t-u)} \right) \\
&\quad - \frac{\nu_1 \nu_2 \eta e^{a_2 t}}{a_1} \left(\frac{1}{a_2} \left(1 - e^{-a_2 (t-u)} \right) - \frac{e^{-a_1 (T-t)}}{a_1 + a_2} \left(1 - e^{-(a_1 + a_2)(t-u)} \right) \right) \\
&\quad - \frac{\nu_2^2 e^{a_2 t}}{a_2^2} \left(\left(1 - e^{-a_2 (t-u)} \right) - \frac{e^{-a_2 (T-t)}}{2} \left(1 - e^{-2 a_2 (t-u)} \right) \right) \\
&\quad + \nu_2 \eta \int_u^t e^{bs} dW_{1s}^T + \nu_2 \sqrt{1-\eta^2} \int_u^t e^{bs} dW_{2s}^T .
\end{aligned}
$$

Multiplying both sides of the above equation by $e^{-a_2 t}$, we find that the closed form expression of X_{2t} is

$$
\begin{aligned}
X_{2t} &= X_{2u}e^{-a_2(t-u)} + \bar{x}_2\left(1-e^{-a_2(t-u)}\right) \\
&\quad -\frac{\nu_1\nu_2\eta}{a_1}\left(\frac{1}{a_2}\left(1-e^{-a_2(t-u)}\right) - \frac{e^{-a_1(T-t)}}{a_1+a_2}\left(1-e^{-(a_1+a_2)(t-u)}\right)\right) \\
&\quad -\frac{\nu_2^2}{a_2^2}\left(\left(1-e^{-a_2(t-u)}\right) - \frac{e^{-a_2(T-t)}}{2}\left(1-e^{-2a_2(t-u)}\right)\right) \\
&\quad +\nu_2\eta e^{-a_2t}\int_u^t e^{a_2s}dW_{1s}^T + \nu_2 e^{-a_2t}\sqrt{1-\eta^2}\int_u^t e^{a_2s}dW_{2s}^T.
\end{aligned} \tag{4.10}
$$

The first two moments of the joint distributions shown in Proposition 4.2.1 follow immediately from equations 4.9 and 4.10.

□

4.3 Model solution

We solve the model by using a backward recursive algorithm based on Clenshaw-Curtis integration. The Clenshaw-Curtis integration approach consists of approximating the integrand by its truncated Chebyshev series of order m, and using the coefficients of the series to evaluate the integral in closed form. The algorithm works for any number of factors and is as follow:

1. Initialization

   (a) For each factor i, $i = 1,...,N$ define the intervals $[x_{i\min}, x_{i\max}]$, such that the probability of the factor getting out of this interval over the life of the bond is very small.

   (b) Choose the degree of truncation of the Chebyshev series m, and for each factor i, $i = 1,...,N$, define the Gauss-Lobatto coordinates

$$x_{ij} = 0.5\left(x_{i\max} + x_{i\min}\right) + 0.5\left(x_{i\max} - x_{i\min}\right)\cos\left(\frac{\pi j}{m}\right), j = 0, 1, ..., m.$$

(c) Denote by Y_i the finite set of the Gauss-Lobatto coordinates for factor i, $i = 1, ..., N$. Define the set of interpolation points as

$$Y = \prod_{i=1}^{N} Y_i.$$

(d) For $t = t_q - \delta$, calculate the callable bond value $C_t(x)$ on every point of Y using Formula 4.1.

2. Recursion. Assume that for a given $k = q^*, ..., q-1$, the value of the callable bond $C_{t_{k+1}-\delta}(x)$ is known on every point $x \in Y$.

   (a) Denote by $f_{x,t}(z, T)$, the transition density function, i.e. the density function of the factors at time T under the forward measure evaluated at z, conditional on the information that the value of the factors at time $t < T$ is x.

   (b) For every point (x, z) of $Y \times Y$, evaluate the function

$$g_{x,t_k-\delta}(z, t_{k+1} - \delta) = f_{x,t_k-\delta}(z, t_{k+1} - \delta) \times C_{t_{k+1}-\delta}(z).$$

   (c) Define the set of multidimensional indices

$$J = \{0, 1, ..., m\}^N.$$

   (d) Using FFT, find the multidimensional truncated Chebyshev series of order m of the function $g_{x,t_k-\delta}(u, t_{k+1} - \delta)$ viewed as a function of u.

$$g_{x,t_k-\delta}(u, t_{k+1} - \delta) = \sum_{l \in J} b(l) I_l(w),$$

   where w is defined by

$$w(i) = -\frac{x_{i\max} + x_{i\min}}{x_{i\max} - x_{i\min}} + \frac{2u(i)}{x_{i\max} - x_{i\min}},$$

   $b(l)$ is the coefficients of order l of the multidimensional Chebyshev series, and I_l is the multidimensional Chebyshev polynomial of order l defined by

$$I_l(w) = \prod_{i=1}^{N} \cos(l(i) \arccos(w(i))), \text{ for } w(i) \in [-1,1].$$

(e) Define the sets J^1 and D as:

$$\begin{aligned} J^1 &= \begin{cases} \{0,2,...,m\}^N, \text{ if } m \text{ is even,} \\ \{0,2,...,m-1\}^N, \text{ if } m \text{ is odd,} \end{cases} \\ D &= \prod_{i=1}^{N} [x_{i\min}, x_{i\max}]. \end{aligned}$$

(f) For every point $x \in Y$, evaluate the holding value $H_{t_k-\delta}(x)$ using Formula 4.3 and the Clenshaw-Curtis integration for the forward expectation:

$$\begin{aligned} \mathbb{E}^{Q^\Delta}_{t_{k-1}-\delta,x}\left[C_{t_{k+1}-\delta}\right] &\simeq \int_D C_{t_{k+1}-\delta}(u) f_{x,t_k-\delta}(u, t_{k+1}-\delta)\, du, \\ &= \int_D g_{x,t_k-\delta}(u, t_{k+1}-\delta)\, du, \\ &\simeq \sum_{l \in J^1} b(l) \prod_{i=1}^{N} \frac{2}{1-(l(i))^2} \times \frac{(x_{i\max} - x_{i\min})}{2}. \end{aligned}$$

(g) For every point $x \in Y$, calculate the callable bond value at time $t = t_k - \delta$, $C_{t_k-\delta}(x)$ using Formula 4.2.

3. Termination. Use steps 2.a to 2.f to calculate the forward expectation in Formula 4.4, and use Formula 4.4 to calculate the callable bond value at the initial date.

Proof. The above algorithm is a numerical approximation to the exact solution of the valuation problem. The approximation occurs in the computation of the forward expectations. There are two levels of approximations, that are made explicit in Step 2.f. The first level is the truncation of the integration domain from $\Re^N$ to the compact domain D. The second level is the computation of the integral over the domain D by the Clenshaw-Curtis numerical integration scheme.

To prove that the algorithm converges to the exact value, it suffices to prove that the approximation error converges to zero as D gets closer to $\Re^N$ and the degree of truncation of the Chebyshev series m goes to infinity.

We first show that, as m goes to infinity, the approximation error of the integral over the domain D converges to zero. Actually, this result is a direct consequence of the continuity of the value function $C_{t_k}(x)$, for any $k = 0, ..., q$ and of the fact that the Chebyshev series of a function defined over a compact domain converges to the exact function when the function is continuous [54]. It is easy to prove by recurrence that the value function $C_{t_k}(x)$ is continuous for any $k = 0, ..., q$.

Second we prove that the approximation error due to the truncation of the integration domain from $\Re^N$ to D converges to zero as D gets larger and larger. First recall that

$$D = \prod_{i=1}^{N} [x_{i\min}, x_{i\max}].$$

It suffices to prove that as $x_{i\min} \to -\infty$ and $x_{i\max} \to +\infty$, for all $i = 1, 2, ..N$, the integrand vanishes to zero. Notice that the multivariate Gaussian density function $f_{x,t_k-\delta}(u, t_{k+1} - \delta)$ is the exponential of a negative quadratic term in u, while the ex-coupon callable bond price is bounded below by 0 and bounded above by the strike times the discount factor, which is the exponential of a linear function in u. So as $u(i) \to -\infty$ or $u(i) \to +\infty$ for all $i = 1, 2, ..N$, the quadratic term dominates and the integrand $g_{x,t_k-\delta}(u, t_{k+1} - \delta)$ vanishes to zero.

Note that the accumulation of errors due to the recursion is not a problem because we control the error made at each step of the recursion. As a result, the overall error can be made as small as required for m large enough and D large (in the senses of sets) enough. □

Notice that the above algorithm gives the callable bond price for factor values corresponding to the Chebyshev nodes. The callable bond value for arbitrary values of the factors in the domain D can be recovered using the truncated Chebyshev series of the callable bond price, which can be swiftly computed by FFT. Thus a single run of our algorithm gives the callable bond price for all values of the factors in the domain D, at all decision dates and at the pricing date.

4.4 Numerical illustrations

For numerical illustrations, we consider a 4.25% callable bond issued by the Swiss Confederation, over the period 1987-2012. At the pricing date $t_0 = 0$, December 23,1991, the time to maturity was $t_q = 20.172$ years, with $q = 21$, a principal scaled to 1, a coupon $c = 0.0425$ paid once a year with the first coupon paid at time $t_1 = 0.172$, a notice period of 2 months which means $\delta = 0.1666$, and a protection period of $t_{q^*} = 10.172$ years, with $q^* = 11$. The call prices are $K_{11} = 1.025$, $K_{12} = 1.020$, $K_{13} = 1.015$, $K_{14} = 1.010$, $K_{15} = 1.005$, $K_{16} = K_{17} = ... = K_{21} = 1$.

The prices of this bond with and without the embedded option have been calculated under the one-factor Vasicek model by several authors using different numerical techniques [2, 8, 22]. The considered risk-neutral parameters for the one factor Vasicek model are

$$\begin{aligned} \bar{x}^{VAS} &= 0.098397028 \\ a^{VAS} &= 0.44178462 \\ v^{VAS} &= 0.13264223. \end{aligned} \tag{4.11}$$

Although we consider the 2FV model for the interest rate dynamics, it is possible to replicate the prices obtained under the Vasicek model under a given parametrization of the 2FV model. The insight behind this replication is that the 2FV model is equivalent to the Vasicek model if the two factors are perfectly correlated. For bond prices to be the same under the 2FV and the Vasicek model, it suffices to choose:

$$
\begin{aligned}
a_1 &= a_2 = a^{VAS}, && (4.12)\\
\bar{x}_1 &= \bar{x}_2 = \bar{x}^{VAS}/2, && (4.13)\\
v_1 &= v_2 = \frac{v^{VAS}}{2},\\
\eta &= 1,\\
X_1 &= X_2 = \frac{X^{VAS}}{2}.
\end{aligned}
$$

This correspondence between the 2FV and the Vasicek model may be verified by using Equations 4.6, 4.8 and 4.12. For our numerical exercises, however, we cannot set η to 1 because this would imply a determinant of zero for the covariance matrix, and the bivariate gaussian distribution would not be defined. As a result, when we seek to compare our 2FV prices with the Vasicek prices, we set η to 0.98. We therefore expect to have a slight difference between the Vasicek prices and the corresponding 2FV prices.

We define the range of variation of each factor i by its long term average times a multiple κ of its standard deviation over the bond life:

$$
\begin{aligned}
x_{i\,\min} &= \bar{x}_i - \kappa\sqrt{\frac{v_i^2}{2a_i}\left(1-e^{-2a_i t_q}\right)}. && (4.14)\\
x_{i\,\max} &= \bar{x}_i + \kappa\sqrt{\frac{v_i^2}{2a_i}\left(1-e^{-2a_i t_q}\right)}.
\end{aligned}
$$

The default value of κ is 6.

4.4.1 Plain vanilla

We first consider the bond without its call option. In this case, the bond price is known in closed form. This allows us to compare the prices obtained by our solution method to the closed form price.

We first set the parameters of the 2FV model using equations 4.11 and 4.12 except that η is set to 0.98. For increasing values of m, we calculate our bond prices and the closed form price for $r = 0.01, 0.02, ..., 0.1$. Table 4.I shows the results. This table shows that a 5 digit precision on the price is reached with $m = 100$. The values for $m = 200$ are virtually the same as those for $m = 100$, indicating that convergence has occurred.

Second, using this value of $m = 100$, we study the accuracy of our vanilla prices for different combination of parameters of the 2FV model. For the various combination of parameters considered, Table 4.II shows that the pricing error is of the order of $1E-07$.

Table 4.I: Convergence of our solution method for plain vanilla bonds for different values of r

r	$m=30$	$m=40$	$m=50$	$m=60$	$m=70$	$m=80$	$m=90$	$m=100$	$m=200$	Closed form
0.01	56.665139	3.433102	1.161971	0.952217	0.926411	0.923556	0.923312	0.923296	0.923296	0.923296
0.02	55.401610	3.358660	1.138262	0.933194	0.907964	0.905173	0.904934	0.904919	0.904918	0.904919
0.03	54.166284	3.285859	1.115063	0.914575	0.889909	0.887181	0.886947	0.886932	0.886932	0.886932
0.04	52.958531	3.214665	1.092361	0.896353	0.872238	0.869570	0.869342	0.869327	0.869327	0.869327
0.05	51.777734	3.145042	1.070148	0.878518	0.854941	0.852333	0.852110	0.852096	0.852095	0.852095
0.06	50.623293	3.076955	1.048411	0.861062	0.838012	0.835462	0.835244	0.835230	0.835229	0.835229
0.07	49.494619	3.010369	1.027140	0.843976	0.821442	0.818949	0.818735	0.818722	0.818721	0.818721
0.08	48.391137	2.945252	1.006326	0.827254	0.805223	0.802785	0.802576	0.802563	0.802563	0.802563
0.09	47.312284	2.881571	0.985958	0.810887	0.789347	0.786964	0.786760	0.786747	0.786747	0.786747
0.1	46.257511	2.819294	0.966027	0.794866	0.773808	0.771479	0.771279	0.771266	0.771266	0.771266

Table 4.II: Accuracy of our solution method for plain vanilla bonds for different parametrizations of the 2FV model

Case	Solution method	$r = 0.01$	$r = 0.02$	$r = 0.03$	$r = 0.04$	$r = 0.05$	$r = 0.06$	$r = 0.07$	$r = 0.08$	$r = 0.09$	$r = 0.1$
Default	Closed form	1.7231860	1.6897869	1.6570573	1.6249836	1.5935526	1.5627514	1.5325670	1.5029872	1.4739997	1.4455924
Default	Our approach	1.7231857	1.6897866	1.6570570	1.6249833	1.5935523	1.5627511	1.5325668	1.5029869	1.4739994	1.4455921
$\bar{x}_1$ down 50 bps	Closed form	1.8394515	1.8036650	1.7685972	1.7342333	1.7005591	1.6675608	1.6352245	1.6035370	1.5724850	1.5420557
$\bar{x}_1$ down 50 bps	Our approach	1.8394511	1.8036647	1.7685968	1.7342329	1.7005588	1.6675605	1.6352242	1.6035367	1.5724847	1.5420554
$\bar{x}_2$ down 50 bps	Closed form	1.8463516	1.8104249	1.7752197	1.7407212	1.7069153	1.6737878	1.6413250	1.6095135	1.5783401	1.5477919
$\bar{x}_2$ down 50 bps	Our approach	1.8463512	1.8104246	1.7752193	1.7407209	1.7069149	1.6737874	1.6413247	1.6095132	1.5783398	1.5477917
a_1 down 0.1	Closed form	1.9627434	1.9174896	1.8733161	1.8301968	1.7881063	1.7470198	1.7069131	1.6677626	1.6295452	1.5922386
a_1 down 0.1	Our approach	1.9627427	1.9174889	1.8733154	1.8301961	1.7881056	1.7470191	1.7069124	1.6677620	1.6295446	1.5922380
a_2 down 0.1	Closed form	1.9176385	1.8773861	1.8380059	1.7994788	1.7617862	1.7249099	1.6888320	1.6535351	1.6190020	1.5852163
a_2 down 0.1	Our approach	1.9176380	1.8773856	1.8380054	1.7994783	1.7617858	1.7249094	1.6888315	1.6535346	1.6190016	1.5852158
ν_1 up 200 bps	Closed form	2.0181132	1.9786511	1.9399832	1.9020933	1.8649658	1.8285850	1.7929360	1.7580038	1.7237740	1.6902323
ν_1 up 200 bps	Our approach	2.0181126	1.9786505	1.9399826	1.9020927	1.8649652	1.8285845	1.7929354	1.7580033	1.7237735	1.6902317
ν_2 up 200 bps	Closed form	1.9260592	1.8884918	1.8516796	1.8156072	1.7802598	1.7456225	1.7116811	1.6784213	1.6458295	1.6138921
ν_2 up 200 bps	Our approach	1.9260587	1.8884913	1.8516791	1.8156068	1.7802593	1.7456221	1.7116807	1.6784209	1.6458291	1.6138917
η -0.5	Closed form	1.2815320	1.2572037	1.2333585	1.2099867	1.1870788	1.1646255	1.1426177	1.1210463	1.0999026	1.0791781
η -0.5	Our approach	1.2815319	1.2572036	1.2333585	1.2099867	1.1870788	1.1646255	1.1426176	1.1210462	1.0999026	1.0791781

The default case corresponds to $\bar{x}_1 = 0.01$, $\bar{x}_2 = 0.03$, $a_1 = 0.4$, $a_2 = 0.6$, $\nu_1 = 0.06$, $\nu_2 = 0.1$, $\eta = 0.5$

4.4.2 Callable bond

Now we consider the bond with its call option.

We first set the 2FV parameters using Equations 4.11 and 4.12. We know that with these parameters, our prices should be very close to those obtained by Ben-Ameur et al. [2], Buttler and Waldvogel [8], D'Halluin et al. [22] in the one-factor case. We calculate the callable bond prices using our approach for $r = 0.01, 0.02, ..., 0.1$ and for increasing values of m. Table 4.III shows that our prices are indeed very close to those obtained by Ben-Ameur et al. [2] and D'Halluin et al. [22]. The slight difference between our prices and the one-factor prices are attributable to our approximation of the one factor world by a two-factor world with a quasi-perfect correlation of 0.98. In order to quantify the magnitude of the error due to this approximation, we also display in Table 4.III the difference between the plain vanilla prices under the 2FV model and the Vasicek model using their respective closed form solution. We find a difference in the order of 0.001, similar to the difference between our callable bond prices and those obtained using the Vasicek model.

Next, we ensure that the observed convergence is not specific to the above set of parameters. We therefore consider several different combinations of parameters and for each combination, we compute the callable bond price for increasing values of m. The results are report in Table 4.IV. These results show that, for all combination parameters, a 4 digits precision on the price is reached for $m = 100$.

Table 4.III: Convergence of our solution method for different values of r, comparison with other approaches

r	$m=80$	$m=90$	$m=100$	$m=200$	DFVL	BBKL	$(m=200)$-BBKL	Closed form Vanilla 2FV, $\eta=0.98$	Closed form Vanilla, Vasicek	Closed form Error on Vanilla
0.01	0.84069	0.84058	0.84056	0.84056	0.84282	0.84285	0.00226	0.92330	0.92742	0.00412
0.02	0.82418	0.82407	0.82406	0.82406	0.82627	0.82630	0.00221	0.90492	0.90895	0.00403
0.03	0.80803	0.80792	0.80791	0.80790	0.81010	0.81009	0.00219	0.88693	0.89088	0.00395
0.04	0.79221	0.79211	0.79210	0.79209	0.79420	0.79423	0.00210	0.86933	0.87318	0.00385
0.05	0.77673	0.77663	0.77661	0.77661	0.77868	0.77871	0.00207	0.85210	0.85587	0.00377
0.06	0.76157	0.76147	0.76146	0.76146	0.76348	0.76351	0.00202	0.83523	0.83892	0.00369
0.07	0.74673	0.74664	0.74662	0.74662	0.74860	0.74862	0.00198	0.81872	0.82233	0.00361
0.08	0.73221	0.73211	0.73210	0.73210	0.73403	0.73406	0.00193	0.80256	0.80609	0.00353
0.09	0.71799	0.71789	0.71788	0.71788	0.71977	0.71980	0.00189	0.78675	0.79019	0.00344
0.1	0.70406	0.70397	0.70396	0.70396	0.70578	0.70583	0.00182	0.77127	0.77464	0.00337

Table 4.IV: Convergence of our solution method for general parametrizations of the 2FV model

Cases	$m = 50$	$m = 60$	$m = 70$	$m = 80$	$m = 90$	$m = 100$	$m = 200$	$(m = 200) - (m = 100)$
Default	1.19788	1.19771	1.19783	1.19776	1.19778	1.19779	1.19778	-0.00001
$\bar{x}_1$ down 50 bps	1.24331	1.24355	1.24339	1.24349	1.24342	1.24346	1.24344	-0.00002
$\bar{x}_2$ down 50 bps	1.24767	1.24791	1.24775	1.24785	1.24778	1.24783	1.24781	-0.00002
a_1 down 0.1	1.24047	1.24053	1.24051	1.24052	1.24051	1.24051	1.24051	0.00000
a_2 down 0.1	1.24199	1.24199	1.24199	1.24198	1.24198	1.24198	1.24198	0.00000
ν_1 up 200 bps	1.28029	1.28034	1.28032	1.28033	1.28032	1.28033	1.28032	0.00000
ν_2 up 200 bps	1.26017	1.26019	1.26018	1.26018	1.26018	1.26019	1.26019	0.00000
η -0.5	1.05454	1.05423	1.05437	1.05431	1.05431	1.05433	1.05432	-0.00002

The default case corresponds to $\bar{x}_1 = 0.01$, $\bar{x}_2 = 0.03$, $a_1 = 0.4$, $a_2 = 0.6$, $\nu_1 = 0.06$, $\nu_2 = 0.1$, $\eta = 0.5$

4.4.3 Numerical stability and effect of model parameter values on the callable bond price

In this subsection, we study the effect of model parameters on the embedded call option price. The default value of the model parameters are $\bar{x}_1 = 0.01$, $\bar{x}_2 = 0.03$, $a_1 = 0.4$, $a_2 = 0.6$, $\nu_1 = 0.06$, $\nu_2 = 0.1$, $\eta = 0.5$. We vary one parameter at the time while fixing the other parameters at their default values and we compute the callable bond price, the plain vanilla bond price and the call option price as the difference between the first two. The results are reported in Figures 4.1 and 4.2.

As we can see, the callable and the plain vanilla bond prices always move in the same direction. This implies that it is not obvious to anticipate the direction in which the embedded option price will move; The graphs show that the embedded call price also moves in the same direction as the bond prices, implying that the plain vanilla bond price is more sensitive to the model parameter values than the callable bond price. This intuition is confirmed by comparing the slopes of the plain vanilla and the callable bond prices on the graphs.

Regarding the specific impacts of the parameters, we can see that the embedded call option price declines monotonically with the long run means of the factors, and with their mean reversion speeds. On the other hand, the call option price increases with each factor volatility and with their correlation.

The long run mean effect is explained by the fact that higher factor values imply higher short rates, which in turn imply lower plain vanilla bond prices. The lower plain vanilla bond prices imply lower call option values because call option values decline with the value of their underlying instrument.

The impacts of volatility and correlation are explained by two effects. The first effect is that higher volatility or correlation of the factors increases the plain vanilla bond price (see Formula 4.8) and therefore the call option price. The other effect is the traditional underlying volatility effect. Higher volatility or correlation of the factors increases the volatility of the short rate and the likelihood of the call option being exercised, so that the option value increases.

The effect of the speed of mean reversion is explained by the fact that lower speed of mean reversion implies more freedom for the factors in their evolution, which in turn implies higher interest rate volatility, higher plain vanilla value and higher call option value.

Finally the graphs show that the computed callable bond prices are smooth functions of the model parameters. This implies that our numerical method is stable.

Figure 4.1: Effect of long run mean and speed of mean reversion on callable bond price

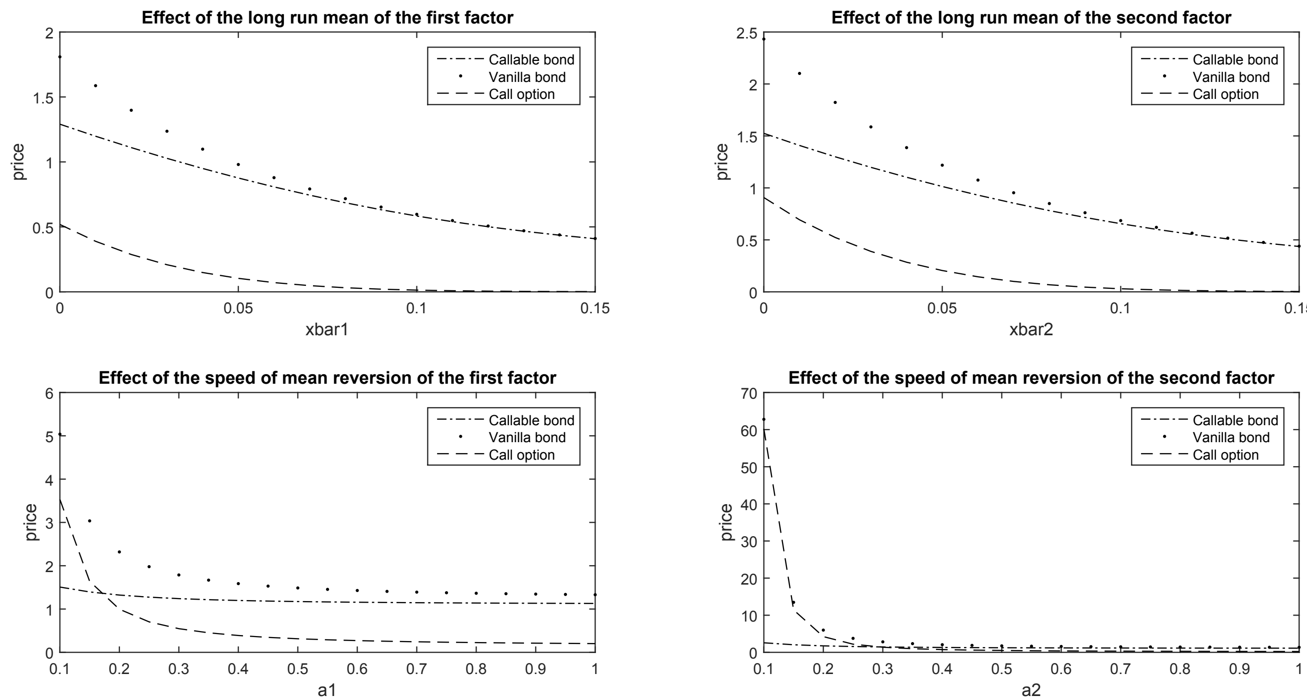
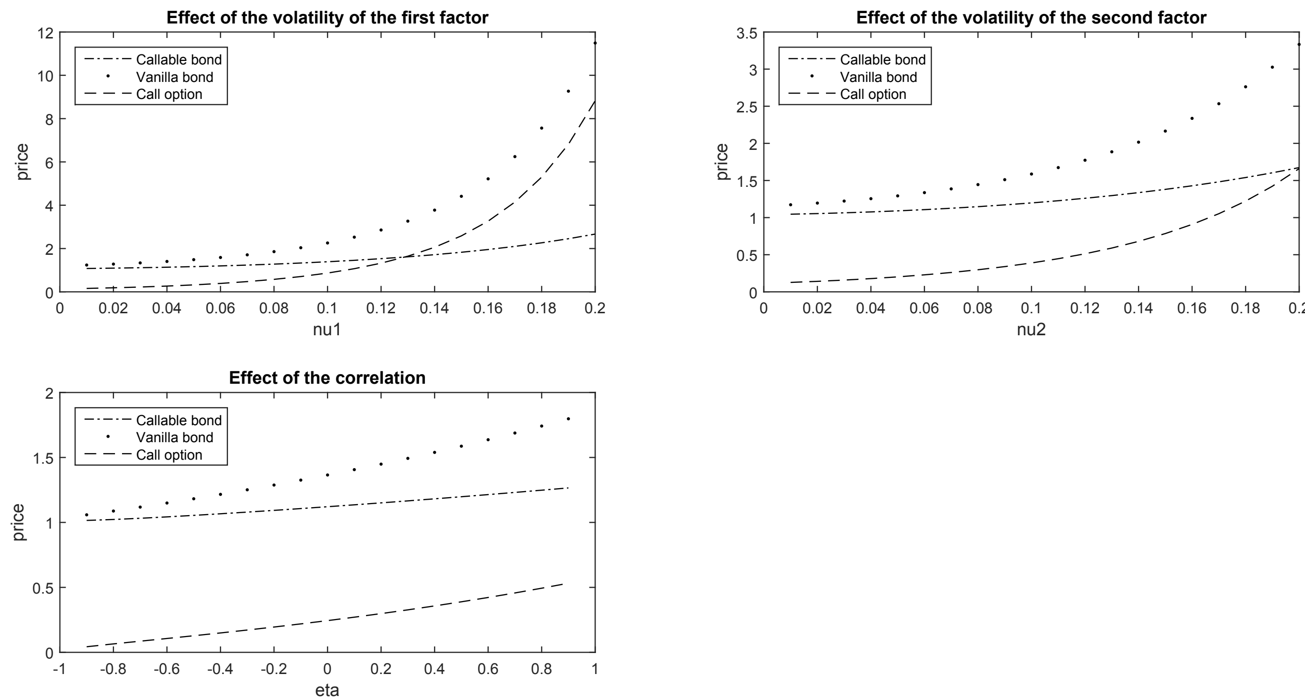


Figure 4.2: Effect of volatility and correlation on the callable bond price



4.4.4 Computing time and grid size

Our code is run using parallel programming in MatlabR2014a on a computer with 48Go RAM and 10 cores of 2GHz each. The computing time is around 60 seconds for $m = 100$.

Regarding the grid size, we have recomputed all the prices reported in the tables of results for $\kappa = 7$, thus increasing the truncation domain of the factors. We found virtually the same prices. This indicates that the truncation domain associated with $\kappa = 6$ is large enough.

4.5 Conclusion

In this essay, we proposed a new approach for the valuation of callable bonds under the two-factor Vasicek model. The approach is based on the distribution of the two factors under the forward measure, on the Chebyshev series expansion of the option value and on the Clenshaw-Curtis integration method. The advantage of our approach compared to binomial trees, trinomial trees, and partial differential equation approaches is its numerical stability. On the other hand, compared to simulation approaches, the advantage of our approach is that it can reach a high precision on the price in a reasonable computing time. We illustrated the performance of the approach on a Swiss Confederation callable bond and we also studied the impact of model parameters such as long run means, speeds of mean reversion, volatilities and correlation on the computed embedded call price.

Although we focused on the two-factor Vasicek model in this essay, our algorithm is very general and may accommodate more complicated models, such as the two-factor CIR model, models with jumps or models with more factors. Future research may focus on applying our algorithm on these models. In addition, callable bonds often include other options, such as convertibility or credit risk. Extending our approach to accommodate these additional options may be another direction for further research. Finally, our numerical solution method is inherently a solution to the valuation problem of any multi-factor Bermudan option. As such it may find applications in the valuation of other interest rate derivatives such as swaptions or equity options. As an example, our approach may be directly used to evaluate American type swaptions under the two-factor Vasicek model, since it is well known that swaptions may be viewed as call or put options embedded in a coupon bond [46].

CHAPTER 5

CONCLUSION

In this thesis we have dealt with three issues in the valuation of bonds. The first issue is the impact of heterogeneity on the average spreads predicted by structural models. The second is the estimation of the share of default risk in corporate bond spreads. The third issue is the valuation of callable bonds under multifactor interest rate models.

In the first essay, we have shown that heterogeneity affects average spreads. We distinguished between cross-sectional and time-series heterogeneity. In the case of cross-sectional heterogeneity, we showed that, in addition to the first moment of the distribution of heterogeneous fundamentals, one has to consider higher order moments, such as standard deviation and skewness, in order to explain to explain the difference in average spreads between different groups of bonds. In the case of time-series heterogeneity, we showed that dynamic leverage, stochastic volatility and stochastic interest rates are important determinants of the average spreads predicted by the Merton model. Even though the model does not price either dynamic leverage, stochastic volatility or stochastic interest rates, future revisions of the model inputs according to the observed changes in leverage, volatility or interest rates affect the time-series average spread in non-trivial ways. In both cases, we studied the biases caused by ignoring heterogeneity. We showed that these biases may be positive or negative, large or small, depending on what parameters are heterogeneous, on the distribution of the heterogeneous parameters, and on the levels of the parameters. Our findings show that calibration to average default rates generally reduces the heterogeneity biases, especially in the case of time-series heterogeneity.

In the second essay, after reviewing existing estimation methods of structural models, we proposed a new estimation method that aims at accurately capturing the share of default risk in the spreads. The main difference between our approach and other approaches proposed in the literature is that we estimate and calibrate the model to historical default rates in a way that recognizes the heterogeneity of bonds. We compared the performance of our approach to that of the standard in the literature. We showed, using simulation experiments, that in comparison to the standard approach, which may lead to large errors on the estimated share of default risk in the spreads, our approach is very accurate. We then applied our approach on a sample of about 3 millions firm-days observations and found that the share of default risk in BBB bonds is around 40%. The figure is smaller for high quality bonds and larger for lower quality bonds. We performed several robustness checks on these findings, regarding several assumptions, including uncertainty on historical default rates, period of study and the choice of a structural model. We computed the heterogeneity biases due to ignoring heterogeneity and we found that they may be large or small, negative or positive, depending on the rating category and on the maturity. These findings provide some empirical validation for our theoretical study of heterogeneity in the first essay.

In the third essay, we presented and justified our choice of the two-factor Vasicek model and derived in closed forms the joint distributions of the two-factors under the forward measure. We then described our numerical pricing algorithm and showed its convergence mathematically. The algorithm is a dynamic program that discretizes the state space and uses Clenshaw-Curtis integration, combined with Chebyshev series expansion by FFT, to compute the continuation value of a callable bond. We considered a benchmark callable bond that has been priced by other approaches in the literature in the case of the one-factor Vasicek model, and showed that we are able to replicate the prices reported in the literature when the two factors in our model are perfectly correlated. We also used numerical illustrations to show that our approach converges to the exact price, which is known in closed form for plain vanilla bonds. Finally, we studied the impact of changes in the model inputs, such as the correlation between the two factors, the speeds of mean reversion, the factors' volatilities and their long term averages, on the computed prices. Our results show that the algorithm is numerically stable and computationally efficient.

Future research arising from the first essay could consist of examining how well the Merton model with heterogeneity fits observed average spreads. The empirical analysis may consider different groupings than the grouping by rating categories, such as grouping by economic sectors or multidimensional clustering by firm fundamentals. The analysis may also aim at demonstrating empirically that the variation in average spreads across groups is related to higher order moments than the first moments of the distributions of firm fundamentals. Another research direction could consists in studying the impact of heterogeneity on the average spreads predicted by more elaborated models than the Merton model. For instance in a stochastic volatility model, considering heterogeneity may imply that one studies the *volatility of the volatility of the volatility*. A third research direction could be the analysis of the impact of heterogeneity on other aggregate measures, such as the average optimal leverage of a group of firms. Such a study may have implications for the low leverage puzzle. A fourth research avenue is to examine the linkage between heterogeneity and default correlations. Default correlation poses significant risks to the bond market and one may find that heterogeneity plays a role in reducing the risk of simultaneous defaults.

Regarding the second essay, future research could examine whether replacing our two-stage approach by a one stage approach where the default boundary is added as an additional variable in the estimation, in such a way that we have three equations for three unknowns, would improve our results. Future work may also use TRACE data in order to conduct the empirical study at the bond level rather than at the firm level. This would allow an estimation of the share of default risk in the spreads at the individual bond level and at each point in time and would allow a richer empirical analysis of the determinants of this default share. Another direction is to include jumps, stochastic volatility and time-varying risk premia in the estimations. Another direction is to relate the non-default share by our approach to the other potential factors such as taxes, individual bond liquidity, market liquidity and macro-economic risk.

Finally, future research stemming from the third essay could use our pricing algorithm for an empirical study of the number of factors relevant for pricing and hedging callable bonds. Such a study may compare the two-factor Vasicek model and its deterministic shift version to the one-factor Vasicek model and its deterministic version. Another direction may consist of extending the approach to other interest rate models, such as the two-factor CIR model or models with jumps. A third direction could consist of illustrating the versatility of our numerical algorithm by showing how it may be used to value several other multifactor derivatives.

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