**HEC MONTRÉAL**

École affiliée à l'Université de Montréal

# Collateral Management and Structured Product Valuation

par

**Mbaye Ndoye**

Thèse présentée en vue de l'obtention du grade de Ph.D en administration

(option Méthodes quantitatives)

Décembre 2015

# HEC MONTRÉAL

École affiliée à l'Université de Montréal

Cette thèse intitulée :

**Collateral Management and Structured Product Valuation**

Présentée par :

**Mbaye Ndoye**

a été évaluée par un jury composé des personnes suivantes :

**Sylvain Perron**
HEC Montréal
Président-rapporteur

**Michèle Breton**
HEC Montréal
Directrice de recherche

**Chantal Labbé**
HEC Montréal
Membre du jury

**Matt Davison**
University of Western Ontario
Examinateur Externe

**Mathieu Fournier**
HEC Montréal
Représentant du directeur de HEC Montréal

# Résumé

Cette thèse apporte diverses contributions sur la gestion du risque de contrepartie et sur l'évaluation des produits structurés. En ce qui concerne le risque de contrepartie, nous proposons de nouvelles méthodes souples et efficaces pour la gestion de biens offerts en garantie, en utilisant des stratégies basées sur l'un ou l'autre de deux objectifs différents qui sont la minimisation des pertes espérées et la maximisation de l'utilité espérée des contreparties. Cette approche est flexible et prend en compte le risque de corrélation (propriété de "Wrong (right) way risk") ainsi que les coûts de transaction. Les stratégies résultantes sont comparées à des stratégies conservatrices existantes à travers quelques exemples de base. En ce qui concerne l'évaluation des produits structurés, nous proposons une méthode basée sur la programmation dynamique qui s'applique à tous les types de produits. Cette méthode d'évaluation est très générale et s'applique à tout type de dynamique de l'actif de référence. Nous proposons également une formule analytique de type Black-Scholes pour l'évaluation des notes rachetables à capital garanti avec une seule date de rachat anticipé lorsque l'actif de référence est modélisé par une dynamique à changement de régime.

**Mots clés** : Gestion de risques, couverture de risques, gestion de collateral, programmation dynamique, fonction d'utilité, coûts de transaction, produits structurés, swap de taux d'intêret.

# Summary

This thesis provides various contributions on the subjects of counterparty risk and structured product valuation. Concerning counterparty risk, we propose new flexible and effective methods to handle collateral management, by using strategies based on either one of two different objectives, that is, the minimization of the expected losses and the maximization of the expected utility of the final wealth of the counterparties. This approach is flexible and can account for wrong (and right) -way risk and transaction costs. The resulting strategies are compared with the existing conservative collateral management strategy through basic examples. Concerning structured product valuation, we propose a dynamic programming-based valuation method that can be used for all types of structured products. The pricing approach is general and can be applied to any type of dynamics for the reference underlying value or return. We also provide a Black-Scholes type formula for the pricing of principal protected callable notes with a single early redemption date under a two state Markov regime-switching log-normal model.

**Keywords** : Risk management, collateral management, dynamic programming, transaction costs, structured products, utility function, interest rate swap.

# Contents

# List of tables

# List of figures

# Acknowledgments

Foremost, I would like to express my sincere gratitude to my supervisor Prof. Michèle Breton for the continuous support of my Ph.D study and research, for her patience, motivation, and enthusiasm. Her guidance helped me in all the time of research and writing of this thesis, and I could not have imagined having a better supervisor and mentor for my Ph.D study. Thank you Michele.

Besides my supervisor, I would like to thank the rest of my committee: Prof. Chantal Labbé, Prof. Matt Davison, Prof. Sylvain Perron, and Prof. Benjamin Croitoru, for their insightful comments, and interesting questions.

My sincere thanks also goes to Prof. Georges Dionne, Prof. Louis Doray, Prof. Giles Pisier, Prof. Serigne Aliou Lo, Prof. Ibrahima Thioub, Prof. Mamadou Sangharé and Prof. Mamour Sankhé who have greatly influenced my academic career.

I thank the Institut Finance Mathématiques de Montréal (IFM2) and the Natural Sciences and Engineering Research Council of Canada (NSERC) for their financial support.

Last but not the least, I would like to thank my family: my wife Fatou S. Diouf, my son, my mother Fatou, my aunts Mbene, Soda, my uncles Ibrahima, Moussa, my cousins Astou Mbene, Modou, Mbaye, my friends of here and abroad and my fellow Ph.D candidates Rim, Malek, Saad, and friendly Diane Brousseau and Jennifer Caron.

# Chapter 1

# General introduction

Following the financial crisis of 2008, with the collapse of giants, such as Lehman Brothers and Bear Stearns, and the contagion effect that ensued, financial institutions are investing more in research and innovation to develop their business practices in order both to reduce counterparty risk and to increase their capital reserves and liquidity. With respect to counterparty risk, the Basel III Committee strongly recommends the use of collateral transfer; on the other hand, to increase capital reserves and liquidity, financial institutions can issue structured products. However, note that each of these operations requires to take specific decisions that have their share of risk, and that it is important that these decisions be taken optimally.

Whether in the literature or in practice, the current approach in collateral management consists of asking for a transfer of additional collateral when the exposure exceeds a certain pre-established threshold at fixed observation dates (for instance, on a daily or weekly basis), and the amount required is such that the exposure is brought back below the threshold. This approach, known as conservative, instantaneous or local, eliminates the instantaneous risk; nevertheless, it does not take into account fees related to the transfer of collateral, default probabilities and wrong (right)-way risk[1]. In addition, like any local risk hedging method,

[1] This notion will be defined latter

this approach is myopic in the sense that it ignores events that may occur between two successive observation dates, and after the near observation date. In this thesis, we propose a more general approach using strategies that account for the whole horizon of the problem.

For financial institutions, structured products constitute an effective financing method, at a cost often lower than their funding rate. Indeed, the average yield of structured products is, in most cases, lower than the yield on bonds of the issuer, especially as these products are often issued at prices higher than their intrinsic values. This may explain the rapid growth of the structured product market in recent years. It is then particularly interesting to have an efficient valuation method that can take into account several important aspects, such as the default probability of the issuer, the various types of reference assets and the diversity of optional clauses in the contract.

Currently, there exist four main approaches to evaluate structured products, that is, Monte Carlo simulation, decomposition, numerical integration and partial differential equation (PDE). However, each of these approaches has its share of shortcomings. For example, Monte Carlo simulation is easy to implement, but requires a large amount of computation time and memory. The decomposition approach is only applicable to a limited number of products and models as its effectiveness depends on the availability of analytical formulas for the components. The numerical integration and the PDE approaches are only efficient when the final payoff does not depend on the reference asset path. We propose a valuation method, based on dynamic programming coupled with finite element interpolation, that can be applied to all types of structured products and any dynamics of the reference asset.

This thesis is composed of four main parts. Chapter 2 is an introduction to dynamic programming that recalls the basic ideas underlying this approach. In Chapter 3, we propose a new flexible and efficient method to handle collateral management, by using strategies based on two possible objectives, that is, the minimization of the expected losses or the maximization of the expected utility of the final wealth of counterparties. This approach is flexible and can take into account wrong (and right) -way risk and transaction costs. The resulting strategies are compared with the conservative collateral management strategy through basic examples. In Chapter 4, we provide a Black-Scholes type formula for the pricing of compound options and principal protected callable notes with a single early redemption date under a two states Markov regime-switching log-normal model. In Chapter 5, we propose a dynamic programming-based valuation method that can be used for all types of structured products. This method is general and can be applied to any dynamics used to model the reference underlying value or return.

In summary, this thesis provides various contributions on the topics of counterparty risk and structured product valuation. With respect to counterparty risk, our numerical illustrations show that the dynamic programming-based approach is beneficial for both the collateral sender and receiver. In fact, compared to the conservative approach, the DP strategy generally requires less collateral adjustments, reduces the loss in case of default, and is able to adapt dynamically. With respect to structured product valuation, in addition to the closed formula derived for callable products with a single early redemption date and compound options under regime switching model, the dynamic programming-based valuation approach that we proposed is more efficient and more general than existing valuation methods.

# Chapter 2

# Introduction to dynamic programming

**Summary**

Most of the techniques used in this thesis are based on a dynamic programming approach; in this introductory chapter, we briefly present the basic principles of such an approach in financial engineering applications. We also present a few techniques used to solve the resulting dynamic programming recursion.

**Keywords** : dynamic programming, finite-element approximation.

## 2.1 Introduction

Dynamic programming (DP) is a general approach used to solve a wide range of problems that present the features of overlapping sub-problems and optimal substructure. The overlapping sub-problems feature is present when the problem can be broken down into smaller sub-problems, and the optimal substructure feature, also called the *Bellman principle of optimality,* means that the solution of the whole problem can be obtained recursively from the optimal solutions of the sub-problems. The DP approach takes its origins in the 1950s with the work of Richard Bellman (Bellman 1953); DP is much more efficient than naive approaches, such as brute-force enumeration. It has been used in many application fields, including economics and financial engineering.

In economics, the DP approach has been applied to a wide range of problems. For instance, Merton (1973) proposes a DP-based method to solve an inter-temporal portfolio choice problem in a continuous time framework. Stokey *et al.* (1989) and Ljungqvist and Sargent (2012) provide several DP-based methods that can be used to solve theoretical problems in applied economics, including economic growth, business investment, industrial organization, policy, and labor economics.

In financial engineering, it is now recognized that DP is probably the most simple and efficient way to handle evaluation problems involving intermediate decisions, such as, for instance, the pricing of derivatives with countable exercise opportunities (e.g. Bermudian options). The DP approach can be used for any underlying asset dynamics, provided that the risk neutral dynamics and parameters can be recovered from that of the model under the physical measure, a condition that is verified by almost all models used in the financial literature.

Applications of DP for the evaluation of financial products can be found in Ben-Ameur, Breton and L'Écuyer (2002), where the authors use DP combined with finite element approximation to price American-style Asian options, Ben-Ameur, Breton and François (2006), where DP is used to evaluate installment options, Ben-Ameur *et al.* (2007), where DP is applied to price options embedded in bonds, Ben-Ameur, Breton and Martinez (2009), to price options under a GARCH specification, Ben Ameur, Brigo and Errais (2009), to price credit default swaps (CDS) and CDS options, Breton, Frutos and Serghini-Idrissi (2010) who focus on spectral approximation methods under the GARCH framework, and Boudhina and Breton (2013), for the pricing of long-maturity American put options.

In this chapter, we briefly present the basic principles of the DP approach in a discrete-time framework and recall a few techniques used to solve the resulting dynamic programming recursion. The chapter is organized as follows: Section 2.2 presents the key components of the discrete-time dynamic program, Section 2.3 provides solution methods for the DP recursion, and Section 2.4 concludes.

## 2.2 Key components of the DP approach

In this section, we present the most important characteristics of dynamic programming in a discrete-time framework. Let $\mathbb{T} = \{t_0, t_1, \ldots, t_n\}$ be a set of *observation dates*, where $t_0$ is the initial date, and $t_n = T$ is the time horizon. For simplicity, we assume that the time elapsed, $t_{m+1} - t_m$, between two successive dates is constant and equal to 1 unit of time (day, week, year, etc), for $m = 1, 2, \ldots, n-1$, so that date $t_m$ is represented by $m$. Consider a problem which presents the features of overlapping sub-problems and optimal substructure, and for which a decision is taken at each observation date in the set $\mathbb{T}$. For example, in the case of the evaluation of a Bermudan option, sub-problem $m$ consists of finding the best strategy between either exercising immediately at date $m$ or holding the option at least until the next exercise date $m+1$. In this particular case, it is convenient to solve the problem backwards in time (backward recursion is applied for most problems where the division into sub-problems is made along the time dimension). Indeed, if we know the value of this option at the next date $m+1 \in \mathbb{T}$, the optimal strategy at the current date $m$ can be found by comparing the discounted expectation of this future value with the current exercise value.

At each decision date, the dynamic programming approach requires the observation of the value of *state variables,* which represent the information that is necessary to solve the current sub-problem. In most cases, the state variable vector includes all relevant information about state and decision history. In the case of derivative pricing, the state could consist of the current characteristics of the underlying asset (current price and eventually volatility), and could also include past characteristics for contracts with path-dependent payoffs.

In the case of collateral management, the state vector could include the market value of the portfolio of transactions, the available amount of collateral and eventually the current wealth of a counterparty. We summarize the state variable through a discrete-time (possibly multidimensional) process $\mathbb{X} = \{X_m\}_{m \in \mathbb{T}}$.

A third key component of the DP formulation is the *decision space*, denoted hereafter by $\mathbf{\Delta}$, which describes all eligible decisions. For example, the available decisions could be "exercise or wait", or an amount of assets to buy or sell. These decisions are taken in order to optimize a given criterion, which can be maximizing an expected payoff (in the case of options), minimizing a global expected loss (in risk management), or maximizing a final utility.

At each decision date, when the state is observed and a decision is taken, a *transition scheme* specifies how the next state is determined from the decision taken in the current state. The transition scheme from date $m$ to date $m+1$ can be represented by a function $f_m$, which generally depends on the current state $s$, the decision $d$, and the date $m$, that is:

$$X_{m+1} = f_m(x, d) \text{ conditional on } X_m = x \text{ and decision is } d \in \mathbf{\Delta} \text{ at date } m. \tag{2.1}$$

In several cases, this transition scheme is stochastic but the distribution of the state variable at the next observation date is known. For example, when the state variable is an asset price modeled using a geometric Brownian motion, the next state follows a log-normal distribution.

The immediate impact of a decision $d \in \mathbf{\Delta}$ at date $m$ in state $x$ can be represented by a reward (or cost) function $R_m$, which depends on $s$ and $d$.

When all components listed above are provided, the formulation of sub-problem $m$ for all $m \in \mathbb{T}$ can be written as follows:

$$W_m(x) = \text{opt}_{d \,\in\, \boldsymbol{\Delta}}\{R_m(x,d) + \beta \mathbb{E}_{m,x}\left[W_{m+1}(f_m(x,d))\right]\}, \tag{2.2}$$

where the function $W_n$ at the final date is known on the state space, where $\mathbb{E}_{m,x}[\cdot]$ denotes the expectation under the risk-neutral measure conditional on information available at date $m$ , and where $\beta$ is the one-period discount factor. $W_m$ is the value function corresponding to sub-problem $m$. Equation (2.2) is called the the dynamic programming functional equation or the DP recursion, which formalizes the *Bellman principle of optimality.* This recursion is simple but sufficiently general to describe several problems in finance and economics. One of the advantages of the DP approach is the fact that, by providing the collection of functions $W_m$ for $m = 0, 1, \ldots, n$, it yields the solution of the problem at all decisions dates for all possible realizations of the state variable (An interpolation is often needed to get the value functions in the whole state space).

Computational complexity of the DP approach increases exponentially with the dimension of the state space. More precisely, the requirements on time and memory increase exponentially with the dimension of the state space. This phenomenon known as the "curse of dimensionality," is the most common limitation of the DP approach. Because of that, DP is better suited to problems written on low-dimensional observable state space.

## 2.3 Approximation procedure

In general, the DP value functions cannot be obtained in closed-form, and therefore a numerical procedure is required to solve the DP recursion. The numerical challenge of the backward numerical procedure is the determination of the expected value of a function that is only known on a finite set of points, that is, the computation of $\mathbb{E}_{m,x}\left[W_{m+1}\left(f_m(x,d)\right)\right]$. In most cases, the easiest way is to use a backward interpolation procedure. The general idea of backward interpolation is to start from the value function $W_n$ that is defined on the whole state space, use the DP recursion (2.2) to compute the value function $W_{n-1}$ on a finite number of state points (a grid), and finally interpolate these values to get an approximation $\widehat{W}_{n-1}$ of $W_{n-1}$ on the whole state space. Similarly, starting from the approximation $\widehat{W}_{n-1}$ of $W_{n-1}$, we get an approximation $\widehat{W}_{n-2}$ of $W_{n-2}$. This procedure is repeated until getting an approximation $\widehat{W}_0$ of $W_0$.

Interpolation is used so that the conditional expectation in Equation (2.2) can be approximated by $\mathbb{E}_{m,x}\left[\widehat{W}_{m+1}\left(f_m(x,d)\right)\right]$. Polynomial interpolation is a standard choice, and several possibilities have been investigated in the literature. These include spectral interpolation or finite element interpolation, for instance linear-quadratic, bi-linear or spline interpolation. The approximation $\widehat{W}_{m+1}$ is then a polynomial function of the state variable $X_{m+1}$, and consequently the conditional expectation $\mathbb{E}_{m,x}\left[\widehat{W}_{m+1}\left(f_m(x,d)\right)\right]$ is a linear combination of terms of the form $\mathbb{E}_{m,x}\left[X_{m+1}^q\right]$ where $q$ is an integer. Closed-form formulas are available for such conditional expectations for several dynamics of the state variable, such as, for instance, geometric Brownian motion and Markov log-normal regime switching models for stock prices, and Vasicek models for interest rates. Numerical integration can be used in any other case where a closed form is not available.

In this thesis, we use finite-element interpolation by piecewise polynomials. Consider a state space of dimension $N \geq 1$. The first step in the interpolation scheme is the choice of a grid $\Upsilon$. Here, we define $\Upsilon = \Upsilon_1 \times \cdots \times \Upsilon_N$, a subset of $\mathbb{R}^N$ with

$$\Upsilon_k = \{a_i^k, i = 0, \ldots, n_k + 1\}, k = 1, \ldots, N$$

where the $\{a_i^k\}$ are strictly increasing sequences of real numbers (it is often assumed that $a_0^k = 0$ or $-\infty$ and $a_{n_k+1} = +\infty$). Denote by $J_k = \{0, 1, \ldots, n_k\}, k = 1, \ldots, N$, the index sets, $J = J_1 \times \cdots \times J_N$, and by $R_j = [a_{j_1}^1, a_{j_1+1}^1] \times [a_{j_2}^2, a_{j_2+1}^2] \times \cdots \times [a_{j_N}^N, a_{j_N+1}^N]$ for all vector $j = (j_1, j_2, \ldots, j_N) \in J$. Let $G$ be a function of $N$ variables defined on $\bigcup_{j \in J} R_j$, the union of all hyper-rectangles $R_j$. The piecewise polynomial interpolation of $G$ consists of finding a function $\psi$ that has a polynomial form $\omega_j$ inside each hyper-rectangle $R_j, \forall j \in J$, and which coincides with $G$ on $\Upsilon$. The terms $\omega_j$ are called polynomial basis functions and we can write:

$$\psi(x) = \sum_{j\in J} I_j(x)\omega_j(x),$$

where $I_j$ is the indicator function of the hyper-rectangle $R_j$. Note that for points outside $\bigcup_{j\in J} R_j$, an extrapolation technique is used to define the interpolating function $\psi$. For instance, a nearest hyper-rectangle or a null second order derivative argument can be considered.

This definition of piecewise polynomial interpolation is sufficiently general to encompass piecewise bilinear, linear-quadratic (cubic), bicubic and spline interpolations. For example, when $N = 2$, the basis polynomials of the piecewise bilinear interpolation are given by:

$$\omega_j(x_1, x_2) = \frac{a^2_{j_2+1} - x_2}{a^2_{j_2+1} - a^2_{j_2}} p_{j_1}(x_1) + \frac{x_2 - a^2_{j_2}}{a^2_{j_2+1} - a^2_{j_2}} p_{j_1+1}(x_1) \tag{2.3}$$

where

$$\begin{aligned} p_{j_1}(x_1) &= \frac{a^1_{j_1+1} - x_1}{a^1_{j_1+1} - a^1_{j_1}} G(a^1_{j_1}, a^2_{j_2}) + \frac{x_1 - a^1_{j_1}}{a^1_{j_1+1} - a^1_{j_1}} G(a^1_{j_1+1}, a^2_{j_2}) \text{ and} \\ p_{j_1+1}(x_1) &= \frac{a^1_{j_1+1} - x_1}{a^1_{j_1+1} - a^1_{j_1}} G(a^1_{j_1}, a^2_{j_2+1}) + \frac{x_1 - a^1_{j_1}}{a^1_{j_1+1} - a^1_{j_1}} G(a^2_{j_1+1}, a^2_{j_2+1}), \end{aligned}$$

and $x = (x_1, x_2)$. The difference between piecewise linear and piecewise cubic or quadratic-linear interpolation is that, in the second and third cases, the basics functions $p_j$ are univariate polynomials of second and third order, respectively. Another example is bicubic spline interpolation, which can be defined as follows:

$$\omega_j(x, y) = \sum_{0\leq q,p\leq 3} \alpha_{qp}(x - a^1_{j_1})^q (y - a^2_{j_2})^p \tag{2.4}$$

where for all pairs $(j_1, j_2)$, coefficients $\alpha_{qp}$ are obtained using coincidence and smooth conditions. This type of interpolation is already programmed in several softwares such as Matlab, Java or $C$, providing the interpolating coefficients automatically. In this thesis, we use the bilinear, linear cubic, and trilinear interpolation.

## 2.4 Conclusion

This chapter briefly outlines and discusses the dynamic programming approach. Details about the general formulation and implementation of DP with backward interpolation are provided. The techniques presented in this chapter are used in the following chapters with some additional details pertaining to the specific application considered.

# Chapter 3

# Collateral management

**Summary**

This chapter proposes a new flexible and efficient approach to manage the collateral for a portfolio of transactions between two entities. We determine collateral management strategies optimizing either one of two different objectives. The first objective is the minimization of the sum of management costs and expected losses, while the second objective is the maximization of the expected utility of the final wealth of the collateral receiver. The optimal collateral management strategy is found using a procedure that combines dynamic programming, multi-objective optimization, and finite-element interpolation. Optimal strategies account for wrong (and right)-way risk and for transaction costs. We compare these strategies with existing conservative collateral management methods through three examples. In the first example, we assume that the market value of the portfolio of transactions follows a jump-diffusion model; in the second example, we consider an interest rate swap; in the third example, the market value of the portfolio of transactions follows a log-normal distribution and the collateral receiver has a constant relative risk aversion utility function.

**Keywords**: Collateral management, expected loss, dynamic programming, multi-objective optimization, utility maximization, finite-element interpolation.

## 3.1 Introduction

Counterparty credit risk (CCR) is the risk that a counterparty involved in a bilateral over-the-counter (OTC) derivative transaction will default before the final settlement of the transaction. This type of risk has increased exponentially over the last two decades, particularly because of the rapid development and the growing complexity of the OTC derivatives market. At this date, CCR is experiencing a revolution and several financial institutions are giving increasing attention to this topic. In fact, following the 2008 financial turmoil with the collapse of giants such as Lehman Brothers, and Bear Sterns, and the contagion effect that ensued, financial institutions are investing heavily in research to develop strong business practices in order to mitigate counterparty credit risk.

There are diverse ways of mitigating CCR, of which the most popular is probably *credit insurance*. To be more explicit, consider an OTC derivative transaction between two counterparties $\mathbb{A}$ and $\mathbb{B}$. A credit insurance for counterparty $\mathbb{A}$ consists of buying insurance from another entity $\mathbb{C}$ that will cover a part or the total losses incurred in the event following a default of $\mathbb{B}$. Despite its popularity, this technique brings additional risk by involving a third party. Indeed, in addition to being exposed to the default of $\mathbb{B}$, counterparty $\mathbb{A}$ is also exposed to the default of the insurance seller $\mathbb{C}$. An example of credit insurance consist of buying a credit default swap (see Giglio 2011 for more details).

CCR mitigating techniques also include *credit or counterparty valuation adjustment* (CVA) technique which consists in adjusting the prices of the trades in a way that takes into account the exposure due to the possibility of default of the counterparty. With this technique the fair prices of the trades are the difference between their default-free prices and the risk-neutral expectation of the discounted loss (called CVA). Note that, CVA method does not involve a third party. For more details about this approach, one can refer Pykhtin and Zhu (2007), Sorensen and Bollier (2003), Brigo and Masetti (2005), Brigo and Capponi (2008) and to Alavian *et al.* (2008) for an overview.

.

A third variety of CCR mitigating techniques includes *netting and collateral agreements*. A netting agreement is a contract that allows aggregation in value of all transactions between the involved names. So, under a netting agreement, a portfolio of contracts between two names can be considered as a single contract (in the sense of one market value). A collateral agreement consists of asking one or both names involved in the agreement to post collateral when the need arises, for example when exposure of one name exceeds a given threshold. Collateral represents all assets, shares or cash given as security by one entity to another in order to cover counterparty credit risk resulting from bilateral transactions between these two entities. In case of default of the collateral sender, the collateral holder has the right to retain the assets given as collateral in order to compensate its financial losses. When only one of the names has the right to request collateral, the agreement is called unilateral, whereas it is bilateral when both can request collateral. As for the CVA approach, an advantage of netting and collateral agreement is that these techniques do not involve a third party, limiting CCR to the entities implicated in the agreement.

The Collateral agreement structure was used for the first time by Bankers Trust and Salomon Brothers in the 1980s for the purpose of credit exposure reduction, but its first widespread use dates back to 1997 with the Russia default and the Asian crisis. As reported in the ISDA Margin Survey (2008), this widespread use has not ceased since 1998 and has experienced a sizable peak in 2009 during the U.S. sub-prime mortgage crisis. The growing volumes in today's securities market partly explains this growth; however the main reason is probably the fact that financial institutions are increasingly aware of the possibility of sudden default events, even from high profile entities.

In practice, major collateral agreements are included in the Credit Support Annex of an International Swaps and Derivatives Association's Master Agreement, which is a set of legal agreements governing transactions between two or several counterparties (see Zepeda

2013 for a detailed description). The Credit Support Annex provides all standard terms that apply to all the transactions entered into by involved counterparties. For example, it includes the procedure for collateral computation and the circumstances under which additional collateral is requested.

Besides reducing counterparty credit exposure, collateral can also reduce the capital requirement. For example Bliss and Kaufman (2006) show that the use of collateral can reduce market exposure by up to around 93%. Cherubini (2005) arrives at a similar conclusion, stating that "for derivative transactions with corporate counterparty, the most effective risk mitigating technique appears to be the use of collateral." In the same vein, Ghosh *et al.* (2008) reports that collateralization is able to reduce overall exposure by approximately 40 to 50%. In summary, as noted by Gregory (2010), when properly specified, collateral agreement can reduce counterparty credit risk more than any other technique and can be an important tool for asset optimization. With all these advantages, it is clear that collateral management will take an increasing importance in the coming years. For example, in Thought (2012), the survey magazine of J.P. Morgan, it is reported that "collateral optimization will become a key competitive arena, as asset managers look to deploy the right collateral, for the right duration, and in the right place."

Despite its advantages, the use of collateral can give rise to additional risk, including operational, liquidity and market risk. For example, an inadequate quantity of collateral or non-liquid collateral may not be sufficient to cover losses in the event of a sudden default. More details about the risks involved in a collateral management system can be found in the introductory survey by Chandrashekar (2008). For a detailed discussion about the current trends and the risk involved in collateral management, see the newly published white paper of the Depository Trust and Clearing Corporation (DTCC 2014).

The basic components of a collateral agreement are very simple. To begin with, the collateral agreement must specify the types of products that can be used as collateral, for instance, cash, government or agency securities, equity, corporate bonds, or mortgage-backed securities. However, non-cash collateral may be subject to liquidity problems and may cause additional risk due to their non-zero volatility and correlation with other products and organizations. Generally speaking, regardless of the type of asset used as collateral, it must be at least sufficiently liquid to be exchanged quickly, it must have small correlation with other assets and organizations, and it should be easy to evaluate if necessary. In other words, it should be a high-quality liquid asset. The agreement must also specify if collateral can be substituted, passed as collateral to others counterparties or used for any other purposes. Usually, two entities that enter into a collateral agreement have more than one bilateral contract and therefore it is necessary that the agreement clearly specify what contracts are subject to the collateralization. A common procedure is to allow the netting of all transactions between the two entities except those concerning products that are complex or hard to evaluate.

Another important parameter that has to be established in a collateral agreement is the *margin call frequency*, which is the minimum delay between two successive margin calls. The margin call frequency may depend on the type of contract included in the agreement, on the type of collateral, and also on the features of the involved counterparties. In practice, a margin call frequency of ten days is often used, but for ordinary products that are easy to evaluate, the frequency can be daily. Nonetheless, note that a small margin call frequency may cause an increase of operational workload risk and cost.

A collateral agreement should also specify all conditions under which a counterparty may request or return collateral. Currently the common practice is to request additional collateral at predefined dates (for example, every ten days), when the *uncollateralized exposure* exceeds a pre-established *threshold*. The uncollateralized exposure corresponds to the positive value of the difference between the market value of the portfolio of contracts and the collateral held by the entity requesting the collateral. Usually, the threshold depends on the credit rating of the counterparty, and it represents the level of unsecured exposure that the other counterparty is willing to accept.

In many cases, when signing a collateral agreement, counterparties specify a *minimum transfer amount* and an upfront amount of collateral, also called the *independent amount*. The role of the minimum transfer amount is primarily to reduce the frequency of collateral exchanges and avoid the transfer of paltry, while the independent amount serves as a cushion against a sudden jump of the market value of contracts in a short time period. Finally, the collateral agreement must also specify how to evaluate additional posted or returned collateral. This is the most difficult step and it must be handled carefully in order to reduce the risk of disputes. A common practice is to leave this part to an external entity who will be responsible for all evaluations. The primary role of this entity, often called *Third-party valuation agent* or *Central Counter Party* (CCP) is to record the collateral agreement terms and conditions, evaluate collateral asset movement requirements and trade exposures, and to resolve disputes. The Basel Committee on Banking Supervision (2010) recommends the use of CCPs, which are often central banks or large financial institutions. More details about the role of CCPs can be found in Chapter fourteen of Gregory (2010).

In practice, there exists a delay between the time when collateral is requested and the time when it is received. Generally a discount (*haircut*) is applied to the requested amount of collateral in order to take into account this posting delay.

In this chapter, we propose a new flexible and efficient approach to manage collateral for a portfolio of transactions between two entities. We determine collateral management strategies optimizing either one of two different objectives. The first objective is the minimization of the sum of management costs and expected losses, while the second objective is the maximization of the expected utility of the final wealth of the collateral receiver. The optimal collateral management strategy is found using a procedure that combines dynamic programming, multi-objective optimization, and finite-element interpolation. Optimal strategies account for wrong (and right)-way risk and for transaction costs. We compare these strategies with existing conservative collateral management methods through three examples.

The chapter is organized as follows. Section 3.2 presents the existing practices in collateral management and the basic framework of our approach. Sections 3.3 and 3.4 describe the dynamic programming and multiobjective optimization-based strategies for unilateral and bilateral agreements. Section 3.5 presents the utility maximization-based approach. Section 3.6 reports on numerical results obtained for several applications. More specifically, as an example of unilateral agreements, we consider an application where the market value of the portfolio of contracts is modeled with a jump diffusion process; to illustrate bilateral agreements, we consider the case of an interest rate swap with Vasicek dynamics for the floating interest rate; finally, we illustrate the utility maximization-based approach by considering a simple application where the market value of the portfolio of contracts is modeled with a log-normal process. Section 3.7 concludes.

## 3.2 Framework and existing collateral management strategies

This section presents the market and default specifications, along with the existing collateral management strategies for unilateral and bilateral agreements.

### 3.2.1 Market and default models

This subsection introduces notation, the market framework and the procedures used to model the default time of a given counterparty. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\mathcal{G}_t, t \geq 0\}$, a filtration of sub-$\sigma$-algebras of $\mathcal{F}$. We assume that for any date $t$, $\mathcal{G}_t$ contains all the null sets of $\mathcal{F}$ and is right continuous, ie: $\mathcal{G}_t = \bigcap_{t' \geq t} \mathcal{G}_s$. We characterize the market state by a discrete-time stochastic process $\mathbb{X} = \{X_m\}_{m \in \mathbb{T}}$, where $\mathbb{T} = \{0, 1, \ldots, n\}$ is a set of successive observation dates, as defined in Chapter 2. The one period risk-free rate $r$ is assumed constant, and for any real number $y$, we define:

$$y^+ = \max\{0, y\} \text{ and } y^- = \max\{0, -y\}. \tag{3.1}$$

In this market, we assume that the collateral agreement concerns a portfolio of contracts between two entities, called the bank and the counterparty. The net market value of the portfolio of contracts at date $m$ from the bank's point of view is denoted by $V_m$ and is assumed to depend on $x$, the realized value of the process $\mathbb{X}$ at date $m$, according to the following relation:

$$V_m(x) = \sum_i V_m^i(x) = \Psi(x, m, \Theta) \tag{3.2}$$

where $V_m^i$ is the market value of contract $i$ at date $m$ from the bank's point of view, and $\Theta$ is a vector of parameters associated with the portfolio of contracts. For instance, $\Theta$ could contain the exercise prices and maturity dates if the portfolio includes options, or coupon rates and principals if the portfolio includes bonds.

We characterize the default time of an entity $j$ with a $\{\mathcal{G}_t\}$-double stochastic stopping time $\tau^j$. Denote by $\{\lambda_t^j\}_{t\geq 0}$, the default intensity process. Therefore, conditional on no default prior and at the current date $m$, the survival probability at any future date $m'$ is given by the following equation:

$$\mathbb{P}\left(\tau^j \geq m' \mid \mathcal{G}_m, \tau^j \geq m\right) = \mathbb{E}\left[e^{-\int_m^{m'} \lambda_u^j du} \mid \mathcal{G}_m\right]. \tag{3.3}$$

For bilateral transactions, losses can be caused by several events, among which the most likely are essentially changes in the default-free market value of the transactions and the possible default of one counterparty. In a significant number of cases, these two components are closely linked in the sense that the market value of the portfolio of transactions is correlated with the counterparties' default probability, adding complexity in the determination of the expected losses. This situation is known as wrong-way risk when the correlation is positive and right-way risk when the correlation is negative. An effective approach to take into account situations of wrong- and right-way risk was proposed by Hull and White (2012). Their approach consists of modeling the hazard rate as a function of the state process $\mathbb{X}$. As an illustration, assume that the process $\mathbb{X}$ solves the following stochastic differential equation:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dZ_t \tag{3.4}$$

where $\{Z_t, t \geq 0\}$ is a $\{\mathcal{G}_t\}$- standard Brownian motion, $\mu$ is a square-root or linear function of the process $\mathbb{X}$, sufficiently regular to allow for the existence of a solution to equation (3.4). This formulation of the market state process includes several types of dynamics used in financial modeling, as, for instance, the Gaussian Ornstein-Uhlenbeck and the CIR (Cox, Ingersoll and Ross 1985) models for interest rates, the Geometric Brownian motion (Black and Scholes 1976) and stochastic volatility models (Heston 1993) for equity prices. Now assume that $\lambda_t^j = g^j(x)$, where $x = X_t$ and $g$ is a positive measurable function. Then, following Duffie (2005), we can write:

$$\mathbb{P}\left(\tau^j \geq t' \mid \tau^j > t, \mathcal{G}_t\right) = \mathbb{E}_{t,x}\left[e^{-\int_t^{t'} g(X_u)du} \mid \tau^j > t\right] \equiv G_{t'}^j(x,t), \text{ with } j \in \{a,b\} \tag{3.5}$$

where $G_{t'}^j$ solves the following partial differential equation:

$$\mu(x)\frac{\partial G_{t'}^j(x,t)}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2 G_{t'}^j(x,t)}{\partial x^2} - \frac{\partial G_{t'}^j(x,t)}{\partial t} - g^j(x)G_{t'}^j(x,t) = 0, \tag{3.6}$$

with $G_{t'}^j(x,t') = 1$.

Below, we will consider an application with a linear default intensity, which is very attractive since it allows for closed-form solution for the Partial Differential Equation (3.6) as well as wrong (and right)-way risk, if we assume that the pricing function $\Psi$ is monotonic with respect to the market state process value $\mathbb{X}$.

### 3.2.2 Metrics for credit exposure

In this subsection, we provide definitions of measures used in this chapter to quantify the exposure of a given entity (the bank or the counterparty). For all the definitions given here, we refer to Gregory (2010). Let $m$ denotes the current date and $t'$ a future date. We denote by $C_m^b$ the amount of collateral held by the bank at date $m$, and by $\xi^b$ its recovery rate. We define the expected exposure of the bank as the amount expected to be lost by the latter if a default of the counterparty occurs at date $t'$, that is:

$$E^b(m,t',x,c) = (1-\xi^b)\mathbb{E}_{m,x,c}\left[\left(V_{t'} - C_{t'}^b\right)^+\right], \tag{3.7a}$$

where the operator $\mathbb{E}_{m,x,c}$ denotes the expectation at date $m$ conditional on $X_m = x$ and available collateral at date $m$ is $C_m^b = c$. Note that the available collateral $C_{t'}^b$ at future date $t'$ is a function of $c$ and the additional collateral posted at date $m$. The term $\left(V_{t'} - C_{t'}^b\right)^+$ is called the uncollateralized exposure of the bank at date $t'$. Similarly, for the counterparty the expected exposure is given by the following equation:

$$E^a(m,t',x,c) = (1-\xi^a)\mathbb{E}_{m,x,c}\left[(-V_{t'} - C_{t'}^a)^+\right], \tag{3.7b}$$

where $\xi^a$ denotes the recovery rate of the counterparty.

The expected positive exposure is the average of the expected exposure over some predefined time period. Accordingly, the expected positive exposure of the bank and of the counterparty over the time period $[m, m+1]$ for $m = 0, 1, \ldots, n-1$ are given by the following equation:

$$\Pi_m^j(x,c) = \int_m^{m+1} E^j(m,t',x,c)dt', \text{ for } j \in \{a,b\}. \tag{3.8}$$

We assume that to receive an amount of collateral adjustment $h$, the bank (the counterparty) has to pay transaction costs that are a function of $h$, denoted by $q$ defined as follows:

$$q(h) = \begin{cases} q_0 + q_1 h & \text{if } h > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{3.9}$$

These transaction costs represent all fees related to the receipt of collateral, and they are composed of a fixed amount $q_0$, and an amount proportional to the collateral received.

### 3.2.3 Existing collateral management strategies

This subsection presents a brief review of existing collateral management strategies for both unilateral and bilateral agreements. To our knowledge, all existing collateral management strategies can be expressed in the form of what we call herein the *conservative strategy* (see Pykhtin 2009). Denote by $H^b$ and $H^a$ the thresholds of the bank and the counterparty respectively, with the convention that $H^b \geq 0$ and $H^a \leq 0$. For the bank (resp. the counterparty), the *conservative strategy* consists of asking an additional amount of collateral equal to the difference between its uncollateralized exposure and the threshold $H^b$ (resp. $H^a$) whenever this difference exceeds some minimum transfer amount, which we denote $M$ and assume positive and the same for both entities. In practice, the uncollateralized exposure is determined at fixed dates, for instance, every day, week or fortnight. Here we assume that the uncollateralized exposures are determined on the set of observation dates $\mathbb{T}$. According to these assumptions, the conservative strategy can be summarized by the following steps:

At each observation date $m \in \mathbb{T}$:

- If one entity has defaulted during the last time period, close out all positions and compensate the other entity for its losses (if any) using available collateral (if any).
- Otherwise, observe or compute the value of the portfolio of contracts $V_m$ and the net available amount of collateral $C_m$ with the convention that $C_m$ is positive if held by

the bank and negative otherwise.

- Compute the adjustment of collateral:

$$h_m = \begin{cases} V_m - C_m^+ - H^b & \text{if } V_m - C_m^+ > H^b \\ V_m + C_m^- - H^a & \text{if } V_m + C_m^- < H^a \\ 0 & \text{otherwise} \end{cases} \tag{3.10}$$

- if $h_m > M$, the counterparty must post additional collateral equal to $h_m$;
- if $h_m < -M$, the bank must post additional collateral equal to $-h_m$;
- Update the available collateral by adding to $C_m$ the adjustment $h_m$ as soon as it is received.

• Otherwise, wait until the next observation date and repeat the first step.

For unilateral agreements, only the bank can request collateral, and this situation corresponds to the case where $H^a = -\infty$.

The conservative strategy has the advantage of being easy to implement and interpret, making it very attractive in practice. However, this strategy is only based on current exposure, and as such does not take into account the possibility of a sudden downgrading in exposure or default probability[1]. In addition, the strategy requires an appropriate choice of parameters (minimum transfer amount and threshold), which have significant impacts on the effectiveness of the strategy (Gregory 2010, Section 3.7), especially the threshold, which defines the minimum acceptable level of uncollateralized exposure. In the next section, we present a collateral management strategy based on a dynamic programming approach.

[1] Assuming that these features are not taken account in the pricing of the transactions.

## 3.3 Expected loss minimization for unilateral agreements

In this section, we assume that the collateral agreement is unilateral in the sense that only the bank has the right to request additional collateral. This situation occurs for instance when the market value of the portfolio of contracts is always positive from the point of view of the bank (for instance: debt or options), or, more generally, when only the default of the counterparty has an impact on transactions settlement. For unilateral agreements, a basic collateral management strategy consists of finding the amount of collateral that must be posted by the counterparty at each margin date, in order to optimize a pre-specified criterion. The criterion could be the maximization of the bank's utility or the minimization of any kind of risk measure. In this section, the criterion is the minimization of the sum of the transaction costs and the expected loss of the bank for the remaining term of the agreement. With this objective, collateral management becomes an obvious example of a dynamic decision problem.

At each observation date $m \in \mathbb{T}$, we characterize the state of the system (the DP state) by the vector $(x,c)$, where $x = X_m$ is the current value of the underlying state process $\mathbb{X}$, and $c = C_m$ is the current amount of collateral held by the bank. The control variable $h$ is the adjustment of collateral requested from the counterparty. Conditional on the state $(x,c)$ and the control $h$ at date $m$, the state at date $m+1$ is given by:

$$f_m(x,c,h) = (X_{m+1}, c+h). \tag{3.11}$$

In this transition scheme, $c+h$ is the available amount of collateral at date $m+1$. No interest is applied to this amount since, we assume that interest are returned to the counterparty. Denote by $W_m$ the sum of the transaction costs and the expected loss of the bank from date $m$ to the end of the horizon $T = n$, assuming that the control variable is optimally chosen. For any state $(x,c)$ and control $h$ at date $m$, the sum $R_m(x,c,h)$ of the transaction cost and the expected loss over the time interval $(m, m+1]$ is given by the following equation:

$$R_m(x,c,h) = q(h) + (1 - G^a_{m+1}(x,m))\Pi^b_m(x,c+h), \tag{3.12}$$

where $G^a_{m+1}(x,m)$ is the survival function of the counterparty, $\Pi^b_m(x,c+h)$ is the expected exposure of the bank, and $R_m$ represents the immediate impact of decision $h$ taken at time $m$ as introduced in Chapter 2. Assembling all these components, the value functions $W_m$ are defined by the following dynamic programming recursion:

$$W_n(x,c) = 0 \tag{3.13a}$$

and for any intermediate date $m$,

$$W_m(x,c) = \min_{h \in [-c,\infty)} \{R_m(x,c,h) + e^{-r} G^a_{m+1}(x,m)\mathbb{E}_{m,x,c}\left[W_{m+1}(f_m(x,c,h))\right]\}, \tag{3.13b}$$

where the optimal adjustment of collateral at date $m$ corresponds to the value of the control variable $h$ that solves the minimization problem of Equation (3.13b). We allow the adjustment of collateral to be negative, meaning that the bank may return a part of the collateral held (with no transaction cost).

Define the following function:

$$F_{m,x,c}(h) = R_m(x,c,h) + e^{-r} G^a_{m+1}(x,m)\mathbb{E}_{m,x,c}\left[W_{m+1}(f_m(x,c,h))\right]. \tag{3.14}$$

$F_m$ is the sum of transaction costs paid by the bank and the discounted expected total loss due to the possible default of the counterparty. To reduce its exposure, the bank ask for collateral, so the more collateral it requires posting, the more his exposure is reduced. However, since receiving this collateral has a cost (transaction costs) which increases with the requested amount. In addition, asking asking higher amount may leads to litigation with the counterparty. So, there is clearly a tradeoff between reducing exposure and paying transaction costs. This tradeoff is made by minimizing function $F_m$.

The following proposition shows that the minimization problem in Equation (3.13b) has a unique solution.

**Proposition 3.3.1.** *At each date $m \in \mathbb{T}$, and for any state $(x,c)$, the univariate function $F_{m,x,c}$ is convex.*

**Proof 3.3.1.** *To prove this proposition, it suffices to show that $R_m(x,c,h)$ is convex with respect to $h$, and the value function $W_{m+1}$ is convex with respect to the available amount of collateral. First of all, note that for any pair of positive real numbers $(K_1, K_2)$, and $\alpha \in [0,1]$, we have:*

$$(V_{t'} - (\alpha K_1 + (1-\alpha)K_2))^+ \leq \alpha\,(V_{t'} - K_1)^+ + (1-\alpha)\,(V_{t'} - K_2)^+$$

*In addition, the conditional expectation operator $\mathbb{E}_{m,x,c}$ preserves the convexity property (because of its linearity property), implying that the expected exposure is convex with respect to the available amount of collateral. On the other hand, the integral is a linear operator, and the transfer cost $q(h)$ as well as the future available amount of collateral $(c+h)e^r$ are linear and increasing with respect to $h$ and $c$. Therefore, $R_m(x,c,h)$ is convex with respect to $h$ and $c$.*

*To show that the value functions $W_m$ are convex with respect to the second argument $c$, we use backward induction. More specifically, since $W_n$ is identically equal to zero, the convexity holds at date $n$. Assume that there exists a stage $m+1 \leq n$, such that $W_{m+1}$ is convex with respect to its second argument, then $F_{m,x,c}$ is convex with respect to $c$ for all pairs $(x,h)$. Therefrom, by definition, $W_m$ is convex with respect to $c$. Finally, since $(c+h)e^r$ is linear and increasing with respect to $h$, then $F_{m,x,c}$ is convex.* $\square$

The DP recursion can be solved using the interpolation procedure discussed in Chapter 2. The convexity of $F_{m,x,c}$ implies that the minimization problem of Equation (3.13b) admits a global unique solution and this holds for any dynamics of the market value process

$\{V_m\}_{m\in\mathbb{T}}$. However, note that this convexity property does not necessarily hold for any type of approximation procedure used to solve the DP recursion. Examples of approximations that preserve the convexity property are the Bernstein polynomials (Bernstein 1912) or linear interpolation. In our numerical illustration (see Section 3.6), we use linear interpolation, which preserves both the monotonicity and convexity properties of the value function, and we provide more details about the evaluation of the conditional expectation in Equation (3.13b).

## 3.4 Expected loss minimization for bilateral agreements

In this section, we propose a multiobjective dynamic programming-based collateral management strategy for bilateral agreements. The main challenge for bilateral agreements is to find, at each margin date, an adjustment of collateral that minimizes the cumulative expected loss of both the bank and the counterparty. This means that we are facing a multiobjective optimization problem. Multiobjective optimization concerns mathematical optimization problems involving more than one objective function to be optimized simultaneously. It is widely used in several areas, as for instance, in equilibrium and welfare theories, in game theory, and in fundamental mathematics. A general formulation of a two-dimensional example of such problem is as follows:

$$\underset{d\,\in\,\boldsymbol{\Delta}}{\mathrm{Min}}\, F(d) = (F_1(d), F_2(d)) \tag{3.15}$$

where $\boldsymbol{\Delta}$, the decision space, describes a set of constraints that the control (or decision) variable $d$ must verify. In practice, multiobjective optimization is an issue in any situation involving multiple decision-makers with conflicting objectives. Typically in this type of problem there is no global solution, and an alternative trade-off solution is often chosen with a predetermined criterion. The most commonly used criterion is the concept of Pareto (1906) optimality, which is defined as follows:

**Definition 3.4.1.** *A point $d^* \in \boldsymbol{\Delta}$ is said to be Pareto optimal if and only if there is no other* *point* $d \in \mathbf{\Delta}$ *such that* $F(d) \le F(d^*)$ *and* $F_i(d) < F_i(d^*)$ *for at least one index* $i = 1,2$.

Accordingly, a Pareto optimal point is such that there is no other point that can improve one of the objective functions without deteriorating another one, so that a Pareto optimal point can be seen as a compromise solution. There exist several approaches to find Pareto optimal points; in this chapter, we focus on the *scalarization criterion* approach, which consists of combining the objective functions into a single scalar function. Our choice to use the scalarization criterion is motivated by the fact that the optimum of a global criterion function which is increasing with respect to each objective function is a Pareto optimal point (see Stadler 1988 or more recently the survey of Marler and Arora 2004). An example of scalarization consists of defining the global criterion function as a weighted sum of the different objective functions.

We now formulate the multiobjective and DP-based collateral management strategy problem. Similarly to the previous section, we characterize the state of the system by the vector $(x, c)$, where $x = X_m$ is the current value of the underlying state process $\mathbb{X}$. However, the second argument $c = C_m$ is now the net amount of collateral held by one party at date $m$, with the convention that $c > 0$ means that the collateral is held by the bank (or equivalently, that the bank holds more collateral than the counterparty), and $c < 0$ means that the collateral is held by the counterparty.

The control variable is $h$, the amount of collateral exchanged, with the assumption that a positive (resp. negative) value of $h$ means that adjustment is received by the bank (resp. the counterparty).

The expected positive exposures $\Pi_m^b$ and $\Pi_m^a$ of the bank and the counterparty are given by Equations (3.7a),(3.7b) and (3.8), with:

$$C_{t'}^b = c^+ + h^+ \text{ and } C_{t'}^a = c^- + h^-.$$

Define:

$$R_m(x,c,h) = \omega_b R_m^b(x,c,h) + \omega_a R_m^a(x,c,h) \tag{3.16}$$

where

$$R_m^b(x,c,h) = q(h^+) + (1 - G_{m+1}^a(x,m))\Pi_m^b(x, c^+ + h^+) \tag{3.17a}$$

and

$$R_m^a(x,c,h) = q(h^-) + (1 - G_{m+1}^b(x,m))\Pi_m^a(x, c^- + h^-) \tag{3.17b}$$

denote the corresponding immediate impact function for the bank and the counterparty respectively. The weights $\omega_b$ and $\omega_a$ are positive real numbers summing to one, and $G_{m+1}^b$ and $G_{m+1}^a$ denote the survival probability functions of the bank and the counterparty in time period $(m, m+1]$. Assembling all these ingredients, the value function is defined by the following dynamic programming recursion:

$$W_n(x,c) = 0 \tag{3.18a}$$

and, for any intermediate date $m$,

$$W_m(x,c) = \min_{h \in \mathbb{R}} \{R_m(x,c,h) + e^{-r} G_{m+1}(x,m) \mathbb{E}_{m,x,c}[W_{m+1}(f_m(x,c,h))]\} \tag{3.18b}$$

where $f_m(x,c,h) = (X_{m+1}, c+h)$, and $G_{m+1} = G_{m+1}^b G_{m+1}^a$ represents the survival function in time interval $(m, m+1]$, that is, the probability that that no entity defaults during that time interval. The optimal adjustment of collateral at date $m$ corresponds to the value of the control variable $h$ that solves the minimization problem of Equation (3.18b). The value function $W_m$ represents the weighted sum of the transaction costs and expected loss of the bank and the counterparty from date $m$ to maturity. Similarly to the unilateral case, the argument function in the minimization problem of Equation (3.18b) is convex with respect to the control variable (it is the weighted sum of two convex functions with positives weights).

## 3.5 Final utility maximization

In this section, we develop a collateral management strategy based on a final utility maximization approach. The strategy consists of finding the sequence of collateral adjustments that maximizes the utility of the final wealth of the bank. We focus on unilateral agreements, but the generalization to bilateral agreements is straightforward, and can be done similarly as in Section 3.4.

We consider the framework of Section 3.3, and we denote by $U$ a concave utility function of wealth, capturing the bank's risk aversion. At each observation date $m \in \mathbb{T}$, we characterize the state of the system with the triplet $(x,c,y)$, where $x = X_m$ is the realized value of the underlying state process, $c = C_m$ is the amount of collateral held by the bank, and $y = Y_m$ is the wealth level of the bank. We assume that transaction costs paid by the bank reduce this wealth; accordingly, when the counterparty does not default during the time interval $[m, m+1]$, the transition scheme of the state variable $(x,c,y)$ is given by:

$$f_m(x,c,y,h) = \left(X_{m+1}, c+h, (y-q(h))e^{r}\right), \tag{3.19a}$$

while, if a default of the counterparty occurs at date $t'$ between dates $m$ and $m+1$, the wealth of the bank at date $t'$ is given by:

$$Y^d_{t'} = \begin{cases} (y-q(h))e^{r(t'-m)} + V_{t'} & \text{if } V_{t'} < C_{t'} \\ (y-q(h))e^{r(t'-m)} + C_{t'} + \xi^b \left(V_{t'} - C_{t'}\right) & \text{otherwise,} \end{cases} \tag{3.19b}$$

where $C_{t'} = c+h$ is the collateral held by the bank at the default date $t'$ and $q(h)$ denotes the transaction costs paid by the bank. The top row on the right hand side of Equation (3.19b) describes the scenarios where the amount of collateral held by the bank exceeds the market value of the portfolio of contracts. In that situation, the bank retains $V_{t'}$ and returns the remaining value $C_{t'} - V_{t'}$ to the counterparty. The bottom row describes the scenarios where the market value of the portfolio of contracts exceeds the collateral held by the bank at the time of default; in that case, the bank retains the whole collateral $C_{t'}$ and recovers $\xi^b(V_{t'} - C_{t'})$.

With this transition scheme, the utility maximization problem is almost similar to a dynamic portfolio choice problem, where at each decision date, the bank must find the amount of collateral that has to be requested in order to maximize the utility of its final wealth. There exists a large literature on dynamic portfolio choice; the most common approach to this problem is Taylor approximation (Arrow 1964, Pratt 1964 and more recently Garlappi and Skoulakis 2011). Unfortunately, in the present case, the time horizon is not a constant (it is $\min\{T, \tau\}$), and the transition scheme depends on the default of the counterparty, so that the Taylor approximation approach cannot be used.

We propose here a dynamic programming approach. As previously, let $R_m(x,c,y,h)$ denote the immediate utility at date $m$ when decision $h$ is taken in state $(x,c,y)$. This function is defined by the following equation:

$$R_m(x,c,y,h) = (1 - G^a_{m+1}(x,m)) \int_m^{m+1} \mathbb{E}_{m,x,c,y}\left[e^{-r(t'-m)} U\left(Y^d_{t'}\right)\right] dt' \tag{3.20}$$

where $Y^d_{t'}$ is given by Equation (3.19b).

Denote by $W_m(x,c,y)$, the maximum expected final utility of the bank at date $m$, if the observed state is $(x,c,y)$. Then, $W_m$ is defined by the following dynamic programming recursion:

$$W_n(x,c,y) = U(y + V_n(x)) \tag{3.21a}$$

and for any intermediate date $m$,

$$W_m(x,c,y) = \max_{h \in [-c,\infty)} \left\{R_m(x,c,y,h) + e^{-r} G^a_{m+1}(x,m) \mathbb{E}_{m,x,c,y}\left[W_{m+1}\left(f_m(x,c,y,h)\right)\right]\right\} \tag{3.21b}$$

where the optimal adjustment of collateral at date $m$ corresponds to the value of the control variable $h$ that solves the maximization problem of Equation (3.21b).

In Section 3.6.3, we provide a detailed description of the algorithm used to solve this three-dimensional DP.

## 3.6 Numerical illustrations

This section is devoted to an empirical investigation of the collateral management strategies proposed above, through several basic examples. For all these examples, we study the performance of the DP-based and the conservative strategies for various parameter values. In each case, we provide details about the computation of the expected exposures and the default probabilities. We also provide additional details about the approximation procedure used to solve the dynamic programming recursions.

### 3.6.1 Example 1: Unilateral agreement in a jump model

In this example, the agreement is unilateral and we assume that the total market value of the portfolio of contracts between the bank and the counterparty evolves according to a jump model. More specifically, under risk-neutral measure, we assume that $X \equiv V$ and:

$$X_{t'} = X_m \exp\{(\mu - \frac{1}{2}\sigma^2)(t'-m) + \sigma Z\sqrt{t'-m} + pJ\} \tag{3.22}$$

where for all pair $(m,t')$ with $m < t' \leq m+1$,

$$p = \begin{cases} 1 \text{ with probability } \gamma(t'-m) \\ 0 \text{ with probability } 1-\gamma(t'-m), \end{cases}$$

where $\gamma$ is the jump intensity, $Z$ is a standard Gaussian random variable independent of $J$ that is also a Gaussian variable with mean $\mu_J$ and standard deviation $\sigma_J$. This simplified jump model assumes that the maximum number of jumps during time period $(m,m+1]$ is one. We also assume that the counterparty's default intensity is a constant $\lambda$, implying that the survival function is:

$$G^a_{m+1}(x,m) = e^{-\lambda}.$$

We first provide details about the computation of the bank's expected exposure (Equation (3.7a)).

#### 3.6.1.1 Computation of the expected exposure

To derive closed formulas for the expected exposure of the bank defined by Equation (3.7a), we start with the computation of the general form $\mathbb{E}_{m,x}\left[(V_{t'}-K)^+\right]$ for a positive real number $K$, and $t', m < t' \leq m+1$. The computation of this general term is done in two steps. First, we have:

$$\mathbb{E}_{m,x}\left[(V_{t'}-K)^+\right] = (1-\gamma(t'-m))\mathbb{E}_{m,x}\left[(V_{t'}-K)^+ I_{p=0}\right] + \gamma(t'-m)\mathbb{E}_{m,x}\left[(V_{t'}-K)^+ I_{p=1}\right]. \tag{3.23}$$

The first conditional expectation in the right-hand side of Equation (3.23) is given by a Black-Scholes type formula, that is:

$$\mathbb{E}_{m,x}\left[(V_{t'}-K)^+ I_{p=0}\right] = xe^{\mu(t'-m)}\Phi(D_1) - K\Phi(D_1 - \sigma\sqrt{t'-m}) \tag{3.24}$$

where $\Phi$ is the standard normal cumulative distribution function, and

$$D_1 = \frac{ln(x/K) + (\mu + \frac{1}{2}\sigma^2)(t'-m)}{\sigma\sqrt{t'-m}}.$$

To derive the second conditional expectation in the right-hand side of Equation (3.23), we proceed as follows. Since the Gaussian variables $Z$ and $J$ are independent, the variable $J' = \sigma Z\sqrt{t'-m} + J$ is a Gaussian variable with mean $\mu_J$ and variance $\sigma_J^2 + \sigma^2(t'-m)$. So this case ($p = 1$) is similar the previous case ($p = 0$) with $Z$ replaced by $J'$. From this case, we have:

$$E_{m,x}\left[(V_{t'}-K)^+ I_{p=1}\right] = xe^{\mu(t'-m)+\mu_J+\sigma_J^2}\Phi(D_2) - K\Phi(D_2 - \sqrt{\sigma^2(t'-m)+\sigma_J^2}) \tag{3.25}$$

where

$$D_2 = \frac{ln(x/K) + (\mu + \frac{1}{2}\sigma^2)(t' - m) + \mu_J + \sigma_J^2}{\sqrt{\sigma^2(t' - m) + \sigma_J^2}}.$$

Finally, to obtain a closed formula for the expected exposure of the bank (Equation (3.7a)), it suffices to replace the constant $K$ by $C_{t'}^b = c + h$ and multiply the resulting conditional expectation $\mathbb{E}_{m,x}\left[(V_{t'} - K)^+\right]$ by $1 - \xi^b$.

#### 3.6.1.2 Additional details on the approximation procedure

The general idea of the approximation procedure used to solve a DP recursion has already been formulated in Chapter 2. In this example, we use a bilinear interpolation scheme, where the value function is approximated with polynomial forms which are linear with respect to the state components $X = V$ and $C$. The choice of a bilinear interpolation scheme is motivated by its simplicity, but also by the fact that the expected exposure presents an almost linear form with respect to $V$ and $C$ as shown in Equations (3.24) and (3.25). Following notations of Section 2.3, we consider a two dimensional grid $\Upsilon = \Upsilon_1 \times \Upsilon_2$. Since the transition scheme with respect to the second state component $C$ is not stochastic, the conditional expectations $\mathbb{E}_{m,x,c}\left[\widehat{W}_{m+1}(f_m(x,c,h))\right]$ are linear combinations of $\mathbb{E}_{m,x}\left[X_{m+1}^q I_{[a_{j_1}^1, a_{j_1+1}^1]}(X_{m+1})\right]$, for $q = 0, 1$, and $a_{j_1}^1 \in \Upsilon_1$. Furthermore, we have:

$$\mathbb{E}_{m,x}\left[X_{m+1}^q I_{[a_{j_1}^1, a_{j_1+1}^1]}(X_{m+1})\right] = \mathbb{E}_{m,x}\left[X_{m+1}^q I_{[a_{j_1}^1, \infty]}(X_{m+1})\right] - \mathbb{E}_{m,x}\left[X_{m+1}^q I_{[a_{j_1+1}^1, \infty]}(X_{m+1})\right].$$

Finally, by repeating the procedure used in the computation of the expected exposure, we can easily show that:

$$\mathbb{E}_{m,x}\left[X_{m+1}^q I_{[K,\infty]}(X_{m+1})\right] = \left[xe^{\mu + \frac{q-1}{2}\sigma^2}\right]^q \left[(1-\gamma)\Phi(D_{q,K}) + \gamma e^{q\mu_J + \frac{1}{2}q^2\sigma_J^2}\Phi(D_{q,K}^{'})\right]$$

where

$$D_{q,K} = \frac{ln(x/K) + \mu + \frac{2q-1}{2}\sigma^2}{\sigma} \text{ and } q, K^{'} = \frac{ln(x/K) + \mu + \frac{2q-1}{2}\sigma^2 + \mu_J + q\sigma_J^2}{\sqrt{\sigma_J^2 + \sigma^2}}.$$

At this point, we are able to compute the expected positive exposures and approximate the conditional expectations in the DP recursion Equation (3.13b). In other words, we have completely described the solution of the dynamic program.

#### 3.6.1.3 Numerical results

We fix the margin frequency and time unit at two days and we consider a set of transactions of which the longest maturity $n = 90$ (6 months). Table 3.1 contains the base case values of the parameters for this illustrative example.

| $x_0$ | $c_0$ | $\mu$ | $\sigma$ | $\lambda$ | $r$ | $\xi^b$ | $\mu_J$ | $\sigma_J$ | $\gamma$ | $q_0$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0.05 | 20% | 0.5 | 0.02% | 50% | -0.24 | 12% | 2 | 0.001 |

Table 3.1: Base case parameter values, jump-diffusion model

We want to compare the efficiency of the DP-based collateral management strategy with that of the conservative strategy. The first step of our investigation is to optimally design the conservative strategy before comparing it with the DP-based strategy. In order to do so, we find the threshold and the minimum transfer amount that are optimal for the conservative strategy for the selected application example. Consider the following recursion:

$$W_n^{H,M}(x,c) = 0 \tag{3.26a}$$

and, for any intermediate date $m$,

$$\begin{aligned} W_m^{H,M}(x,c) &= R_m(x,c,h(x,c,H,M) \\ &\quad + e^{-r} G_{m+1}^a(x,m) \mathbb{E}_{m,x,c}\left[W_{m+1}^{H,M}(X_{t_{m+1}}, c + h(x,c,H,M))\right] \end{aligned} \tag{3.26b}$$

where $h(x,c,H,M)$ corresponds to the adjustment of collateral in the conservative strategy (see Section 3.2.3) given the state $(x,c)$, the threshold $H^b = H$ and the minimum transfer amount $M$. The value function $W_m^{H,M}$ defined by the above recursion corresponds to the sum of expected losses and transaction costs when the conservative strategy is used for given levels $H$ and $M$. Given an initial state $(x_0, c_0)$ at date 0, we define the optimal threshold $H^*$ and minimum transfer amount $M^*$ as follows:

$$(H^*, M^*) = \underset{H,M}{\text{Argmin}}\{W_0^{H,M}(x_0, c_0))\}. \tag{3.27}$$

The optimal conservative strategies for three contrasting cases according to the variable transaction cost rate are given in Table 3.2.

| $q_1$ | $M^*$ | $H^*$ |
|---|---|---|
| 2% | 0.2195 | 0.0000 |
| 4% | 0.2443 | 0.0173 |
| 6% | 0.2571 | 0.0200 |

Table 3.2: Optimal conservative strategy as a function of the transaction costs

In a second step, we compare the collateral adjustment and the total cost (expected losses plus transaction costs) of the conservative and DP-based approaches for different parameter values. Comparisons at a given $(m, x, c)$ are performed using the conservative strategy $(H^*, M^*)$ that is optimal at $(0, x_0, c_0)$.

Figures 3.1 and 3.2 represent the sum of the expected loss and transaction costs and the strategies at $m = 0$ as functions of the initial market value of the contract for various level of initial collateral and two contrasting variable transaction cost rates $q_1$. The dotted lines correspond to the DP-based strategy while the full lines correspond to the conservative strategy (which differs according to $q_1$). Other parameter values used to obtain these results are reported in Table 3.1. One observes that the DP-based strategy yields a smaller total expected cost for all initial conditions, including $(x_0 = 1, c_0 = 0)$ for which the conservative strategy has been optimized, and that the difference becomes larger when the market value increases or when the transaction cost are higher.These results are representative of what is obtained with various sets of parameter values.

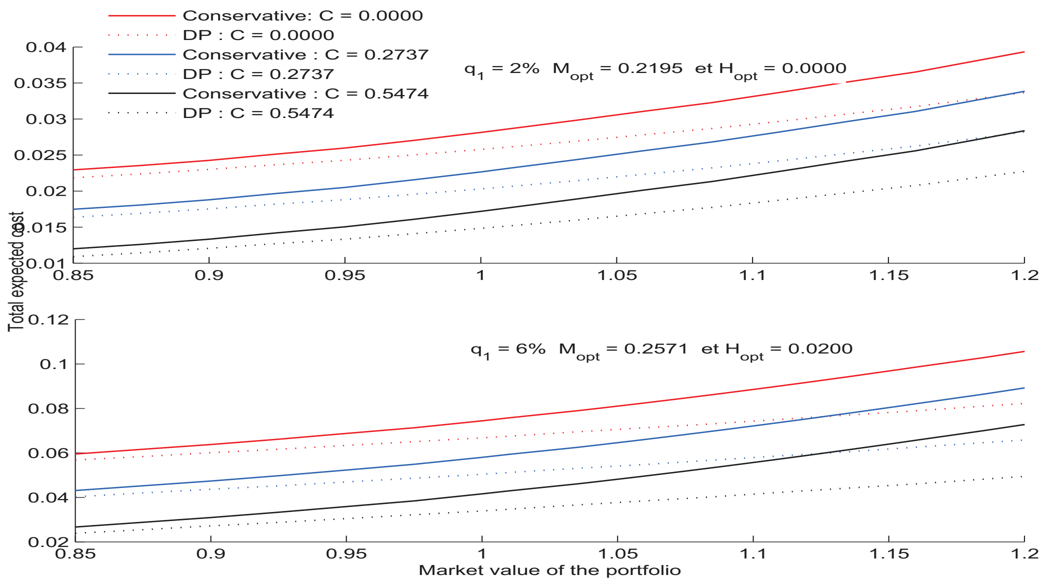


Figure 3.1: Total expected cost as a function of the market value at date 0, for various initial levels of collateral $C$ and variable transaction costs $q_1$. Other parameter values are reported in Table 3.1.

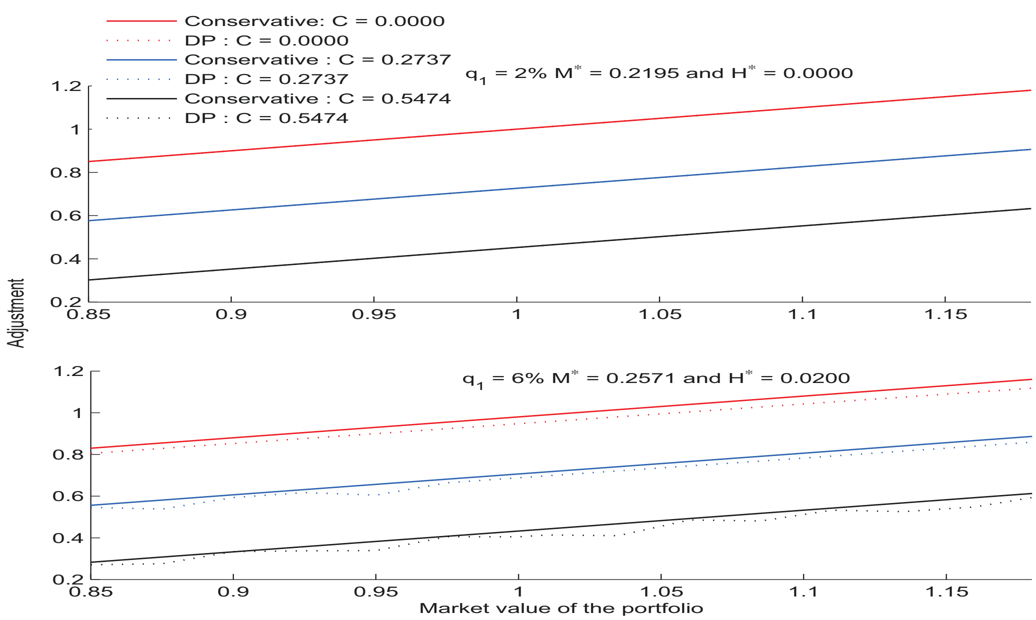


Figure 3.2: Strategy as a function of the market value at date 0, for various initial levels of collateral $C$ and variable transaction costs $q_1$. Other parameter values are reported in Table 3.1.

The fact that, the total expected cost grow faster than linear with respect to the market value of the portfolio is explained by the convexity of the expected exposure with respect to this market value. Indeed, from Equations (3.24) and (3.25), the expected exposure can be seen as the value of call options on this market value, so that, the second derivatives are positive.

| $m = 30$ (end of the second month) | | | | $q_1 = 2\%$ | | $M^* = 0.2195$ | | $H^* = 0.0000$ | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | **Panel A : Adjustment** | | | | | | | | | |
| $V_m = X_m$ | 0.9 | | 0.95 | | 1 | | 1.06 | | 1.11 | |
| $C_m$ | CS | DP | CS | DP | CS | DP | CS | DP | CS | DP |
| 0.0000 | 0.9000 | 0.9000 | 0.9500 | 0.9500 | 1.0000 | 1.0000 | 1.0600 | 1.0600 | 1.1100 | 1.1100 |
| 0.1368 | 0.7631 | 0.7632 | 0.8131 | 0.8132 | 0.8631 | 0.8632 | 0.9231 | 0.9232 | 0.9731 | 0.9732 |
| 0.2737 | 0.6263 | 0.6263 | 0.6763 | 0.6763 | 0.7263 | 0.7263 | 0.7863 | 0.7863 | 0.8363 | 0.8363 |
| 0.4105 | 0.4894 | 0.4895 | 0.5394 | 0.5395 | 0.5894 | 0.5895 | 0.6494 | 0.6495 | 0.6994 | 0.6995 |
| 0.5474 | 0.3526 | 0.3526 | 0.4026 | 0.4026 | 0.4526 | 0.4526 | 0.5126 | 0.5126 | 0.5626 | 0.5626 |
| | **Panel B : Total expected cost** | | | | | | | | | |
| $V_m$ | 0.9 | | 0.95 | | 1 | | 1.06 | | 1.11 | |
| $C_m$ | CS | DP | CS | DP | CS | DP | CS | DP | CS | DP |
| 0.0000 | 0.0224 | 0.0217 | 0.0238 | 0.0228 | 0.0256 | 0.0242 | 0.0284 | 0.0260 | 0.0310 | 0.0277 |
| 0.1368 | 0.0196 | 0.0190 | 0.0211 | 0.0201 | 0.0229 | 0.0214 | 0.0256 | 0.0232 | 0.0283 | 0.0249 |
| 0.2737 | 0.0169 | 0.0162 | 0.0183 | 0.0174 | 0.0202 | 0.0187 | 0.0229 | 0.0205 | 0.0255 | 0.0222 |
| 0.4105 | 0.0142 | 0.0135 | 0.0156 | 0.0146 | 0.0174 | 0.0160 | 0.0201 | 0.0177 | 0.0228 | 0.0195 |
| 0.5474 | 0.0114 | 0.0108 | 0.0128 | 0.0119 | 0.0147 | 0.0132 | 0.0174 | 0.0150 | 0.0201 | 0.0167 |

Table 3.3: Adjustment and expected cost at the end of the second month for $q_1 = 2\%$.

| $m = 30$ (end of the second month) | | | | $q_1 = 4\%$ | | $M^* = 0.2443$ | | $H^* = 0.0173$ | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | **Panel A : Adjustment** | | | | | | | | | |
| $V_m = X_m$ | 0.9 | | 0.95 | | 1 | | 1.06 | | 1.11 | |
| $C_m$ | CS | DP | CS | DP | CS | DP | CS | DP | CS | DP |
| 0.0000 | 0.8827 | 0.9000 | 0.9327 | 0.9000 | 0.9827 | 0.9474 | 1.0427 | 1.0042 | 1.0927 | 1.1100 |
| 0.1368 | 0.7459 | 0.7230 | 0.7959 | 0.7704 | 0.8459 | 0.8177 | 0.9059 | 0.8746 | 0.9559 | 0.9219 |
| 0.2737 | 0.6090 | 0.5934 | 0.6590 | 0.6407 | 0.7090 | 0.6881 | 0.7690 | 0.7449 | 0.8190 | 0.7923 |
| 0.4105 | 0.4722 | 0.4637 | 0.5222 | 0.4827 | 0.5722 | 0.5584 | 0.6322 | 0.6153 | 0.6822 | 0.6627 |
| 0.5474 | 0.3353 | 0.3341 | 0.3853 | 0.3602 | 0.4353 | 0.4050 | 0.4953 | 0.4857 | 0.5453 | 0.5330 |
| | **Panel B : Expected total loss** | | | | | | | | | |
| $V_m$ | 0.9 | | 0.95 | | 1 | | 1.06 | | 1.11 | |
| $C_m$ | CS | DP | CS | DP | CS | DP | CS | DP | CS | DP |
| 0.0000 | 0.0412 | 0.0400 | 0.0441 | 0.0421 | 0.0474 | 0.0444 | 0.0525 | 0.0475 | 0.0577 | 0.0502 |
| 0.1368 | 0.0357 | 0.0345 | 0.0386 | 0.0367 | 0.0420 | 0.0389 | 0.0470 | 0.0419 | 0.0522 | 0.0447 |
| 0.2737 | 0.0303 | 0.0290 | 0.0331 | 0.0312 | 0.0365 | 0.0334 | 0.0415 | 0.0364 | 0.0468 | 0.0392 |
| 0.4105 | 0.0248 | 0.0235 | 0.0277 | 0.0257 | 0.0310 | 0.0280 | 0.0360 | 0.0309 | 0.0413 | 0.0337 |
| 0.5474 | 0.0193 | 0.0180 | 0.0222 | 0.0203 | 0.0255 | 0.0225 | 0.0306 | 0.0255 | 0.0358 | 0.0282 |

Table 3.4: Adjustment and expected cost at the end of the second month for $q_1 = 4\%$.

| $m = 30$ (end of the second month) | | | $q_1 = 6\%$ | | | $M^* = 0.2571$ | | $H^* = 0.0200$ | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | **Panel A : Adjustment** | | | | | | | | | |
| $V_m = X_m$ | 0.9 | | 0.95 | | 1 | | 1.06 | | 1.11 | |
| $C_m$ | CS | DP | CS | DP | CS | DP | CS | DP | CS | DP |
| 0.0000 | 0.8800 | 0.7579 | 0.9300 | 0.7500 | 0.9800 | 0.7895 | 1.0400 | 0.8926 | 1.0900 | 0.9347 |
| 0.1368 | 0.7432 | 0.6025 | 0.7932 | 0.6420 | 0.8432 | 0.6814 | 0.9032 | 0.7288 | 0.9532 | 0.7683 |
| 0.2737 | 0.6064 | 0.4615 | 0.6564 | 0.4983 | 0.7064 | 0.5352 | 0.7664 | 0.6208 | 0.8164 | 0.6162 |
| 0.4105 | 0.4695 | 0.0000 | 0.5195 | 0.0000 | 0.5695 | 0.4033 | 0.6295 | 0.4786 | 0.6795 | 0.5154 |
| 0.5474 | 0.3327 | 0.0000 | 0.3827 | 0.0000 | 0.4327 | 0.0000 | 0.4927 | 0.0000 | 0.5427 | 0.3553 |
| | **Panel B : Expected total loss** | | | | | | | | | |
| $V_m$ | 0.9 | | 0.95 | | 1 | | 1.06 | | 1.11 | |
| $C_m$ | CS | DP | CS | DP | CS | DP | CS | DP | CS | DP |
| 0.0000 | 0.0596 | 0.0563 | 0.0638 | 0.0594 | 0.0688 | 0.0627 | 0.0763 | 0.0669 | 0.0843 | 0.0705 |
| 0.1368 | 0.0514 | 0.0481 | 0.0556 | 0.0512 | 0.0606 | 0.0545 | 0.0681 | 0.0587 | 0.0761 | 0.0623 |
| 0.2737 | 0.0432 | 0.0399 | 0.0474 | 0.0430 | 0.0524 | 0.0463 | 0.0599 | 0.0504 | 0.0679 | 0.0541 |
| 0.4105 | 0.0349 | 0.0314 | 0.0392 | 0.0348 | 0.0441 | 0.0380 | 0.0517 | 0.0422 | 0.0596 | 0.0459 |
| 0.5474 | 0.0267 | 0.0228 | 0.0310 | 0.0261 | 0.0359 | 0.0296 | 0.0435 | 0.0339 | 0.0514 | 0.0377 |

Table 3.5: Adjustment and expected cost at the end of the second month for $q_1 = 6\%$.

Tables 3.3 to 3.5 and the corresponding Figure 3.3 to 3.5 illustrate the impact of transaction costs on the conservative and DP-based collateral adjustment strategies and on the total expected costs, for various levels of the portfolio market value and available collateral. Results are reported at $m = 30$, which corresponds to the end of the second month. One can observe that the DP-based strategy always yields a lower total cost, and that the difference is increasing with both transaction costs and market value, ranging from 3% to 27% of the total expected cost of the conservative strategy. The collateral adjustment required by the DP approach may be higher or lower than that of the conservative approach. When transaction costs are low, strategies are close, but they increasingly differ with the level of transaction costs. In some cases when transaction costs and collateral level are sufficiently high, no adjustment of collateral is required by the DP strategy while the conservative strategy always requires an adjustment as long as it is higher than the minimum transfer amount $M^*$. It is interesting to note that, contrary to the conservative strategy, the DP adjustment is not a simple linear function of $x$ and $c$.

Tables 3.6 and 3.7 and Figures 3.6 and 3.7 illustrate the impact of time in collateral management strategies. More precisely, we compare the required adjustment and expected total

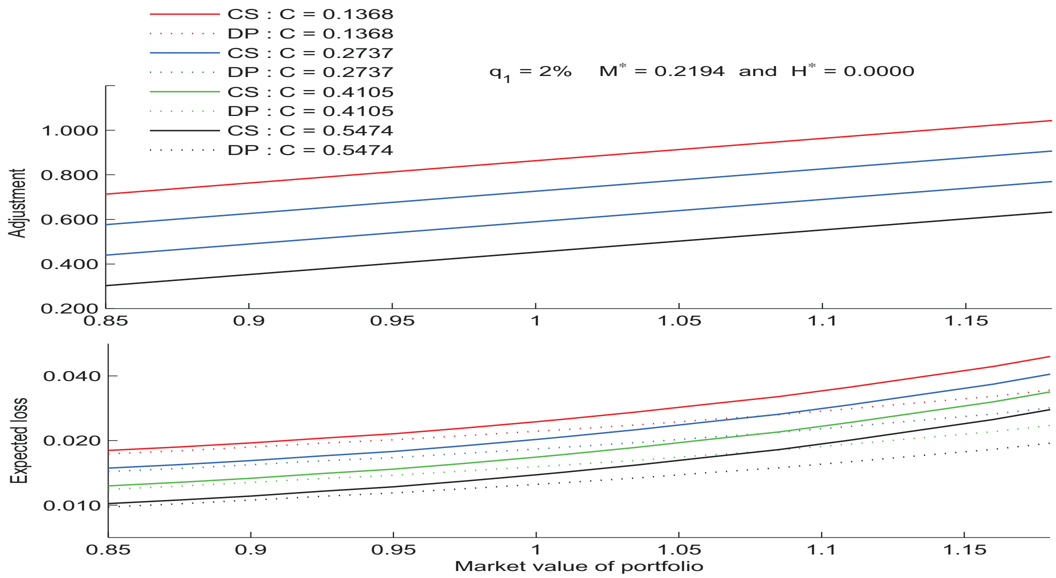


Figure 3.3: Strategy as a function of the market value of the portfolio at the end of the second month, for various initial levels of collateral $C$ and $q_1 = 2\%$. Other parameter values are reported in Table 3.1.

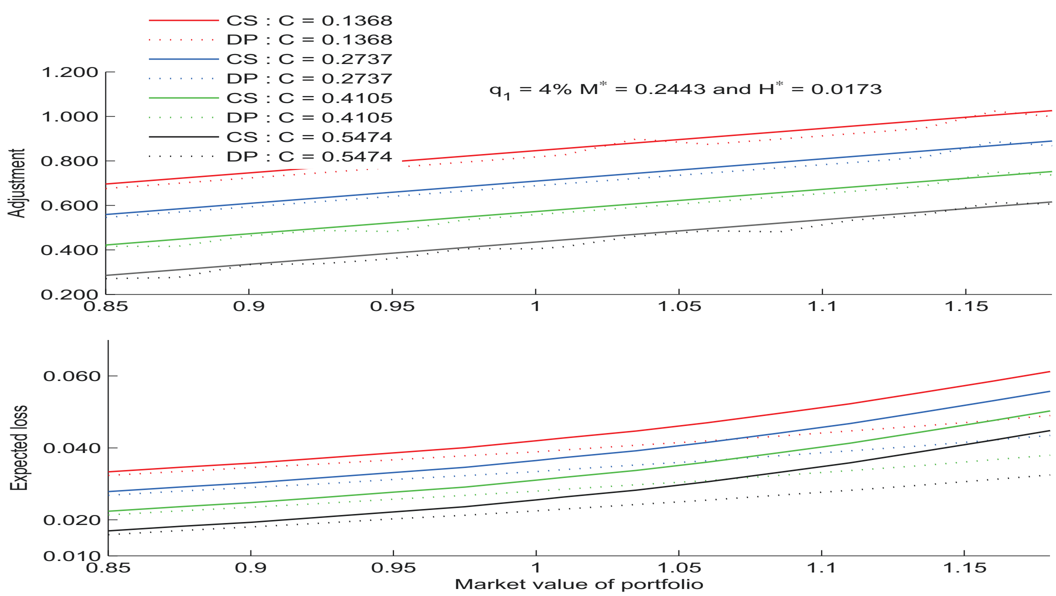


Figure 3.4: Strategy as a function of the market value of the portfolio at the end of the second month, for various initial levels of collateral $C$ and $q_1 = 4\%$. Other parameter values are reported in Table 3.1.

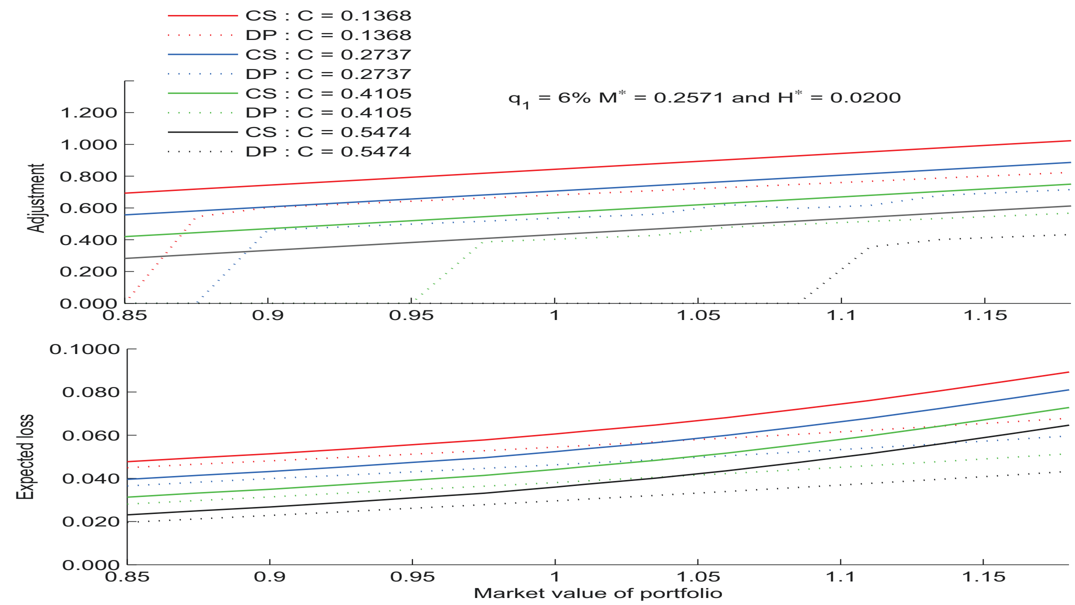


Figure 3.5: Strategy as a function of the market value of the portfolio at the end of the second month, for various initial levels of collateral $C$ and $q_1 = 6\%$. Other parameter values are reported in Table 3.1.

cost at three different dates, $m \in \{15, 30, 45, 60\}$, which corresponding respectively to the end of the first, the second, the third and the fourth month. Two levels of the transaction costs rate are considered. Notice that the conservative strategy is independent of time, contrary to the DP strategy, which decreases the required adjustment as maturity is approaching, and this explains why the difference in expected costs between the two strategies may be significant at some dates where these strategy are close. The expected cost is always lower using a DP-based approach, with a difference increasing with transaction costs and market value, attaining 39% at the end of the third month for high values of the transaction cost, available collateral and portfolio value.

| | Panel A: $m = 15$ (end of first month) $q_1 = 2\%$ $M^* = 0.2195$ $H^* = 0.0000$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $V_m = X_m$ | 0.9 | | 0.95 | | 1.00 | | 1.06 | | 1.11 | |
| | CS | DP | CS | DP | CS | DP | CS | DP | CS | DP |
| $C_m$ | Adjustment | | | | | | | | | |
| 0.1368 | 0.7631 | 0.7632 | 0.8131 | 0.8132 | 0.8631 | 0.8632 | 0.9231 | 0.9232 | 0.9731 | 0.9732 |
| 0.2737 | 0.6263 | 0.6263 | 0.6763 | 0.6763 | 0.7263 | 0.7263 | 0.7863 | 0.7863 | 0.8363 | 0.8363 |
| 0.4105 | 0.4894 | 0.4895 | 0.5394 | 0.5395 | 0.5894 | 0.5895 | 0.6494 | 0.6495 | 0.6994 | 0.6995 |
| 0.5474 | 0.3526 | 0.3526 | 0.4026 | 0.4026 | 0.4526 | 0.4526 | 0.5126 | 0.5126 | 0.5626 | 0.5626 |
| | Expected total loss | | | | | | | | | |
| 0.1368 | 0.0206 | 0.0197 | 0.0222 | 0.0209 | 0.0243 | 0.0223 | 0.0271 | 0.0242 | 0.0298 | 0.0260 |
| 0.2737 | 0.0179 | 0.0169 | 0.0195 | 0.0181 | 0.0215 | 0.0196 | 0.0244 | 0.0215 | 0.0270 | 0.0233 |
| 0.4105 | 0.0152 | 0.0142 | 0.0167 | 0.0154 | 0.0188 | 0.0168 | 0.0217 | 0.0188 | 0.0243 | 0.0206 |
| 0.5474 | 0.0124 | 0.0115 | 0.0140 | 0.0127 | 0.0160 | 0.0141 | 0.0189 | 0.0160 | 0.0216 | 0.0178 |
| | Panel B: $m = 30$ (end of second month) $q_1 = 2\%$ $M^* = 0.2195$ $H^* = 0.0000$ | | | | | | | | | |
| | Adjustment | | | | | | | | | |
| 0.1368 | 0.7631 | 0.7632 | 0.8131 | 0.8132 | 0.8631 | 0.8632 | 0.9231 | 0.9232 | 0.9731 | 0.9732 |
| 0.2737 | 0.6263 | 0.6263 | 0.6763 | 0.6763 | 0.7263 | 0.7263 | 0.7863 | 0.7863 | 0.8363 | 0.8363 |
| 0.4105 | 0.4894 | 0.4895 | 0.5394 | 0.5395 | 0.5894 | 0.5895 | 0.6494 | 0.6495 | 0.6994 | 0.6995 |
| 0.5474 | 0.3526 | 0.3526 | 0.4026 | 0.4026 | 0.4526 | 0.4526 | 0.5126 | 0.5126 | 0.5626 | 0.5626 |
| | Expected total loss | | | | | | | | | |
| 0.1368 | 0.0196 | 0.0190 | 0.0211 | 0.0201 | 0.0229 | 0.0214 | 0.0256 | 0.0232 | 0.0283 | 0.0249 |
| 0.2737 | 0.0169 | 0.0162 | 0.0183 | 0.0174 | 0.0202 | 0.0187 | 0.0229 | 0.0205 | 0.0255 | 0.0222 |
| 0.4105 | 0.0142 | 0.0135 | 0.0156 | 0.0146 | 0.0174 | 0.0160 | 0.0201 | 0.0177 | 0.0228 | 0.0195 |
| 0.5474 | 0.0114 | 0.0108 | 0.0128 | 0.0119 | 0.0147 | 0.0132 | 0.0174 | 0.0150 | 0.0201 | 0.0167 |
| | Panel C: $m = 45$ (end of third month) $q_1 = 2\%$ $M^* = 0.2195$ $H^* = 0.0000$ | | | | | | | | | |
| | Adjustment | | | | | | | | | |
| 0.1368 | 0.7631 | 0.7632 | 0.8131 | 0.8132 | 0.8631 | 0.8632 | 0.9231 | 0.9232 | 0.9731 | 0.9732 |
| 0.2737 | 0.6263 | 0.6263 | 0.6763 | 0.6763 | 0.7263 | 0.7263 | 0.7863 | 0.7863 | 0.8363 | 0.8363 |
| 0.4105 | 0.4894 | 0.4895 | 0.5394 | 0.5395 | 0.5894 | 0.5895 | 0.6494 | 0.6495 | 0.6994 | 0.6995 |
| 0.5474 | 0.3526 | 0.3526 | 0.4026 | 0.4026 | 0.4526 | 0.4526 | 0.5126 | 0.5126 | 0.5626 | 0.5626 |
| | Expected total loss | | | | | | | | | |
| 0.1368 | 0.0186 | 0.0182 | 0.0199 | 0.0193 | 0.0215 | 0.0205 | 0.0239 | 0.0221 | 0.0264 | 0.0236 |
| 0.2737 | 0.0159 | 0.0155 | 0.0171 | 0.0165 | 0.0187 | 0.0178 | 0.0211 | 0.0194 | 0.0237 | 0.0209 |
| 0.4105 | 0.0132 | 0.0128 | 0.0144 | 0.0138 | 0.0160 | 0.0150 | 0.0184 | 0.0166 | 0.0209 | 0.0182 |
| 0.5474 | 0.0104 | 0.0100 | 0.0116 | 0.0111 | 0.0132 | 0.0123 | 0.0157 | 0.0139 | 0.0182 | 0.0154 |
| | Panel D: $m = 60$ (end of fourth month) $q_1 = 2\%$ $M^* = 0.2195$ $H^* = 0.0000$ | | | | | | | | | |
| | Adjustment | | | | | | | | | |
| 0.1368 | 0.7631 | 0.7632 | 0.8131 | 0.8132 | 0.8631 | 0.8632 | 0.9231 | 0.9232 | 0.9731 | 0.9732 |
| 0.2737 | 0.6263 | 0.6263 | 0.6763 | 0.6763 | 0.7263 | 0.6881 | 0.7863 | 0.7449 | 0.8363 | 0.8363 |
| 0.4105 | 0.4894 | 0.4895 | 0.5394 | 0.5395 | 0.5894 | 0.5584 | 0.6494 | 0.6153 | 0.6994 | 0.6995 |
| 0.5474 | 0.3526 | 0.3341 | 0.4026 | 0.4026 | 0.4526 | 0.4288 | 0.5126 | 0.4857 | 0.5626 | 0.5626 |
| | Expected total loss | | | | | | | | | |
| 0.1368 | 0.0177 | 0.0174 | 0.0187 | 0.0184 | 0.0200 | 0.0196 | 0.0219 | 0.0209 | 0.0241 | 0.0222 |
| 0.2737 | 0.0149 | 0.0147 | 0.0160 | 0.0157 | 0.0173 | 0.0168 | 0.0192 | 0.0182 | 0.0214 | 0.0195 |
| 0.4105 | 0.0122 | 0.0119 | 0.0133 | 0.0130 | 0.0145 | 0.0141 | 0.0165 | 0.0154 | 0.0187 | 0.0168 |
| 0.5474 | 0.0095 | 0.0092 | 0.0105 | 0.0102 | 0.0118 | 0.0113 | 0.0137 | 0.0127 | 0.0159 | 0.0140 |

Table 3.6: Adjustment and expected cost at various dates for $q_1 = 2\%$.

| | Panel A: $m = 15$ (end of first month) $q_1 = 6\%$ $M^* = 0.2571$ $H^* = 0.0200$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $V_m = X_m$ | 0.9 | | 0.95 | | 1.00 | | 1.06 | | 1.11 | |
| | CS | DP | CS | DP | CS | DP | CS | DP | CS | DP |
| $C_m$ | Adjustment | | | | | | | | | |
| 0.1368 | 0.7432 | 0.6828 | 0.7932 | 0.7276 | 0.8432 | 0.7723 | 0.9032 | 0.8260 | 0.9532 | 0.8707 |
| 0.2737 | 0.6064 | 0.5604 | 0.6564 | 0.6051 | 0.7064 | 0.6499 | 0.7664 | 0.7035 | 0.8164 | 0.7483 |
| 0.4105 | 0.4695 | 0.4122 | 0.5195 | 0.4827 | 0.5695 | 0.4964 | 0.6295 | 0.5469 | 0.6795 | 0.6258 |
| 0.5474 | 0.3327 | 0.2784 | 0.3827 | 0.3391 | 0.4327 | 0.3573 | 0.4927 | 0.4047 | 0.5427 | 0.4738 |
| | Expected total loss | | | | | | | | | |
| 0.1368 | 0.0535 | 0.0502 | 0.0582 | 0.0535 | 0.0636 | 0.0569 | 0.0715 | 0.0612 | 0.0794 | 0.0650 |
| 0.2737 | 0.0453 | 0.0421 | 0.0500 | 0.0453 | 0.0554 | 0.0487 | 0.0633 | 0.0530 | 0.0712 | 0.0567 |
| 0.4105 | 0.0371 | 0.0338 | 0.0418 | 0.0371 | 0.0472 | 0.0405 | 0.0551 | 0.0448 | 0.0629 | 0.0485 |
| 0.5474 | 0.0289 | 0.0256 | 0.0336 | 0.0288 | 0.0390 | 0.0323 | 0.0469 | 0.0366 | 0.0547 | 0.0403 |
| | Panel B: $m = 30$ (end of second month) $q_1 = 6\%$ $M^* = 0.2571$ $H^* = 0.0200$ | | | | | | | | | |
| | Adjustment | | | | | | | | | |
| 0.1368 | 0.7432 | 0.6025 | 0.7932 | 0.6420 | 0.8432 | 0.6814 | 0.9032 | 0.7288 | 0.9532 | 0.7683 |
| 0.2737 | 0.6064 | 0.4615 | 0.6564 | 0.4983 | 0.7064 | 0.5352 | 0.7664 | 0.6208 | 0.8164 | 0.6162 |
| 0.4105 | 0.4695 | 0.0000 | 0.5195 | 0.0000 | 0.5695 | 0.4033 | 0.6295 | 0.4786 | 0.6795 | 0.5154 |
| 0.5474 | 0.3327 | 0.0000 | 0.3827 | 0.0000 | 0.4327 | 0.0000 | 0.4927 | 0.0000 | 0.5427 | 0.3553 |
| | Expected total loss | | | | | | | | | |
| 0.1368 | 0.0514 | 0.0481 | 0.0556 | 0.0512 | 0.0606 | 0.0545 | 0.0681 | 0.0587 | 0.0761 | 0.0623 |
| 0.2737 | 0.0432 | 0.0399 | 0.0474 | 0.0430 | 0.0524 | 0.0463 | 0.0599 | 0.0504 | 0.0679 | 0.0541 |
| 0.4105 | 0.0349 | 0.0314 | 0.0392 | 0.0348 | 0.0441 | 0.0380 | 0.0517 | 0.0422 | 0.0596 | 0.0459 |
| 0.5474 | 0.0267 | 0.0228 | 0.0310 | 0.0261 | 0.0359 | 0.0296 | 0.0435 | 0.0339 | 0.0514 | 0.0377 |
| | Panel C: $m = 45$ (end of first month) $q_1 = 6\%$ $M^* = 0.2571$ $H^* = 0.0200$ | | | | | | | | | |
| | Adjustment | | | | | | | | | |
| 0.1368 | 0.7432 | 0.0000 | 0.7932 | 0.0000 | 0.8432 | 0.0000 | 0.9032 | 0.0000 | 0.9532 | 0.0000 |
| 0.2737 | 0.6064 | 0.0000 | 0.6564 | 0.0000 | 0.7064 | 0.0000 | 0.7664 | 0.0000 | 0.8164 | 0.0000 |
| 0.4105 | 0.4695 | 0.0000 | 0.5195 | 0.0000 | 0.5695 | 0.0000 | 0.6295 | 0.0000 | 0.6795 | 0.0000 |
| 0.5474 | 0.3327 | 0.0000 | 0.3827 | 0.0000 | 0.4327 | 0.0000 | 0.4927 | 0.0000 | 0.5427 | 0.0000 |
| | Expected total loss | | | | | | | | | |
| 0.1368 | 0.0492 | 0.0370 | 0.0529 | 0.0395 | 0.0571 | 0.0421 | 0.0639 | 0.0454 | 0.0716 | 0.0483 |
| 0.2737 | 0.0410 | 0.0304 | 0.0447 | 0.0329 | 0.0489 | 0.0355 | 0.0557 | 0.0388 | 0.0634 | 0.0417 |
| 0.4105 | 0.0328 | 0.0238 | 0.0365 | 0.0263 | 0.0407 | 0.0289 | 0.0475 | 0.0322 | 0.0552 | 0.0351 |
| 0.5474 | 0.0246 | 0.0173 | 0.0283 | 0.0197 | 0.0325 | 0.0223 | 0.0393 | 0.0256 | 0.0469 | 0.0285 |
| | Panel D: $m = 60$ (end of fourth month) $q_1 = 6\%$ $M^* = 0.2571$ $H^* = 0.0200$ | | | | | | | | | |
| | Adjustment | | | | | | | | | |
| 0.1368 | 0.7432 | 0.0000 | 0.7932 | 0.0000 | 0.8432 | 0.0000 | 0.9032 | 0.0000 | 0.9532 | 0.0000 |
| 0.2737 | 0.6064 | 0.0000 | 0.6564 | 0.0000 | 0.7064 | 0.0000 | 0.7664 | 0.0000 | 0.8164 | 0.0000 |
| 0.4105 | 0.4695 | 0.0000 | 0.5195 | 0.0000 | 0.5695 | 0.0000 | 0.6295 | 0.0000 | 0.6795 | 0.0000 |
| 0.5474 | 0.3327 | 0.0000 | 0.3827 | 0.0000 | 0.4327 | 0.0000 | 0.4927 | 0.0000 | 0.5427 | 0.0000 |
| | Expected total loss | | | | | | | | | |
| 0.1368 | 0.0476 | 0.0264 | 0.0509 | 0.0282 | 0.0544 | 0.0300 | 0.0599 | 0.0322 | 0.0667 | 0.0343 |
| 0.2737 | 0.0394 | 0.0217 | 0.0426 | 0.0235 | 0.0462 | 0.0253 | 0.0517 | 0.0275 | 0.0584 | 0.0295 |
| 0.4105 | 0.0311 | 0.0170 | 0.0344 | 0.0188 | 0.0380 | 0.0205 | 0.0435 | 0.0228 | 0.0502 | 0.0248 |
| 0.5474 | 0.0229 | 0.0123 | 0.0262 | 0.0140 | 0.0298 | 0.0158 | 0.0353 | 0.0180 | 0.0420 | 0.0201 |

Table 3.7: Adjustment and expected cost at various dates for $q_1 = 6\%$.

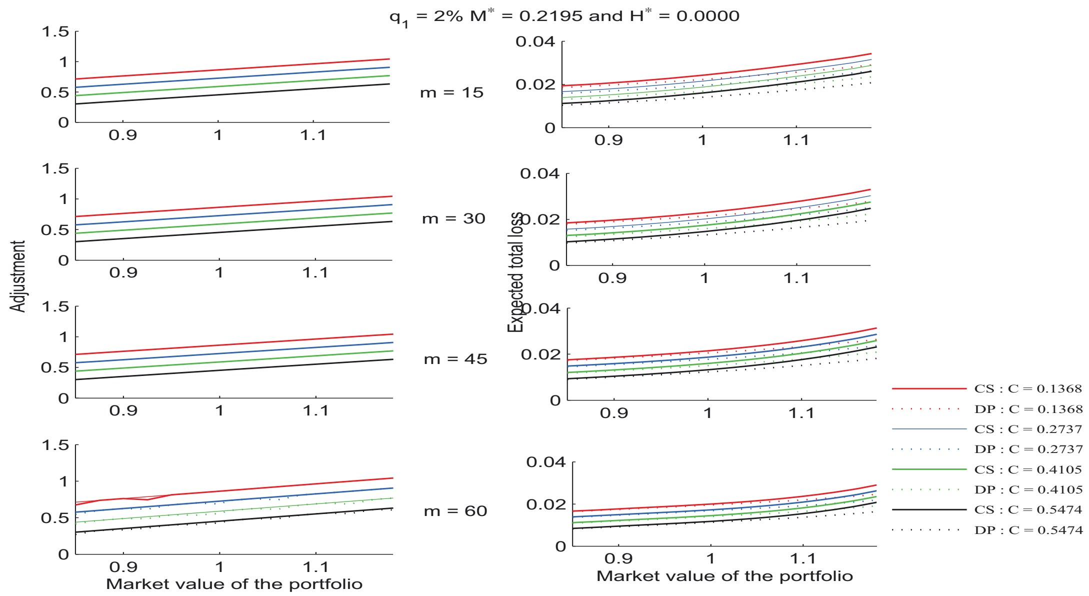


Figure 3.6: Strategy as a function of the market value of the portfolio at different dates for various initial levels of collateral $C$ and $q_1 = 2\%$. Other parameter values are reported in Table 3.1.

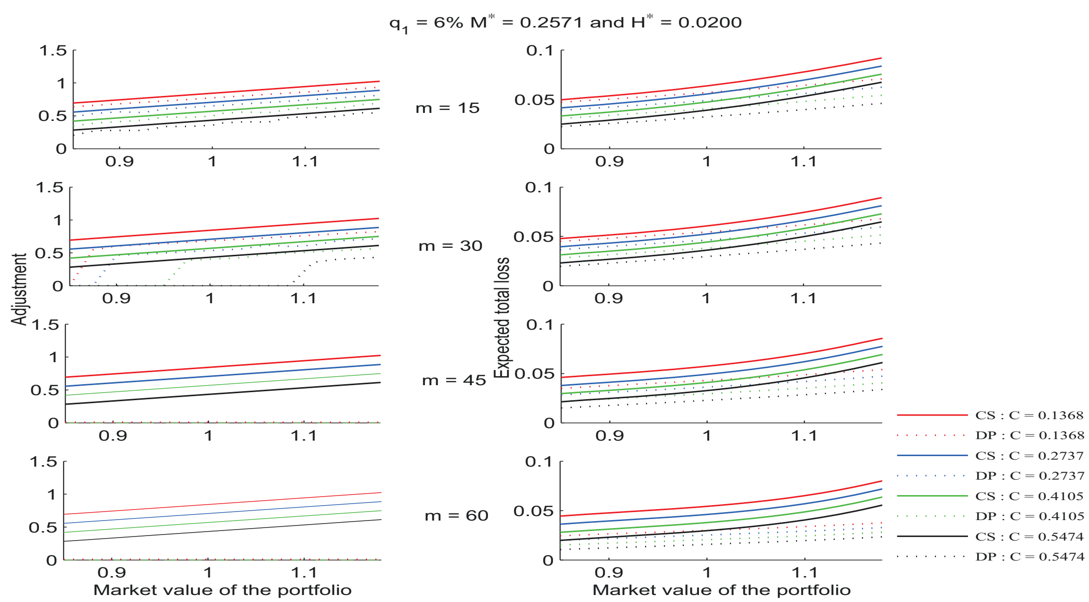


Figure 3.7: Strategy as a function of the market value of the portfolio at different dates for various initial levels of collateral $C$ and $q_1 = 6\%$. Other parameter values are reported in Table 3.1.

In view of the results obtained in these numerical investigations, we can conclude that the dynamic programming-based collateral management strategy is significantly more efficient than the conservative strategy for unilateral agreements, and that it adapts better to the time remaining and to the state vector. It is worthwhile mentioning that the optimization of the two parameters of the conservative strategy is computationally demanding; when no optimization is performed, differences in relative performances between the two approaches may be much larger.

### 3.6.2 Example 2: Bilateral agreement under the Vasicek model

This second example illustrates the case of bilateral agreements, where we consider a simple contract portfolio consisting of a single vanilla swap with a principal of 1. We assume that default intensity is linearly related with the floating interest rate. The state process $\mathbb{X}$ corresponds to this rate (e.g. the London Interbank Offer Rate), and it is described by the Vasicek model. Accordingly, we assume that, under the risk-neutral measure,

$$dX_t = \kappa(\mu - X_t)dt + \sigma dW_t \tag{3.28}$$

where $\kappa, \mu$ and $\sigma$ are non-negative parameters. Conditional on information available at date $m$, the solution of Equation (3.28) at any future date $t'$ is given by the following equation:

$$X_{t'} = \mu + (X_m - \mu)e^{-\kappa(t'-m)} + \sigma\sqrt{\frac{1-e^{-2\kappa(t'-m)}}{2\kappa}}Z \tag{3.29}$$

where $Z$ is a standard Gaussian variable. Under this interest rate model, the value at date $m$ of a zero-coupon bond with principal equal to 1 and maturity $T = n$ is given by the following equation:

$$P(X_m, m, n) = A_{n-m}\exp\{-X_m B_{n-m}\} \tag{3.30}$$

where

$$B_m = \frac{1-e^{-\kappa m}}{\kappa} \text{ and } A_m = \exp\left\{(\mu - \frac{\sigma^2}{2\kappa^2})(B_m - m) - \frac{\sigma^2}{4\kappa}B_m^2\right\}.$$

We denote by $T_1, \ldots, T_l = n$, the coupon dates and by $r_f$ the swap rate, which is determined so that the value of the swap at the inception date is zero. Therefrom, at any date $m$, the value of the swap from the point of view of the fixed-rate payer (the bank) is given by the following formula:

$$V_m = 1 - P(X_m, m, n) - r_f \sum_{j=j_m}^{l} (T_j - T_{j-1}) P(X_m, m, T_j), \tag{3.31}$$

where $T_{j_m}$ corresponds to the first coupon date after date $m$, so that $T_{j_m - 1} = m$.

#### 3.6.2.1 Deriving the expected exposure of the bank

Similarly to the previous example, the derivation of the expected exposure of the bank starts with the computation of the general form $\mathbb{E}_{m,x}\left[(V_{t'} - K)^+\right]$ where $K$ is a positive real number and $t' \in [m, m+1]$. Since the value of the swap at any date is continuous and strictly increasing with respect to the interest rate (because the bond price is continuous and strictly decreasing with respect to the interest rate), we have:

$$V_{t'} > K \Longleftrightarrow X_{t'} > X_{Kt'}$$

where $X_{Kt'}$ corresponds to the level of the rate $X$ such that the swap value at date $t'$ is equal to $K$. Consequently, we can write:

$$\mathbb{E}_{m,x}\left[(V_{t'} - K)^+\right] = \mathbb{E}_{t,x}\left[(V_{t'} - K) I_{\{X_{t'} > X_{Kt'}\}}\right]. \tag{3.32}$$

Using Equation (3.31), the conditional expectation in the right-hand side of Equation (3.32) is given by:

$$\begin{aligned}\mathbb{E}_{m,x}\left[(V_{t'}-K)I_{\{X_{t'}>X_{Kt'}\}}\right] &= (1-K)\mathbb{E}_{m,x}\left[I_{\{X_{t'}>X_{Kt'}\}}\right]-\mathbb{E}_{m,x}\left[P(X_{t'},t',T)I_{\{X_{t'}>X_{Kt'}\}}\right]\\ &\quad -r_f\sum_{j=j_m}^{l}(T_j-T_{j-1})\mathbb{E}_{m,x}\left[P(X_{t'},t',T_j)I_{\{X_{t'}>X_{Kt'}\}}\right].\end{aligned}$$

Equation (3.29) implies that:

$$\mathbb{E}_{m,x}\left[I_{\{X_{t'}>X_{Kt'}\}}\right]=\mathbb{P}(Z>d_{Kt'})=1-\Phi(d_{Kt'}), \tag{3.33}$$

where

$$d_{Kt'}=\frac{X_{Kt'}-\mu-(x-\mu)e^{-\kappa}}{\sigma\sqrt{\frac{1-e^{-2\kappa}}{2\kappa}}}.$$

Using the density of the standard Gaussian variable $Z$, we can show that the conditional expectation of the bond price is given by:

$$\mathbb{E}_{m,x}\left[P(X_{t'},t',n)I_{\{X_{t'}>X_{Kt'}\}}\right]=L_s\left[1-\Phi\left(d_{Kt'}+\sigma B_{n-t'}\sqrt{\frac{1-e^{-2\kappa}}{2\kappa}}\right)\right], \tag{3.34}$$

where

$$L_{t'}=A_{n-t'}\exp\left\{-\left[\mu+(x-\mu)e^{-\kappa}\right]B_{n-t'}+\frac{1}{2}\left[\frac{1-e^{-2\kappa}}{2\kappa}\right]\sigma^2B_{n-t'}^2\right\}.$$

Finally, the expected exposure of the bank corresponds to the case where $K=c^{+}+h^{+}$.

#### 3.6.2.2 Deriving the expected exposure of the counterparty

To derive the expected exposure of the counterparty, it suffices to use the procedure employed previously for the bank. The starting point is the computation of the general form $\mathbb{E}_{m,x}((-V_{t'}-K)^{+})$. Again, with the monotonicity property of the swap value with respect to the floating interest rate, we have:

$$V_{t'}<-K\iff X_{t'}<X_{Kt'}$$

where $X_{Kt'}$ corresponds to the level of the floating rate such that the swap value $V_{t'}$ at date $t'$ is equal to $-K$. Therefrom,

$$\mathbb{E}_{m,x}\left[(-V_{t'}-K)^{+}\right]=\mathbb{E}_{m,x}\left[(-V_{t'}-K)I_{\{X_{t'}<X_{Kt'}\}}\right]. \tag{3.35}$$

This expression is a combination of $\mathbb{E}_{m,x}\left[I_{\{X_{t'}<X_{Kt'}\}}\right]$ and $\mathbb{E}_{m,x}\left[P(X_{t'},t',n)I_{\{X_{t'}<X_{Kt'}\}}\right]$ which are given by:

$$\mathbb{E}_{m,x}\left[I_{\{X_{t'}<X_{Kt'}\}}\right]=\mathbb{P}(Z<d_{Kt'})=\Phi(d_{Kt'}) \tag{3.36}$$

and

$$\mathbb{E}_{m,x}\left[P(X_{t'},t',n)I_{\{X_{t'}<X_{Kt'}\}}\right]=L_{t'}\Phi\left(d_{Kt'}+\sigma B_{n-t'}\sqrt{\frac{1-e^{-2\kappa}}{2\kappa}}\right), \tag{3.37}$$

where the expressions of $d_{Kt'}$ and $L_{t'}$ remain the same as for the bank. Finally, the expected exposure of the counterparty is obtained by replacing $K$ with $(c^{-}+h^{-})e^{r(t'-m)}$.

To solve the DP recursion in this example, we approximate the value function with polynomial functions that are bi-linear with respect to the available amount of collateral and to the market value of the swap. Similarly to the previous example, the challenge is in determining the conditional expectations $\mathbb{E}_{m,x,c}\left[\widehat{W}_m(f_m(x,c,h))\right]$ which are again linear combinations of $\mathbb{E}_{m,x}\left[X^q_{m+1}I_{[a^1_{j_1},a^1_{j_1+1}]}(X_{m+1})\right]$ for $q=0,1$ and where $\{a^1_{j_1}\}_{j_1}$ denote the grid points for the swap value. To determine these latter conditional expectations, it suffices to remark that:

$$\mathbb{E}_{m,x}\left[X^q_{m+1}I_{[a^1_{j_1},a^1_{j_1+1}]}(X_{m+1})\right]=\mathbb{E}_{m,x}\left[X^q_{m+1}I_{[a^1_{j_1},\infty)}(X_{t_{m+1}})\right]-\mathbb{E}_{m,x}\left[X^q_{m+1}I_{[a^1_{j_1+1},\infty)}(X_{m+1})\right]$$

and compute the expectations in the right hand side by using the mechanism employed previously for the expected exposures computation.

#### 3.6.2.3 Computation of the survival functions

We assume that the default intensity of the bank (the counterparty) is linear with respect to the floating interest rate, that is:

$$g^i(x) = \max\left\{0, g_0^i + g_1^i x\right\},\ i \in \{a, b\} \tag{3.38}$$

where $g_0^i$ and $g_1^i$ are constants determined from credit-spread data of counterparty $i$ (for more details see Hull and White 2012). A positive value of $g_1^b$ coincides with a situation of wrong way-risk for the bank. In fact, since the counterparty pays float, an increase of the floating rate implies an increase of the bank's exposure to the counterparty. To solve the Partial Differential Equation (3.6), we follow Duffie (2005) by assuming that:

$$G_{t'}(x,t) = \exp\{\varphi(t'-t) + \psi(t'-t)x\}$$

where $\varphi(.)$ and $\psi(.)$ are deterministic functions satisfying $\varphi(0) = \psi(0) = 0$. Deriving $G_{t'}(.)$ with respect to $t$ and $x$, the PDE (3.6) becomes:

$$\kappa(\mu - x)\psi(z) + \frac{1}{2}\sigma^2\psi^2(z) + \varphi'(z) + x\psi'(z) - (g_0 + g_1 x) = 0 \text{ for all } x \text{ and } z = t' - t.$$

Separating variables, we obtain:

$$\kappa\mu\psi(z) + \frac{1}{2}\sigma^2\psi^2(z) + \varphi'(z) - g_0 \ = \ 0 \tag{3.39}$$

$$-\kappa\psi(z) + \psi'(z) - g_1 \ = \ 0. \tag{3.40}$$

Equation (3.40) is a simple first degree differential equation whose solution is:

$$\psi(z) = \frac{g_1}{\kappa}(e^{\kappa z} - 1) \tag{3.41}$$

Equation (3.39) is equivalent to:

$$\varphi'(z) = g_0 - \kappa\mu\psi(z) - \frac{1}{2}\sigma^2\psi^2(z),$$

implying that:

$$\varphi(z) = g_0 z - \kappa\mu \int_0^z \psi(z')dz' - \frac{1}{2}\sigma^2 \int_0^z \psi^2(z')dz'. \tag{3.42}$$

With Equation (3.41), we can show that:

$$\int_0^z \psi(z')dz' = \frac{g_1}{\kappa^2}(e^{\kappa z} - 1) - \frac{g_1}{\kappa} z \tag{3.43}$$

and

$$\int_0^z \psi^2(z')dz' = \frac{g_1^2}{2\kappa^3}(e^{2\kappa z} - 1) - \frac{2g_1^2}{\kappa^3}(e^{\kappa z} - 1) + \frac{g_1^2}{\kappa^2} z. \tag{3.44}$$

#### 3.6.2.4 Numerical results

The numerical illustrations presented here concern a seven-months swap with coupon dates every two months and a principal normalized to 1. The recovery rate of the bank and the counterparty are fixed at respectively $\xi^b = 60\%$ and $\xi^a = 65\%$. We consider a margin frequency of one week, corresponding to the time unit, and we set the weights $\omega_b$ and $\omega_a$ at 0.5. Default intensities are given by Equation (3.38) and the one-period risk-free rate is $r = 0.0205\%$, corresponding to an annualized risk-free rate of 1.5%. For simplicity, we assume that the threshold of the bank and the counterparty is zero and that the minimum transfer amount is $M = \frac{q_0}{1-q_1}$, the minimum adjustment of collateral that exceeds the corresponding transaction cost. In practice, these parameters are the result of negotiation. Other parameter values are given in Table 3.8.

| $\kappa$ | $r_0$ | $\sigma$ | $\mu$ | $g_0^b$ | $g_1^b$ | $g_0^a$ | $g_1^a$ | $q_0$ |
|---|---|---|---|---|---|---|---|---|
| 0.5 | 6.5% | 17% | 0.065 | 3 | 0.01 | 4 | 0.03 | 0.0001 |

Table 3.8: Basis parameters (used in Figures 3.8 and 3.9, and Tables 3.9 and 3.10)

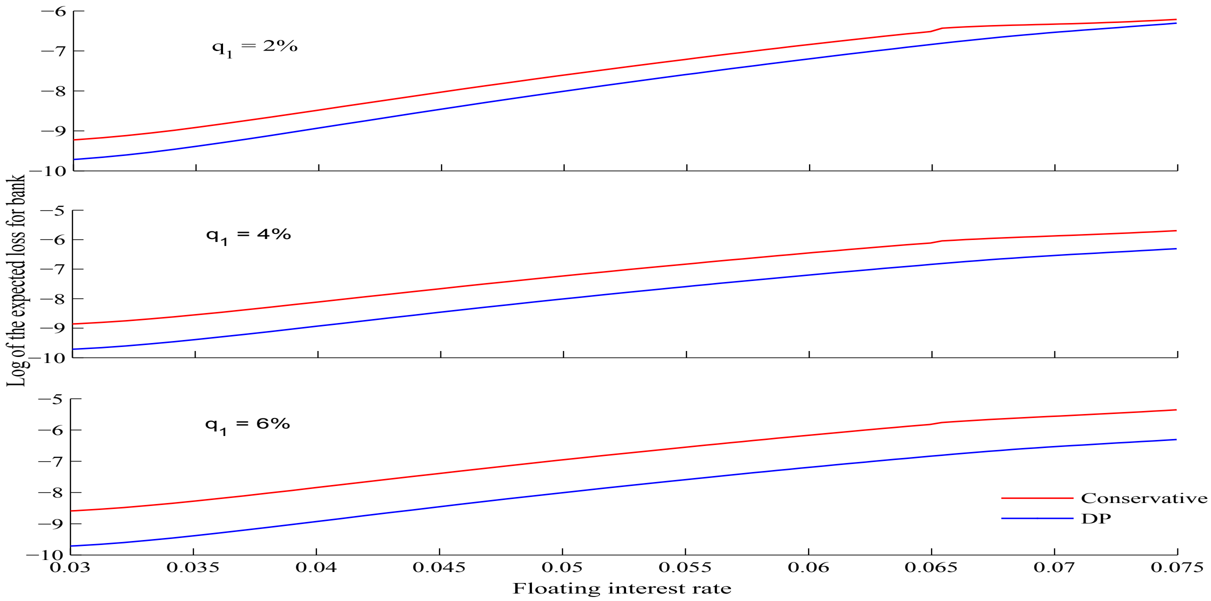


Figure 3.8: Comparing the logarithm of the expected cost of the bank for various level of $q_1$; Parameter values are given in Table 3.8.

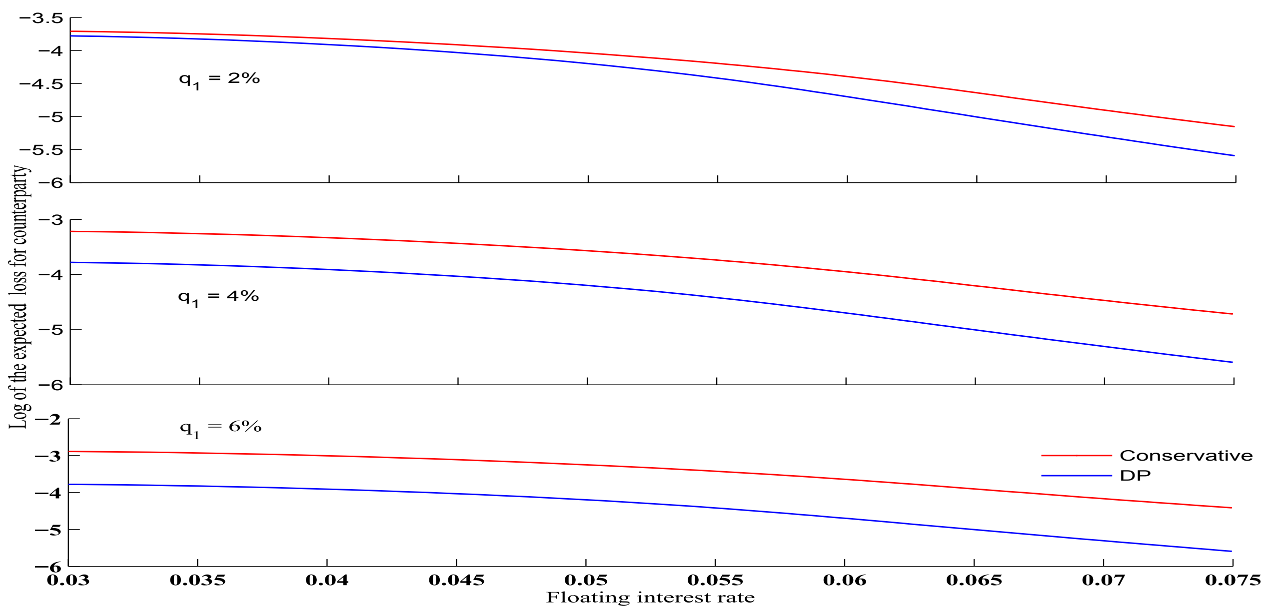


Figure 3.9: Comparing the logarithm of the expected cost of the counterparty for various level of $q_1$; Parameter values are given in Table 3.8.

Figures 3.8 and 3.9 display the logarithm of the expected total cost (expected loss plus transaction costs) for the bank and the counterparty respectively, corresponding to the DP and conservative approaches, as a function of the floating interest rate and for various levels of the transaction cost rate $q_1$. The parameter values used to obtain these results are reported in Table 3.8 where $(g_0^b, g_1^b)$ and $(g_0^a, g_1^a)$ denote the coefficients of the default intensity functions of the bank and the counterparty respectively. Note that these default intensity parameters correspond to a situation of wrong-way risk for the bank and right-way risk for the counterparty. The corresponding swap rate is $r_f = 6.23\%$.

Several observations can be made from these results. Notice that, as expected, the expected total loss is increasing for the bank and decreasing for the counterparty with respect to the floating rate. As for the previous illustrative example, one observes that the total expected cost is lower using the DP approach, for all levels of the floating interest and transaction cost rates, and for both the bank and the counterparty. One also observes that the difference in expected costs between the two approaches increases with the transaction cost rate $q_1$.

**Floating interest rate**

| | 6.3% | | 6.9% | | 7.1% | | 7.5% | | 7.8% | |
|---|---|---|---|---|---|---|---|---|---|---|
| | CS | DP | CS | DP | CS | DP | CS | DP | CS | DP |
| **Collateral** | **Panel A:variable transaction cost rate $q_1 = 2\%$** | | | | | | | | | |
| -0.0733 | 0.0008 | 0.0004 | 0.0016 | 0.0008 | 0.0019 | 0.0010 | 0.0023 | 0.0014 | 0.0025 | 0.0017 |
| 0.0000 | 0.0013 | 0.0008 | 0.0015 | 0.0009 | 0.0015 | 0.0010 | 0.0015 | 0.0011 | 0.0016 | 0.0011 |
| 0.0524 | 0.0017 | 0.0010 | 0.0015 | 0.0008 | 0.0014 | 0.0008 | 0.0014 | 0.0007 | 0.0013 | 0.0007 |
| | **Panel B:variable transaction cost rate $q_1 = 4\%$** | | | | | | | | | |
| -0.0733 | 0.0012 | 0.0005 | 0.0025 | 0.0011 | 0.0030 | 0.0014 | 0.0036 | 0.0021 | 0.0040 | 0.0026 |
| 0.0000 | 0.0019 | 0.0011 | 0.0022 | 0.0013 | 0.0022 | 0.0014 | 0.0024 | 0.0015 | 0.0026 | 0.0016 |
| 0.0524 | 0.0026 | 0.0012 | 0.0023 | 0.0009 | 0.0022 | 0.0008 | 0.0021 | 0.0008 | 0.0020 | 0.0008 |
| | **Panel C:variable transaction cost rate $q_1 = 6\%$** | | | | | | | | | |
| -0.0733 | 0.0016 | 0.0006 | 0.0033 | 0.0013 | 0.0040 | 0.0017 | 0.0050 | 0.0025 | 0.0056 | 0.0032 |
| 0.0000 | 0.0024 | 0.0013 | 0.0029 | 0.0015 | 0.0030 | 0.0017 | 0.0033 | 0.0019 | 0.0036 | 0.0021 |
| 0.0524 | 0.0034 | 0.0012 | 0.0031 | 0.0009 | 0.0030 | 0.0009 | 0.0028 | 0.0008 | 0.0028 | 0.0008 |

Table 3.9: Comparing weighted expected costs using the DP and conservative approach for different values of the transaction cost $q_1$. Other parameter values are given in Table 3.8.

Tables 3.9 and 3.10 report on the weighted sum of the expected costs of both counter-

| Collateral | 6.3% CS | 6.3% DP | 6.9% CS | 6.9% DP | 7.1% CS | 7.1% DP | 7.5% CS | 7.5% DP | 7.8% CS | 7.8% DP |
|---|---|---|---|---|---|---|---|---|---|---|
| | **Panel A: volatility parameter $\sigma = 17\%$** | | | | | | | | | |
| -0.0733 | 0.0008 | 0.0004 | 0.0016 | 0.0008 | 0.0019 | 0.0010 | 0.0023 | 0.0014 | 0.0025 | 0.0017 |
| 0.0000 | 0.0013 | 0.0008 | 0.0015 | 0.0009 | 0.0015 | 0.0010 | 0.0015 | 0.0011 | 0.0016 | 0.0011 |
| 0.0524 | 0.0017 | 0.0010 | 0.0015 | 0.0008 | 0.0014 | 0.0008 | 0.0014 | 0.0007 | 0.0013 | 0.0007 |
| | **Panel B:volatility parameter $\sigma = 25\%$** | | | | | | | | | |
| -0.0733 | 0.0016 | 0.0010 | 0.0020 | 0.0013 | 0.0021 | 0.0014 | 0.0023 | 0.0016 | 0.0024 | 0.0018 |
| 0.0000 | 0.0016 | 0.0010 | 0.0016 | 0.0011 | 0.0016 | 0.0011 | 0.0017 | 0.0012 | 0.0017 | 0.0012 |
| 0.0524 | 0.0015 | 0.0010 | 0.0014 | 0.0009 | 0.0014 | 0.0008 | 0.0013 | 0.0008 | 0.0013 | 0.0008 |

Table 3.10: Comparing weighted expected costs using the DP and conservative approach for two contrasting values of the volatility $\sigma$.Other parameter values are given in Table 3.8.

parties for different levels of the floating rate and the available amount of collateral and for various values of the transaction cost rate $q_1$ and volatility level $\sigma$. In all cases, the weighted sum of total expected costs is lower using a DP collateral management strategy than using the conservative approach. Differences in cost can be significant, in some instances yielding a cost that is 3.5 times lower under the DP strategy.

### 3.6.3 Example 3: CRRA utility under a log-normal model

In this last example, we provide an illustration of the utility-based model by assuming that the market value of the portfolio evolves according to a log-normal model under risk-neutral measure. More specifically, we assume that $X \equiv V$, and:

$$X_{m'} = X_m \exp\{\mu(m'-m) + \sigma\sqrt{(m'-m)}Z\}, \tag{3.45}$$

where $Z$ is a standard Gaussian random variable. The survival function is given by:

$$G_{m'}(x,m) = \exp\{-\lambda(m'-m)\},$$

where $\lambda$ is the constant default intensity of the counterparty. We assume that $U$ is a constant relative risk aversion (CRRA) utility function, that is:

$$U(y) = \frac{1}{1-\eta} y^{1-\eta} \tag{3.46}$$

where $y$ is the level of wealth and $\eta$ is a positive constant, strictly less than 1, that measures the degree of relative risk aversion of the bank. Notice that the conditional expectation in the right-hand side of Equation (3.20) is an integral that cannot be computed in closed form. To solve the DP recursion in this example, we use a three-dimensional interpolation scheme combined with numerical integration. More specifically, the value function $W_{m+1}$ is interpolated with a polynomial $\widehat{W}_{m+1}$ cubic with respect to the market value of the portfolio of contracts, the available amount of collateral, and the wealth. To do so, we consider a three-dimensional grid $G_1 \times G_2 \times G_3$ where $G_1 = \{x_1, x_2, \ldots, x_{n_1}\}, G_2 = \{y_1, y_2, \ldots, y_{n_2}\}$ and $G_3 = \{c_1, c_2, \ldots, c_{n_3}\}$ represent respectively the market value of the portfolio of contracts, the wealth, and the available amount of collateral. At each intermediate stage $m$ the future value function $W_{m+1}$ is approximated with a three-variate polynomial:

$$P_{ijk}(x,y,c) = \sum_{i_1=1}^{3} \sum_{j_1=1}^{3} \sum_{k_1=1}^{3} a_{i_1 j_1 k_1} (x - x_i)^{i_1} (y - y_j)^{j_1} (c - c_k)^{k_1} \tag{3.47}$$

inside each small cube $[x_i, x_{i+1}] \times [y_j, y_{j+1}] \times [c_k, c_{k+1}]$. The conditional expectations $\mathbb{E}_{m,x,c,y}\left[P_{ijk}\left(f_m(x,c,y,h)\right)\right]$ are computed by means of a Gauss-Hermite quadrature. Our numerical illustrations are done with a portfolio of contracts with a longest maturity of six months where the margin frequency is set to one week. For the shake of simplicity, we assume that the threshold is zero and the minimum transfer amount $M = \frac{q_0}{1-q_1}$, corresponding to the minimum adjustment $h$ such that $h \geq q(h)$. Parameters $\mu, \sigma$ and $r$ are assumed constant and reported in Table 3.11, as well as the recovery rate $\xi^b$ and the fixed transaction cost.

| $\mu$ | $\sigma$ | $r$ | $\xi^b$ | $q_0$ | $\eta$ | $\lambda$ |
|---|---|---|---|---|---|---|
| 0.015 | 24% | 0.9% | 65% | $10^{-3}$ | 0.3 | 4.0 |

Table 3.11: Parameter values for the CRRA utility model.

We consider various levels of the transaction cost rates $q_1$. Figure 3.10 compares the DP and conservative approaches by plotting the relative difference $\frac{U_{DP}-U_{CS}}{U_{CS}}$ of expected final utility between these two approaches in percentage of $U_{CS}$ the expected final utility in the conservative approach, as a function of the market value of the portfolio. The initial available amount of collateral is $C_0 = 0$ and three levels of the initial wealth $Y_0$ and the transaction cost rate $q_1$ are considered. In all cases, the expected final utility is always higher within the DP strategy, and the difference is increasing with the market value of the portfolio.

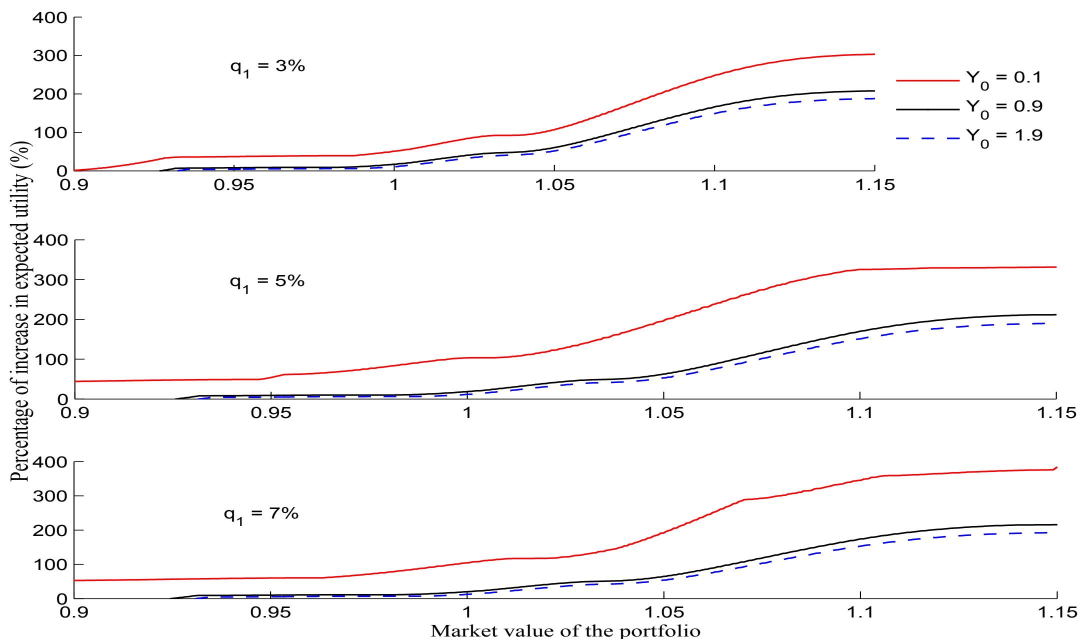


Figure 3.10: Relative difference in expected final utilities using the DP or conservative strategies, as a function of the portfolio value, for various values of the initial wealth and transaction cost. Parameter values are given in Table 3.11.

Table 3.12 reports on the adjustment of collateral and the expected final utilities in the

DP and conservative approaches for different levels of the transaction cost rate $q_1$, the market value $V$ of the portfolio, and the available amount of collateral $C$ at date 0. The initial wealth is fixed at $Y_0 = 2$ and all other parameters are taken from Table 3.11. In all the cases, we observe that the required adjustment of collateral is smaller with the DP approach, and that the expected utility is larger, with relative differences ranging from 4% to 25%.

| | **Panel A : $q_1 = 3\%$ and $Y_0 = 2$** | | | | | |
|---|---|---|---|---|---|---|
| Market value | 0.9500 | | 1.0000 | | 1.0500 | |
| | CS | DP | CS | DP | CS | DP |
| Collateral | Adjustment of collateral | | | | | |
| 0.2000 | 0.7500 | 0.4026 | 0.8000 | 0.4211 | 0.8500 | 0.4895 |
| 0.4000 | 0.5500 | 0.3579 | 0.6000 | 0.3553 | 0.6500 | 0.4211 |
| 0.6000 | 0.3500 | 0.2947 | 0.4000 | 0.3158 | 0.4500 | 0.3553 |
| Collateral | Expected final utility | | | | | |
| 0.2000 | 0.2628 | 0.2879 | 0.2681 | 0.3342 | 0.2726 | 0.3460 |
| 0.4000 | 0.2610 | 0.2871 | 0.2662 | 0.2900 | 0.2707 | 0.3347 |
| 0.6000 | 0.2592 | 0.2862 | 0.2644 | 0.2892 | 0.2689 | 0.2942 |
| | **Panel B : $q_1 = 5\%$ and $Y_0 = 2$** | | | | | |
| Collateral | Adjustment of collateral | | | | | |
| 0.2000 | 0.7500 | 0.4026 | 0.8000 | 0.4474 | 0.8500 | 0.4895 |
| 0.4000 | 0.5500 | 0.3579 | 0.6000 | 0.4026 | 0.6500 | 0.4474 |
| 0.6000 | 0.3500 | 0.2947 | 0.4000 | 0.3579 | 0.4500 | 0.4026 |
| Collateral | Expected final utilities | | | | | |
| 0.2000 | 0.2598 | 0.2755 | 0.2748 | 0.2881 | 0.2816 | 0.3306 |
| 0.4000 | 0.2573 | 0.2712 | 0.2718 | 0.2866 | 0.2785 | 0.2902 |
| 0.6000 | 0.2548 | 0.2708 | 0.2688 | 0.2818 | 0.2755 | 0.2885 |

Table 3.12: Collateral adjustment and expected utilities according to the conservative and DP-based strategies. Parameter values are given in Table 3.11.

## 3.7 Conclusion

In this chapter, we discussed collateral management strategies for both unilateral and bilateral agreements. We examined existing collateral management strategies, and compared them to strategies based on dynamic programming, multiobjective optimization and utility maximizing approaches. These strategies are able to take into account several features, as for instance, wrong (right)-way risk and transaction costs. The dynamic programming strategy is also able to adapt to the state (portfolio value and collateral level) and to the time remaining until maturity. For unilateral agreements, we compared the performance of both types of strategies in an example where the market value of the portfolio of contracts between the bank and the counterparty is modeled with a jump diffusion model. For bilateral agreements, we compared their performance using an interest rate swap. We finally compared the performances of both types of strategies in a final utility maximization setting. Our numerical experiments all point out to a significant improvement in collateral management when using DP-based strategies.

# Chapter 4

# Analytical Valuation of Compound Options Under Regime Switching Dynamics

**Summary**

The aim of this chapter is to propose an analytical formula for the evaluation of compound options when the underlying asset is described by a two-states Markov regime-switching log-normal model. One specific application of interest of such a formula is the pricing of principal protected callable notes with an early redemption feature. This approach provides practitioners with a Black-Scholes type formula under a realistic assumption about market prices behavior. A numerical illustration is provided for principal protected callable notes issued by the National Bank of Canada.

**Keywords**: principal protected callable notes, regime-switching model.

## 4.1 Introduction

A *compound option*, also named *split-free option,* is an option written on another option, meaning that it gives its owner the right to buy or sell another option, called the *underlying option.* According to the nature of the compound and the underlying option, there are four types of compound options: Call on Call, Call on Put, Put on Call and Put on Put. A compound option has two maturity dates $(T_1, T_2)$ and two exercise prices $(K_1, K_2)$, where $T_1$ and $K_1$ denote respectively the maturity date and exercise price of the compound option while $T_2$ and $K_2$ denote the maturity date and exercise price of the underlying option, with $T_1 \leq T_2$.

Compound options are not traded in financial markets; however, many traded financial products can be expressed or approximated as a combination of compound options and other simpler instruments. For instance, it is the case of convertible bonds (Gong et al. 2006), American options (Geske and Johnson 1984), and more particularly some widespread types of principal protected callable notes. The first and most popular paper dedicated to the valuation of compound options is undoubtedly that of Geske (1979) where, assuming a geometric Brownian motion for the dynamics of the underlying asset of the underlying option, the author derives a Black-Scholes type formula for the value of a Call on Call, using Fourier integration and some characteristics of the bivariate normal distribution. Another derivation based on the risk-neutral principle is suggested by Rubinstein (1991) and implemented in Lajeri-Chaherli (2002). In addition to providing an elegant proof, this last paper also gives an interesting interpretation of the different components that make up the formula. Based on the formula derived in Geske (1979), Geske and Johnson (1984) obtain an analytical approximation for the price of an American put option. This approximation method requires the infinite summation of multivariate normal integrals, which is not easy to implement and very time consuming. Selby and Stewart (1987) propose a technique that reduces the number of integrals to be evaluated for the implementation of methods that use multinomial distributions, as it is the case for the Geske formula. Agliardi and Agliardi (2003) provide a generalization of the Geske formula when the interest rate and the volatility are time-dependent. This generalization seems more realistic in practice, but the resulting formula is more complex to implement. Fouque and Han (2005) use a perturbation technique to derive an approximation of compound option prices under a two-factor stochastic volatility model. Their methodology can be summarized into two main steps; First the underlying option's value is approximated with a Black-Scholes term (with constant volatility) plus a perturbation term, and then a Taylor expansion of the compound option's payoff around the constant term of the underlying option's approximation is used to produce the risk-neutral discounted conditional expectation of the result. This method can be easily implemented in practice, but it requires calibration based on observed prices of compound options, which unfortunately are not exchanged in financial markets.

## 4.2 Compound options valuation under regime switching

In this section, we extend the Geske formula to the two-states Markov regime-switching log-normal (MRSLN2) model of Hardy (2001). The choice of such dynamics is motivated essentially by the fact that this model is able to capture several empirical properties of long-term financial stock returns, such as heteroscedasticity, autocorrelation and fat tails (see Hamilton 1989 and Hardy 2001). For instance, Hardy (2001) finds that, compared to several econometric models, a MRSLN2 model describes better the behavior of the TSE-300 and the S&P-500 indexes. We first start by recalling the Geske formula for the geometric Brownian motion model.

### 4.2.1 Notations

We first introduce the notation and several functions that will be used in this chapter. Denote by $N(.)$ and $N_2(.,\rho)$ the cumulative distribution functions of respectively the univariate standard normal and the bivariate normal distribution with correlation matrix

$$\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix},$$

where $\rho \in [-1,1]$.

For any vector $(r,\sigma,K,T)$ of positive real numbers, define:

$$\begin{aligned} d_{r,\sigma,K,T}(s) &= \frac{ln(s/K)+(r+\sigma^2/2)T}{\sigma\sqrt{T}} \\ d'_{r,\sigma,K,T}(s) &= d_{r,\sigma,K,T}(s)-\sigma\sqrt{T}. \end{aligned}$$

Denote by $C(s;r,\sigma,K,T)$ the value of a European call option with time to maturity $T$ and strike $K$ where $\sigma$ is the volatility of the underlying asset, and $r$ the risk-free rate under the Geometric Brownian Motion model. This function is given by the Black and Scholes (1973) formula, that is:

$$C_B(s;r,\sigma,K,T) = sN(d_{r,\sigma,K,T}(s)) - Ke^{-rT}N(d'_{r,\sigma,K,T}(s)). \tag{4.1}$$

### 4.2.2 The Geske formula

Within the Black-Scholes framework, Geske (1979) showed that the value at date $t=0$ of a Call on Call with exercise prices $(K_1,K_2)$ and maturity dates $(T_1,T_2)$ when the underlying asset price is $s$ is given by:

$$\begin{aligned} C_G(s;r,\sigma,K_1,K_2,T_1,T_2) = sN_2(d_1,d_2;\rho) &- K_2e^{-rT_2}N_2(d'_1,d'_2;\rho) \\ &- K_1e^{-rT_1}N(d'_1) \end{aligned} \tag{4.2a}$$

where

$$
\begin{aligned}
d_1 &= d_{r,\sigma,K_2,T_2}(s),\ \ d_2' = d'_{r,\sigma,K_2,T_2}(s) && (4.2b)\\
d_2 &= d_{r,\sigma,K_1,T_1}(s^*),\ \ d_1' = d'_{r,\sigma,K_1,T_1}(s^*) && (4.2c)\\
\rho &= \sqrt{\frac{T_1}{T_2}}, && (4.2d)
\end{aligned}
$$

and where $s^*$ is the underlying asset level such that the payoff of the compound option at maturity $T_1$ is zero. In other words, $s^*$ is the value of $s$ that solves the following equation:

$$C_B(s;r,\sigma,K_2,T_2-T_1) = K_1. \tag{4.3}$$

Formula (4.2a) has an analogue for a Call on Put, Put on Call and a Put on Put, and also for options on stocks that pay dividends. Notice that the Geske formula, while analytic, is not in closed form since it requires the numerical solution of implicit Equation (4.3). To solve (4.3), one can use for instance the Newton-Raphson algorithm, which is guaranteed to converge regardless of the starting point because of the monotonicity and convexity of the Black-Scholes function $C_B$.

We now assume that the underlying asset price dynamics is described by the MRSLN2 model of Hardy (2001). Accordingly, during a time interval $[m, m+1]$ where the regime is $k \in \{1,2\}$, the log-return of the underlying asset follows a Gaussian distribution with mean and variance indexed on $k$, that is:

$$\ln \frac{S_{m+1}}{S_m} \sim \mathbb{N}(\mu_k, \sigma_k^2),\ \ k \in \{1,2\},$$

where $S_m$ denotes the value of the underlying asset price at date $m$. Regime transitions are modeled by a discrete-stage Markov chain. We denote by $p_{kl}$ the transition probability from regime $k$ to regime $l$, $k,l \in \{1,2\}$. The MRSLN2 model is fully characterized by the vector $\Theta = (\mu_1, \mu_2, \sigma_1, \sigma_2, p_{12}, p_{21})$. Notice that we assume that the time unit corresponds to the time elapsed between two successive stages of the Markov chain.

To specify a risk-neutral measure, one can follow Bollen (1998), who assumes that the additional risk brought by the possibility of regime changes is not priced in the market, so that the risk-neutral dynamics of assets is obtained by replacing $\mu_j$ with $r - \sigma_j^2/2$.

Consider a European call option, where the underlying asset price is modeled with a MRSLN2 as described above, with exercise price $K$ and maturity $T = n$. Given the MRSLN2 parameter vector $\Theta$, Hardy (2001) shows that the price of such an option at $t = 0$ when the underlying asset price is $s$ is given by[1]

$$C_H(s; r, \Theta, K, T) = \sum_{i=0}^{n} C_B(s; r, \overline{\sigma}_{in}, K, T) P(i; n) \tag{4.4}$$

where

$$\overline{\sigma}_{in} = \sqrt{i\sigma_1^2 + (n-i)\sigma_2^2} \tag{4.5}$$

and $P(\cdot; n)$ is the probability of the number of time steps spent in Regime 1 during $n$ time steps. Following Hardy (2001), denote by $Q_m(j|k; n)$ the probability that the total sojourn (measured in time steps) in Regime 1 during time period $[m, n)$ be $j$ if the prevailing regime in time period $[m-1, m)$ is $k \in \{1, 2\}$, where $m = 0, 1, \ldots, n-1$. We then have:

$$\begin{aligned} Q_m(j|k; n) &= p_{k1} Q_{m+1}(j-1|1; n) + p_{k2} Q_{m+1}(j|2; n), \quad m < n-1 && \text{(4.6a)} \\ Q_{n-1}(1|k; n) &= p_{k1} && \text{(4.6b)} \\ Q_{n-1}(0|k; n) &= p_{k2}, && \text{(4.6c)} \end{aligned}$$

and the probabilities $P(i; n)$ are given by

$$P(i; n) = \pi_1 Q_0(i|1; n) + \pi_2 Q_0(i|2; n) \tag{4.7}$$

[1]Notice that the Hardy formula does not depend on the current regime, which is assumed unobservable.

where $\pi_1$ and $\pi_2$ describe the unconditional regime probability distribution for the process $\{S_t\}$, and are given by:

$$\pi_1 = 1 - \pi_2 = p_{21}/(p_{12} + p_{21}). \tag{4.8}$$

Now consider a compound option issued at date $t = 0$ with exercise prices $(K_1, K_2)$ and maturity dates $(T_1, T_2)$, where the underlying asset price is modeled with a MRSLN2 as described above. Assume that $T_1 = n_1$ and $T_2 = n = n_1 + n_2$. The price of the compound Call on Call option is given by the following proposition.

**Proposition 4.2.1.** *The value of the Call on Call under a MRSLN2 model is given by:*

$$CC_0(s; r, \Theta, K_1, K_2, T_1, T_2) = \sum_{i=0}^{n_1} C_1(s|i; r, \Theta, K_1, K_2, T_1, T_2) P(i; n_1), \tag{4.9a}$$

*where $C_1(s|i; r, \Theta, K_1, K_2, T_1, T_2)$ is the value of the compound option conditional on $i$, the number of time steps spent in Regime 1 during period $[0, T_1)$. For $i = 0, 1, \ldots, n_1$, this value is given by:*

$$\begin{aligned} C_1(s|i; r, \Theta, K_1, K_2, T_1, T_2) = & - K_1 e^{-rn_1} N(d_2(i)) \\ & + \sum_{j=0}^{n_2} C_2(s|i,j; r, \Theta, K_2, T_2) P(j; n_2) \end{aligned} \tag{4.9b}$$

*with*

$$C_2(s|i,j; r, \Theta, K_2, T_2) = s N_2(d_1(i,j), d_2(i); \rho_{ij}) - K_2 e^{-rn_2} N_2(d_1'(i,j), d_2'(i); \rho_{ij}) \tag{4.9c}$$

*where*

$$d_1(i,j) = d_{r,\overline{\sigma}_{i+j,n} K_2, T_2}(s), \;\; d_1'(i,j) = d'_{r,\overline{\sigma}_{i+j,n} K_2, T_2}(s) \tag{4.9d}$$

$$d_2(i) = d_{r,\overline{\sigma}_{i,n_1}, K_1, T_1}(s^*), \;\; d_2'(i) = d'_{r,\overline{\sigma}_{i,n_1}, K_1, T_1}(s^*) \tag{4.9e}$$

$$\rho_{ij} = \frac{\overline{\sigma}_{i,n_1}}{\overline{\sigma}_{i+j,n}} \tag{4.9f}$$

*and where $s^*$ is the value of the underlying asset price such that the compound option is at* *the money at maturity, that is, $s^*$ solves the following equation:*

$$C_H(s;r,\Theta,K_2,T_2-T_1)=K_1, \tag{4.9g}$$

*where $C_H$ is given by formula (4.4).*

*Proof.* At date $T_1$, if the underlying asset price is $s$, the value of the Call on Call is given by

$$CC_{T_1}(s;r,\Theta,K_1,K_2,T_1,T_2)=[C_H(s;r,\Theta,K_2,T_2-T_1)-K_1]^+.$$

Now, denote by $s^*$ the value of $S_{T_1}$ such that $C_H(S_{T_1};r,\Theta,K_2,T_2-T_1)=K_1$. Since the Hardy function $C_H$ is increasing in terms of $s$, we have:

$$CC_{T_1}(s;r,\Theta,K_1,K_2,T_1,T_2)=[C_H(s;r,\Theta,K_2,T_2-T_1)-K_1]\,\mathbb{I}_{[s^*,\infty)}(s)$$

where $\mathbb{I}_A$ is the indicator function of the set $A$. At issuance, the value of the compound option is therefore

$$\begin{aligned} CC_0 &= CC_0(s;r,\Theta,K_1,K_2,T_1,T_2) \\ &= e^{-rn_1}\mathbb{E}_s\left[(C_H(S_{T_1};r,\Theta,K_2,T_2-T_1)-K_1)\mathbb{I}_{[s^*,\infty)}(S_{T_1})\right] \end{aligned} \tag{4.10}$$

On the other hand, as shown by Hardy (2001), conditional on $i$, the number of time steps spent in Regime 1 during the time interval $[0,T_1)$, the asset price $S_{T_1}$ at date $T_1$ satisfies

$$S_{T_1}=S_0\exp\left((r-\frac{\overline{\sigma}^2_{i,n_1}}{2})T_1+\overline{\sigma}^2_{i,n_1}\sqrt{T_1}\mathbb{Z}\right)$$

where $\mathbb{Z}$ is a standard normal random variable. Thereby, to compute the conditional expectation in Equation (4.10), we condition on $i$, the number of time units spent in Regime 1 during time interval $[0, T_1)$, yielding

$$\begin{aligned} CC_0 &= CC_0(s; r, \Theta, K_1, K_2, T_1, T_2) \\ &= e^{-rn_1} \sum_{i=0}^{n_1} \mathbb{E}_{s,i} \left[ (C_H(S_{T_1}; r, \Theta, K_2, T_2 - T_1) - K_1) \mathbb{I}_{[s^*,\infty)}(S_{T_1}) \right] P(i; n_1) \end{aligned} \quad (4.11)$$

where $\mathbb{E}_{s,i}$ denotes the expectation conditional on $S_0 = s$ and on the number of time units spent in Regime 1. Using

$$S_{T_1} \geq s^* \Longleftrightarrow \mathbb{Z} \geq -d'_{r,\overline{\sigma}_{i,n_1} s^*, T_1}(s),$$

we then have:

$$\begin{aligned} E &= \mathbb{E}_{s,i} \left[ (C_H(S_{T_1}; r, \Theta, K_2, T_2 - T_1) - K_1) \mathbb{I}_{[s^*,\infty)}(S_{T_1}) \right] \\ &= \mathbb{E}_{s,i} \left[ C_H(S_{T_1}; r, \Theta, K_2, T_2 - T_1) \mathbb{I}_{[s^*,\infty)}(S_{T_1}) \right] - \mathbb{E}_{s,i} \left[ K_1 \mathbb{I}_{[s^*,\infty)}(S_{T_1}) \right] \\ &= \mathbb{E}_{s,i} \left[ C_H(S_{T_1}; r, \Theta, K_2, T_2 - T_1)) \mathbb{I}_{[s^*,\infty)}(S_{T_1}) \right] - K_1 N(d'_{r,\overline{\sigma}_{i,n_1} s^*, T_1}(s)). \end{aligned}$$

To compute the last conditional expectation, we replace $C_H$ with its expression in equation (4.4), which leads to:

$$C_H(S_{T_1}; r, \Theta, K_2, T_2 - T_1)) \mathbb{I}_{[s^*,\infty)}(S_{T_1}) = \sum_{j=0}^{n_2} C_B(S_{T_1}; r, \overline{\sigma}_{j,n_2}, K_2, n_2) P(j; n_2) \mathbb{I}_{[S^*,\infty)}(S_{T_1})$$

with

$$C_B(S_{T_1}; r, \overline{\sigma}_{j,n_2}, K_2, n_2) = S_{T_1} N(d_{r,\overline{\sigma}_{j,n_2}, K_2, n_2}(S_{T_1})) - K_2 e^{-rn_2} N(d'_{r,\overline{\sigma}_{j,n_2}, K_2, n_2}(S_{T_1})).$$

Now,

$$
\begin{aligned}
E' &= \mathbb{E}_{s,i}\left[S_{T_1}N(d_{r,\overline{\sigma}_{j,n_2},K_2,n_2}(S_{T_1}))\mathbb{I}_{[s^*,\infty)}(S_{T_1})\right] \\
&= se^{\left(r-\frac{\overline{\sigma}^2_{i,n_1}}{2}\right)n_1}\mathbb{E}_s\left[e^{\overline{\sigma}_{i,n_1}\sqrt{n_1}\mathbb{Z}}N\left(\frac{d_{r,\overline{\sigma}_{i+j,n},K_2,T}(s)+\rho_{ij}(\mathbb{Z}-\overline{\sigma}_{i,n_1}\sqrt{n_1})}{\sqrt{1-\rho^2_{ij}}}\right)\mathbb{I}_{[-d'_{r,\overline{\sigma}_{i,n_1}s^*,T_1}(s),\infty)}(\mathbb{Z})\right] \\
&= se^{rn_1}\int_{-d'_{r,\overline{\sigma}_{i,n_1}s^*,T_1}(s)}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(z-\overline{\sigma}_{i,n_1}\sqrt{n_1})^2}N\left(\frac{d_{r,\overline{\sigma}_{i+j,n},K_2,T}(s)+\rho_{i,j}(z-\overline{\sigma}_{i,n_1}\sqrt{n_1})}{\sqrt{1-\rho^2_{ij}}}\right)dz \\
&= se^{rn_1}\int_{-d'_{r,\overline{\sigma}_{i,n_1}s^*,T_1}(s)+\overline{\sigma}_{i,n_1}}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}N\left(\frac{d_{r,\overline{\sigma}_{i+j,n},K_2,T}(s)+\rho_{i,j}z}{\sqrt{1-\rho^2_{ij}}}\right)dz \\
&= se^{rn_1}N_2(d_{r,\overline{\sigma}_{i+j,n},K_2,T}(s),d_{r,\overline{\sigma}_{i,n_1}s^*,T_1}(s);\rho_{ij}).
\end{aligned}
$$

The same procedure can be used to show that:

$$
\mathbb{E}_{s,i}\left[N(d'_{r,\overline{\sigma}_{j,n_2},K_2,n_2)}(S_{T_1}))\mathbb{I}_{[s^*,\infty)}(S_{T_1})\right] = N_2(d_{r,\overline{\sigma}_{i+j,n},K_2,T_2)}(s),d_{r,\overline{\sigma}_{i,n_1}s^*,T_1}(s);\rho_{ij}).
$$

It suffices to replace these expressions in (4.11) to obtain the result. $\square$

Formula (4.9a) is easy to handle and can be easily extended to the case of Call on Put, Put on Call and Put and Put. This semi-analytic formula can be very helpful to evaluate long-term contracts which can be expressed as a combination of compound options and simpler instruments. In the next section, we explore a specific application of compound options, that is, principal protected callable notes.

## 4.3 Principal protected callable notes (PPCN)

A principal protected callable note or PPCN is a structured product with principal protection and call feature. More specifically, an investor holding such a derivative will receive at maturity at least the principal amount initially invested, provided the note has not been redeemed by the issuer, who has the right to do so at some predefined dates. Like most structured products, the potential return of a PPCN is linked to the performance of a given risky asset (e.g. an index, a fund, equity or a portfolio), called herein the *reference underlying*. In addition to its reference underlying, a PPCN is also characterized by its early redemption date(s) and prices, or adjusted potential return. According to the number and the nature of the intermediate decisions and to the way that the potential return is linked to the reference underlying performance, one can distinguish several types of PPCNs. In this chapter, we focus on PPCNs with a single redemption date where the issuer has the option of redeeming the contract at a pre-determined redemption price, while if the contract reaches maturity, the holder receives an amount linked to the total return of the reference underlying. Examples of such contracts are the Canadian Blue Chip III Deposit Notes issued by National Bank of Canada, the Callable Canadian Equity Deposit Notes issued by the Canadian Imperial Bank of Commerce, and the Callable Index-Linked Notes, linked to the Eurostoxx 50 Index and issued by HSBC Holdings plc.

More specifically, denote by $T_1$ the early redemption date, by $K_1$ the corresponding early redemption price, and by $D$ the initial investment of the note holder. The contract is then described by the following events:

- At issuance date $t = 0$, the issuer receives from the holder an initial investment $D$, which is linked to the value $S_0$ of the reference underlying. Generally, this initial investment does not includes management fees eventually charged by the issuer.
- At redemption date $T_1$, since the issuer has the option to call the PPCN for redemption at price $K_1$.
- If it has not been redeemed earlier, the holder receives at maturity date $T_2$ the maximum between his initial investment $D$ and the value $S_{T_2}$ of the reference underlying.

To the best of our knowledge, the valuation of PPCNs is not directly addressed in the literature. Monte Carlo simulation is generally used in practice to evaluate such products. In this section, we show that the valuation of PPCNs with a single early redemption date can be linked to the value of a compound option. More precisely, we show that holding a

PPCN is equivalent to holding a long position on a European call option and a Treasury bond and a short position on a Call on Call. Using the results of the previous section, an analytical pricing formula can therefore be obtained under the Black-Scholes framework and the MRSLN2 assumptions. Notice that this approach can be generalized for PPCNs with multiple early redemption dates, which can be expressed as a combination of multi-fold sequential compound options, but the resulting numerical scheme becomes rapidly inefficient. The valuation of more complex structured notes will be analyzed in a subsequent chapter of this thesis.

In the sequel, we assume that the initial investment $D$ is equal to the initial value of the reference underlying. This assumption does not change the procedure, but it simplifies the notation. The following proposition provides an analytical formula for the initial value of such principal protected callable note.

**Proposition 4.3.1.** *When there is a single early redemption date, the value at issuance of the PPCN is given by:*

$$V_0(D;r,\Theta,T_1,T_2,K_1) = \beta_0\beta_1 D + C(D;D;T_2;\Theta) - CC(D;K_1-\beta_1 D;D;T_1,T_2;\Theta), \tag{4.12}$$

*where $\beta_0 = e^{-rT_1}, \beta_1 = e^{-r(T_2-T_1)}\}$ and $\Theta$ is a vector containing the parameters of the model used for the dynamics of the reference underlying. For example, $\Theta$ is the volatility for the geometric Brownian motion model, while it contains the transition probabilities and the high and low volatility levels for the MRSLN2. Given the underlying asset initial price $D$, $C(D;D;T_2;\Theta)$ is the value of a European call option with maturity $T_2$ and exercise price $D$, while $CC(D;K_1-\beta_1 D;D;T_1,T_2;\Theta)$ is the price of a Call on Call with maturity dates $(T_1,T_2)$ and exercise prices $(K_1-\beta_1 D,D)$.*

*Proof.* If the note reaches maturity, its final value is given by:

$$V_2 = \max\{D;S_{T_2}\} = D + (S_{T_2}-D)^+.$$

At the early redemption date, the value of the note is given by:

$$\begin{aligned}
V_1 &= \min\{K_1; \beta_1 \mathbb{E}_{T_1}[V_2]\} \\
&= \min\left\{K_1; \beta_1 D + \beta_1 \mathbb{E}_{T_1}\left[(S_{T_2}-D)^+\right]\right\} \\
&= \beta_1 D + \beta_1 \mathbb{E}_{T_1}\left[(S_{T_2}-D)^+\right] - \left(\beta_1 \mathbb{E}_{T_1}\left[(S_{T_2}-D)^+\right] - (K_1 - \beta_1 D)\right)^+. \qquad (4.13)
\end{aligned}$$

Finally, the value of the note at issuance date is:

$$\begin{aligned}
V_0(D; r, \Theta, T_1, T_2, K_1) &= \beta_0 \mathbb{E}_0[V_1] \\
&= \beta_0 \beta_1 D + \beta_0 \mathbb{E}_0\left[\beta_1 \mathbb{E}_{T_1}\left[(S_{T_2}-D)^+\right]\right] \\
&\quad - \beta_0 \mathbb{E}_0\left[\left(\beta_1 \mathbb{E}_{T_1}\left[(S_{T_2}-D)^+\right] - (K_1 - \beta_1 D)\right)^+\right] \\
&= \beta_0 \beta_1 D + C(D; D; T_2; \Theta) - CC(D; K_1 - \beta_1 D; D; T_1, T_2; \Theta),
\end{aligned}$$

where $\mathbb{E}_t[\cdot]$ is the expectation at date $t$ conditional on $S_t$. □

The previous proposition shows that an analytical formula can be obtained for the value of a PPCN with a single early redemption date when analytical formulas are available, for the reference underlying dynamics model, for the price of a European call option and a Call on Call. This is the case, for instance, when the reference underlying is modeled with a geometric Brownian motion or a MRSLN2. For the Geometric Brownian motion model, functions $C$ and $CC$ are given by $C_B$ and $C_G$ in Equations (4.1) and (4.2a) respectively, and for the MRSLN2, they are given by the Hardy formula (4.4) and Equation (4.9a) respectively.

When issuing such a contract, a financial institution faces the problem of fixing the early redemption prices so that the value of the contract is equal to its nominal initial investment. Alternatively, one may want to determine the fair value of the initial investment corresponding to a given early redemption price. In both cases we end up solving the fixed point problem:

$$V_0(D; r, \Theta, T_1, T_2, K_1) = D. \qquad (4.14)$$

The Newton-Raphson algorithm can be used to solve (4.14) for either $K_1$ or $D$.

## 4.4 Numerical illustrations

In this section, we report on numerical experiments for the evaluation of principal protected callable notes in order to evaluate the impact of the choice of the reference underlying dynamics model.

### 4.4.1 Principal protected notes under the Black-Scholes framework

In this subsection we analyze the behavior of PPCNs with respect to the risk free rate (eventually including a counterparty risk spread of the issuer) and to the volatility of the reference underlying under the Black-Scholes framework. The PPCN value is computed using Equation (4.12). The experiment concerns principal protected callable notes similar to those issued by the National Bank of Canada (National Bank of Canada 2012), with a maturity $T$, a single redemption date at mid-term, $T_1 = T/2$, where the value of the reference underlying at issuance date is normalized at $S_0 = 1$, and where the early redemption price corresponds to a given annual return $R$. Accordingly, if the number of time steps in a year is equal to $\delta$, $K_1 = (1+R)^{\frac{T}{2\delta}}$, with $T$ expressed in term of time steps.

| $R$ | $r$ | $T$(months) | $S_0$ | $\delta$ | $\sigma(\%)$ |
|---|---|---|---|---|---|
| 8% | 7% | 96 | 1 | 12 | 20 |

Table 4.1: Base case parameter values

Figures 4.1, 4.2 and 4.3 plot the value of the PPCN with respect to the reference underlying volatility, for various levels of the risk-free rate $r$, early redemption price $K_1$, and maturity $T$. Base case parameter values are given in Table 4.1. In each panel, one parameter is varied while the others are fixed at their base case values. A first observation is that the note value is decreasing and sensitive to the volatility of the reference underlying, with an average impact varying between 6% and 27%. These percentages correspond to the minimum

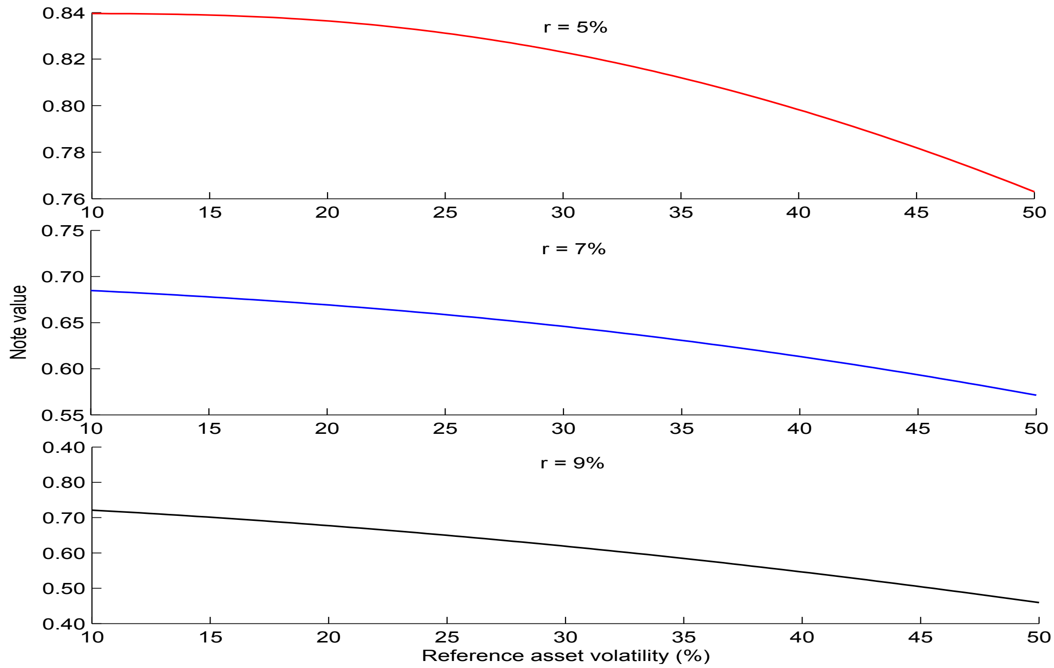


Figure 4.1: Impact of the reference underlying volatility on the callable note value for various levels of the risk-free rate. Other parameter values are given in Table 4.1

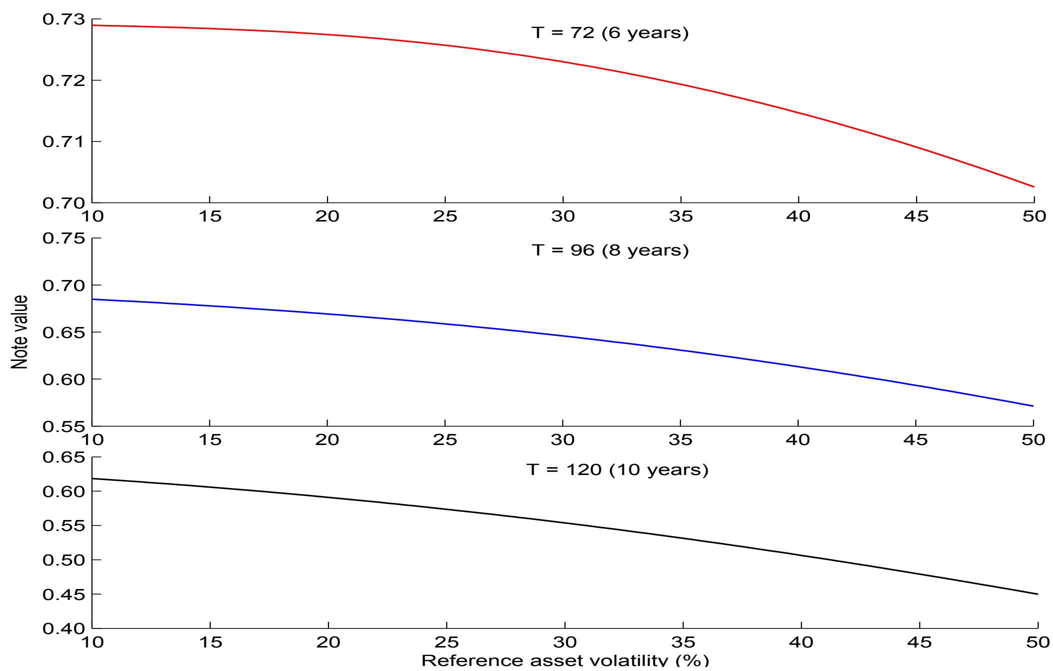


Figure 4.2: Impact of the reference underlying volatility on the callable note value for various levels of the maturity. Other parameter values are given in Table 4.1

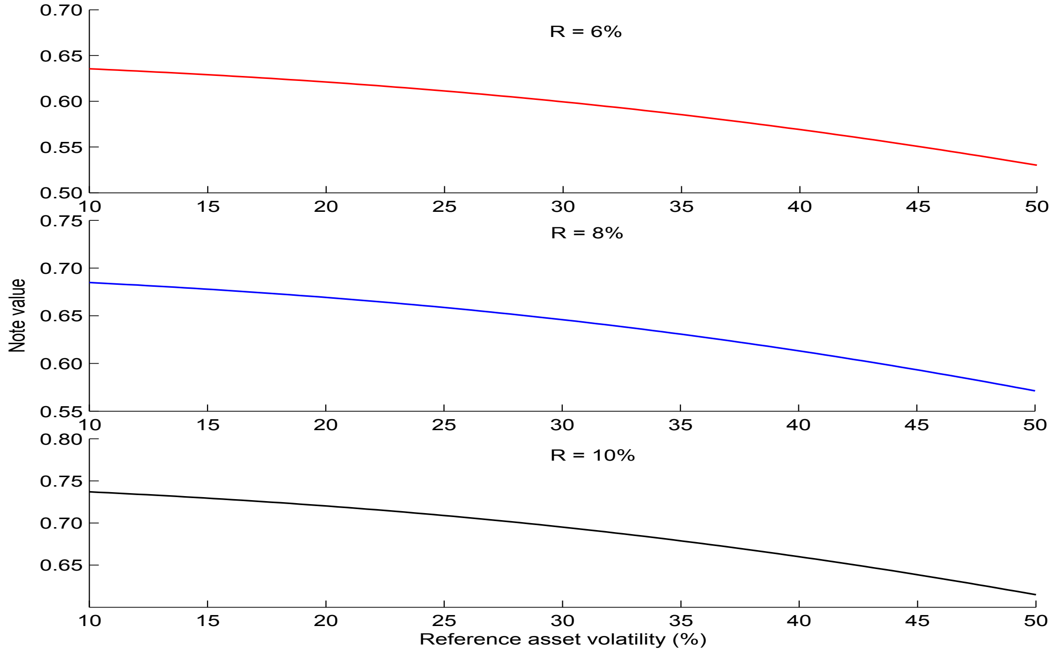


Figure 4.3: Impact of the reference underlying volatility on the callable note value for various levels of the early redemption price. Other parameter values are given in Table 4.1

and maximum change when the volatility varies from 10 to 50% in Figures 4.1, 4.2 and 4.3. This comes from the fact that, the callable note considered here is a combination of bond and options, and these latter are sensitive to volatility (think of volatility smile). The note value is also decreasing and very sensitive to variations of the risk-free rate; for instance, a variation of 1% in the base-case value of the risk-free rate decreases the note value by 10%. This can be explained by the presence of the long bond component and the discount factor which are both decreasing with respect to this rate. Table 4.2 and corresponding Figure 4.4 illustrate the dependence of the note value to the volatility and risk-free rate.

The dependence of the PPCN issuance value with respect to the reference volatility suggests that the assumption of constant volatility in the Black-Scholes model may cause pricing distortions in a market with high and low volatility episodes.

| | Volatility of the reference underlying (%) | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| risk-free rate (%) | 10.0% | 15.0% | 20.0% | 25.0% | 30.0% | 35.0% | 40.0% | 45.0% | 50.0% |
| 4.0% | 0.9314 | 0.9352 | 0.9362 | 0.9338 | 0.9283 | 0.9200 | 0.9090 | 0.8955 | 0.8795 |
| 5.0% | 0.8396 | 0.8389 | 0.8364 | 0.8312 | 0.8230 | 0.8120 | 0.7983 | 0.7819 | 0.7630 |
| 6.0% | 0.7579 | 0.7535 | 0.7477 | 0.7395 | 0.7289 | 0.7156 | 0.6997 | 0.6812 | 0.6603 |
| 7.0% | 0.6848 | 0.6779 | 0.6692 | 0.6587 | 0.6459 | 0.6308 | 0.6133 | 0.5934 | 0.5713 |
| 8.0% | 0.6193 | 0.6106 | 0.6000 | 0.5876 | 0.5732 | 0.5567 | 0.5382 | 0.5175 | 0.4950 |
| 9.0% | 0.5605 | 0.5506 | 0.5387 | 0.5250 | 0.5095 | 0.4922 | 0.4731 | 0.4522 | 0.4297 |
| 10.0% | 0.5076 | 0.4969 | 0.4842 | 0.4698 | 0.4537 | 0.4360 | 0.4168 | 0.3960 | 0.3740 |
| 11.0% | 0.4603 | 0.4488 | 0.4358 | 0.4210 | 0.4047 | 0.3870 | 0.3679 | 0.3476 | 0.3263 |
| 12.0% | 0.4185 | 0.4057 | 0.3926 | 0.3778 | 0.3615 | 0.3440 | 0.3254 | 0.3058 | 0.2854 |
| 13.0% | 0.3824 | 0.3670 | 0.3540 | 0.3394 | 0.3234 | 0.3064 | 0.2884 | 0.2696 | 0.2503 |

Table 4.2: Impact of the risk-free rate and of the volatility of the reference underlying on the PPN value. Other parameter values are those of Table 4.1.

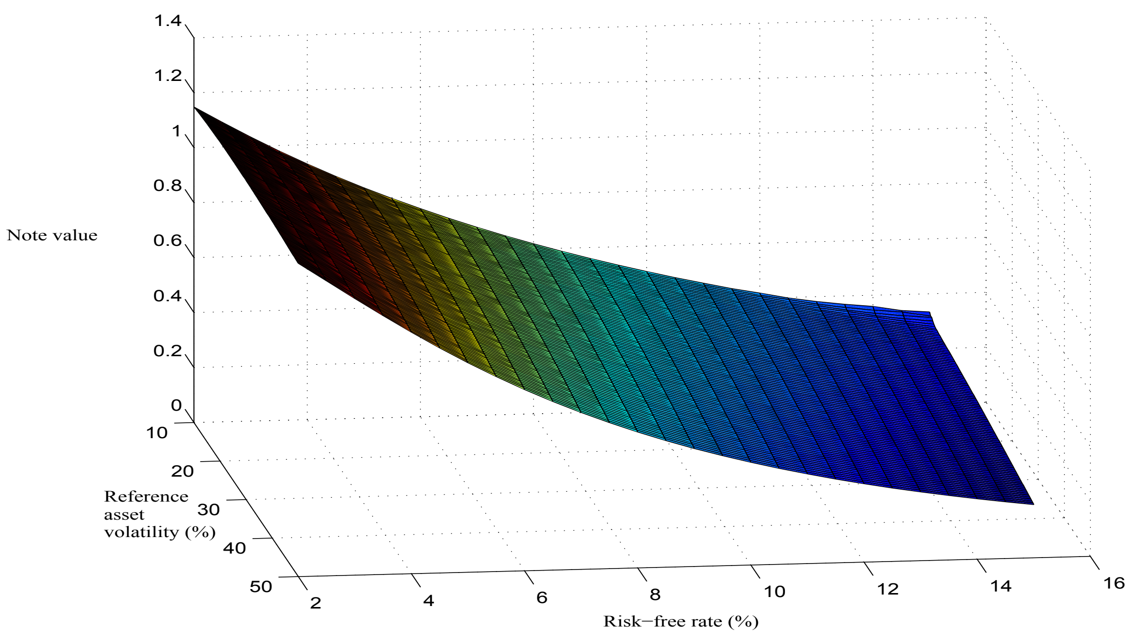


Figure 4.4: Impact of the risk-free rate and of the volatility of the reference underlying on the PPN value. Other parameter values are those of Table 4.1

### 4.4.2 Principal protected callable notes under MRSLN2 model

In the first part of this numerical illustration, we compare the value of principal protected notes under Black-Scholes and MRSLN2 frameworks in order to assess the pricing difference due to a constant volatility specification. Accordingly, we fix the constant volatility in the Black-Scholes model to the annualized value of the mean of the two volatilities in the MRSLN2 model, taking into account the number of time steps spent in Regime 1 during the $n$ total time steps, that is,

$$\sigma_{bls} = \sum_{i=0}^{n} \overline{\sigma}_{in} P(i; n) \tag{4.15}$$

where $\overline{\sigma}_{in}$ is given by Equation (4.5) and $P(\cdot; n)$ is the probability of the number of time steps spent in Regime 1 during $n$ time steps, already defined in Subsection 4.2.2. Volatilities are annualized and comparison is done for various levels of the transition probabilities $p_{12}$ and $p_{21}$; Other parameter values are those of the base case in Table 4.3.

| $R$ | $r$ | $T$(months) | $S_0$ | $\delta$ | $p_{12}$ | $p_{21}$ | $\sigma_1$ | $\sigma_2$ |
|---|---|---|---|---|---|---|---|---|
| 8% | 7% | 96 | 1 | 12 | 2.5% | 5.83% | 15% | 25% |

Table 4.3: Base case parameter values

Table 4.4 reports the Black-Scholes and MRSLN2 values of a principal protected note for different values of $\sigma_1$ (low volatility) and $\sigma_2$ (high volatility) that are increasingly distant, and different sets of transition probabilities corresponding to decreasing unconditional probability of the low volatility regime and increasing turbulence. In Panel A, the market is in the low volatility regime 80% or the time in Panel B, 70% or the time, and in Panel C, 60% of the time.

One observes that the Black-Scholes value is significantly smaller than the MRSLN2 value in all cases, meaning that there exists a non negligible model impact in terms of pricing. As expected, this model impact is increasing with the difference $\sigma_2 - \sigma_1$.

| Volatilities (%) | | | Note value | |
|---|---|---|---|---|
| $\sigma_1$ | $\sigma_2$ | $\sigma_{bls}$ | Black-Scholes | MRSLN2 |
| **Panel A:** $p_{12} = 1.00\%, p_{21} = 4.00\%$ and $\pi_1 = 80.00\%$ | | | | |
| 10.0 | 20.0 | 18.1 | 0.8936 | 0.9217 |
| 12.5 | 25.0 | 22.6 | 0.8793 | 0.9065 |
| 15.0 | 30.0 | 27.1 | 0.8680 | 0.8944 |
| 17.5 | 35.0 | 31.7 | 0.8583 | 0.8845 |
| 20.0 | 40.0 | 36.2 | 0.8494 | 0.8761 |
| 35.0 | 35.0 | 35.0 | 0.8517 | 0.8517 |
| **Panel B:** $p_{12} = 2.50\%, p_{21} = 5.83\%$ and $\pi_1 = 70.00\%$ | | | | |
| 10.0 | 20.0 | 17.2 | 0.8970 | 0.9150 |
| 12.5 | 25.0 | 21.5 | 0.8825 | 0.9000 |
| 15.0 | 30.0 | 25.8 | 0.8712 | 0.8881 |
| 17.5 | 35.0 | 30.1 | 0.8615 | 0.8785 |
| 20.0 | 40.0 | 34.4 | 0.8529 | 0.8702 |
| 35.0 | 35.0 | 35.0 | 0.8517 | 0.8517 |
| **Panel C:** $p_{12} = 4.00\%, p_{21} = 6.00\%$ and $\pi_1 = 60.00\%$ | | | | |
| 10.0 | 20.0 | 16.2 | 0.9007 | 0.9094 |
| 12.5 | 25.0 | 20.3 | 0.8861 | 0.8945 |
| 15.0 | 30.0 | 24.4 | 0.8746 | 0.8829 |
| 17.5 | 35.0 | 28.4 | 0.8651 | 0.8733 |
| 20.0 | 40.0 | 32.5 | 0.8566 | 0.8650 |
| 35.0 | 35.0 | 35.0 | 0.8517 | 0.8517 |

Table 4.4: Comparing the PPN value under the Black-Scholes and MRSLN2 models for various volatilities and transition probabilities. Other parameter values are as in Table 4.3.

In the second part of this specific illustration, we study the impact of the regime parameters on the note value under the MRSLN2 framework. Base case values are again those of Table 4.3.

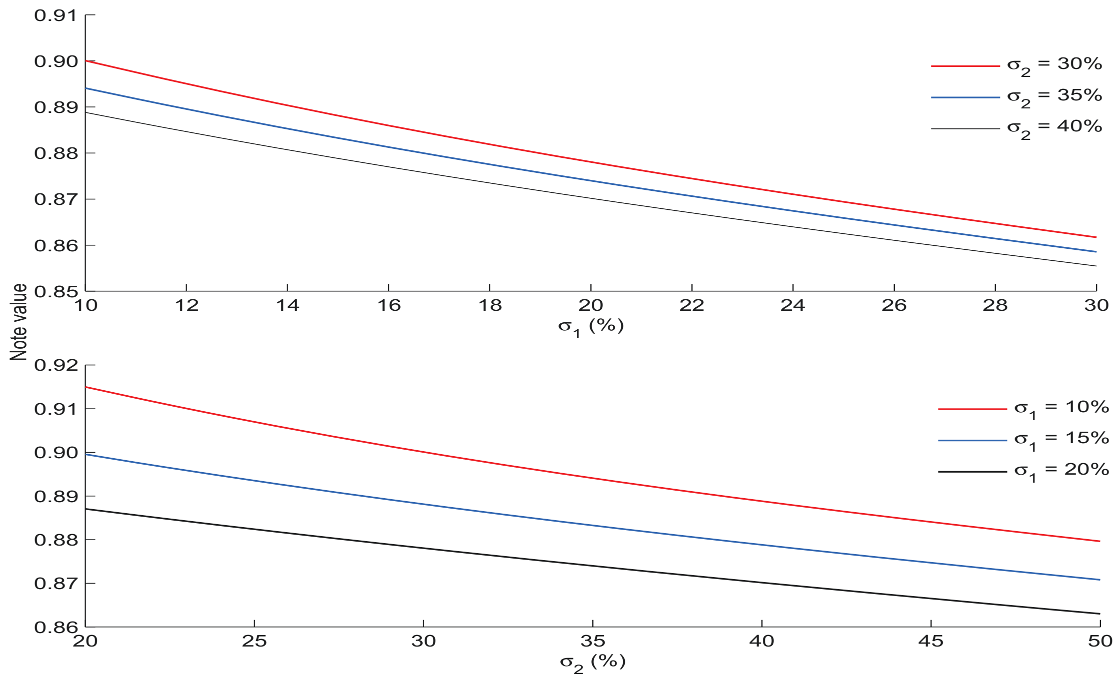


Figure 4.5: Impact of the volatility parameters on the note value.Other parameter values are as in Table 4.3.

Figure 4.5 displays the value of the note for different combinations of the volatility parameters. In the top panel, we represent the note value as a function of $\sigma_1$ for three different levels of $\sigma_2$. This illustration shows that the note value is decreasing with respect to $\sigma_1$ with relatively small variations. For example, when $\sigma_1$ passes from 10% to 30%, an increase of 200%, the note value decreases by only by 3.33% when $\sigma_2 = 30\%$, by 3.37% when $\sigma_2 = 40\%$. In the bottom panel, where $\sigma_1$ is fixed, we observe even smaller variations. Comparing these two panels, one remarks that the impact of the small volatility $\sigma_1$ is more pronounced.

Figure 4.6 displays the note value for different with respect to the regime transition prob-

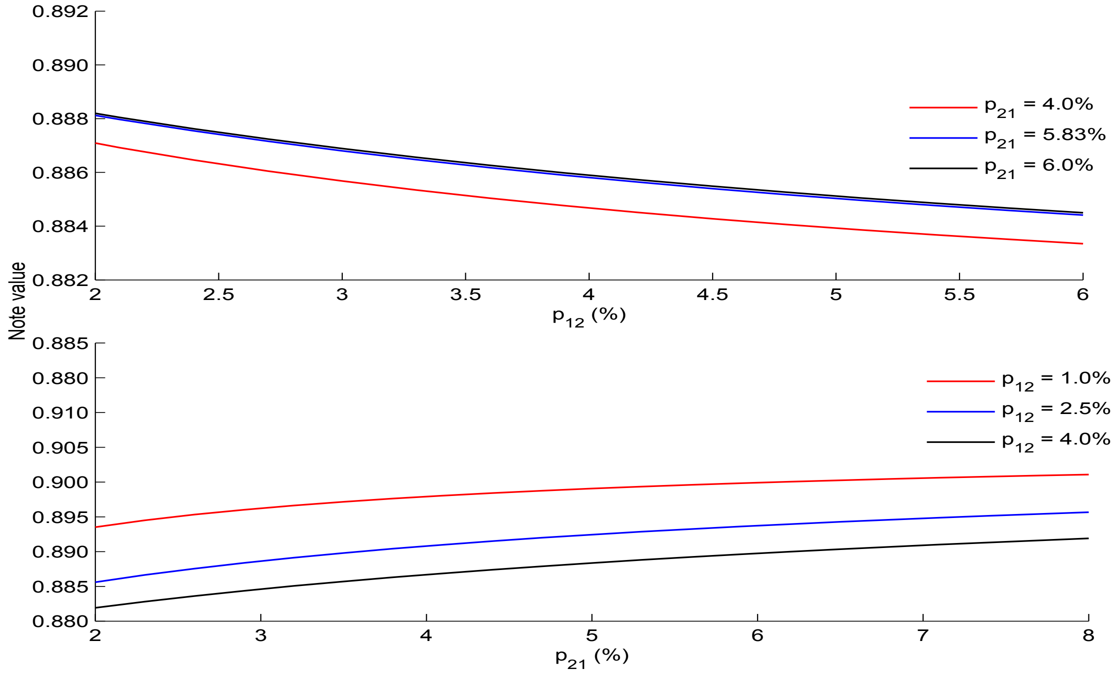


Figure 4.6: Impact of the regime transition probabilities on the note value. Other parameter values are as in Table 4.3.

abilities. In the top panel, we represent the note value as a function of $p_{12}$, the probability to pass from the low to the high volatility regime, for three different level of $p_{21}$. The fact that the note value is decreasing with respect to $p_{12}$ is because when this probability increases, we have more chance to go in high volatility regime where the note value is smaller. The bottom panel represents the note value as a function of $p_{21}$ for different level of the small volatility $p_{12}$. These results show relatively small variations of the note value with respect to the regime transition probabilities.

## 4.5 Conclusion

In this chapter, a semi-analytical formula is derived for compound options under a regime-switching log-normal model. This formula can be used for any product that possesses compound option components. For example, we use it in a specific application to derive an analytical formula for a callable structured product with one early redemption date when the reference underlying is modeled with a regime-switching log-normal model. Numerical applications are presented to show the impact of parameter values for a representative callable note such as the protected callable notes issued by National Bank of Canada.

# Chapter 5

# Valuation of Structured Products

**Summary**

This chapter proposes a dynamic programming-based valuation method that can be used for structured products. The method is developed using a general model for the dynamics of the relevant underlying asset characteristics. We first propose a classification of structured products into three categories and derive a general DP recursion that can apply for all categories. The efficiency of the valuation method is tested using several examples of structured products, under both the geometric Brownian motion and the two states Markov log-normal regime switching models for the dynamics of the reference underlying asset.

**Keywords** : Structured products, Monte Carlo simulation, dynamic programming, finite-element interpolation.

## 5.1 Introduction

Structured products (SP), also called structured notes, are highly customized derivatives that are tied to an asset commonly called the reference underlying. The complex fashion that their potential return is related to the performance of the reference underlying makes them very different from other financial instruments, like options and bonds. SP are very popular in the European market, where most of them are listed on the *SIX Structured Products Exchange*. In recent years, the North American SP market has experienced a phenomenal growth and a large number of innovations, making these products even more interesting for issuers and attractive for investors. Indeed, according to the Bloomberg Structured Notes Brief (BSB 2011, 2012, 2014) and Review (2012, 2013, 2014) of the last years, more than $170 billion of SP have been issued on the U.S. market only. For instance, on May 20, 2014, J.P. Morgan Chase & Co. sold more than $5 million of 18-month SP linked to the Mexican peso relative to the EURO, while the Goldman Sachs Group Inc. sold $21.6 million of three-year notes tied to a basket including the S&P 500 Index and the MSCI EAFE Index, and more than $30 million of three-month notes tied to the Topix index. In Canada, almost all banks issue SP (for instance, the Royal Bank of Canada sold more than $72 million of SP tied to the Euro Stoxx 50 Index in 2013).

The overall growth of the SP market can be explained partly by the fact that these derivatives offer to their issuers an advantageous and flexible mean of getting cash-flow. For instance, Hernandez *et al.* (2007) performed an economic and empirical analysis of 7426 reverse exchangeable bonds and reported that these SP provided significant positive profits to their issuers. On the other hand, a wide range of investors are interested by structured products, making these easy to sell. Indeed, SP might be suitable for at least three categories of investors. A first category includes conservative investors who cannot afford to lose capital over the short-term, but who are interested by the possibility of obtaining a higher return than that of fixed-income products. A second category contains the outright speculators who may see in a SP a simple way of diversifying their portfolio and of obtaining significant earnings with a controlled risk profile. A third category includes investors who want to invest in a wide range of high-return assets, but might not have the right or the means to do this directly. Finally, the transaction costs and fees involved when trading SP are usually higher compared to other derivatives.

Each SP has its particular features. For instance, payoffs may depend on the level or the path of the reference underlying, and possibly on particular choices of the issuer. Other characteristics that differentiate SP are the various types of reference underlying, principal protection features, participation rate, coupon payments and European, Bermudan or American exercise features. Below, a description of different elements that characterize the majority of structured products.

a) The underlying asset can be a single stock, an index, a debt instrument, an equity or a basket of assets. For instance, in May 2014, UBS AG proposed structured products linked to the Robotics & Drones Index (a new index composed of Kuka AG, a German company, Fanuc Corp. in Japan, and Aerovironment Inc., a U.S drones maker) while ING Groep NV issued a structured product linked to the Mongolian bank debt. However, note that more than half of SP issued in the North American market are linked to an equity or a portfolio of equities.

b) Principal protection refers to the fact that an investor who holds the SP is guaranteed to get back at least a given percentage of his initial investment (for instance, a 50% principal protection means that the final payoff of the SP will be at least 50% of the initial investment, while a 100% principal protection means that the investor will recover at least his initial investment (if the issuer does not default).

c) The participation rate determines the extent of the reference underlying's participation to the final payoff of the SP; for instance, a participation rate of 60% at maturity means that the payoff to the investor would represent 60% of the return of the reference underlying, if positive.

d) Structured products linked to debt instruments or interest rates usually offer coupon payments to their holder; coupons are paid at pre-specified dates, at a fixed or variable rate that may depend on the reference underlying's level.

e) Exercise features pertain to decisions concerning the SP, which can be made only at maturity (European), at pre-specified dates (Bermudan) or at any time during the life of the SP (American). The majority of SP present European or Bermudan features.

The literature on structured product valuation is very diverse; almost every paper deals with a very specific product, and most of them focus on the overpricing of such product. For instance, Chen and Kensinger (1990) propose an empirical analysis of Market Index Certificates of Deposit, which provide a guaranteed minimum return linked to the performance of the S&P 500 index. Comparing the option component of these products with S&P 500 index options, these authors report significant differences between theoretical and market values. Chen and Sears (1990) study the S&P 500 Index Notes issued by Salomon Brothers and also find significant differences between market and model prices for these products. Baubonis *et al.* (1993) arrive at a similar conclusion with equity-linked certificates of deposit, as well as Wilkens *et al.* (2003) and Stoimenov and Wilkens (2005) for products issued on the German market. Hernandez *et al.* (2007) report significant mispricing of reverse convertibles. Szymanowska *et al.* (2009) find that Dutch plain vanilla and knock-in reverse convertible bonds were, on average, issued at almost 6% over their fair price. Deng *et al.* (2010) report that the average fair value of more than 2000 reverse convertibles issued between 2001 and 2010 was around 93% of the average issue price. Henderson and Pearson (2010) report that the $2 billion short-term SPARQS reverse convertibles issued from 2001 to 2005 by Morgan Stanley were purchased on average at 8% over their fair value.

Recent papers on specific products include Deng *et al.* (2011b) on the modeling of autocallable SP and Deng *et al.* (2013) on structured certificates of deposit. Publications that consider a wide range of SP and focus on valuation approaches include Das (2000), who proposes a comprehensive introduction on major types of structured products; Christl (2004), who provides a handbook for the valuation of a large number of SP; Shahid (2011), who treats the valuation of almost all types of commodity-linked notes; and Deng *et al.* (2014), who presents current valuation approaches for all types of structured notes.

As frequently remarked in the literature, most of SP are issued above their fair price. Beside, even when they include a principal protection feature, SP are not risk-free; as pointed out by the U.S Securities and Exchange Commission, “any promise to repay some or all of the money you invest will depend on the creditworthiness of the issuer meaning that you could lose all of your money if the issuer of your note goes bankrupt.” For instance, one may recall the collapse, in 2008, of the largest issuer of SP (Lehman Brothers). Another risk associated with SP is the non-existence of a liquid organized secondary market for most of the cases.

In this chapter, we propose a dynamic programming based valuation method that can be used for all types of structured products and any model for the dynamics of the reference underlying value, and that can account for the issuer default risk. The chapter is organized as follows. In Section 5.2, we provide a classification of SP, along with several examples. Section 5.3 highlights the current valuation approaches, and Section 5.4 presents the dynamic programming pricing approach. In Section 5.5, we provide numerical illustrations and empirical investigations for several structured products, and Section 5.6 concludes.

## 5.2 Description and classification of structured products

Structured products include a very large variety of contracts, with many particular features. A classification of SP was proposed by Stoimenov and Wilkens (2005) for German market equity-linked SP; they proposed two categories, the first contains SP including a plain vanilla option component; the second contains SP that have an exotic option component. Products of each category are further classified into sub-categories. Notice however that there exists SP that contain both a plain vanilla and an exotic option component[1]. In this chapter, we propose a new classification of SP based on early-redemption provisions (by

[1]For instance, the CANADIAN Blue Chip III Deposit Notes$^{TM}$, Series 34, issued by National Bank of Canada, is equivalent to a a long position on a bond and a European call option and a short position on a compound option.

the issuer). In fact, for all structured products, only three situations may arise:

a) the product cannot be redeemed;

b) the product can be redeemed on the basis of a predefined condition verified by the reference underlying;

c) the product can be redeemed at the whim of the issuer.

This observation allows to classify structured products into three categories. The first, hereafter named **non-callable,** includes structured products that cannot be redeemed by the issuer. The second category, named **auto-callable,** includes structured products that are automatically redeemed by the issuer when some characteristic of the reference underlying reaches a pre-specified level.The last category, named **callable,** contains all structured products that can be redeemed at the whim of the issuer at predefined dates.

For a general description of structured products, assume that the reference underlying value is described by the process $\mathbb{X} = \{S_m, Y_m\}$, observed at discrete equally space dates $m \in \mathbb{T} = \{0, 1, \ldots, n = T\}$, where $S_m$ is the level (possibly multi-dimensional), $Y_m$ contains other relevant characteristics that are observable or computable (for instance, volatility or regime), $T$ is the maturity, $m = 0$ is the inception date, and $\{1, \ldots, n-1\}$ are intermediate (if any) decision, coupon and/or payoff update dates.

Denote by $c_m$ the coupon rate paid by the issuer at date $m$. Since this coupon rate may depend on the current characteristics of the reference underlying, assume that at date $m$

$$c_m = g_m(s) \text{ with } S_m = s \tag{5.1}$$

where the $\{g_m\}_{m=1,\ldots,n}$ are predefined functions that may depend on certain parameters like a threshold or a participation rate, and where $g_m = 0$ if the product does not pay a coupon at date $m$.

Denote by $\Pi$ the final payoff to the investor. For some SP, the final payoff may depend on a criterion based on the entire path of the reference underlying value. For instance, this criterion could be that the reference underlying level has ever crossed some given barriers, or that the final payoff was updated at some intermediate date. These features can be described with a component $k_m$ that summarizes the criterion at date $m \in \mathbb{T}$. This component evolves according to the following transition:

$$k_m = h_m(s, k_{m-1}), \text{ with } s = S_m, 1 \leq m \leq n, \; k_0 \text{ given}, \tag{5.2}$$

where $\{h_m\}_{m=1,\ldots,n}$ are predefined functions that may depend on certain parameters.

An example of SP that present such feature is the TD Accumulator Notes, Series $1, 2, \ldots, 6$ issued by the Toronto-Dominion Bank on February, 2008. These notes provide a 100% principal protection at maturity and their return is linked to a diversified basket of assets, described by the issuer as follows: "The Variable Return, if any, payable at maturity will equal the Principal Amount multiplied by the sum of the Asset Returns (each of which may be positive or negative, subject to a maximum of 5.25% for each Asset) of the twelve assets comprising the Basket. An asset return will be "locked-in" every six months following the issuance of the Notes and will equal the percentage increase or decrease of the best performing asset in the basket on each semi-annual valuation date. After an asset return is determined on a Semi-Annual Valuation Date, that asset will be removed from the basket and subsequent performance of such asset will not factor into the determination of the Variable Return payable under the Notes". In this example, $k_m$ represents the sum of locked-in asset returns and the basket composition at date $m$.

Finally, define the subset $\mathbb{T}^* \subseteq \mathbb{T}$ (possibly empty) of early redemption dates. Denote by $\delta_m$ the indicator function

$$\delta_m = \begin{cases} 1 \text{ if the product is redeemed at } m \\ 0 \text{ otherwise} \end{cases}$$

and by $R_m$ the early redemption price if $\delta_m = 1$. This redemption price may be fixed, or may depend on some characteristics of the reference underlying. For a non-callable product, $\delta_m = 0$ for all $m$. For auto-callable products, $\delta_m = 1$ if the redemption condition $s \in C_m$ is satisfied at a redemption date $m$, while for callable products, at any redemption date $m$, the issuer decides on the value of $\delta_m \in \{0, 1\}$.

Using this notation, the mechanism of structured products can be summarized according to the following points:

- At issuance date, the investor pays an amount $D$ to the issuer, where $D$ corresponds to net amount after the application of selling fees.
- The initial value $k_0$ of component $k$ (if any) is fixed or determined.
- At each intermediate date $m \in \{1, 2, \ldots, n-1\}$, if the product has not already been redeemed:
  - The current level $S_m$ of the reference underlying and the other relevant characteristics $Y_m$ are observed.
  - The coupon rate $c_m$ is computed using Equation (5.1) and the investor receives a coupon $c_m D$ from the issuer.
  - If $m$ is a redemption date,
    - the early redemption price $R_m$ is computed
    - the redemption condition (if any) is verified
    - the issuer decides (if he has this possibility) to redeem the product or not.

– If $\delta_m = 1$, the investor receives $R_m$ and the contract is closed.

– Otherwise, the component $k_m$ (if any) is updated using Equation (5.2).

- If the product reaches maturity $T$ the investor receives the final payoff.

### 5.2.1 Non-callable structured products

This category includes all structured products that cannot be redeemed by the issuer. For products of this category, intermediate dates correspond either to coupon payments or final return condition updates.

Popular examples of non-callable structured products are *barrier reverse convertibles* (BRC) which are short-term path dependent products, often linked to an index, a single stock or a basket of stocks. An investor holding a BRC receives a series of fixed coupon payments, and gets back his initial investment at maturity if the reference underlying value does not fall below a pre-specified barrier. More specifically, denote by $H$ the barrier (also called the trigger price) and $\alpha$ the constant coupon rate. BRCs corresponds to the case where the component $k$ represents the minimum level of the reference underlying, that is: $k_0 = S_0$ and for $m = 1, 2, \ldots, n-1$, $g_m \equiv \alpha$, $h_m \equiv \min$, $\delta_m = 0$ and the final payoff is:

$$\Pi(s,k) = \begin{cases} D \text{ if } \min\{s,k\} > H \\ \frac{s}{S_0} D \qquad \text{otherwise.} \end{cases} \tag{5.3}$$

BRCs remain principal protected as long as the reference underlying does not cross the barrier; however, once the barrier is attained, the investor could lose a part of his initial investment if the final value of the reference underlying is strictly below the initial value $S_0$. Notice that removing the path dependent feature of this example yields the well known standard *reverse convertibles*. Major issuers of (barrier) reverse convertibles are, among others, UBS, Credit Suisse Group, Goldman Sachs, Citigroup and Royal Bank of Canada (more than 4000 versions of reverse convertibles can be fount at https://www.rbccm.com/usstructurednotes/cid-208193.html).

Other examples of non-callable structured products with coupon payments are the *RBC Principal Protected Guaranteed Return BlueChip Yield LEOS, Series* 69, issued by the Royal Bank of Canada, and the *TD Canadian Companies -Linked* 1.25% *Coupon Plus Growth Notes, Series* 1, issued by the Toronto Dominion Bank. This last product provides principal protection and semi-annual coupons equal to 0.625% of the principal amount. At maturity, the investor holding this product will get back the principal plus, if positive, a participation rate of 50% of the total return of the reference underlying, an equally-weighted basket of Canadian common shares. An example of non-callable structured product without intermediate coupon payments is the BNS $S\&P/TSX60$ Index Deposit Notes, Series 7, issued by the Bank of Nova Scotia. This product provides principal protection and variable return paid only at maturity. The variable return corresponds to the total return of the $S\&P/TSX60$ index, subject to a participation rate of 65%.

### 5.2.2 Auto-callable structured products

These products can present all features of non-callable, with the added characteristic of being redeemed automatically, whenever the value or the return of the reference underlying verifies a given criterion at pre-specified dates (for instance, when it reaches a specified barrier).

Basic examples of auto-callable SP are the classical Express Certificates, which have a maturity of 3 to 6 years. At each observation date (set initially), if the level of the reference underlying is at or above a specified threshold, the certificate is redeemed prematurely by the issuer at a price which is the sum of the initial investment and a premium corresponding to an annual total return lying between 5 to 10%. If the level of the reference underlying is below the threshold, the certificate continues to run until the next observation date. Finally, if the certificate reaches maturity, the investor gets back the initial investment if the reference underlying level is at least at a pre-specified safety threshold. Usually, Express Certificates do not pay intermediate coupons. An example of such product is the Deutsche Bank Express Certificate (for instance, see https://xmarkets-uat.dbxuat.com/LU/Product_Detail/DE000DB9ZFS2). Other examples of auto-callable SP are the Autocallable Reverse Convertible Notes issued by Royal Bank of Canada with various reference assets and payoff features (for instance, see: https://http://www.rbcnotes.com).

### 5.2.3 Callable structured products

This category refers to all structured products that can be redeemed at pre-specified dates by the issuer, without any conditions. Examples of callable SP are Callable Barrier Reverse Convertibles, which are standard barrier reverse convertibles with an early redemption feature. For example such product was issued on April 2013 by Credit Suisse, where the reference underlying is a basket containing the Nikkei 225, Swiss Market, S&P 500 and the EURO STOXX 50 Indexes (https://derivative.credit-suisse.com/get.cfm?id=B1503490-EE59-4F27-AF9A-D66CA07933B8). This SP promises a fixed coupon of 6% p.a. paid semi-annually and the issuer has the right to redeem the product semi-annually at its issuance price.

## 5.3 Current valuation approaches

Currently, there exist four main valuation methods for structured products, however none of them seems appropriate for all product categories.

The first method is (Quasi) Monte Carlo simulation (referring to Glasserman 2003 or McLeish 2004), which is particularly popular among practitioners because of its simplicity. However, this brute force approach can be very expensive in time and memory requirements, especially when the product is path dependent, which is the case of most callable SP. In addition, the computation of sensitivity measures (Greeks) could require additional intensive computations.

For non-callable SP, the Monte Carlo simulation approach consists of the following steps: first, simulate a large number $M$ of paths of the reference underlying level at intermediate dates, say $\mathbb{S}^j = \{S^j_m\}_{m \in \mathbb{T}}, j = 1, 2, \ldots, M$. Second, for each path $\mathbb{S}^j$, compute the current value $V^j$ of the product, combining the discounted value of coupons and of the final payoff, that is:

$$V^j = \beta^0_n \Pi(S^j_n, k^j_{n-1}) + D \sum_{m=1}^{n-1} \beta^0_m g_m(S^j_m) \tag{5.4}$$

where $\beta^i_m = e^{-r(m-i)}$ is the discount factor corresponding to the time elapsed between dates $i$ and $m$, $r$ is the discount rate (eventually including a credit risk component as cds spread when counterparty risk is present), and $k^j_m$ corresponds to the final return at date $m$ for path $\mathbb{S}^j$. Finally, the fair price of the SP is obtained by averaging all $V^j$, that is:

$$V_0 = \frac{1}{M} \sum_{j=1}^{M} V^j.$$

For auto-callable SP, an additional step is required; that is, for each path $\mathbb{S}^j$, we have to record the first time $\tau_j$ that the reference underlying value verifies the early redemption criterion. More precisely, define:

$$\tau_j \quad = \quad \min\{m \in [1, n] : \delta_m = 1 \text{ for path } \mathbb{S}^j\}. \tag{5.5}$$

The current value $V^j$ of the product for path $\mathbb{S}^j$ is then given by:

$$V^j = \beta^0_{\tau_j} R_{\tau_j} I_{\{\tau_j < n\}} + \beta^0_m \Pi(S^j_n, k^j_{n-1}) I_{\{\tau_j = n\}} + D \sum_{m=1}^{\tau_j} \beta^0_m g_m(S^j_m). \tag{5.6}$$

Notice that the Monte Carlo simulation method cannot be applied directly to callable SP, since it would require a simulation of the issuer's decision (a decomposition in simpler instruments is usually necessary).

The second method is a numerical integration approach that consists of computing the discounted expected payoff of the structured product, by integrating directly the payoff using the density function of the reference underlying value or return; this can be done for instance with the Simpson quadrature rule (McKeeman 1962) or the adaptive Lobatto method (Ueberhuber 1997). According to Glasserman (2003), the numerical integration approach is faster and more accurate than the Monte Carlo simulation method. However, it can only work when the distribution of the reference underlying return is continuous and integrable, and when the structured product is non-callable. For example, assume that the density $\phi$ of the distribution of the reference underlying level is known for a given non-callable SP. If the promised final return of this product depends only on the reference underlying level at maturity, its value is given by the following equation:

$$V_0 = \beta_n^0 \int_0^\infty \Pi(s)\phi(s)ds + D \sum_{m=1}^{n} \beta_m^0 \int_0^\infty g_m(s)\phi(s)ds, \tag{5.7}$$

where the first term in the right hand side corresponds to the discounted expected value of the final payoff, while the second term is the discounted expected value of the coupons. This method cannot be used when the final return is path-dependent, namely for callable or auto-callable products, without any technique to transform the path dependent aspect (Carr and Chou 1997).

The third method is the partial differential equation (PDE) approach, which consists of finding and solving the Black-Scholes type PDE verified by the structured product value. This PDE is derived using Ito's lemma (Ito 1951) and the PDE of the reference underlying value. The implementation of this method is far from being easy, especially with the possibility of having more than one boundary condition. For most structured products, the corresponding PDE can only be solved using numerical methods, which might bring additional complexity and approximation errors (see Deng *et al.* 2014 for more details).

The fourth method is the decomposition approach, which consists of expressing the SP as a combination of simpler instruments that are easy to price (for instance, with a Black-Scholes type formula), such as bonds, vanilla, barrier, and exotic options. This method can be applied whenever SP payoffs are combinations of simple instruments' payoffs, and most non-callable structured products can be priced using the decomposition method. However, for path dependent non-callable SP, the decomposition may involve complex exotic instruments. For instance as shown by Hernandez *et al.* (2007), several reverse convertibles can be expressed as combination of bonds and exotic options, such as up-and-out and down-and-in options. On the other hand, this method is difficult or even impossible to implement for callable SP. In the previous chapter, we showed that a decomposition can be obtained using vanilla and compound options for auto-callable SP when there is only one early redemption date.

## 5.4 Dynamic programming valuation approach

We now propose a general DP valuation approach that can accommodate the three categories of structured products presented above. In our model, we adopt the notation of Section 5.2. We assume that the reference underlying process is characterized by the vector process $X_t = (S_t, Y_t)$. At each observation date $t_m$, the state of the system is characterized by the vector $(x, y, k)$ where $x = X_m, y = Y_m$ and $k$ is the path criterion component (if any). Denote by $f_m$ the state transition function at date $t_m$, we then have:

$$f_m(s, y, k) = (S_{m+1}, Y_{m+1}, h_m(s, k)). \tag{5.8}$$

The value function $W_m$ represents the value of the SP at date $m$ and it is given by the following dynamic programming recursion:

$$W_m(x, y, k) = \begin{cases} \Pi(s, k) & \text{if } m = n \\ g_m(s)\, D + \delta_m R_m(s) + (1 - \delta_m) W_m^h(s, y, k) & \text{if } m < n, \end{cases} \tag{5.9a}$$

where

$$W_m^h(s, y, k) = \beta \mathbb{E}_{m,x} \left[ W_{m+1}(f_m(x, y, k)) \right] \tag{5.9b}$$

is the discounted expected future value of the structured product. The top row of Equation (5.9a) represents the final payoff of the SP when it reaches maturity date. The first component $g_m(s)D$ of the second row is the coupon payment made at date $m$ and the second component $\delta_m R_m(s)$ is the early redemption value of the product.

The dynamic programming recursion of Equation (5.9a) is very simple but it is enough general to apply to all categories of structured products, under all dynamics of the reference underlying.

For non-callable SPs, $\delta_m$ is equal to zero for all stages $m$ and the value function is the sum of the coupon and the discount expected future values. For callable SPs, the early redemption decision depends on the whim of the issuer; the valuation of these products is then made under the issuer's optimal redemption strategy, that is, redemption happens when the early redemption value $R_m$ exceeds the discounted expected future value; that is:

$$C_m = \Big\{(s, y, k) \text{ such that } R_m(s) \geq W_m^h(s, y, k)\Big\}. \tag{5.10}$$

The DP recursion finds the optimal early redemption decision region. Finally, for auto-callable SPs, the early redemption criterion is defined at issuance, so that at any date $m$, the value of $\delta_m$ is known once the reference underlying value $s$ is observed.

| Type | Name | MC | DC | NI | PDE | DP |
|---|---|---|---|---|---|---|
| Non-callable | Barrier Reverse convertible | Yes(slow) | Yes(hard) | Yes* | Yes* | Yes |
| | Reverse convertible | Yes | Yes* | Yes* | Yes* | Yes |
| | Reverse convertible | Yes | Yes* | Yes* | Yes* | Yes |
| | BlueChip Yield LEOS of RBC | Yes(slow) | Yes(hard) | Yes* | Yes* | Yes |
| | BNS $S\&P/TSX60$ Index Deposit | No | Yes* | Yes | Yes | Yes |
| Auto-callable | Autocallable Reverse Convertible | Yes(very slow) | No | No | No | Yes |
| | Deutsche Bank Express Certificate | Yes(very slow) | No | No | No | Yes |
| Callable | Callable Barrier Reverse Convertibles | No | No | No | No | Yes |

Table 5.1: Practicable valuation approach for different product

Table 5.1 reports on the most feasible approaches for different examples of structured product and for different dynamics of the reference underlying[2]. Not that the DP approach is applicable to all products and for all dynamics, contrary to the other methods that could be very hard to implement specially when the reference underlying is modeled with dynamics differing of the geometric Brownian motion (GBM).

## 5.5 Examples and numerical illustrations

In this section, we compare the DP pricing approach with other existing approaches if applicable, in term of convergence and CPU time, for several SPs. We consider three types of dynamics for the reference underlying level, that is geometric Brownian motion, the two states Markov log-normal regime switching and the generalized autoregressive conditional heteroscedasticity (GARCH, Bollerslev (1986)) models.

### 5.5.1 Example 1 : Principal protected note of National Bank of Canada

In this first example, we consider the principal protected callable note issued by National Bank of Canada. This product is well described in Chapter 4 where an analytical formula was proposed when the reference underlying is modeled with a two states Markov log-normal regime switching model (MRSLN2). In this numerical illustration, we compare that analytical formula with the DP valuation.

The DP recursion program for this product is given by a simplified version of Equations (5.9a) and (5.9b). More specifically, we have:

$$W_m(s) = \begin{cases} D\max\{1, \frac{s}{s_0}\} & \text{if } m = n \text{ (maturity)} \\ \min\{R_m(s), \beta_m \mathbb{E}_{m,s}\left[W_{m+1}(S_{m+1})\right]\} & \text{if } m \text{ is an early redemption date} \\ \beta_m \mathbb{E}_{m,s}\left[W_{m+1}(S_{m+1})\right] & \text{otherwise} \end{cases} \tag{5.11}$$

[2]Yes* means only for geometric Brownian motion

where the DP value function $W_m$ is the value of the product at date $m$. Herein we assume that regimes are not directly observable and the transition scheme for the reference underlying value is obtained by conditioning on the number of periods spent in Regime 1, as in Subsection 4.2.2.

We consider principal protected callable notes with a single redemption date, where the reference underlying is the S&P 500 index. To estimate parameters of the MRSLN2 model, we use maximum likelihood based on the Baum-Welch algorithm (Baum *et al.* (1970)). The estimation procedure is applied to a sample of the S&P 500 index monthly log-returns from January 1, 2000 to June 30, 2014 and parameter estimates are reported in Table 5.2. We use a time step of one week in the DP equation since these parameters are obtained with weekly returns of the reference underlying. The risk-free rate is $r = 7\%$, and the initial value of the reference underlying is $s_0 = 1$. We consider several PPCNs with different maturities, and early redemption price corresponding to a return of 8.0% per year.

| Annualized volatilities (%) | | Transition probabilities (%) | |
|---|---|---|---|
| $\sigma_1$ | $\sigma_2$ | $p_{12}$ | $p_{21}$ |
| 11.53 | 28.90 | 2.49 | 6.20 |

Table 5.2: Estimated parameters of the log-normal regime switching model

Table 5.3 reports on the error and the computational time of the DP approach for different maturities dates. $N$ is the grid size, $W_0$ is the note value at issuance date and CPU is the computational time in seconds. In Figure 5.1, we represent the absolute difference between the note values computed by DP and by the analytical approach (the error) as a function of the DP computational time for PPCNs with different maturities (6, 8, and 10 years). The computational time for the dynamic programming approach includes the computation of transition tables and the resolution of the fixed point problem (4.14) which are the most time consuming. As expected, one observes that the error and computational time

| $N$ | $T=6$ $W_0$ | Error ($\times 10^4$) | CPU | $T=8$ $W_0$ | Error ($\times 10^4$) | CPU | $T=10$ $W_0$ | Error ($\times 10^4$) | CPU |
|---|---|---|---|---|---|---|---|---|---|
| 31 | 0.9257 | 28.78 | 2.83 | 0.9093 | 41.28 | 3.07 | 0.8778 | 41.53 | 3.96 |
| 51 | 0.9298 | 4.67 | 4.66 | 0.9058 | 12.38 | 5.44 | 0.8824 | 21.17 | 6.91 |
| 71 | 0.9281 | 3.90 | 7.02 | 0.9040 | 6.02 | 7.96 | 0.8838 | 7.52 | 10.27 |
| 91 | 0.9287 | 1.89 | 9.21 | 0.9054 | 2.66 | 10.91 | 0.8842 | 3.56 | 13.99 |
| 111 | 0.9287 | 1.33 | 12.60 | 0.9054 | 2.19 | 14.05 | 0.8844 | 1.78 | 18.29 |
| 131 | 0.9283 | 1.05 | 15.71 | 0.9050 | 1.65 | 17.63 | 0.8844 | 0.86 | 22.54 |
| 151 | 0.9286 | 0.98 | 16.87 | 0.9051 | 1.43 | 21.82 | 0.8845 | 0.41 | 27.48 |
| 171 | 0.9285 | 0.63 | 19.82 | 0.9053 | 0.62 | 26.06 | 0.8845 | 0.17 | 32.76 |
| 191 | 0.9285 | 0.30 | 23.12 | 0.9052 | 0.42 | 30.42 | 0.8845 | 0.06 | 38.38 |
| 211 | 0.9286 | 0.16 | 25.69 | 0.9051 | 0.24 | 33.87 | 0.8845 | 0.01 | 43.13 |
| 231 | 0.9284 | 0.09 | 29.09 | 0.9052 | 0.12 | 38.01 | 0.8845 | 0.00 | 68.55 |

Table 5.3: Convergence of the DP approach for different maturity dates.

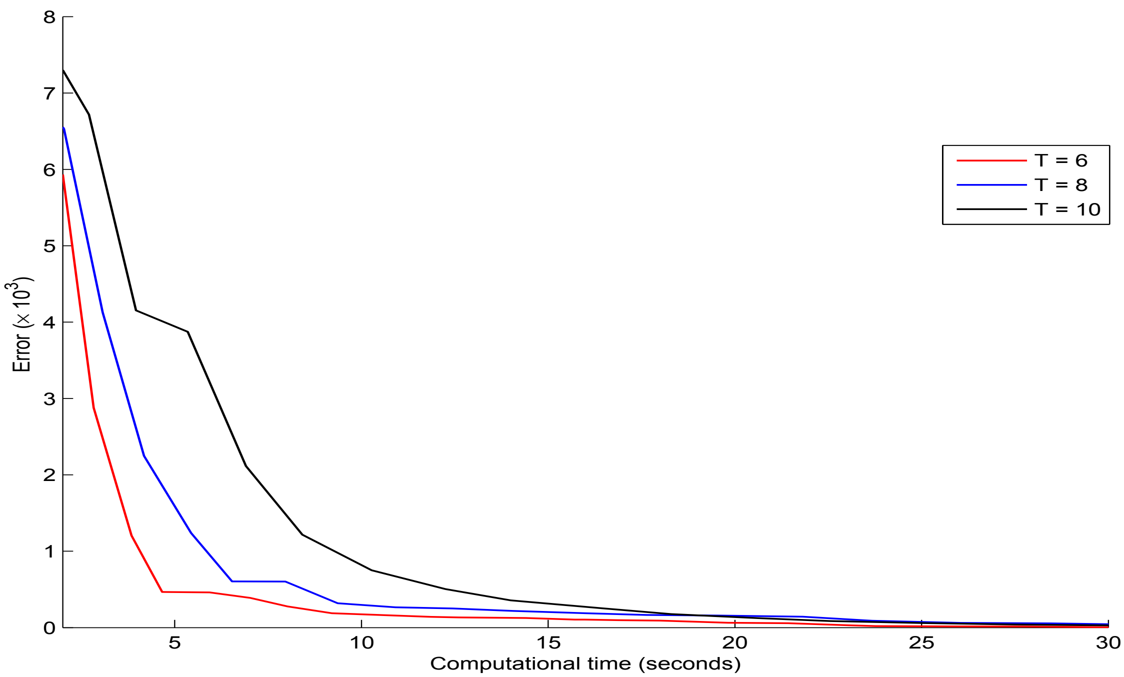


Figure 5.1: DP error with respect to the computational time. Other parameter values are as in Table 5.2

are sensitive to maturity (both increase with maturity). In all cases considered, one observes that a three digits convergence is obtained in less than 10 seconds. This shows that the DP approach is almost comparable to the analytical approach, which takes around 8 seconds. These results illustrate the stability and efficiency of the DP approach, even compared with an analytical formula.

Now, we assume that the reference asset follows a discrete-time GARCH dynamics as in Duan (1995). Without loss of generality, we assume that the process is observed at discrete time intervals normalized to 1. The reference asset price level under the risk-neutral measure follows:

$$\ln(\frac{S_{m+1}}{S_m}) = r - \frac{1}{2}H_{m+1} + \sqrt{H_{m+1}}\varepsilon_{m+1} \tag{5.12a}$$

while the conditional variance of the log return is given by

$$H_{m+1} = b_0 + b_1 H_m + b_2 H_m \left(\varepsilon_m - \theta - \lambda\right)^2 \tag{5.12b}$$

where $\varepsilon_m$ is a standard normal random variable. In this model, $\lambda$ is the risk premium, $\theta$ is a leverage parameter that determines the nature of the correlations between innovations $\varepsilon_t$ and $r$ is the one period risk-free rate. The auto-regressive parameters $b_0, b_1$ and $b_2$ are assumed to satisfy the following conditions: $b_0 > 0, b_1, b_2 \geq 0$ and $b_1 + b_2(1+\theta^2) < 1$.

At date $m$, the state vector in this model is $(S_m, H_{m+1})$ and the state space is $[0,\infty[\times[\underline{\sigma},\infty[$ where $\underline{\sigma} = \min\left\{\beta_0; \frac{\beta_0}{1-\beta_1-\beta_2(1+(\theta+\lambda)^2}\right\}$ is a lower bound on the value of the conditional volatility.

The adaptation of the DP model (5.9a) and (5.9b) under the GARCH assumption is:

$$W_m(s,h) = \begin{cases} D\max\{1, \frac{s}{s_0}\} & \text{if } m = n \text{ (maturity)} \\ \min\{R_m(s), \beta_m \mathbb{E}_{m,s,h}\left[W_{m+1}(S_{m+1}, H_{m+2})\right]\} & \text{if } m \text{ is an early redemption date} \\ \beta_m \mathbb{E}_{m,s,h}\left[W_{m+1}(S_{m+1}, H_{m+2})\right] & \text{otherwise.} \end{cases} \tag{5.13}$$

Note that under the GARCH dynamics, none of methods presented in Table 5.1 can be used to evaluate the principal protected callable note, except DP. To solve this DP recursion, we use linear-cubic spline approximation, where the interpolation function is continuous, twice differentiable, piecewise linear in $h$ and piecewise cubic in $s$. The reference asset is a portfolio of 7 single stocks listed in Table 5.4 below.

| Component | Weight |
|---|---|
| BCE Inc | 14% |
| Bank of Montreal | 13% |
| Barrick Gold Corporation | 15% |
| Enbridge Inc | 18% |
| Imperial Oil Ltd | 17% |
| Royal Bank of Canada | 11% |
| Toronto-Dominion Bank | 12% |

Table 5.4: Reference asset composition

To estimate the GARCH parameters, we use weekly returns of reference asset components between December 2001 and December 2012 as shown in Figure 5.2. The GARCH estimated parameters are given in Table 5.5 below.

The risk-free rate is fixed at $r = 7.0\%$, the initial value of the reference portfolio is fixed at $s_0 = 1$, the early redemption price corresponds to an annual return of 8% and the initial conditional variance $h_0$ is the unconditional long-term volatility of the stock return, that is:

| $b_0$ | $b_1$ | $b_2$ | $\lambda$ | $\theta$ |
|---|---|---|---|---|
| $1.1345 \times 10^{-5}$ | 0.7914 | 0.1091 | 0.2064 | 0.3180 |

Table 5.5: Estimated GARCH parameters for the reference portfolio

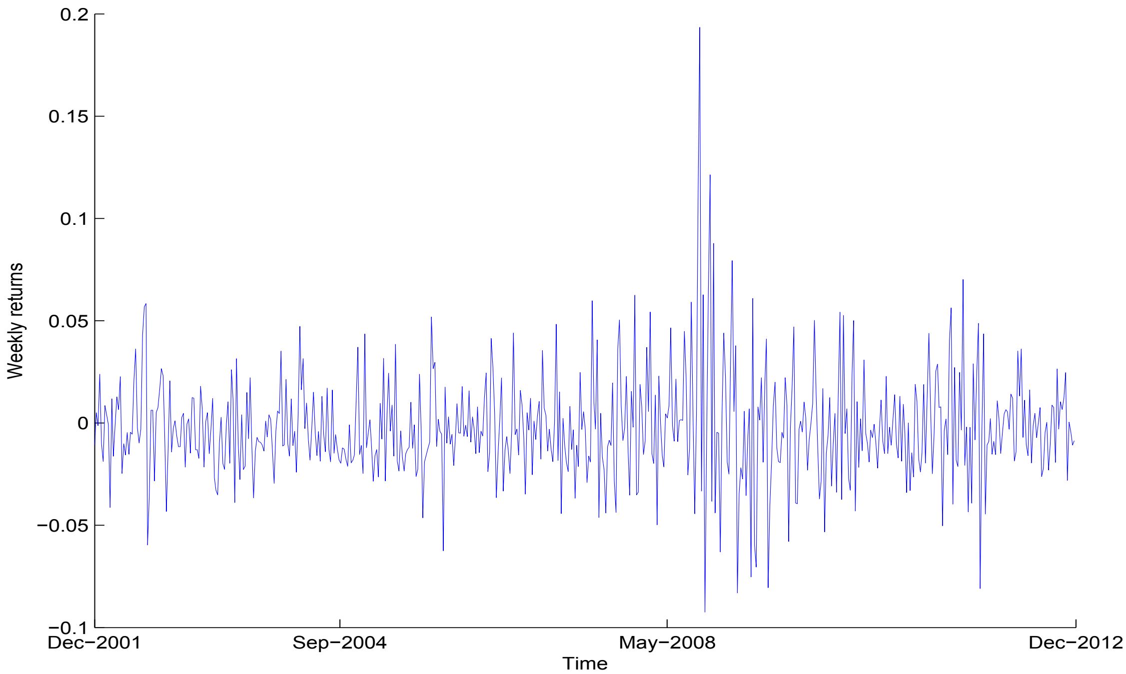


Figure 5.2: Times series of the reference portfolio weekly returns

$h_0 = \frac{b_0}{1-b_1-b_2(1+(\lambda+\theta)^2)} = 1.9353 \times 10^{-4}$.

| | $T=6$ | | | $T=8$ | | | $T=10$ | | |
|---|---|---|---|---|---|---|---|---|---|
| $N \times N$ | $W_0$ | Error ($\times 10^4$) | CPU | $W_0$ | Error ($\times 10^4$) | CPU | $W_0$ | Error ($\times 10^4$) | CPU |
| $11 \times 11$ | 0.8472 | 0.0809 | 4.8 | 0.8080 | 0.0994 | 11.7 | 0.6744 | 0.1331 | 8.2 |
| $21 \times 21$ | 0.8978 | 0.0303 | 8.9 | 0.8676 | 0.0399 | 19.9 | 0.7277 | 0.0798 | 13.6 |
| $31 \times 31$ | 0.9145 | 0.0136 | 12.8 | 0.8887 | 0.0188 | 30.6 | 0.7752 | 0.0323 | 21.1 |
| $41 \times 41$ | 0.9140 | 0.0141 | 18.7 | 0.8940 | 0.0135 | 44.3 | 0.7706 | 0.0369 | 31.6 |
| $51 \times 51$ | 0.9258 | 0.0023 | 25.6 | 0.9004 | 0.0070 | 63.9 | 0.7922 | 0.0153 | 47.0 |
| $61 \times 61$ | 0.9264 | 0.0018 | 38.0 | 0.9042 | 0.0033 | 87.7 | 0.8061 | 0.0015 | 63.7 |
| $71 \times 71$ | 0.9272 | 0.0009 | 49.9 | 0.9045 | 0.0029 | 117.2 | 0.7974 | 0.0101 | 84.8 |
| $81 \times 81$ | 0.9257 | 0.0024 | 68.9 | 0.9064 | 0.0010 | 162.9 | 0.8071 | 0.0004 | 112.4 |
| $91 \times 91$ | 0.9278 | 0.0004 | 106.3 | 0.9068 | 0.0006 | 239.0 | 0.8072 | 0.0003 | 167.1 |
| $101 \times 101$ | 0.9279 | 0.0002 | 118.0 | 0.9071 | 0.0003 | 280.1 | 0.8074 | 0.0001 | 202.1 |
| $111 \times 111$ | 0.9280 | 0.0001 | 163.2 | 0.9073 | 0.0001 | 382.3 | 0.8075 | 0.0000 | 273.2 |
| $121 \times 121$ | 0.9281 | 0.0000 | 228.8 | 0.9074 | 0.0000 | 530.0 | 0.8075 | 0.0000 | 396.4 |
| $131 \times 131$ | 0.9281 | 0.0000 | 359.2 | 0.9074 | 0.0000 | 792.3 | 0.8075 | 0.0000 | 515.3 |

Table 5.6: DP convergence with linear-cubic spline interpolation under GARCH dynamic.

Table 5.6 reports on the values, the error and the computational time of the PPCN for different maturities, early redemption prices and different sizes of the interpolation grid. The error corresponds to the absolute difference between the note value for the corresponding grid size and the convergence value which is the note value when the grid size is $131 \times 131$. This error is also represented in Figure 5.3 as a function of the DP computational time for different maturities. $M$ and $N$ are respectively the number of points in $s$ (reference asset value) and in $h$ (conditional variance) used in the linear-cubic spline interpolation. The CPU column displays the average computational time required to solve the DP recursion. For example for a grid of 71 by 71, it takes around 50 seconds to obtain the value of the note with a precision of three digits, at all stages and for all states.

### 5.5.2 Example 2 : Buffered PLUS note of Morgan Stanley

In this example, we consider the Buffered PLUS issued by Morgan Stanley on April 3, 2013. This SP which is linked to the S&P 500 index is non-callable with a maturity of two years. It does not pay coupons and at maturity, if the index level is higher than its initial level $S_0$, the note pays a return equal to the percentage increase in the index multiplied by participation rate $p$, up to a cap $\gamma$. If at maturity, the index level is below $S_0$ but above a

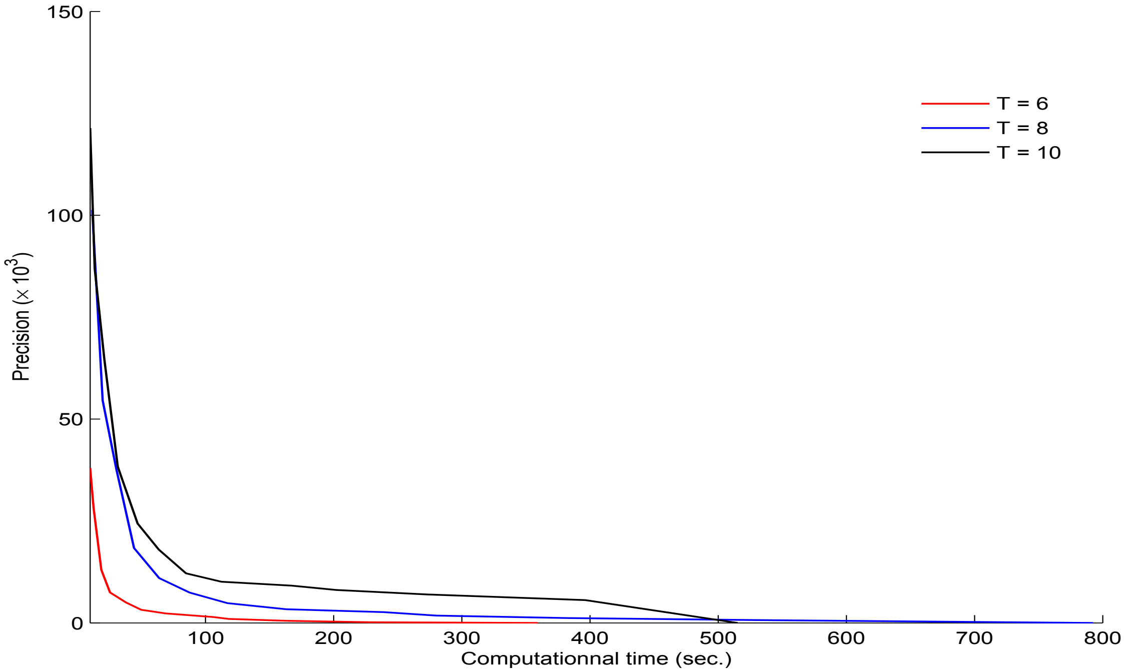


Figure 5.3: DP error with respect to the computational time.

minimum barrier $\overline{F}$, the note will pay a fixed amount $D$ equal to the invested capital. If at maturity the index level is below the minimum barrier $\overline{F}$, the final payment is equal to the fixed amount $D$ minus the difference between the minimum barrier $\overline{F}$ and the final level of the index (this difference is taken as a percentage of the initial index level). Mathematically, this means that the final payoff is given by the following equation:

$$\begin{aligned}\Pi(S_T) &= D\left(1+\min\{pR_T,\gamma\}-\min\{pR_T,0\}+\min\{R_T+K,0\}\right) \\ &= D((1+\gamma)+\frac{p}{S_0}(S_0-S_T)^+-\frac{1}{S_0}(\overline{F}-S_T)^+-\frac{p}{S_0}(S_0+\frac{S_0\gamma}{p}-S_T)^+), \quad (5.14)\end{aligned}$$

where $R_T = \frac{S_T-S_0}{S_0}$ is the total return of the reference underlying and $K = \frac{S_0-\overline{F}}{S_0}$ is the buffer level. Under geometric Brownian motion or regime switching dynamics, this product can be evaluated with the decomposition approach. Indeed, the right hand side of the bottom line of Equation (5.14) is composed of the payoff of a zero-coupon bond with a face value of $D(1+\gamma)$, a number of long at-the-money put option, a number of short in-the-money put options with strike price $\overline{F}$, and a number of short out-of the-money put options with strike price $S_0 + \frac{S_0\gamma}{p}$ which can be evaluated with analytical formulas under geometric Brownian motion and regime switching dynamics.

The PDE approach consists of solving the Black and Scholes partial differential equation with the following boundary conditions:

$$V(T,S) = \Pi(S_T); V(t,0) = KD\beta_t^0 \text{ and } \lim_{S \longrightarrow \infty} V(t,S) = D(1+\gamma). \tag{5.15}$$

These boundary conditions follow from Equation (5.14).

Following Deng et al. (2014), the numerical integration approach consists of estimating the following integral:

$$\int_{-1}^{\infty} f(R_T)\phi(R_T)dR_T, \tag{5.16}$$

where $\phi$ is the density function of the log-return $\log(R_T)$ and $f(R_T) = \Pi(S_T)$ is given by the bottom line of Equation (5.14).

The DP recursion program for this product is given by a simplified version of Equations (5.9a) and (5.9b). More specifically, we have:

$$W_m(s) = \begin{cases} \Pi(s) & \text{if } m = n \text{ (maturity)} \\ \beta_m \mathbb{E}_{m,s}\left[W_{m+1}(S_{m+1})\right]\} & \text{if } m < n. \end{cases} \tag{5.17}$$

First, we assume that the reference underlying (the S&P 500 index) follows a geometric Brownian motion, implying that log-return $\log(R_T)$ is normally distributed. More specifically, we assume that:

$$dS_t = rS_t dt + \sigma S_t dZ_t, \tag{5.18}$$

where $\{Z_t\}_{t\geq 0}$ is a standard Brownian motion. In the numerical illustration, we will use parameter values of Table 5.7 as considered by the issuer[3].

| $S_0$ | $\sigma$ | r | T | $H$ | p | $\gamma$ |
|---|---|---|---|---|---|---|
| 1569.2 | 17.56% | 1.35% | 30 months | 1412.27 | 2.0 | 20.75% |

Table 5.7: Note and model parameters

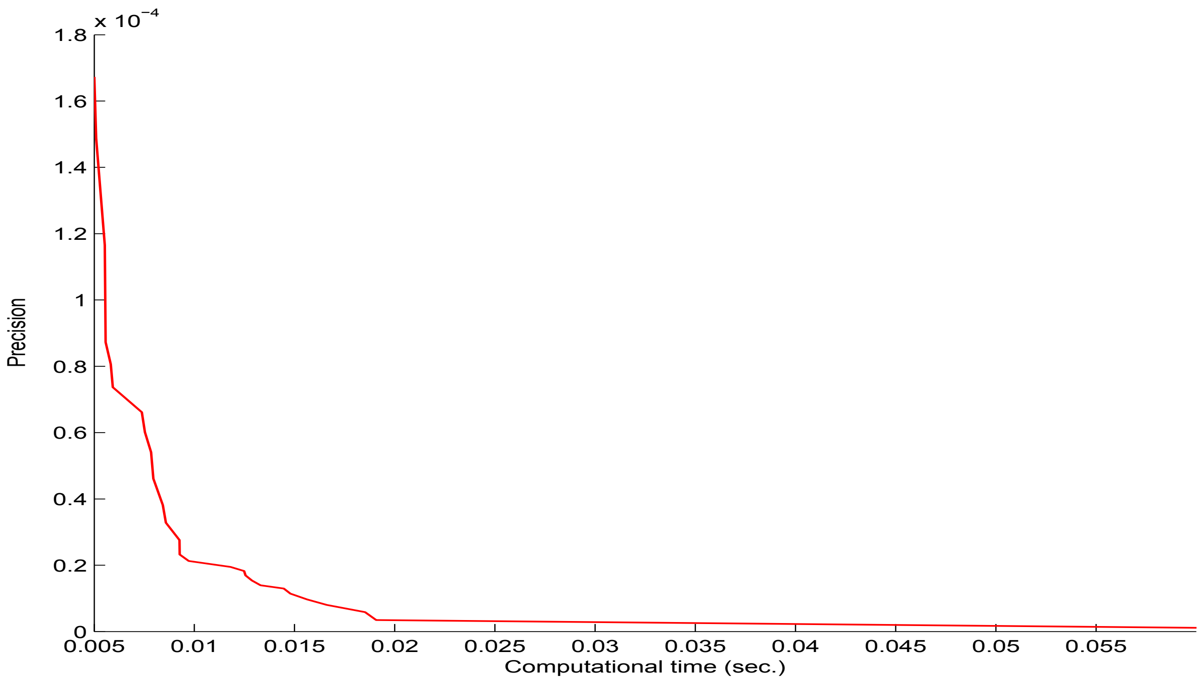


Figure 5.4: Precision as a function of the computational time in the DP approach. Other parameter values are as in Table 5.7

In Figure 5.4, we represent the precision as a function of the computational time required in the DP approach. The precision if defined as the absolute difference between the note value in the DP approach and its exact value obtained with the decomposition approach. This result shows that a four digits convergence can be obtained in the DP approach within less than a hundredth of a second. Note that the corresponding grid size is 151 points.

[3] Available at : http://slcg.com/pdf/tearsheets/61761M698.pdf

| $S_0$ | Decomposition | PDE | Numerical Integration | DP method |
|---|---|---|---|---|
| 1412.3 | 9.4306 | 9.4311 | 9.4306 | 9.4306 |
| 1443.7 | 9.5406 | 9.5410 | 9.5405 | 9.5406 |
| 1475.0 | 9.6396 | 9.6401 | 9.6397 | 9.6396 |
| 1506.4 | 9.7287 | 9.7291 | 9.7287 | 9.7287 |
| 1537.8 | 9.8086 | 9.8090 | 9.8087 | 9.8086 |
| 1569.2 | 9.8803 | 9.8807 | 9.8803 | 9.8803 |
| 1600.6 | 9.9445 | 9.9449 | 9.9446 | 9.9445 |
| 1632.0 | 10.0019 | 10.0023 | 10.0019 | 10.0019 |
| 1663.3 | 10.0532 | 10.0535 | 10.0533 | 10.0532 |
| 1694.7 | 10.0989 | 10.0992 | 10.0989 | 10.0989 |
| 1726.1 | 10.1396 | 10.1399 | 10.1397 | 10.1396 |
| Average CPU(sec.) | 0.0238 | 0.5947 | 0.0046 | 0.0044 |

Table 5.8: Comparing pricing approaches. Other parameter values are as in Table 5.7

Table 5.8 reports on the value of the Buffered Plus obtained by the four pricing methods for different values of the reference underlying at issuance date. The DP results are for a grid of 151 points. In term of values, the four approaches are very close, however one observe significant differences between methods in term of average computational time. For the DP approach, the average computational time includes the note values for all points in the grid and for the other approaches, this average computational time is just for the computation of one value. So it becomes clear that the DP is more advantageous in term of computational time.

Now assume that the reference underlying is modeled with the two states Markov log-normal regime switching model (MRSLN2) as described in Chapter 4. The MRSLN2 model parameter values are reported in Table 5.9 and other parameter values are that of Table 5.7.

Under these dynamics, the decomposition approach is still applicable, as well as numerical

| Annualized volatilities (%) | | Transition probabilities (%) | |
|---|---|---|---|
| $\sigma_1$ | $\sigma_2$ | $p_{12}$ | $p_{21}$ |
| 10.00 | 30.00 | 3.00 | 6.00 |

Table 5.9: Parameters of the log-normal regime switching model

integration. For the numerical integration, the distribution of the reference underlying is obtained by conditioning on the number of periods spent in Regime 1, as in Subsection 4.2.2. More precisely, when the number of periods spent in Regime 1 is $i$, we have:

$$S_T^i = S_0 \exp\left((r - \frac{\overline{\sigma}_{i,n}^2}{2})T + \overline{\sigma}_{i,n}^2\sqrt{T}\mathbb{Z}\right) \tag{5.19}$$

where $n$ is the total number of periods (months in this case) and $\overline{\sigma}_{i,n}$ is given by Equation Equation 4.5. The note value in the numerical integration approach is then given by the following equation:

$$V_{NI}(S_0) = \sum_{i=0}^{n} P(i;n) \int_{-1}^{\infty} f(R_T^i)\phi(R_T^i) dR_T^i \text{ with } R_T^i = \frac{S_T^i - S_0}{S_0} \tag{5.20}$$

where $P(\cdot;n)$ is the probability of the number of time steps spent in Regime 1 during $n$ time steps, defined in Subsection 4.2.2.

In Figure 5.5, we represent the precision as a function of the computational time required in the DP approach. As in the previous case, this shows that a four digits convergence can be obtained using the DP approach within less than a tenth of a second and a two digits convergence is obtained within less than a thousandth of a second.

Table 5.10 reports on the value of the Buffered Plus obtained by the decomposition, the numerical integration and the DP pricing methods for different values of the reference underlying at issuance date. The DP results are for a grid of 201 points. In term of values, approaches are again very close, however one observe significant differences between methods

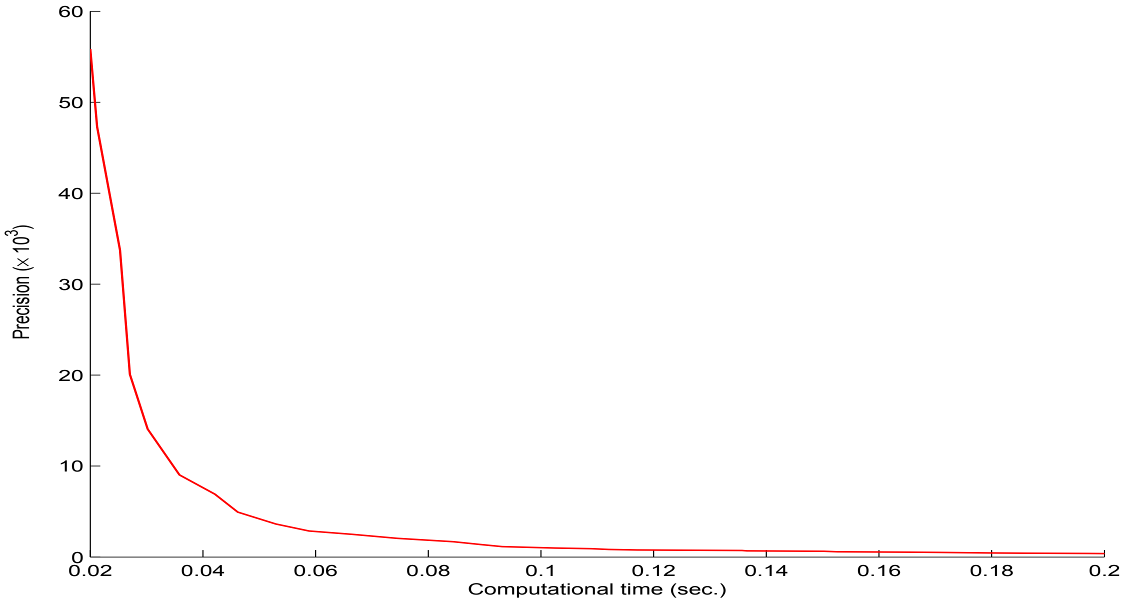


Figure 5.5: Precision as a function of the computational time in the DP approach. Other parameter values are as in Table 5.9

| $S_0$ | Decomposition | Numerical Integration | DP method |
|---|---|---|---|
| 1412.3 | 9.8699 | 9.8609 | 9.8701 |
| 1443.7 | 9.9771 | 9.9699 | 9.9771 |
| 1475.0 | 10.0711 | 10.0588 | 10.0710 |
| 1506.4 | 10.1533 | 10.1517 | 10.1534 |
| 1537.8 | 10.2247 | 10.2178 | 10.2248 |
| 1569.2 | 10.2867 | 10.2746 | 10.2866 |
| 1600.6 | 10.3403 | 10.3356 | 10.3403 |
| 1632.0 | 10.3866 | 10.3791 | 10.3865 |
| 1663.3 | 10.4264 | 10.4189 | 10.4264 |
| 1694.7 | 10.4607 | 10.4604 | 10.4607 |
| 1726.1 | 10.4901 | 10.4830 | 10.4901 |
| Average CPU(sec.) | 0.0318 | 0.0296 | 0.0131 |

Table 5.10: Comparing pricing approaches. Other parameter values are as in Table 5.9

in term of average computational time, while the DP approach is slightly less time consuming while yielding the value at all dates and all possible states of the world.

### 5.5.3 Example 3 : TD EURO STOXX 50 Index-Linked Autocallable Notes

This five years term auto-callable structured product is tied to the EURO STOXX 50 Index (called the Index hereafter) and it provides to the holder possible payment of a variable return linked to the price performance of the Index. At each valuation date $T_i, i = 1, 2, \ldots, n = 5$ (each annual anniversary of the note issuance date) the notes will be automatically called for early redemption by the issuer if the closing Index level is greater than or equal to the auto-call level, which is the level $S_0$ of the Index at issuance date, and the holder will receive the initially invested amount plus a variable return. The variable return, if any, is equal to a predefined fixed return $R_i$ plus an excess return equal to $p = 5\%$ of the index total return in excess of the fixed return $R_i$. If at maturity the note has not yet been redeemed and the Index level is less than the auto-call level but greater than the barrier level, which is $B = 70\%$ of the Index level at issuance, the holder will receive the initially invested amount. Finally, if at maturity the note has not yet been redeemed and the Index level is less than the auto-call level and the barrier level $B$, the holder will receive the initially invested amount minus the index return. Predefined fixed return are reported in Table 5.11.

| | **Note annual anniversary (year)** | | | | |
|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 |
| Fixed return (%) | 7.50 | 15.00 | 21.00 | 26.00 | 30.00 |

Table 5.11: Note fixed return

Mathematically, the payoff of this note can be summarized as follows:

$$f_i(S) = \begin{cases} D(1+R_i+p(\frac{S-S_0}{S_0}-R_i)^+ & \text{if } S \geq S_0 \text{ for } i=1,2,\ldots,5 \\ 0 & \text{if } S < S_0 \text{ for } i=1,2,\ldots,4 \\ D & \text{if } B < S < S_0 \text{ for } i=5 \\ D\frac{S}{S_0} & \text{if } S < B \text{ for } i=5. \end{cases} \tag{5.21}$$

This final payoff suggests that the SP is not principal protected. In fact, if at maturity the value of the Index is less than the barrier level $B$, the SP holder receives a final payment less than the initial investment. We assume that the value of the reference Index is characterized by a two states Markov log-normal regime switching (MRSLN2) model as described previously. The annual risk-free rate is $r = 7.0\%$ and the initial value of the reference underlying is $S_0 = 3424.30$, corresponding to the observed value on August 19, 2015. The corresponding barrier level is $B = 2397.01$.

To estimate parameters of the MRSLN2 model, we use a maximum likelihood based on the Baum-Welch algorithm. The estimation procedure is applied to a sample of the Euro Stoxx 50 Index monthly log-returns from December 29, 2000 to August 3, 2015 and parameter estimates are reported in Table 5.12.

| Annualized volatilities (%) | | Transition probabilities (%) | |
|---|---|---|---|
| $\sigma_1$ | $\sigma_2$ | $p_{12}$ | $p_{21}$ |
| 19.31 | 30.78 | 2.50 | 3.88 |

Table 5.12: Parameters estimates of the regime switching model (monthly)

The DP recursion program for this product is given by a simplified version of Equations

(5.9a) and (5.9b). More specifically, we have:

$$W_m(s) = \begin{cases} f_m(s) & \text{if } m = n \text{ (maturity)} \\ f_m(s)I_{\{s>s_0\}}(s) + (1 - I_{\{s>s_0\}}(s))\beta_m \mathbb{E}_{m,s}\left[W_{m+1}(S_{m+1})\right] & \text{if } m < n \end{cases} \tag{5.22}$$

where the DP value function $W_m$ is the value of the product at date $m$.

Under the MRSLN2 model, the TD EURO STOXX 50 Index-Linked Autocallable Notes can be evaluated by using the Monte Carlo simulation approach. To perform the Monte-Carlo simulation we first simulate a large number of paths of the index level and second, for each path we compute the note value as the discounted value of the final payoff which corresponds to the early redemption value if the early redemption criterion is verified at a point of the path and the note value at maturity otherwise. Finally, we take the note value as the average of those computed values.

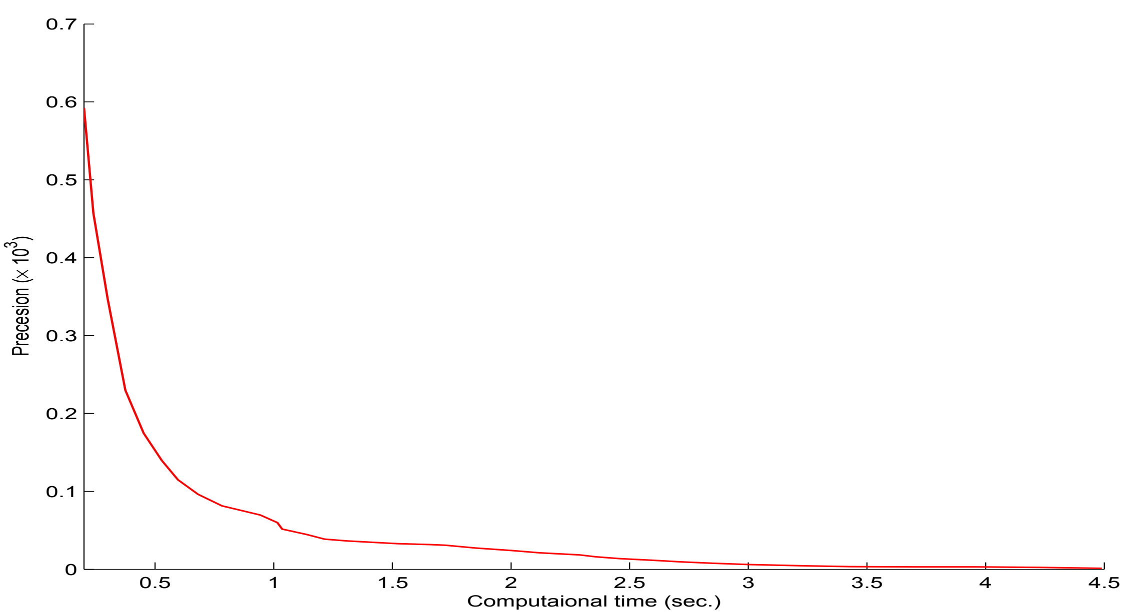


Figure 5.6: Precision as a function of the computational time in the DP approach.

In Figure 5.6, we represent the precision as a function of the computational time required in the DP approach. As in the previous dynamic, this figure shows that a four digits convergence can be obtained in the DP approach within less than a half second.

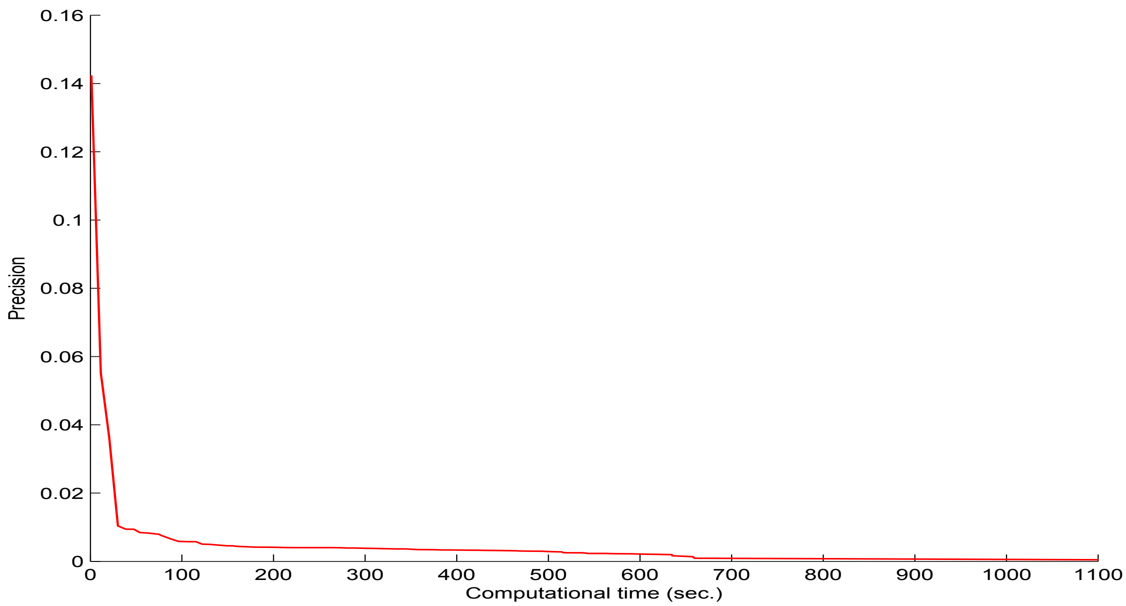


Figure 5.7: Precision as a function of the computational time in the Monte Carlo simulation approach.

In Figure 5.7, we represent the precision as a function of the computational time required in the Monte Carlo simulation approach. Compared to the DP approach, this figure shows that a two digits convergence cannot be obtained within less than a minute.

## 5.6 Conclusion

In this chapter, we have presented a general valuation method based on a dynamic programming approach and finite element approximation for structured products. The main advantage of this method is that it can be used for any type of structured products and for any dynamics of the underlying reference. We have classified structured products into three classes, based on their redemption features. We have tested the efficiency of the method with several examples of SPs issued in the financial market. Numerical results show that the

DP recursion converges within a relatively small CPU time. In addition, the DP approach is more suitable than almost all other valuation methods and can be applied when other methods fail.

# Chapter 6

# General conclusion

In this thesis, we addressed two main areas of financial engineering. The first is collateral management, used to reduce counterparty risk, and the second is the valuation of derivatives. For each of these subjects, we provide theoretical algorithms based on dynamic programming and approximation by finite elements. Numerical applications considered in each case show that these new algorithms appear to be more effective than those currently used in practice.

In Chapter 3 we address collateral management for a series of transactions between two entities. We propose collateral management strategies based on dynamic programming, multi-objective optimization and utility maximization approaches for both unilateral and bilateral agreements. These strategies are able to take into account several important factors such as wrong (right) -way risk and transaction costs. In the case of a unilateral agreement, we compared the DP strategy with the existing conservative strategy when the value of the portfolio of contracts is characterized by a jump diffusion model. The numerical results of this basic example shows that the DP strategy is more effective in reducing the risk measure. In the case of bilateral agreement, the DP strategy we propose is based on a combination of dynamic programming and multi-objective optimization approaches. Compared with the conservative strategy, the DP is more advantageous for both entities.

In Chapter 4, we provide practitioners with a Black-Scholes (1973) type formula for the pricing of compound options and principal protected callable notes with one early redemption date under a two states Markov regime-switching log-normal model.

Chapter 5 is devoted to the evaluation of structured products. In this chapter, we propose a general approach based on dynamic programming combined with finite element approximation. The main advantage of this method of valuation is that it can be used for all types of structured products and for most dynamics of the reference asset. We tested the effectiveness of the method with several examples of products issued in financial markets and the numerical results show that our method is generally more suitable than existing methods in terms of CPU time and feasibility.

# Bibliography


[1] Agliardi, E., Agliardi, R., "A generalization of Geske formula for compound options" *Mathematical Social Sciences (2003)* 45, p.75-82.

[2] Alavian, S., Ding, J., Whitehead, P., and Laudicina L., "Counterparty Valuation Adjustment (CVA)." *Available at ssrn.com, 2008*

[3] Arrow, K.J., "The role of securities in the optimal allocation of risk-bearing" *The Review of Economic Studies, April, 1964)* Vol. 31, No.2, p.91-96.

[4] Basel Committee on Banking Supervision (2010), "A Global Regulatory Framework for More Resilient Banks and Banking Systems (2010)" *Available at : http://www.bis.org/publ/bcbs189dec2010.pdf, December 2010*

[5] Baubonis, C., Gastineau, G.L., and Purcell, D., "The Banker's Guide to Equity-Linked Certificates of Deposit" *The Journal of Derivatives, Winter 1993* Vol. 1, No.2, p.87-95.

[6] Baum, L.E., Petrie, T., Soules, and G.,Weiss, N., "A maximization technique occurring in the statistical analysis of probabilistic functions of Markov chains" *The Annals of Mathematical Statistics (1970)* (41). p.164-171.

[7] Bellman, R., "An introduction to the theory of dynamic programming" *RAND Corp. Report (1953)*

[8] Ben-Ameur, H., Breton, M. and Martinez, J.M., "Dynamic Programming Approach for Valuing Options in the GARCH Model" *Management Science (2009)* 55, p.252-266.

[9] Ben-Ameur, H., Breton, M. and Francois, P., "A dynamic programming approach to price installment options" *European Journal of Operational Research (2006)*, v.169, i.2, p.667-676.

[10] Ben-Ameur, H., Breton, M. and L'Ecuyer, P., "A dynamic programming procedure for pricing American-style Asian options" *Management Science (2002)* 48, p.625-643.

[11] Ben-Ameur, H., Breton, M. and L'Ecuyer, P., "A dynamic programming approach for pricing options embedded in bonds" *Journal of Economic Dynamics and Control (2006)*

[12] Ben-Ameur, H., Brigo, D. and Errais, E., "A dynamic programming approach for pricing CDS and CDS options" *Quantitative Finance (2009)* 9, p. 717-726.

[13] Bernstein, S., "Démonstration du théorème de Weierstrass fondée sur le calcul des probabilities" *Comm. Soc. Math. Kharkov (1912)*, 13,p.1-2.

[14] Black, F. and Scholes, M., "The pricing of options and corporate liabilities" *J. Political Economics (1976)* 81, p.637-654.

[15] Bliss, R., and Kaufman, G., G., "Derivatives and Systematic Risk: Netting, Collateral and Closeout" *Journal of Financial Stability (2006)* 2, p.55-70.

[16] Bloomberg Structured Notes Brief, *Available at:http://www.bloombergbriefs.com/structured-notes/*

[17] Bollen, N.P.B, "Valuing options in regime-switching models" *Journal of Derivatives. (1998)* 6.

[18] Boudhina, A. and Breton, M., "An Efficient Numerical Method for Pricing Long-maturity American Put Options" *Cahiers du GERAD G-2013-30, April 2013.*

[19] Breton, M., De Frutos,J. and Serghini-Idrissi, S., "Pricing options under GARCH dynamic in a dynamic programming spectral approximation framework" *GERAD working paper (2010).*

[20] Brigo, D., and Capponi, A., "Bilateral counterparty risk with application to CDSs" *Risk Magazine, March 2010. Extended Version Under Review. Preprint from 2008*

[21] Brigo D., and Masetti M., "Risk Neutral Pricing of Counterparty Risk" *In: Pykhtin, M. (2005), Counterparty Credit Risk Modeling: Risk Management, Pricing and Regulation. Risk Books, London*

[22] Carr, P., and Chou, A., "Breaking barriers" *RISK (1997)* 10(9), p.139-145.

[23] Chen, A.H., and Kensinger, J.W., "An Analysis of Market-Index Certificates of Deposit." *Journal of Financial Services Research, July 1990* Vol. 4, No. 2, p.93-110.

[24] Chen, A.H., and Sears, R.S., "Pricing the SPIN" *Financial Management, Summer 1990* Vol. 19, No. 2, p.36-47.

[25] Christl, J., "Financial instruments : structured products handbook" *Oesterreichische Nationalbank, Vienna, 2004*

[26] Das, S., "Structured Products and Hybrid Securities" 2nd ed. New York, NY,John Wiley and Sons 2000

[27] Deng, G., Dulaney, T., Husson, T., and McCann, C., "Structured certificates of deposit: Introduction and valuation" *SLCG working paper, 2013, available at http://www.slcg.com/*

[28] Deng, G., Dulaney, T., Husson, T., and McCann, C., "Structured product based variable annuities" *Journal of Retirement (2014a)* 1.

[29] Deng, G., McCann, C., and O'Neal, E., "What TiVo and JPMorgan teach us about Reverse Convertibles" *SLCG Working Paper (2010)*

[30] Deng, G., Mallett, J., and McCann, C., "Modeling Autocallable structured products" *Journal of Derivatives and Hedge Funds (2011b)* 17, p.326-340.

[31] DTCC, "Trends, risks and opportunities in collateral management: a collateral management white paper" *DTCC, January, 2014, Available at: http://www.dtcc.com/about/managing-risk/collateral-management.aspx*

[32] Duffie, D., "Credit Risk Modeling with Affine Processes" *Journal of Banking and Finance, November 2005* Vol. 29, No. 11, p.2751-2802.

[33] Chandrashekar, K., "Collateral Management: an Introduction" *Wipro Technologies (2008).*

[34] Cherubini, U., "Counterparty Risk in Derivatives and Collateral Policies:The Replicating Portfolio Approach" *In ALM of Financial Institutions, ed. Tilman, L., Institutional Investor Books (2005)*

[35] Cox, J.C., Ingersoll, J.E., and Ross, S.A., "A Theory of the Term Structure of Interest Rates" *Econometrica (1985)* 53, p.385-407.

[36] Fouque,J.P. and Han, C.H., "Evaluation of Compound Options using Perturbation Approximation" *Journal of Computational Finance (2005)* 9, 1.

[37] Garlappi, L., and Skoulakis, G., "Taylor Series Approximations to Expected Utility and Optimal Portfolio Choice" *Mathematics and Financial Economics (2011)*, Vol. 5, No. 2, 2011, pp. 121-156

[38] Geske, R., "The valuation of compound options" *Journal of Financial Economics (1979)* 7, p.63-81.

[39] Geske, R. and Johnson, H.E., "The American put option valued analytically" *Journal of Finance (1984*a*)* 39, p.1511-1524.

[40] Gong, P., Zhiwei He, Z. and Song-Ping Zhub, S.P., "Pricing convertible bonds based on a multi-stage compound-option model" *Physica A (2006)* 366, p.449-462

[41] Ghosh, A., Rennison, G., Soulier, A., Sharma, P., and Malinowska, M., "Counterparty Risk in Credit Markets" *Barclays Capital Research Report (2008)*

[42] Giglio, S., "Credit default swap spreads and systemic financial risk" *Federal Reserve Bank of Chicago, Proceedings 1122 (2011)*

[43] Glasserman, P., "Monte Carlo Methods in Financial Engineering" *Chapman and Hall, Boca Raton, FL, 2003*

[44] Gregory, J., "Counterparty Credit Risk: The New Challenge for Financial Markets" *Wiley, Chichester, 2010*

[45] Hamilton, J.D., "A New Approach to the Economic Analysis of Non-stationary Time Series and the Business Cycle" *Econometrica (1989)* 57 (2), p.357-384.

[46] Hardy, M.R., "A Regime-Switching Model of Long-Term Stock Returns" *North American Actuarial Journal (2001)* 5(2), p.41-53.

[47] Henderson, B., and Pearson, N., (2010) "The Dark Side of Financial Innovation: A Case Study of the Pricing of a Retail Financial Product" *Journal of Financial Economics (2010)* 100(2), p.227-247.

[48] Hernandez, R., Lee, W., and Liu, P., "An Economic Analysis of Reverse Exchangeable Securities: An Option-Pricing Approach" *Working Paper, University of Arkansas (2007)*

[49] Heston, S.L., "A Closed-form Solution for Options with Stochastic Volatility with Applications to Bonds and Currency Options" *The Review of Financial Studies (1993)* 6 (2), p.327-343.

[50] Hull, J., White, A., "CVA and wrong-way risk" *Financial Analysts Journal (2012)*,68, p.58-69.

[51] International Swaps and Derivatives Association, "ISDA Margin Survey (2008)" *Available at: http://www2.isda.org/functional-areas/research/surveys/margin-surveys/*

[52] Ito, K., "On stochastic differential equations" *Memoirs, American Mathematical Society (1951)*, 4, p.1-51.

[53] Lajeri-Chaherli, F., "A note on the valuation of compound options" *Journal of Futures Markets (2002)* 22, p.1103-1115.

[54] Ljungqvist, L. and Sargent, T., "Recursive Macroeconomic Theory (Third ed.)" *MIT Press (2012)*

[55] Marler, R.T., and Arora, J.S., "Survey of Multi-Objective Optimization Methods for Engineering" *Structural and Multidisciplinary Optimization (2004)* 26(6), p.369-395.

[56] McLeish, D.L., "Monte Carlo Simulation and Finance" *Wiley, 2004*

[57] McKeeman, W.M., "Algorithm 145: Adaptive numerical integration by Simpson's Rule" *Communications of the ACM (1962)* 5(12).

[58] Merton, R.C., "An Intertemporal Capital Asset Pricing Model" *Econometrica (1973)*, 41 (5), p.867-887.

[59] National Bank of Canada, "NBC Canadian Blue Chip III Deposit $Notes^{TM}$, Series 34" *Information statement dated March 7, 2012*

[60] Pareto, V., "Manuale di Economia Politica" *Societa Editrice Libraria, Milan (1906)*

[61] Pratt, J.W., "Risk aversion in the small and in the large" *Econometrica (1964)* 32, p.122-136.

[62] Pykhtin, M., "Modeling Credit Exposure for Collateralized Counterparties" *Journal of Credit Risk (2009)*, Vol. 5, No. 4, p.3-27

[63] Pykhtin, M., and Zhu, S., "A Guide to Modelling Counterparty Credit Risk" *GARP Risk Review,July-August 2007*

[64] Rubinstein, M., "Exotic options" *Finance Working chapter No. 220, (1991). Institute of Business and Economic Research, University of California at Berkeley.*

[65] Selby, M.J.P. and Stewart, D., "On the Evaluation of Compound Options" *Management Science (1987)*, Vol. 33, No. 3, p. 347-355.

[66] Shahid J., "Valuation of Structured Products:Pricing of Commodity Linked Notes" *Aarhus School of Business, University of Aarhus, February 2011*

[67] Sorensen, E. H., Bollier, T.F., "Pricing Swap Default Risk" *Financial Analysts Journal (2003)*, Vol.50, No.3, pp.23-33.

[68] Stokey, N., Lucas, R.E., and Prescott, E., "Recursive Methods in Economic Dynamics" *Harvard Univ. Press (1989)*

[69] Stoimenov, P. A., and Wilkens, S., "Are structured products <fairly> priced? An analysis of the German market for equity-linked instruments" *Journal of Banking and Finance (2005)* 29, p.2971-2993.

[70] Szymanowska, M., Ter Horst, J., and Veld, C., "Reverse convertible bonds analyzed" *Journal of Futures Markets (2009)* 29(10), p.895-919.

[71] Though (2012), *Available at: https://www.jpmorgan.com/tss/DocumentForEmail /DownloadtheFullPDF/1320503758248*

[72] Stadler, W., "Fundamentals of multicriteria optimization" *In: Stadler,W. (ed.) Multicriteria Optimization in Engineering and in the Sciences, New York: Plenum Press (1988)* p.1-25.

[73] Ueberhuber, C.W., "Numerical Computation 2: Methods, Software, and Analysis" *Springer-Verlag, Berlin, 1997*

[74] Wilkens, S., Erner, C., and Roder, K., "The pricing of structured products in Germany" *Journal of Derivatives (2003)* 11(1), p.55-69.

[75] Zepeda, R., "The ISDA Master Agreement 2012: A Missed Opportunity?" *Journal of International Banking Law and Regulation (2013)*, p.12-28.