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Essays on Dynamic Games Played Over Event Trees

Par

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Résumé

Cette thèse traite de la théorie des jeux dynamiques et stochastiques. Elle est constituée de trois essais, dont l'objet est le développement et l'application de cette théorie. Tout au long de la thèse, nous supposons que des joueurs interagissent dans le temps faisant face à des aléas représentés par un processus stochastique défini sur un arbre d'évènements, donné d'une manière exogène. Ainsi, la transition entre des nœuds successifs est l'œuvre de la nature et ne peut être influencée par les joueurs.

Dans le premier essai, intitulé "*S-Adapted Equilibria in Games Played Over Event Trees with Coupled Constraints*", nous analysons un jeu non coopératif à n joueurs où certains paramètres sont aléatoires et où les joueurs font faces à des contraintes couplées. Un tel cadre s'applique par exemple à un secteur industriel compétitif faisant face à une demande incertaine et à une régulation limitant les émissions sectorielles de produits polluants. Ainsi, bien que le jeu soit non-coopératif, il comporte quelques éléments de jeu coopératif dans la mesure où les joueurs doivent se coordonner pour satisfaire la contrainte commune à leur activité, et ce, durant chaque période et à chaque nœud de l'arbre d'évènement. La nature mixte d'un tel exercice implique des difficultés tant conceptuelles que computationnelles. Pour résoudre ce problème, nous utilisons le concept d'équilibre de Nash normalisé. L'apport principal de cet article est ainsi la caractérisation d'un tel équilibre pour la classe des jeux dynamiques joués sur un arbre d'évènements (JDAE).

Dans le deuxième essai, "*Incentive Equilibrium Strategies in Dynamic Games Played over Event Trees*", nous concevons des stratégies incitatives afin d'assurer la durabilité de la coopération à travers le temps. Nous caractérisons l'équilibre en stratégies incitatives pour la classe des JDAE, nous assurant ainsi que chaque joueur respectera sa part de l'accord à chaque nœud de l'arbre d'évènements. Nous démontrons que la solution coordonnée maximisant la somme des gains individuels est réalisable en tant qu'équilibre en stratégies incitatives. Nous prêtons une attention particulière à deux classes de jeux très usitées, les jeux dynamiques linéaire dans l'état et les jeux dynamiques qui sont linéaire-quadratiques.

Dans le troisième essai, "*Cost-Revenue Sharing in a Closed Loop Supply Chain Played over Event Trees*", nous appliquons les JDAE à une chaîne logistique formée d'un manufacturier et d'un détaillant. Nous analysons en particulier les stratégies de recyclage des produits usagés par le producteur. En effet, dans la mesure où l'on considère qu'il est intéressant de produire un bien en utilisant des produits recyclés, le manufacturier investit dans un programme environnemental afin d'encourager les consommateurs à retourner les produits usagés en fin de cycle de vie. Nous analysons et comparons deux scenarios. En premier lieu, le détaillant ne s'implique pas dans le programme. Dans le second scenario, ce dernier s'implique en assumant une partie des frais. En échange, le producteur est prêt à baisser son prix de vente en fonction du taux de retour des produits usagés. Les deux scénarios se jouent à la Stackelberg (le détaillant étant le meneur et le manufacturier le suiveur) avec une demande incertaine.

Mots clés: Jeux dynamiques, Arbre d'évènement, Incertitude, Équilibre de Nash normalisé, Contraintes joints, Stratégies incitatives, Coopération, Chaîne logistique, Contrôle de pollution.

Abstract

In this dissertation, presented in three essays, we develop and apply the theory of dynamic games played over event trees. In such games, the stochastic process is exogenously given as an event tree, that is, the transition from one node to another is nature's decision and cannot be influenced by the players' actions.

In the first essay, entitled "*$\mathcal{S}$-Adapted Equilibria in Games Played Over Event Trees with Coupled Constraints*", we consider a game where a set of players are engaged in a non-cooperative game over time, with some parameters being stochastic while the players face joint or coupling constraints. An example of such setting is an industry formed of a set of firms competing in a market described by a stochastic demand law, where a regulator is imposing a global cap on emissions of some pollutants by the industry. This setting presents some conceptual as well computational difficulties, which are due to mixed nature of the problem. Indeed, whereas the game is non-cooperative in its market competition aspect, it has a cooperative flavor as the players need to coordinate to satisfy the joint constraints at each period and each node of the event tree. The relevant solution concept in this context is the so-called normalized, or generalized Nash equilibrium. The main contribution of this essay is in the characterization of this equilibrium in the class of dynamic games played over event trees (DGPET).

The second essay addresses the main issue in cooperative dynamic games on how to sustain cooperation over time, that is, how to ensure that each player will indeed implement her part of the agreement as time goes by. In this article entitled "*Incentive Equilibrium Strategies in Dynamic Games Played over Event Trees*", we design incentive strategies to sustain cooperation. We characterize incentive equilibrium strategies and outcomes for the class of DGPET. We show that the coordinated solution that optimizes the joint payoff can be achieved as an incentive equilibrium, and therefore is self supporting. We focus on two popular classes of dynamic games in applications, namely, linear-state and linear-quadratic games.

In the third essay, entitled "*Cost-Revenue Sharing in a Closed Loop Supply Chain Played over Event Trees*", we consider a supply chain formed of one manufacturer and one retailer.

As producing with used parts is more efficient than producing with exclusively new material, the manufacturer invests in green activities (GA) to encourage consumers to bring back their used products at the end of their useful life. Two scenarios are analyzed and compared, namely, a scenario where the retailer is not involved in GA, and a second where the retailer pays part of the GA cost. In return, the manufacturer reduces the wholesale price by an amount that depends on the return of used products. Both games are played non-cooperatively à la Stackelberg, with the retailer acting as leader and the manufacturer as follower. Also, in both games, we assume that the demand is stochastic.

Keywords: Dynamic games, Event tree, Uncertainty, Normalized equilibrium, Coupled constraint, Incentive equilibria, Cooperation, Closed-Loop supply chain, Pollution control.

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[1]In this section, for reasons of simplicity, we have replaced (t, s^t) by the appropriate node number to show the position on the event tree.

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To: My parents, and Kambiz,
for their unconditional and endless love.
You mean the world to me.

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Here it is, the first page of the thesis but the last to be written and I'd wager, probably the most read. I feel truly blessed with this accomplishment in my life, and I would like to take this opportunity to thank all who made it possible.

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Chapter 1

Introduction

Many problems in management science, economics and engineering involve the following features:

1. Few agents (firms, countries, automata, etc.) having interdependent payoffs, that is, each agent's outcome does not depend only on her own decisions, but also on other agents' actions.

2. The payoffs depend on current as well as on previous decisions. For instance, a production decision is constrained by production capacity, which is the result of some previous investment decisions. This intertemporal aspect of both decisions and payoffs, requires to distinguish in the model between flow (control, action or decision) variables and state (or stock) variables. State variables allow to represent in an adequate way any cumulative processes such as brand reputation, pollution stocks, knowledge, etc., which greatly influence the decision-making process.

3. Some data is not known with certainty. For instance, future demand or cost parameters may vary with the (future) state of the economy or weather conditions. They may also depend on the availability or not of some new technologies.

The first two features invite quite naturally to use dynamic game theory as the methodological framework to model the problems at hand and predict the outcomes that the players may achieve. The three features together require to use stochastic dynamic games to properly account for the inherent uncertainties.

Throughout this thesis, we shall assume that the stochastic process describing the uncertain parameters can be captured by a discrete and finite event tree. This modeling approach is highly intuitive and practical as decision makers attempt to figure out future values following a tree representation of the uncertain quantity. For instance, we may think about future oil prices as being low, medium or high and put a probability on the occurrence of each of these values. Once the model is developed and an algorithm for solving it is available, we can vary at will these probabilities and what is meant by a low or a high price, and simulate the impact of these parameters on the equilibrium strategies and outcomes.

The idea of dynamic games played over event trees was initially put forward by Zaccour [1] and Haurie et al. [2] to determine the equilibrium quantities and prices in the European gas markets, where four producers compete à la Cournot in nine markets. These authors introduced the concept of $\mathcal{S}$-adapted strategies and equilibrium, meaning that decision variables (e.g., quantities, investments in production capacities) are indexed over the nodes of the event tree. Early contributions in this area include [3] and [4] where it was shown that this solution concept is related to the concept of stochastic variational inequality. Haurie and Zaccour in [5] provided a stochastic-control formulation of this class of games, characterized the $\mathcal{S}$-adapted equilibria through maximum principles, and established a link with the theory of open-loop multistage games; see, also [6]. Application wise, the formalism of games played over event trees has been used to study equilibria in energy markets in [7], [8], [9] and [10]. Recently, Reddy et al. in [11] and Parilina and Zaccour in [12] considered dynamic cooperative games played over event trees and defined a node-consistent Shapley value and node-consistent imputations in the core for this class of games.

This three-essay thesis uses and further develops the theory of dynamic games played over event trees (DGPET).

In the first essay, we are interested in characterizing non-cooperative equilibria in DGPET in the context where the players face coupled (or joint) constraints. A coupled constraint involves the decision variables of more than one player, thus players' action spaces become interdependent. Coupling constraints are quite natural in many applications in global environmental problem, energy markets, networks with common capacities, etc. For instance renewable portfolio standard policy which requires electricity supply companies to produce a specified fraction of their electricity from renewable (sustainable) energy sources such as wind, solar, biomass, marine and geothermal power involve coupled constraint. Renewable portfolio standards are legislated in 76 countries, states and provinces around the world including three Canadian provinces (40% by 2020 in both Nova Scotia and New Brunswick and 93% in British Columbia) [13].[1] Another example would be the problem of the firms (producers) in a given industry which must collectively reduce their pollutant emissions by a certain percentage. Government of Canada announces 2030 emissions target

[1] From `http://cansia.ca/`, accessed online on 2015-06.

to reduce its greenhouse gas (GHG) emissions by 30% below 2005 levels by 2030 [14].[2] Another example is when a group of countries, say the European Union, collectively negotiates in a first step its share of pollutant emissions reductions at a global level, and at a second step let the countries decide how to share the total burden.

The problem of dealing with coupled constraint is highly complex because even if the game is played non-cooperatively, the players need to somehow cooperate to enforce the constraints imposed by a regulator. Rosen in [15] dealt with the problem of determining coupled-constraint Nash equilibria in a static context and introduced the concept of normalized equilibrium, which is also known as generalized equilibrium in the literature; see, [16]. In a nutshell, Rosen shows that satisfying the coupled constraints can be achieved by introducing an appropriate weighting scheme that decentralizes the common Lagrange multipliers appended to these coupling constraints.

More specifically, the first essay entitled "*$\mathcal{S}$-Adapted Equilibria in Games Played Over Event Trees with Coupled Constraints*", deals with a policy coordination model where a supranational agent has to induce a set of countries competing on an oligopolistic market to achieve a common global constraint. A game of multiple players (countries) producing a homogeneous good is considered while there is an uncertain fluctuation in the price of nonrenewable resource, i.e., natural gas, commonly used by the agents in their production process. At the same time, the players must keep the total pollutant emissions less than a certain level. The concept of normalized equilibrium is used to solve the problem. Existence and uniqueness conditions for this equilibrium are provided, as well as a stochastic-control formulation of the game and a maximum principle. The problem is also solved through introducing penalty tax rates for the violation of the coupled constraint.

The motivation for the next part of this thesis arises from the widely observed fact that players (firms, union and management, countries, spouses, etc.) commit to long-term agreements. One interesting question is why economic and social agents sign long-term contracts, instead of keeping all of their options open by committing for only one period at the time? A first answer is that negotiation to reach an acceptable arrangement is costly in terms of dollars, time, emotions and feelings, etc. and therefore, it makes sense to avoid frequent renegotiation whenever this is feasible. Second, some problems are inherently dynamic. For instance, curbing polluting emissions in the industrial and transport sectors requires investments in cleaner technologies, changes in consumption habits, etc., which clearly cannot be achieved overnight. If the players have short-planning horizons when they perform their cost-benefit analysis, they may end up constantly postponing relevant decisions concerning the future, and nothing would ever be achieved. This explains

[2] From `http://ec.gc.ca/`, accessed online on 2015-06.

why the parties (countries, provinces, regions, etc.) typically seek long-term environmental agreements.

In this fashion, the players agree to cooperate over a certain period of time, i.e., the parties agree to coordinate their strategies in view of optimizing a collective performance index (profit, cost, welfare, happiness, etc.). Although coordination may induce some loss of freedom to the parties in terms of their choice of actions, its rationale stems, on balance, from the collective and individual gains it generates compared to noncooperation. In this setting, a common observation is that some cooperative programs are abandoned before reaching their maturity. In a dynamic setting, if an agreement breaks down before its intended end date, it is said to be time inconsistent. This means that some parties prefer, payoff-wise, to switch at an intermediate instant of time to non-cooperative mode of play, rather than sticking to the agreement. To put it differently, breakdowns of long-term agreements before their maturity will occur if, either all the parties agree, at an intermediate instant of time, to replace the initial agreement by a new one for the remaining periods, or if one of the players finds it (individually) rational to deviate, that is, to switch to her non-cooperative strategy from that time onward [17]. However, the interest in dealing with such instabilities is not in explaining why they may occur, but in attempting to design mechanisms, schemes, side payments, etc., that would help prevent breakdowns from taking place.

In the second essay of this thesis, we address sustainability in cooperative dynamic games, that is, how to ensure that each player will indeed implement her part of the agreement as time goes by. In the literature, one may find different options to tackle this problem. In some special structural formalism, the cooperative solution may be embodied with an equilibrium property. Hence, the rational players play their cooperative strategy since it is self supported and the issue of durability of the agreement issue is emptied. However, this situation rarely happens unless some special structure is assumed for the game under consideration; see [18], [19], and [20].

To endow the cooperative solution with an equilibrium property, one approach is to use trigger strategies that punish credibly and effectively any player who deviates from the agreement; see [21], [22] and [23]. Trigger strategies may embody large discontinuities, i.e., a slight deviation from an agreed path triggers harsh retaliation generating a very different path than the agreed one. One may also assume a binding agreement. This concept has been used in some early works in cooperative differential games such as in [17].

Another approach is to design a time-consistent cooperative agreement. A cooperative agreement is time consistent at initial date and state, if at any intermediate instant of time the cooperative payoff-to-go of each player dominates, at least weakly, her non-cooperative payoff-to-go; see [24], [25], [26],[27], [28]. To ensure time-consistency of the agreement, the cooperative and non-cooperative payoffs-to-go should be compared along the cooperative state trajectory. A stronger concept is agreeability, which requires the cooperative payoff-to-go to dominate the non-cooperative payoff-to-go along *any* state trajectory; see e.g., [29], [30], [31], [32]. For a survey of time consistency see [33].

In a two-player game setting, another possible option is to support the cooperative agreement by incentive strategies, which is the approach used in this essay; see [34], [35], [36], [37]. Informally, the incentive strategy of each player is a function of possible deviation of the other player with respect to the coordinated solution, i.e., the cooperative agreement. An incentive equilibrium has the property that when both players implement their incentive strategies, the cooperative outcome is realized as an equilibrium. On the contrary, if any player deviates from the coordinated solution, the other player also deviates from the coordinated solution and uses his incentive strategy instead. Therefore, no player should be tempted to deviate from the agreement during the course of the game, provided that incentive strategies are credible. An incentive strategy is credible if it is better for a player who has been cheated to use her strategy rather than sticking to the coordinated solution. Ehtamo and Hämäläinen in [36] and [38] used linear incentive strategies in a dynamic resource game and demonstrated that such strategies are credible when deviations are not too large.

More precisely, in the second essay entitled "*Incentive Equilibrium Strategies in Dynamic Games Played over Event Trees*", we characterize incentive equilibrium strategies and outcomes for the class of dynamic games played over event trees. We show that the coordinated solution that optimizes the joint payoff can be achieved as an incentive equilibrium. We focus on two popular classes of dynamic games in applications namely, linear-state and linear-quadratic games, as they admit closed-form solutions; see, e.g., the books by Engwerda [39] and Haurie et al. [6] and a survey of some applications in [40]. Martín-Herrán and Zaccour in [41] and [42] characterized incentive strategies and their credibility for linear-state and linear-quadratic dynamic games (LQDG), but in a deterministic setting.

In the third part of this thesis, we move slightly from the theoretical to an applied context through an application of dynamic game theory in Supply Chain Management (SCM). SCM was defined as the strategic coordination of the business processes within a company and across different units in the supply chain which includes both forward and backward activities [43]. A precise set of actions from the members of the chain are required to reach optimal performance while self serving focus of each member often results in poor performance. However, optimal performance is achievable through implementing some kind of contract which makes members' objective in line with the chain's objective. One may find a variety of these contracts in the literature, such as revenue sharing contract, lease contract, pay back, etc.; see [44] and [45].

Revenue Sharing Contract (RSC) was defined as a contract under which, in addition to a per unit wholesale price, a retailer pays the manufacturer a percentage of his revenues [46]. In this setting a precise information system is required to ensure the effectiveness of the contract and overcome the complexities of it's implementation and administration. Tirole [47] mentioned mitigation of the double-marginalization effect as the main strength of implementing RSC. The Reverse Revenue Sharing Contract (RRSC), a modified version of RSC, differs from the traditional RSC as the manufacturer is the one who transfers a part of his revenues to the retailer in order to affect his strategy. For some examples of this type of contract see [48], [49], [50] and [51].

Continuing the investigation of the games played over the event trees, in the third part of this thesis, we develop a dynamic game of Closed-Loop Supply Chain (CLSC) played over uncontrolled event trees through introducing a cost-revenue sharing (CRS) program along with a reverse revenue sharing contract (RRSC). In a CLSC forward and reverse activities are combined into a unique system to increase economic, environmental, and social performance [52]. The economic benefits of the CLSC lies in the cost reduction that results from producing by means of used components instead of only with new materials. But management of returned products is known as the most challenging aspect of reverse logistics [53].

Usually in a CLSC the manufacturer is the one who is interested in closing the loop as he appropriates the returns' residual value while other members of the supply chain are excluded from the benefits [54]. On the other hand, Savaskan et al. in [55] mentioned the retailer as the key member of the chain in creating the environmental knowledge and convincing the costumers to return the products. However clearly, the retailer does not find it in his preference to contribute in this process unless he is offered an attractive economic incentive by the manufacturer through an appropriate contract [56]. This reasoning implies that there is room for a two-way incentive scheme in a CLSC, i.e., sharing both revenues and costs. This is the line of thought pursued in this part of the thesis.

In the third essay of this thesis, entitled "*Cost-Revenue Sharing in a Closed Loop Supply Chain Played over Event Trees*", we add the flavor of uncertainty to the game of a CLSC consisting of a manufacturer and a retailer by assuming that the parameters of the model are not fixed over time and vary based on a predetermined event tree. We assume that the manufacturer can influence the return of used products by conducting some "green" activities (GA) such as advertising and communications campaigns about the recycling policies, logistics services, monetary and symbolic incentives, employees-training programs, etc. We characterize and compare strategies and outcomes in two non-cooperative scenarios played à la Stackelberg, where the retailer acts as leader and the manufacturer as follower. In the first scenario, which plays the role of a benchmark, the retailer does not participate financially in the GA program and the manufacturer does not offer any discount on the wholesale price. In the second scenario, the two members of the supply chain implements a cost-revenue sharing contract.

Each essay in this thesis is supported by a numerical example for more elaboration and to provide more insight about the problem under study.

Chapter 2

S-adapted Equilibria in Games Played Over Event Trees with Coupled Constraint[1]

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abstract

This article deals with the general theory of games that are played over uncontrolled event trees, i.e., games where the transition from one node to another is nature's decision and cannot be influenced by the players' actions. The solution concept for this class of games was introduced under the name of S-adapted equilibrium where S stands for sample of realizations of the random process. In this paper, it is assumed that the players also face a coupled constraint at each node. The concept of normalized equilibrium is used to solve the problem. Necessary conditions for optimality of the normalized S-adapted equilibrium

[1]This paper is accepted for publication by Journal of Optimization Theory and Applications.

are presented and the problem is also solved through introducing the penalty tax rates for the violation of the coupled constraint. Furthermore, a simple illustrative example in environmental economics is presented for more elaboration.

Keywords: Dynamic games, Event tree, Normalized equilibrium, Coupled constraint, Pollution control.

2.1 Introduction

This article deals with dynamic games played over uncontrolled event trees, that is, games where the transition from one node to another is nature's decision and cannot be influenced by the players' actions. Zaccour [1] and Haurie *et al.* [2] introduced S-adapted equilibrium as the solution to this class of games, and it was shown later in [3] and [4] that this solution concept is related to the concept of stochastic variational inequality. Haurie and Zaccour [5], see also [6], provided a stochastic-control formulation of this class of games, characterized the S-adapted equilibria through maximum principles, and established a link with the theory of open-loop multistage games. Haurie and Roche [57] compared S-adapted information structure with piecewise open-loop information structure. The formalism of games played over event trees has been used to study equilibria in energy markets in [7], [8], [9] and [10]. Recently, Reddy et al. in [11] considered dynamic cooperative games played over event trees and defined a time-consistent Shapley value for this class of games.

The main contribution of this paper is in extending, in a straightforward manner, the existence and uniqueness results of S-adapted equilibria to a setting where the players face coupled constraints, that is, their action spaces are interdependent. Coupling constraints are quite natural in many applications in, e.g., networks with common capacities, energy markets and global environmental problems. Rosen [15] dealt with the problem of determining coupled-constraint Nash equilibria in static context and introduced the concept of normalized equilibrium, which is also known as generalized equilibrium in the literature (see [16]). In a nutshell, Rosen shows that satisfying the coupled constraints can be achieved by introducing an appropriate weighting scheme that decentralizes the common Lagrange multipliers appended to these coupling constraints. The existence of normalized equilibrium in infinite-horizon dynamic games has been studied in [58], [59], and [60]. Carlson also extended the Rosen's idea to a Hilbert space setting in [61], and considered the same problem in [62], with the dynamics of the game being described by a set of control systems and allowing for more general spaces.

The rest of the paper is organized as follows: In Section 2.2, we recall the general theory of games played over event trees and extend the existence and uniqueness theorems of S-

adapted equilibrium to the case where the players face a coupled constraint. In Section 2.3, we extend the stochastic-control formulation to include a coupling constraint and characterize the S-adapted equilibrium through a maximum principle. In Section 2.4, we provide an illustrative example in environmental economics, and briefly conclude in Section 2.5.

2.2 Normal form, existence and uniqueness of equilibrium

In this section, we recall the main ingredients of dynamic games played over event trees.[2] Let $\mathcal{T} = \{0,\ 1,\ \ldots,\ T\}$ be the set periods, and denote by $(\xi(t) : t \in \mathcal{T})$ the exogenous stochastic process represented by an event tree. This tree has a root node n_0 in period 0 and has a set of nodes $\mathcal{N}^t$ in period $t = 0,\ 1,\ \ldots,\ T$. Each node $n^t \in \mathcal{N}^t$ represents a possible sample value of the history h^t of the $\xi(.)$ process up to time t. We introduce the following notations:

1. $a(n^t) \in \mathcal{N}^{t-1}$ is the unique predecessor of node $n^t \in \mathcal{N}^t$ on the event-tree graph for $t = 0,\ 1,\ \ldots,\ T$;
2. $S(n^t) \in \mathcal{N}^{t+1}$ is the set of all possible direct successors of node $n^t \in \mathcal{N}^t$ for $t = 0,\ 1,\ \ldots,\ T-1$;
3. A path from the root node n_0 to a terminal node n^T is called a *scenario*. Each scenario has a probability and the probabilities of all scenarios sum up to 1. We denote by $\pi(n^t)$ the probability of passing through node n^t, which corresponds to the sum of the probabilities of all scenarios that contain this node. In particular, $\pi(n_0) = 1$ and $\pi(n^T)$ is equal to the probability of the single scenario that terminates in (leaf) node $n^T \in \mathcal{N}^T$.

Let $M = \{1,\ \ldots,\ m\}$ be the set of players, and denote by $u_j^{n^t} \in U_j^{n^t} \subseteq R^{m_j}$ the decision variables of player j at node n^t, where $U_j^{n^t}$ is the control set. For each node $n^t,\ t = 1,\ \ldots .T$, we introduce a transition reward function for player $j \in M$

$$L_j^{n^t}\left(u^{n^t}, u^{a(n^t)}\right), \tag{2.1}$$

where $L_j^{n^t}(.,.)$ is assumed to be twice continuously differentiable.

[2]We heavily draw on Haurie et al. (2012) for the description of this class of games.

Remark 1. *The rewards depend on the decision made at antecedent node* $a\left(n^t\right)$ *in the preceding period, and on the decision made in the current period at node* n^t*. Therefore,* $L_j^{n^t}(.,.)$ *is a transition reward that should be associated with time period* $t - 1$ *and, hence, discounted by factor* β_j^{t-1} *if a discount factor* $\beta_j \in [0,\ 1]$ *were used by player* j*. However, we will not introduce discounting in our formalism, in order to keep the presentation as simple as possible.*

At each terminal node n^Ta terminal reward $\Phi_j^{n^T}(u^{n^T})$ is defined, and is also supposed to be twice continuously differentiable.

The control set $U_j^{n^t}$ of player j at node $n^t \in \mathcal{N}^t$, is described by the following two groups of "regular" constraints for each player, namely:

$$\begin{aligned} f_j^{n^t}\left(u_j^{n^t}\right) &\geq 0, \quad j = 1,\ \ldots, m, \\ g_j^{n^t}\left(u_j^{n^t}, u_j^{a\left(n^t\right)}\right) &\geq 0, \quad j = 1, \ldots, m. \end{aligned} \tag{2.2}$$

Additionally, we introduce a coupling constraint at each node, which is defined as a proper subset $U^{n^t}$of $U_1^{n^t} \times \cdots \times U_j^{n^t} \times \cdots \times U_m^{n^t}$ by a $K-$vector function h^{n^t} where $K \geq 1$, that is,

$$\begin{aligned} h^{n^t}\left(u_1^{n^t}, \ldots,\ u_j^{n^t},\ \ldots,\ u_m^{n^t}\right) \geq 0 & \\ h^{n^t} = h_k^{n^t}, \quad k = 1,\ \ldots, K. & \end{aligned} \tag{2.3}$$

We assume that f_j, g_j and h_k $(\forall j, k)$ are twice continuously differentiable mappings from Euclidean spaces to Euclidean spaces and are concave functions of $u_j^{n^t}$, for $j = 1,\ \ldots,\ m$. Further, we suppose that the constraint sets (2.2) and (2.3) lead to the set $\mathcal{K}$ with nonempty interior. Note that each players' decision $(u_j^{n^t})$ might be multidimensional, which would result in a multidimensional action space $(U_j^{n^t})$ for each player.

Remark 2. *The coupling constraints in (2.3) can be generalized to include ancestors of the current node* n^t*. Again, for the sake of simplicity, we restrict ourselves to the above simple formulation.*

To state the game in normal form, we define the set of admissible strategies and the payoffs in terms of these strategies. Next, we define the S-adapted Nash equilibrium.

An admissible S-adapted strategy under coupled constraint for player j is a vector $\gamma_j = \{u_j^{n^t} : n^t \in\ \mathcal{N}^t,\ t = 0,\ \ldots,\ T - 1\}$ that satisfies the constraint sets (2.2) and (2.3).

We call Γ_j the set of all admissible S-adapted strategies of player j. It follows from the concavity of the functions f_j , g_j and h_j that Γ_j is convex. Associated with an admissible S-adapted strategy vector $\gamma = \{\gamma_j\}_{j \in M}$ are the following payoffs:

$$V_j(\gamma) = \sum_{t=1,\ \dots,\ T-1} \sum_{n^t \in\ \mathcal{N}^t} \pi(n^t) L_j^{n^t} \left(u^{n^t}, u^{a\left(n^t\right)}\right) + \sum_{n^T \in\ \mathcal{N}^T} \pi^{n^T} \Phi_j^{n^T} \left(u^{n^T}\right).$$

An S-adapted equilibrium is an admissible S-adapted strategy vector γ^* such that

$$V_j\left(\left[\gamma_j, \gamma_{-j}^*\right]\right) \leq V_j(\gamma^*), \quad j = 1,\ \dots,\ m,$$

where

$$\left[\gamma_j, \gamma_{-j}^*\right] = (\gamma_1^*,\ \dots,\ \gamma_j,\ \dots,\ \gamma_m^*)$$

represents the unilateral deviation of player j.

In the above game, clearly the payoff function for the jth player, $V_j(\gamma) = V_j\left(\gamma_1, \dots, \gamma_i,\ \dots, \gamma_m\right)$, depends on the strategies of all players. To be able to use directly the results in Rosen [15] for concave games, we make the following assumption:

Assumption: For $\gamma_j \in \Gamma_j$, $V_j(\gamma)$ is continuous in γ and is concave in γ_j for each fixed value of $(\gamma_1, \dots,\ \gamma_{i-1},\ \gamma_{i+1},\ \dots,\ \gamma_m)$.

Assume that players are playing $\left[\gamma_j, \gamma_{-j}^*\right]$,which means that all players, except player j, use their equilibrium strategies γ_i^*, $\forall i \in M \backslash \{j\}$ while player j is using γ_j. So, we are dealing with a single agent optimization problem with concave objective function. If the constraint qualification conditions [63] are satisfied, then there exists a vector of multipliers

$$\mu_j = (\mu_{jk}), \quad k = 1,\ \dots,\ K, \tag{2.4}$$

and multipliers $\eta_j \geq 0$ and $\nu_j \geq 0$ such that the Lagrangian is given by

$$\begin{aligned}
\mathcal{L}_j\left(\mu_j, \eta_j, \nu_j, \left[\gamma_j, \gamma^*_{-j}\right]\right) =& L_j^{n_0}\left(u^{n_0}\right) + \sum_{t=1}^{T-1} \sum_{n^t \in\, \mathcal{N}^t} \pi\left(n^t\right) \left\{ \mathrm{L}_j^{n^t} \left([u_j^{n^t}, u_{-j}^{*n^t}], [u_j^{a\left(n^t\right)}, u_{-j}^{*a\left(n^t\right)}] \right) \right. \\
& + \mu_j\left(n^t\right) h^{n^t}\left(u_1^{*n^t}, \ldots,\ u_j^{n^t},\ \ldots,\ u_m^{*n^t}\right) + \eta_j\left(n^t\right) f_j^{n^t}\left(u_j^{n^t}\right) \\
& \left. + \nu_j(n^t) g_j^{n^t}\left(u_j^{n^t}, u_j^{a\left(n^t\right)}\right)\right\} \\
& + \sum_{n^t \in\, \mathcal{N}^T} \pi(n^T) \Big\{ \tau^i \Phi_j^{n^T}\left([u_j^{n^T}, u_{-j}^{*n^T}]\right) \\
& + \mu_j\left(n^T\right) h^{n^T}\left(u_1^{*n^T}, \ldots,\ u_j^{n^T},\ \ldots,\ u_m^{*n^T}\right) \\
& + \eta_j\left(n^T\right) f_j^{n^T}\left(u_j^{n^T}\right) + \nu_j(n^T) g_j^{n^T}\left(u_j^{n^T}, u_j^{a\left(n^T\right)}\right) \Big\},
\end{aligned} \tag{2.5}$$

and the first-order optimality conditions are as follows:

$$\begin{aligned}
& \frac{\partial}{\partial u_j} \mathcal{L}_j\left(\mu_j, \eta_j, \nu_j, \left[\gamma_j^*, \gamma_{-j}^*\right]\ \right) \leq 0, \\
& u_j^{n^t} \geq 0\, \mu_j \geq 0, \eta_j \geq 0, \nu_j \geq 0, \\
& h^{n^t}\left(u_1^{*n^t}, \ldots,\ u_j^{n^t},\ \ldots,\ u_m^{*n^t}\right) \geq 0, \quad f_j^{n^t}\left(u_j^{n^t}\right) \geq 0, \quad g_j^{n^t}\left(u_j^{n^t}, u_j^{a\left(n^t\right)}\right) \geq 0, \\
& \mu_j\left(n^t\right) h^{n^t}\left(u_1^{*n^t}, \ldots,\ u_j^{n^t},\ \ldots,\ u_m^{*n^t}\right) = \eta_j\left(n^t\right) f_j^{n^t}\left(u_j^{n^t}\right) = \nu_j\left(n^t\right) g_j^{n^t}\left(u_j^{n^t}, u_j^{a\left(n^t\right)}\right) = 0, \\
& u_j^{n^t} . \frac{\partial}{\partial u_j^{n^t}} \mathcal{L}_j\left(\mu_j, \eta_j, \nu_j, \left[\gamma_j^*, \gamma_{-j}^*\right]\ \right) = 0.
\end{aligned} \tag{2.6}$$

Generally, the values of the nonnegative multipliers $\mu_j^{n^t}$, $j = 1,\ \ldots,\ m$, given by the Karush-Kuhn-Tucker conditions at an equilibrium point will not be related to each other. If there exists a common vector $\mu^0\left(n^t\right)$ such that the vector of multipliers for the coupled constraint $\mu_j\left(n^t\right)$ has the form of

$$\mu_j\left(n^t\right) = \frac{\mu^0\left(n^t\right)}{r_j}, \tag{2.7}$$

for some $r_j > 0,\ j = 1,\ \ldots,\ m$, then we call the optimum point a normalized equilibrium.

Theorem 1. *Assume the functions* $L_j^{n^t}\left(u^{n^t}, u^{a\left(n^t\right)}\right)$ *and* $g_j^{n^t}\left(u_j^{n^t}, u_j^{a\left(n^t\right)}\right)$ *are concave in* $\left(u_j^{n^t}, u_j^{a\left(n^t\right)}\right)$, *assume the functions* $f_j^{n^t}\left(u_j^{n^t}\right)$ *and* $h_j^{n^t}\left(u_1^{n^t}, \ldots,\ u_j^{n^t},\ \ldots,\ u_m^{n^t}\right)$ *are concave* *in* $u_j^{n^t}, \forall j$*, and assume the functions* $\Phi_j^{n^T}\left(u_j^{n^T}\right)$ *are concave in* $u_j^{n^T}$*. Assume that the set of admissible strategies is compact. Then for any vector* $r > 0$*, there exists a normalized equilibrium.*

Proof. The result follows from Rosen [15], which is based on Kakutani's fixed point theorem. $\square$

To characterize the conditions under which the normalized S-adapted equilibrium is unique, we define the pseudo-gradients

$$\begin{aligned}
\mathcal{G}^{n^t}(u^{n^t}) &= \left(r_1 \frac{\partial L_1^{n^t}\left(u^{n^t}\right)}{\partial u_1^{n^t}}, \ldots,\ r_j \frac{\partial L_j^{n^t}\left(u^{n^t}\right)}{\partial u_j^{n^t}}, \ldots,\ r_m \frac{\partial L_m^{n^t}\left(u^{n^t}\right)}{\partial u_m^{n^t}}\right), \qquad t = 1,\ \ldots,\ T-1, \\
\mathcal{G}^{n^T}(u^{n^T}) &= \left(r_1 \frac{\Phi_1^{n^T}\left(u^{n^T}\right)}{\partial u_1^{n^T}}, \ldots,\ r_j \frac{\Phi_j^{n^T}\left(u^{n^T}\right)}{\partial u_j^{n^T}}, \ldots\ r_m, \frac{\Phi_m^{n^T}\left(u^{n^T}\right)}{\partial u_m^{n^T}}\right),
\end{aligned}$$

and the Jacobian matrices, which are defined for any fixed $r > 0$ as follows:

$$\begin{aligned}
\mathcal{J}^{n^t}\left(u^{n^t}\right) &= \frac{\partial \mathcal{G}^{n^t}\left(u^{n^t}\right)}{\partial u^{n^t}}, \qquad t = 1,\ \ldots,\ T-1, \\
\mathcal{J}^{n^T}\left(u^{n^T}\right) &= \frac{\partial \mathcal{G}^{n^T}(u^{n^T})}{\partial u^{n^T}}.
\end{aligned}$$

Theorem 2. *If for all* u^{n^t}*, the matrices*

$$\begin{aligned}
\mathcal{Q}^{n^t}\left(u^{n^t}\right) &= \frac{1}{2}\left[\mathcal{J}^{n^t}\left(u^{n^t}\right) + (\mathcal{J}^{n^t}\left(u^{n^t}\right))'\right], \\
\mathcal{Q}^{n^T}\left(u^{n^T}\right) &= \frac{1}{2}\left[\mathcal{J}^{n^T}\left(u^{n^T}\right) + (\mathcal{J}^{n^T}\left(u^{n^T}\right))'\right]
\end{aligned}$$

are negative definite, then the normalized equilibrium is unique.

Proof. Based on Theorem 6 in [15], negative definiteness of $\mathcal{Q}^{n^t}\left(u^{n^t}\right)$ and $\mathcal{Q}^{n^T}\left(u^{n^T}\right)$ implies strict diagonal concavity[3] of the function $\sum_{j=1}^{m} r_j V_j\left(\gamma\right)$, in which $r_j > 0,\ j = 1,\ \ldots,\ m$. This implies uniqueness of normalized equilibrium. $\square$

[3]The function $\sum_{j=1}^{m} r_j V_j\left(\gamma\right)$ is strictly diagonal concave if the Jacobian of the pseudo-gradient, for the given game, is negative definite [15].

2.3 Stochastic-control formulation

In this section, we formulate as a stochastic-control problem the game played over event tree with coupling constraints. This formulation is useful for modeling non-cooperative multistage games, where it is natural to distinguish between flow (input or control) variables and stock (output or state) variables.

Let $X \subseteq R^q$, with q a given positive integer, be a state set. For each node $n^t \in \mathcal{N}^t$, $t = 0, \ldots, T$, define the control set U^{n^t} as a proper subset of $U_1^{n^t} \times \ldots \times U_j^{n^t} \times \ldots \times U_m^{n^t}$ defined by the coupled constraint given by the $K-$vector,

$$h^{n^t}\left(u_1^{n^t}, \ldots, u_j^{n^t}, \ldots, u_m^{n^t}\right) \geq 0. \tag{2.8}$$

This constraint implies that each player's strategy space may depend on the strategy of the other players. A transition function $f^{n^t}(.,.) : X \times U^{n^t} \to X \subseteq R^q$ is associated with each node n^t, and the state equations are given by

$$x\left(n^t\right) = f^{a\left(n^t\right)}\left(x\left(a\left(n^t\right)\right), u\left(a\left(n^t\right)\right)\right), \tag{2.9}$$

$$u\left(a\left(n^t\right)\right) \in U^{a\left(n^t\right)}, \quad n^t \in \mathcal{N}^t, t = 1, \ldots, T. \tag{2.10}$$

At each node $n^t, t = 0, \ldots, T-1$, the reward to player j is a function of the state and of the controls of all players, given by $\mathrm{L}_j^{n^t}(x\left(n^t\right), u\left(n^t\right))$. At a terminal node n^T, the reward to Player j is given by the function $\Phi_j^{n^T}(x\left(n^T\right))$. Now, we can define the following multistage game where $x = \{x\left(n^t\right) : n^t \in \mathcal{N}^t, t = 0, \ldots, T\}$ and $u = \{u\left(n^t\right) : n^t \in \mathcal{N}^t, t = 0, \ldots, T-1\}$ and $J_j\left(x, u\right)$ is the payoff to player j:

$$J_j\left(x, u\right) = \max_{u_j\left(n^t\right)} \sum_{t=0}^{T-1} \sum_{n^t \in \mathcal{N}^t} \pi\left(n^t\right) \mathrm{L}_j^{n^t}\left(x\left(n^t\right), u\left(n^t\right)\right) + \sum_{n^t \in \mathcal{N}^t} \pi\left(n^T\right) \Phi_j^{n^T}\left(x\left(n^T\right)\right), \quad j \in M, \tag{2.11}$$

subject to (2.8), (2.9) and (2.10).

As for open-loop multistage games, we can formulate the necessary conditions for an S-adapted equilibrium, in the form of a maximum principle. For each player j, we form the Lagrangian

$$
\begin{aligned}
&\mathcal{L}_j\left(\lambda_j,\mu_j,x,u\right)=L_j^{n_0}\left(x^{n_0},u^{n_0}\right)+\sum_{t=1}^{T-1}\sum_{n^t\in\mathcal{N}^t}\pi\left(n^t\right)\Big\{\tau^i L_j^{n^t}\left(x\left(n^t\right),u\left(n^t\right)\right)\\
&+\lambda_j\left(n^t\right)\left(f^{a\left(n^t\right)}\left(x\left(a\left(n^t\right)\right),u\left(a\left(n^t\right)\right)\right)-x\left(n^t\right)\right)+\mu_j\left(n^t\right)h^{n^t}\left(u_1^{n^t},\ldots,u_j^{n^t},\ldots,u_m^{n^t}\right)\Big\}\\
&+\sum_{n^t\in\mathcal{N}^T}\pi(n^T)\{\tau^i\Phi_j^{n^T}\left(x\left(n^T\right)\right)+\lambda_j\left(n^T\right)\left(f^{a\left(n^T\right)}\left(x\left(a\left(n^T\right)\right),u\left(a\left(n^T\right)\right)\right)-x\left(n^T\right)\right)\\
&+\mu_j\left(n^T\right)h^{n^T}\left(u_1^{n^T},\ldots,u_j^{n^T},\ldots,u_m^{n^T}\right)\},
\end{aligned}
\tag{2.12}
$$

where $\lambda_j(n^t)$ is the costate variable of the same dimension of x (denoted by q), and $\mu_j(n^t)$ is the Lagrange multiplier of player j associated with the coupled constraint with dimension K, with both being indexed over the event tree.

Theorem 3. If $(x^*(n^t),u^*(n^t))$ is a Nash equilibrium, and if the convex sets X^q and U^{n^t} have nonempty interiors, then there exists a real number $\tau^j\geq 0$, an element $\lambda_j(n^t)\geq 0$ and $\mu_j\left(n^t\right)\geq 0$ for $j=1,\ldots,m$, not all zero, satisfying

$$
\begin{aligned}
-\lambda_j(n^t)=&\sum_{t=1}^{T-1}\sum_{n^t\in\mathcal{N}^t}\pi\left(n^t\right)\tau^j\partial_x L_j^{n^t}\left(x\left(n^t\right),u\left(n^t\right)\right)+\sum_{n^t\in\mathcal{N}^T}\pi(n^T)\tau^j\partial_x\Phi_j^{n^T}\left(x\left(n^T\right)\right)\\
&+\sum_{t=1}^{T-1}\sum_{n^t\in\mathcal{N}^t}\pi\left(n^t\right)\lambda_j\left(n^t\right)\left(\partial_x f^{a\left(n^t\right)}\left(x\left(a\left(n^t\right)\right),u\left(a\left(n^t\right)\right)\right)-1\right)\\
&+\sum_{n^t\in\mathcal{N}^T}\pi(n^T)\lambda_j\left(n^T\right)\left(\partial_x f^{a\left(n^T\right)}\left(x\left(a\left(n^T\right)\right),u\left(a\left(n^T\right)\right)\right)-1\right),
\end{aligned}
$$

$$
\mu_j\left(n^t\right)h_j^{n^t}\left(u_1^{n^t},\ldots,u_j^{n^t},\ldots,u_m^{n^t}\right)=0,
$$

$$
\begin{aligned}
&\sum_{t=1}^{T-1}\sum_{n^t\in\mathcal{N}^t}\pi\left(n^t\right)\{\tau^j\partial_{u_j}L_j^{n^t}\left(x\left(n^t\right),u\left(n^t\right)\right)+\lambda_j\left(n^t\right)\partial_{u_j}f^{a\left(n^t\right)}\left(x\left(a\left(n^t\right)\right),u\left(a\left(n^t\right)\right)\right)\\
&+\mu_j\left(n^t\right)\partial_{u_j}h^{n^t}\left(u_1^{n^t},\ldots,u_j^{n^t},\ldots,u_m^{n^t}\right)\}+\sum_{n^t\in\mathcal{N}^t}\pi\left(n^T\right)\{\tau^j\Phi_j^{n^T}\left(x\left(n^T\right)\right)\\
&+\lambda_j\left(n^T\right)\partial_{u_j}f^{a\left(n^T\right)}\left(x\left(a\left(n^T\right)\right),u\left(a\left(n^T\right)\right)\right)+\mu_j\left(n^T\right)\partial_{u_j}h^{n^T}\left(u_1^{n^T},\ldots,u_j^{n^T},\ldots,u_m^{n^T}\right)\}=0.
\end{aligned}
$$

Proof. See Theorem 1 in [64] or in [62]. □

The above necessary conditions introduce a set of Lagrange multipliers (μ_j) for each player with no relationship between the multipliers of the different players. Intuitively, to extend the idea of normalized equilibrium à la Rosen [15] to games played over event trees, we need to define a relationship between the set of Lagrange multipliers of the different players.

Let us start by defining the combined payoff function $L_r^{n^t} : X^{n^t} \times U^{n^t} \times U^{n^t} \to R$ at node n^t and $\Phi_r^{n^T} : X^{n^T} \to R$ at node n^T using a set of positive weights $r_j \geq 0$, $j = 1, \dots, m$ as

$$L_r^{n^t}\left(x\left(n^t\right), u\left(n^t\right), v\left(n^t\right)\right) = \sum_{j=1}^{m} r_j L_j^{n^t}\left(x\left(n^t\right), v_{-j}\left(n^t\right), u_j\left(n^t\right)\right),$$

$$\Phi_r^{n^T}\left(x\left(n^T\right)\right) = \sum_{j=1}^{m} r_j \Phi_j^{n^T}\left(x\left(n^T\right)\right).$$

The optimization is performed subject to (2.9) and (2.10). Note that in the above formulation, the payoff function of player j, ($L_j^{n^t}$ or $\Phi_j^{n^T}$), depends on the jth element of vector $u \in U^{n^t}$, and on the fixed controls for all other players $v \in U^{n^t}$.

Theorem 4. If $(x^*(n^t), u^*(n^t), v^*(n^t))$ is a Nash equilibrium, and if the convex sets X^q and U^{n^t} have nonempty interiors, there exists a real number $\tau \geq 0$, an element $\lambda(n^t) \geq 0$ with dimension q and $\mu\left(n^t\right) \geq 0$, not all zero, satisfying

$$\begin{aligned}
-\lambda(n^t) = & \sum_{t=1}^{T-1} \sum_{n^t \in \mathcal{N}^t} \pi(n^t) \tau \partial_x L_r^{n^t}\left(x\left(n^t\right), u\left(n^t\right), v\left(n^t\right)\right) + \sum_{n^T \in \mathcal{N}^T} \pi(n^T) \tau \partial_x \Phi_r^{n^T}\left(x\left(n^T\right)\right) \\
& + \sum_{t=1}^{T-1} \sum_{n^t \in \mathcal{N}^t} \pi\left(n^t\right) \lambda\left(n^t\right) \left(\partial_x f^{a\left(n^t\right)}\left(x\left(a\left(n^t\right)\right), u\left(a\left(n^t\right)\right)\right) - 1\right) \\
& + \sum_{n^T \in \mathcal{N}^T} \pi(n^T) \lambda\left(n^T\right) \left(\partial_x f^{a\left(n^T\right)}\left(x\left(a\left(n^T\right)\right), u\left(a\left(n^T\right)\right)\right) - 1\right),
\end{aligned}$$

$$\mu\left(n^t\right) h^{n^t}\left(u_1^{n^t}, \dots, u_j^{n^t}, \dots, u_m^{n^t}\right) = 0,$$

$$\begin{aligned}
& \sum_{t=1}^{T-1} \sum_{n^t \in \mathcal{N}^t} \pi\left(n^t\right) \left\{ \tau \partial_u L_r^{n^t}\left(x\left(n^t\right), u\left(n^t\right), v\left(n^t\right)\right) + \lambda\left(n^t\right) \partial_u f^{a\left(n^t\right)}\left(x\left(a\left(n^t\right)\right), u\left(a\left(n^t\right)\right)\right) \right. \\
& \left. + \mu\left(n^t\right) \partial_u h^{n^t}\left(u_1^{n^t}, \dots, u_j^{n^t}, \dots, u_m^{n^t}\right) \right\} + \sum_{n^T \in \mathcal{N}^T} \pi\left(n^T\right) \left\{ \tau \partial_u \Phi_r^{n^T}\left(x\left(n^T\right)\right) \right. \\
& \left. + \lambda\left(n^T\right) \partial_u f^{a\left(n^T\right)}\left(x\left(a\left(n^T\right)\right), u\left(a\left(n^T\right)\right)\right) + \mu\left(n^T\right) \partial_u h^{n^T}\left(u_1^{n^T}, \dots, u_j^{n^T}, \dots, u_m^{n^T}\right) \right\} = 0,
\end{aligned}$$

with h^{n^t} and $\mu\left(n^t\right)$ being K-vectors.

Proof. See Theorem 3 in [62]. $\square$

The objective function is now defined as the weighted sum of the m-player objective functions, that is, $L_r^{n^t}$ (it was $L_j^{n^t}$ in Theorem 3). Using the following coordinates:

$$
\begin{aligned}
-\lambda_k(n^t) = & \sum_{t=1}^{T-1} \sum_{n^t\in\mathcal{N}^t} \pi\left(n^t\right)\tau r_k \partial_{x_k} L_k^{n^t}\left(x\left(n^t\right), u\left(n^t\right)\right) + \sum_{n^t\in\mathcal{N}^t} \pi(n^T)\tau r_k \partial_{x_k}\Phi_j^{n^T}\left(x\left(n^T\right)\right) \\
& + \sum_{t=1}^{T-1} \sum_{n^t\in\mathcal{N}^t} \pi\left(n^t\right)\lambda_k\left(n^t\right)\left(\partial_{x_k} f^{a\left(n^t\right)}\left(x\left(a\left(n^t\right)\right), u\left(a\left(n^t\right)\right)\right) - 1\right) \\
& + \sum_{n^t\in\mathcal{N}^t} \pi(n^T)\lambda_k\left(n^T\right)\left(\partial_{x_k} f^{a\left(n^T\right)}\left(x\left(a\left(n^T\right)\right), u\left(a\left(n^T\right)\right)\right) - 1\right),
\end{aligned}
$$

$$
\mu\left(n^t\right) h^{n^t}\left(u_1^{n^t}, \ldots, u_j^{n^t}, \ldots, u_m^{n^t}\right) = 0,
$$

$$
\begin{aligned}
& \sum_{t=1}^{T-1} \sum_{n^t\in\mathcal{N}^t} \pi\left(n^t\right)\left\{\tau r_j \partial_{u_j} L_j^{n^t}\left(x\left(n^t\right), u\left(n^t\right)\right) + \lambda_k\left(n^t\right)\partial_{u_j} f^{a\left(n^t\right)}\left(x\left(a\left(n^t\right)\right), u\left(a\left(n^t\right)\right)\right)\right. \\
& \left. + \mu\left(n^t\right)\partial_{u_j} h^{n^t}\left(u_1^{n^t}, \ldots, u_j^{n^t}, \ldots, u_m^{n^t}\right)\right\} + \sum_{n^t\in\mathcal{N}^t} \pi\left(n^T\right)\left\{\tau r_j \partial_{u_j}\Phi_j^{n^T}\left(x\left(n^T\right)\right)\right. \\
& \left. + \lambda\left(n^T\right)\partial_{u_j} f^{a\left(n^T\right)}\left(x\left(a\left(n^T\right)\right), u\left(a\left(n^T\right)\right)\right) + \mu\left(n^T\right)\partial_{u_j} h^{n^T}\left(u_1^{n^T}, \ldots, u_j^{n^T}, \ldots, u_m^{n^T}\right)\right\} = 0,
\end{aligned}
$$

allows us to construct the normalized equilibrium. Indeed, dividing the second group of the above equations by $r_j \geq 0$, we have the set of multipliers $\mu_j = \frac{\mu}{r_j}, j = 1, \ldots, m$, which satisfy the conditions in Theorem 3. Now, the relationship between the Karush-Kuhn-Tucker multipliers, $\mu_j(n^t)$, shows that $\left(x^*\left(n^t\right), u^*\left(n^t\right)\right)$ defined in Theorem 4 is a normalized equilibrium. Once we obtain the value of the multiplier $\mu\left(n^t\right)$ associated with the coupled constraint at node $n^t \in \mathcal{N}^t, t = 1, \ldots, T$, the determination of an S-adapted Rosen equilibrium can be achieved by solving the decoupled game defined as follows:

$$
\begin{aligned}
J_j\left(x, u\right) = & \sum_{t=0}^{T-1} \sum_{n^t\in\mathcal{N}^t} \pi\left(n^t\right)\left\{L_j^{n^t}\left(x\left(n^t\right), u\left(n^t\right)\right) - \frac{\mu\left(n^t\right)}{r_j} h^{n^t}\left(u_1^{n^t}, \ldots, u_j^{n^t}, \ldots, u_m^{n^t}\right)\right\} \\
& + \sum_{n^t\in\mathcal{N}^t} \pi\left(n^T\right)\left\{\Phi_j^{n^T}\left(x\left(n^T\right)\right) - \frac{\mu\left(n^T\right)}{r_j} h^{n^T}\left(u_1^{n^T}, \ldots, u_j^{n^T}, \ldots, u_m^{n^T}\right)\right\}, \quad j \in M
\end{aligned}
$$

subject to (2.9) and (2.10). Observe that the additional term in player j's objective function, namely, $\frac{\mu(n^t)}{r_j} h^{n^t}\left(u_1^{n^t}, \ldots, u_j^{n^t}, \ldots, u_m^{n^t}\right)$, plays the role of a penalty term. Thus, the normalized equilibrium of the game in which the players satisfy the coupled constraints can be implemented by a taxation scheme. This is done by setting the taxes at the appropriate levels, i.e., by choosing them equal to the common Karush-Kuhn-Tucker multipliers while adjusted for each player j by his weighting factor $1/r_j$. To wrap up, solving for a normalized equilibrium requires the determination of the solution of the combined optimization problem, and next to solving, for a given vector of weights $r = (r_1, \ldots, r_m)$, the above uncoupled m-player game. Similar approach (enforcement through taxation) has been used widely in the literature. In [65], the regulator calculates Rosen coupled-constraint equilibrium (normalized Nash), and then uses the coupled-constraint Lagrange multiplier to formulate a threat, under which the agents will play a decoupled Nash game. Carlson and Haurie [58] also used the equilibrium of the relaxed game as a penalizing term for the coupled constraint in the main game. See [66, 67, 68] and [69] for more applications of normalized equilibrium to define penalty tax in environmental games.

2.4 Illustrative example

Coupled constraint games are well suited to study pollution control problems, where there is a cap on the total pollution emitted by firms in an industry. This problem involves a coupled constraint set in the combined strategy space of all agents. To implement a normalized equilibrium, the assumption is that there is an external authority (e.g., a regulator) that can select a particular vector of weights $r = (r_1, \ldots, r_m)$. In international pollution control, such an authority does not exist and the players (countries) need then to agree on a particular vector of weights. In this sense, [66] stated that a game with coupled constraints cannot be considered as a purely non-cooperative game.

The normalized equilibrium concept to deal with environmental games has been first recognized by Haurie [67] and further explored by Haurie and Zaccour [66], Haurie and Krawczyk [68] and Krawczyk [65], [70]. Tidball and Zaccour in [71] and [72] studied a game with environmental constraint in a static and dynamic context, respectively. Conceptually similar studies have been explored in [69], [73], [74], [75], [76], and [77]. The common feature of these studies is that they all deal with games in which the competitive agents maximize their utility functions subject to coupled constraints, which define their joint strategy space.

To illustrate the coupled S-adapted equilibrium, we consider an oligopoly with three players competing à la Cournot in a four-period game. Players must jointly obey a given environmental standard. Denote by $q_j(t,\ s^t)$ and $I_j\left(t,\ s^t\right)$ player j's rate of production and investment decisions in production capacity, respectively, $j=1,2,3$ at stage t, where s^t is the sample value at t of a random perturbation of the demand. Similarly, denote by $e_j(t,\ s^t)$ the rate of emissions resulting from the production of each player. For simplicity, we assume that

$$q_j\left(t,\ s^t\right) = e_j\left(t,\ s^t\right).$$

The stochastic inverse demand law in the market is given by

$$p\left(t,\ s^t\right) = P\left(q_1\left(t,\ s^t\right) + q_2\left(t,\ s^t\right) + q_3\left(t,\ s^t\right),\ s^t\right), \tag{2.13}$$

where $p\left(t,\ s^t\right)$ is the market price at stage t for the realization s^t of the random perturbation. The function $P\left(.,.\right)$ is assumed to be affine and decreasing in total output $Q\left(t,\ s^t\right) = \sum_{j=1}^{3} q_j\left(t,\ s^t\right)$, and the random perturbation is described by an event tree.

Denote by $K_j(t,\ s^t)$ player j's production capacity and by $S(t,\ s^t)$ the pollution stock at node $s^t, s^t \in \mathcal{N}^t, t = 0,\ 1,\ 2,\ 3$. The evolution of these state variables is described by the following difference equations:

$$\begin{aligned} K_j\left(t,\ s^t\right) &= \left(1-\mu\right) K_j\left(t-1,\ a(s^t)\right) + I_j(t-1,\ a(s^t)), & K_j\left(0,\ s^0\right) = K_j^0, \\ S\left(t,\ s^t\right) &= (1-\delta) S\left(t-1,\ a\left(s^t\right)\right) + \sum_{j=1}^{3} e_j\left(t,\ s^t\right), & S\left(0,\ s^0\right) = S^0, \end{aligned}$$

where $\mu > 0$ is the depreciation rate of production capacity, and $\delta > 0$ is nature's absorption rate of pollution.

Denote by $C_j\left(q_j\left(t,\ s^t\right)\right)$ the production cost function, and by $F_j(I_j\left(t,\ s^t\right))$ the investment cost function of player i. We suppose that both functions are convex, increasing and twice continuously differentiable. Assuming profit-maximization behavior, the objective function of player $j \in M$ is given by

$$\begin{aligned} J_j\left(S, K, q, I\right) = &\sum_{t=0}^{T-1} \sum_{n^t \in\ \mathcal{N}^t} \pi\left(n^t\right) \left[p\left(t,\ s^t\right) q_j\left(t,\ s^t\right) - C_j\left(q_j\left(t,\ s^t\right)\right) - F_j(I_j\left(t,\ s^t\right))\right] \\ &+ \sum_{n^T \in\ \mathcal{N}^T} \pi\left(n^T\right) \{\Phi_j\left(K_j\left(n^T\right)\right)\}, \end{aligned} \tag{2.14}$$

where

$$K = \left(K_1, K_2, K_3\right), \quad q = \left(q_1, q_2, q_3\right), \quad I = \left(I_1, I_2, I_3\right),$$

and $\Phi_j^{n^T}\left(K_j\left(n^T\right)\right)$ is the salvage value, assumed to be concave, increasing and twice continuously differentiable. The constraints are given by

$$\begin{aligned}
\text{Pollution accumulation constraint} &: S\left(t,\ s^t\right) \leq \overline{S}, && s^t \in \mathcal{N}^t,\ t = 0, ..., 3 && (2.15)\\
\text{Capacity constraint} &: q_j\left(t,\ s^t\right) \leq K_j\left(t,\ s^t\right), && s^t \in \mathcal{N}^t,\ t = 0, ..., 3 && (2.16)\\
\text{Non-negativity of production} &: q_j\left(t,\ s^t\right) \geq 0, && s^t \in \mathcal{N}^t,\ t = 0, ..., 3 && (2.17)\\
\text{Non-negativity of investment} &: I_j\left(t,\ s^t\right) \geq 0, && s^t \in \mathcal{N}^t,\ t = 0, 1, 2 && (2.18)
\end{aligned}$$

where $\overline{S}$ is a positive parameter corresponding to the maximum accumulated pollution tolerated by a regulating agency. Note that the coupling constraint in (2.15) must be satisfied at each node of the event tree.

To illustrate the S-adapted equilibrium concept with coupling constraints, we retain the following parameter values and functional forms:

$$\begin{aligned}
\overline{S} &= 2, \quad S^0 = 0, \quad \mu = 0.3, \quad \delta = 0.2,\\
K_1^0 &= 0.5, \quad K_2^0 = 0.3, \quad K_3^0 = 0.1,\\
C_1\left(q_1\right) &= 3q_1^2, \quad C_2\left(q_2\right) = 7q_2^2, \quad C_3\left(q_2\right) = 11q_3^2,\\
F_1\left(I_1\right) &= 8I_1^2, \quad F_2\left(I_2\right) = 5I_2^2, \quad F_3\left(I_2\right) = 3I_3^2,\\
\Phi_1\left(K_1\left(n^T\right)\right) &= \frac{7}{2}K_1^2\left(n^T\right), \quad \Phi_2\left(K_2\left(n^T\right)\right) = 5K_2^2\left(n^T\right), \quad \Phi_3\left(K_3\left(n^T\right)\right) = 4K_3^2\left(n^T\right).
\end{aligned}$$

The stochastic inverse demand law introduced in (2.13) is specified as follows:

$$p\left(t, s^t\right) = a\left(t, s^t\right) - b\left(q_1\left(t, s^t\right) + q_2\left(t, s^t\right) + q_3\left(t, s^t\right)\right),$$

i.e., assumed to be linear in its parameters, with fixed slope ($b = 10$) and stochastic intercept. In each node of the event tree, $a\left(t, s^t\right)$ goes up or down by 40% with given probabilities and $a\left(0\right) = 25$. The random demand laws in the market are given as the event tree in Figure 2.1.

We solve the equilibrium problem in three different scenarios, namely:

Benchmark scenario: The players choose production and investment levels without considering the pollution cap (2.15). The results are given in Table 2.1.

Equal weights: The players must satisfy the coupling constraint and are given equal weights by the regulator $r = (1/3, 1/3, 1/3)$. The results are provided in Table 2.2.

Different weights: The players must satisfy the coupling constraint and are given unequal

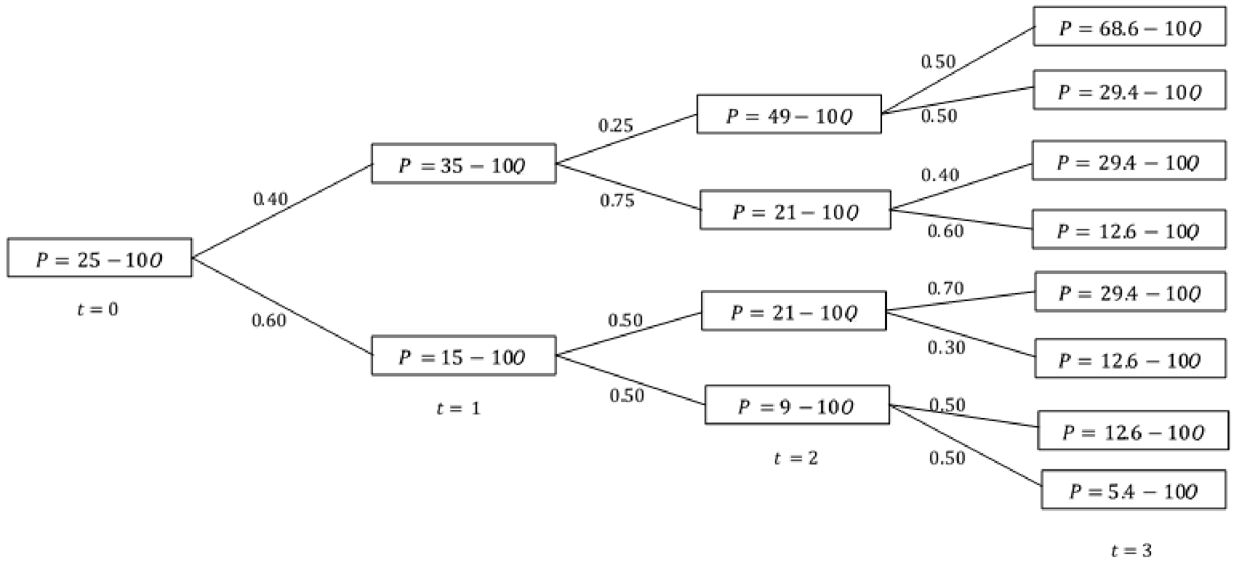

Figure 2.1: Random demand laws in the market

weights by the regulator $r = (0.58, 0.30, 0.12)$. The results are exhibited in Table 2.3. Here the weights are calculated based on each player's relative contribution to the overall pollution level in the benchmark scenario, i.e., r_j corresponds to the proportion of weighted average (by the probabilities) of pollution stock produced by player j to the total weighted average of pollution level (produced by all the players) in the benchmark scenario.

The reported numerical experiments were conducted using different modules in MATLAB programming environment. More technically, using MATLAB symbolic toolbox, the associated complementarity problem has been defined. Finally, LCP and Lemke functions are used to derive the numerical results (See [78]). To illustrate, we provide in the Appendix the details of the linear-complementarity problem for one player. Table 2.4 gives the contribution of the players to the pollution stock at all final nodes, and Table 2.5 summarizes the expected payoffs at the same terminal nodes.

The numerical results can be summarized as follows:

1. The main contributor to the stock of pollution is Player 1, followed by Player 2 and next by Player 3. This is consistent with the ordering of their unit production costs.

2. The highest pollution accumulation is observed in the benchmark scenario, where the players do not face any environmental constraints. A comparison of the two other scenarios shows that the pollution stock is only slightly lower in scenario 3, where the players are assigned different weights, than in scenario 2.

3. The players have lower payoffs in the two constrained scenarios than in the benchmark scenario, which is somehow expected. This is due to a loss in output, which leads to higher price in nodes where the constraint is binding.

4. Player 1, who is punished more (higher r_1) than the two other players in scenario 3, sees his payoff decreasing in that scenario with respect to the equal-weight scenario. The reverse is observed for the two other players, who would surely then argue that an equal weight is not fair in view of the individual contribution to the pollution stock in the benchmark.

2.5 Conclusion

In this study, we extended in a straightforward manner the framework of dynamic games played over event trees to the setting where the players face coupled constraints. The main interest of this class of games lies in its wide range of application in economics and management science, where it is quite natural to assume that some of the problem's data are stochastic. The present paper gives directions for future research. The main challenge for future developments is to compute the more conceptually appealing feedback equilibrium for this class of games. Also, one may seek to compute the S-adapted normalized equilibrium as solution to an extended variational inequality, for which we have shown the existence and uniqueness conditions. This has been already done for uncoupled-constraint S-adapted equilibrium in, e.g., [3] and [4].

Table 2.1: Solution of the benchmark scenario

Node	(n^0)	(n^1)	(n^2)	(n^3)	(n^4)	(n^5)	(n^6)	(n^7)	(n^8)	(n^9)	(n^{10})	(n^{11})	(n^{12})	(n^{13})	(n^{14})
p	16	25.389	8.133	39.517	12.885	13.585	5.636	61.023	25.935	22.765	8.871	20.283	10.242	9.262	3.969
q_1	0.5	0.681	0.406	0.550	0.429	0.453	0.188	0.422	0.199	0.379	0.204	0.488	0.133	0.185	0.079
q_2	0.3	0.21	0.21	0.237	0.227	0.239	0.099	0.200	0.089	0.166	0.101	0.262	0.062	0.089	0.038
q_3	0.1	0.07	0.07	0.160	0.154	0.049	0.049	0.134	0.057	0.117	0.067	0.161	0.040	0.059	0.025
K_1	0.5	0.681	0.808	0.550	0.550	0.666	0.666	0.422	0.422	0.443	0.443	0.549	0.549	0.527	0.527
K_2	0.3	0.21	0.147	0.237	0.237	0.214	0.214	0.200	0.200	0.166	0.166	0.232	0.232	0.227	0.227
K_3	0.1	0.07	0.049	0.160	0.160	0.034	0.034	0.134	0.134	0.187	0.187	0.151	0.151	0.048	0.048
I_1	0.331	0.073	0	0.101	0.07	0.058	0.083								
I_2	0	0.090	0.111	0.034	0	0.081	0.077								
I_3	0	0.111	0	0.022	0.075	0.127	0.024								
S	0.9	1.681	1.406	2.293	2.156	1.866	1.462	2.592	2.181	2.388	2.098	2.405	1.729	1.503	1.312
J_1	7.186	16.575	3.018	21.278	5.244	5.816	0.991	26.145	5.739	9.108	2.440	10.595	2.400	2.639	1.280
J_2	4.485	5.156	1.522	9.192	2.749	3.038	0.511	12.301	2.491	3.829	0.999	5.355	0.889	1.061	0.406
J_3	1.545	1.731	0.542	6.200	1.852	0.628	0.262	8.180	1.548	2.743	0.711	3.218	0.495	0.536	0.106

Table 2.2: Solution of the game with coupled constraint with equal weights for the players

Node	(n^0)	(n^1)	(n^2)	(n^3)	(n^4)	(n^5)	(n^6)	(n^7)	(n^8)	(n^9)	(n^{10})	(n^{11})	(n^{12})	(n^{13})	(n^{14})
p	16	26.229	8.775	41.876	14.246	15.163	5.636	64.521	25.935	25.025	8.871	22.662	10.242	9.262	3.969
q_1	0.5	0.597	0.345	0.404	0.365	0.349	0.188	0.235	0.199	0.245	0.204	0.371	0.133	0.185	0.079
q_2	0.3	0.21	0.207	0.185	0.185	0.185	0.099	0.105	0.089	0.116	0.101	0.186	0.062	0.089	0.038
q_3	0.1	0.07	0.07	0.123	0.125	0.049	0.049	0.068	0.057	0.076	0.067	0.116	0.040	0.059	0.025
K_1	0.5	0.597	0.665	0.462	0.462	0.538	0.538	0.338	0.338	0.372	0.372	0.428	0.428	0.428	0.428
K_2	0.3	0.21	0.147	0.185	0.185	0.209	0.209	0.144	0.144	0.129	0.129	0.222	0.222	0.222	0.222
K_3	0.1	0.07	0.049	0.125	0.125	0.034	0.034	0.101	0.101	0.146	0.146	0.105	0.105	0.048	0.048
I_1	0.247	0.044	0.072	0.015	0.049	0.052	0.052								
I_2	0	0.038	0.106	0.014	0	0.076	0.076								
I_3	0	0.076	0	0.013	0.058	0.081	0.024								
S	0.9	1.597	1.342	1.990	1.953	1.657	1.410	1.999	1.938	1.999	1.935	1.999	1.562	1.462	1.271
J_1	7.380	15.118	2.830	16.672	4.987	5.106	0.995	15.468	5.515	6.532	2.239	8.846	1.984	2.307	0.948
J_2	4.485	5.350	1.639	7.647	2.522	2.672	0.511	6.859	2.394	2.945	0.945	4.356	0.868	1.049	0.395
J_3	1.545	1.800	0.587	5.063	1.691	0.719	0.262	4.390	1.516	1.958	0.655	2.599	0.448	0.536	0.106

Table 2.3: Solution of the game with coupled constraint with non-equal weights for the players

Node	(n^0)	(n^1)	(n^2)	(n^3)	(n^4)	(n^5)	(n^6)	(n^7)	(n^8)	(n^9)	(n^{10})	(n^{11})	(n^{12})	(n^{13})	(n^{14})
p	16	26.383	8.889	41.908	14.176	15.214	5.636	64.397	25.935	24.983	8.871	22.548	10.242	9.262	3.969
q_1	0.5	0.581	0.331	0.362	0.335	0.318	0.188	0.177	0.199	0.202	0.204	0.322	0.133	0.185	0.079
q_2	0.3	0.21	0.21	0.202	0.202	0.211	0.099	0.140	0.089	0.139	0.101	0.218	0.062	0.089	0.038
q_3	0.1	0.07	0.07	0.144	0.144	0.049	0.049	0.102	0.057	0.099	0.067	0.144	0.040	0.0593	0.025
K_1	0.5	0.581	0.639	0.450	0.450	0.518	0.518	0.329	0.329	0.363	0.363	0.413	0.413	0.413	0.413
K_2	0.3	0.21	0.147	0.202	0.202	0.209	0.209	0.157	0.157	0.141	0.141	0.222	0.222	0.222	0.222
K_3	0.1	0.07	0.049	0.144	0.144	0.034	0.034	0.116	0.116	0.168	0.168	0.134	0.134	0.048	0.048
I_1	0.231	0.043	0.071	0.014	0.047	0.050	0.050								
I_2	0	0.055	0.106	0.015	0	0.076	0.076								
I_3	0	0.095	0	0.015	0.067	0.11	0.024								
S	0.9	1.582	1.331	1.974	1.947	1.643	1.401	1.999	1.926	1.999	1.931	1.999	1.550	1.454	1.264
J_1	7.410	14.831	2.758	14.993	4.580	4.680	0.995	11.757	5.495	5.461	2.214	7.706	1.939	2.262	0.903
J_2	4.485	5.378	1.684	8.330	2.723	3.044	0.511	9.108	2.414	3.514	0.961	5.011	0.868	1.049	0.395
J_3	1.545	1.806	0.595	5.940	1.926	0.714	0.262	6.584	1.530	2.549	0.684	3.211	0.475	0.536	0.106

Table 2.4: Contribution of each player to the pollution stock at the final nodes

Benchmark scenario								
Node	(n^7)	(n^8)	(n^9)	(n^{10})	(n^{11})	(n^{12})	(n^{13})	(n^{14})
Player 1	1.554	1.331	1.415	1.240	1.366	1.012	0.852	0.746
Player 2	0.678	0.567	0.636	0.571	0.742	0.541	0.457	0.406
Player 3	0.359	0.282	0.337	0.286	0.296	0.175	0.194	0.160
Total	2.592	2.181	2.388	2.098	2.405	1.729	1.503	1.312
Coupled constraint with equal weights								
Node	(n^7)	(n^8)	(n^9)	(n^{10})	(n^{11})	(n^{12})	(n^{13})	(n^{14})
Player 1	1.196	1.161	1.175	1.134	1.127	0.890	0.812	0.706
Player 2	0.541	0.526	0.552	0.537	0.621	0.496	0.455	0.404
Player 3	0.262	0.252	0.272	0.263	0.251	0.175	0.194	0.160
Total	1.999	1.938	1.999	1.935	1.999	1.561	1.462	1.271
Coupled constraint with different weights								
Node	(n^7)	(n^8)	(n^9)	(n^{10})	(n^{11})	(n^{12})	(n^{13})	(n^{14})
Player 1	1.096	1.117	1.099	1.101	1.045	0.856	0.803	0.697
Player 2	0.590	0.539	0.589	0.551	0.675	0.519	0.457	0.406
Player 3	0.314	0.269	0.311	0.278	0.279	0.175	0.194	0.160
Total	1.999	1.926	1.999	1.931	1.999	1.550	1.454	1.264

Table 2.5: Total expected payoffs at terminal nodes

Benchmark scenario								
Node	(n^7)	(n^8)	(n^9)	(n^{10})	(n^{11})	(n^{12})	(n^{13})	(n^{14})
Player 1	17.252	16.231	16.483	15.829	12.967	10.958	9.691	9.487
Player 2	8.082	7.591	7.832	7.552	7.435	6.390	5.711	5.613
Player 3	3.267	2.935	3.122	2.921	2.734	2.103	2.029	1.965
Total	28.600	26.758	27.437	26.302	23.137	19.452	17.431	17.065
Coupled constraint with equal weights								
Node	(n^7)	(n^8)	(n^9)	(n^{10})	(n^{11})	(n^{12})	(n^{13})	(n^{14})
Player 1	15.869	15.371	15.708	15.327	12.468	10.789	9.723	9.519
Player 2	7.732	7.509	7.735	7.552	7.185	6.348	5.779	5.681
Player 3	2.991	2.847	3.007	2.890	2.659	2.153	2.056	1.992
Total	26.592	25.728	26.451	25.770	22.313	19.292	17.559	17.193
Coupled constraint with non-equal weights								
Node	(n^7)	(n^8)	(n^9)	(n^{10})	(n^{11})	(n^{12})	(n^{13})	(n^{14})
Player 1	15.430	15.117	15.373	15.116	12.088	10.644	9.703	9.499
Player 2	7.925	7.590	7.875	7.626	7.461	6.487	5.806	5.708
Player 3	3.191	2.938	3.151	2.968	2.791	2.159	2.061	1.997
Total	26.546	25.645	26.399	25.711	22.339	19.290	17.571	17.204

2.6 Appendix

2.6.1 LCP to find the normalized equilibrium[4]

Each player $j \in M$ is maximizing his objective function given by

$$J_j(S, K, q, I) = \sum_{t=0}^{T-1} \sum_{n^t \in \mathcal{N}^t} \pi\left(n^t\right) \left[p\left(n^t\right) q_j\left(n^t\right) - C_j\left(q_j\left(n^t\right)\right) - F_j(I_j\left(n^t\right))\right] \\ + \sum_{n^T \in \mathcal{N}^T} \pi\left(n^T\right)\{\Phi_j\left(K_j\left(n^T\right)\right)\}, \tag{2.19}$$

where

$$\begin{aligned} K &= (K_1, K_2, K_3), \quad q = (q_1, q_2, q_3), \quad I = (I_1, I_2, I_3), \\ p\left(n^t\right) &= a\left(n^t\right) - b\left(q_1\left(n^t\right) + q_2\left(n^t\right) + q_3\left(n^t\right)\right), \end{aligned}$$

subject to

$$K_j\left(n^t\right) = (1-\mu) K_j\left(a\left(n^t\right)\right) + I_j(a\left(n^t\right)), \ K_j\left(n^0\right) = K_j^0, \tag{2.20}$$

$$S\left(n^t\right) = (1-\delta) S\left(a\left(n^t\right)\right) + \sum_{j=1}^{3} e_j\left(n^t\right), \ S\left(n^0\right) = S^0, \tag{2.21}$$

and

$$S\left(n^t\right) \leq \overline{S}, n^t \in \mathcal{N}^t, \quad t = 0,\ 1,\ 2,\ 3, \tag{2.22}$$

$$q_j\left(n^t\right) \leq K_j\left(n^t\right), \quad n^t \in \mathcal{N}^t, \quad t = 0,\ 1,\ 2,\ 3, \tag{2.23}$$

$$q_j\left(n^t\right) \geq 0, \quad n^t \in \mathcal{N}^t, \quad t = 0,\ 1,\ 2,\ 3, \tag{2.24}$$

$$I_j\left(n^t\right) \geq 0, \quad n^t \in \mathcal{N}^t, \quad t = 0,\ 1,\ 2. \tag{2.25}$$

Solving (2.20) recursively, we have the following equations for the capacity of player j at each node,

$$K_j(n^0) = k_j^0, \tag{2.26}$$

[4]In this section, for reasons of simplicity, we have replaced (t, s^t) by the appropriate node number to show the position on the event tree.

$$
\begin{aligned}
K_j\left(n^1\right) &= K_j\left(n^2\right) = (1-\mu)\, k_j^0 + I_j\left(n^0\right), \\
K_j\left(n^3\right) &= K_j\left(n^4\right) = (1-\mu)^2 k_j^0 + (1-\mu)\, I_j\left(n^0\right) + I_j\left(n^1\right), \\
K_j\left(n^5\right) &= K_j\left(n^6\right) = (1-\mu)^2 k_j^0 + (1-\mu)\, I_j\left(n^0\right) + I_j\left(n^2\right), \\
K_j\left(n^7\right) &= K_j\left(n^8\right) = (1-\mu)^3 k_j^0 + (1-\mu)^2 I_j\left(n^0\right) + (1-\mu)\, I_j\left(n^1\right) + I_j\left(n^3\right), \\
K_j\left(n^9\right) &= K_j\left(n^{10}\right) = (1-\mu)^3 k_j^0 + (1-\mu)^2 I_j\left(n^0\right) + (1-\mu)\, I_j\left(n^1\right) + I_j\left(n^4\right), \\
K_j\left(n^{11}\right) &= K_j\left(n^{12}\right) = (1-\mu)^3 k_j^0 + (1-\mu)^2 I_j\left(n^0\right) + (1-\mu)\, I_j\left(n^2\right) + I_j\left(n^5\right), \\
K_j\left(n^{13}\right) &= K_j\left(n^{14}\right) = (1-\mu)^3 k_j^0 + (1-\mu)^2 I_j\left(n^0\right) + (1-\mu)\, I_j\left(n^2\right) + I_j\left(n^6\right).
\end{aligned}
$$

Similarly, solving (2.21) recursively, we have the following equations for the pollution stock at each node,

$$S(n^0) = q_1(n^0) + q_2^*(n^0) + q_3^*(n^0), \tag{2.27}$$

$$
\begin{aligned}
S\left(n^1\right) = S\left(n^2\right) &= (1-\delta)\left(q_1(n^0) + q_2^*(n^0) + q_3^*(n^0)\right) \\
&+ \left(q_1(n^i) + q_2^*(n^i) + q_3^*(n^i)\right),
\end{aligned}
$$

$$
\begin{aligned}
S\left(n^3\right) = S\left(n^4\right) &= (1-\delta)^2\left(q_1(n^0) + q_2^*(n^0) + q_3^*(n^0)\right) \\
&+ (1-\delta)\left(q_1(n^1) + q_2^*(n^1) + q_3^*(n^1)\right) + \left(q_1(n^i) + q_2^*(n^i) + q_3^*(n^i)\right),
\end{aligned}
$$

$$
\begin{aligned}
S\left(n^5\right) &= (1-\delta)^2\left(q_1(n^0) + q_2^*(n^0) + q_3^*(n^0)\right) \\
&+ (1-\delta)\left(q_1(n^2) + q_2^*(n^2) + q_3^*(n^2)\right) + \left(q_1(n^i) + q_2^*(n^i) + q_3^*(n^i)\right),
\end{aligned}
$$

$$
\begin{aligned}
S\left(n^7\right) = S\left(n^8\right) &= (1-\delta)^3\left(q_1(n^0) + q_2^*(n^0) + q_3^*(n^0)\right) + (1-\delta)^2\left(q_1(n^1) + q_2^*(n^1) + q_3^*(n^1)\right) \\
&+ (1-\delta)\left(q_1(n^3) + q_2^*(n^3) + q_3^*(n^3)\right) + \left(q_1(n^i) + q_2^*(n^i) + q_3^*(n^i)\right),
\end{aligned}
$$

$$
\begin{aligned}
S\left(n^9\right) = S\left(n^{10}\right) &= (1-\delta)^3\left(q_1(n^0) + q_2^*(n^0) + q_3^*(n^0)\right) + (1-\delta)^2\left(q_1(n^1) + q_2^*(n^1) + q_3^*(n^1)\right) \\
&+ (1-\delta)\left(q_1(n^4) + q_2^*(n^4) + q_3^*(n^4)\right) + \left(q_1(n^i) + q_2^*(n^i) + q_3^*(n^i)\right),
\end{aligned}
$$

$$
\begin{aligned}
S\left(n^{11}\right) = S\left(n^{12}\right) &= (1-\delta)^3\left(q_1(n^0) + q_2^*(n^0) + q_3^*(n^0)\right) + (1-\delta)^2\left(q_1(n^2) + q_2^*(n^2) + q_3^*(n^2)\right) \\
&+ (1-\delta)\left(q_1(n^5) + q_2^*(n^5) + q_3^*(n^5)\right) + \left(q_1(n^i) + q_2^*(n^i) + q_3^*(n^i)\right),
\end{aligned}
$$

$$S\left(n^{13}\right) = S\left(n^{14}\right) = (1-\delta)^3\left(q_1(n^0) + q_2^*(n^0) + q_3^*(n^0)\right) + (1-\delta)^2\left(q_1(n^2) + q_2^*(n^2) + q_3^*(n^2)\right)$$
$$+ (1-\delta)\left(q_1(n^6) + q_2^*(n^6) + q_3^*(n^6)\right) + \left(q_1(n^i) + q_2^*(n^i) + q_3^*(n^i)\right).$$

We have assumed quadratic functional form for the production and investment cost functions, and for the salvage value,

$$C_j\left(q_j\left(n^t\right)\right) = c_j q_j^2\left(n^t\right), \quad F_j\left(I_j\left(n^t\right)\right) = f_j I_j^2\left(n^t\right), \quad \Phi_j\left(K_j\left(n^T\right)\right) = \frac{v_j}{2} K_j^2\left(n^T\right).$$

Consequently, the Lagrangian for player 1 is

$$
\begin{aligned}
L_1 = &\sum_{i=0}^{i=6} \pi(n^i) \left(\left(a(n^i) - b\left(q_1(n^i) + q_2^*(n^i) + q_3^*(n^i) \right) \right) q_1(n^i) - c_1 q_1^2(n^i) - f_1 I_1^2(n^i) \right) \\
&+ \sum_{i=7}^{i=14} \pi(n^i) \left(\left(a(n^i) - b\left(q_1(n^i) + q_2^*(n^i) + q_3^*(n^i) \right) \right) q_1(n^i) - c_1 q_1^2(n^i) \right) \\
&+ m_1^c(n^0)(q_1(n^0) + q_2^*(n^0) + q_3^*(n^0) - \overline{S}) \\
&+ \sum_{i=1,2} \pi(n^i) m_1^c(n^i)(\left(1-\delta\right) \left(q_1(n^0) + q_2^*(n^0) + q_3^*(n^0) \right) + \left(q_1(n^i) + q_2^*(n^i) + q_3^*(n^i) \right) - \overline{S} \big) \\
&+ \sum_{i=3,4} \pi(n^i) m_1^c(n^i) \Big[(1-\delta)^2 \left(q_1(n^0) + q_2^*(n^0) + q_3^*(n^0) \right) \\
&+ (1-\delta) \left(q_1(n^1) + q_2^*(n^1) + q_3^*(n^1) \right) + \left(q_1(n^i) + q_2^*(n^i) + q_3^*(n^i) \right) - \overline{S} \Big] \\
&+ \sum_{i=5,6} \pi(n^i) m_1^c(n^i) \Big[(1-\delta)^2 \left(q_1(n^0) + q_2^*(n^0) + q_3^*(n^0) \right) \\
&+ (1-\delta) \left(q_1(n^2) + q_2^*(n^2) + q_3^*(n^2) \right) + \left(q_1(n^i) + q_2^*(n^i) + q_3^*(n^i) \right) - \overline{S} \Big] \\
&+ \sum_{i=7,8} \pi(n^i) m_1^c(n^i) \Big[(1-\delta)^3 \left(q_1(n^0) + q_2^*(n^0) + q_3^*(n^0) \right) + (1-\delta)^2 \left(q_1(n^1) + q_2^*(n^1) + q_3^*(n^1) \right) \\
&+ (1-\delta) \left(q_1(n^3) + q_2^*(n^3) + q_3^*(n^3) \right) + \left(q_1(n^i) + q_2^*(n^i) + q_3^*(n^i) \right) - \overline{S} \Big] \\
&+ \sum_{i=9,10} \pi(n^i) m_1^c(n^i) \Big[(1-\delta)^3 \left(q_1(n^0) + q_2^*(n^0) + q_3^*(n^0) \right) + (1-\delta)^2 \left(q_1(n^1) + q_2^*(n^1) + q_3^*(n^1) \right) \\
&+ (1-\delta) \left(q_1(n^4) + q_2^*(n^4) + q_3^*(n^4) \right) + \left(q_1(n^i) + q_2^*(n^i) + q_3^*(n^i) \right) - \overline{S} \Big] \\
&+ \sum_{i=11,12} \pi(n^i) m_1^c(n^i) \Big[(1-\delta)^3 \left(q_1(n^0) + q_2^*(n^0) + q_3^*(n^0) \right) + (1-\delta)^2 \left(q_1(n^2) + q_2^*(n^2) + q_3^*(n^2) \right) \\
&+ (1-\delta) \left(q_1(n^5) + q_2^*(n^5) + q_3^*(n^5) \right) + \left(q_1(n^i) + q_2^*(n^i) + q_3^*(n^i) \right) - \overline{S} \Big] \\
&+ \sum_{i=13,14} \pi(n^i) m_1^c(n^i) \Big[(1-\delta)^3 \left(q_1(n^0) + q_2^*(n^0) + q_3^*(n^0) \right) + (1-\delta)^2 \left(q_1(n^2) + q_2^*(n^2) + q_3^*(n^2) \right) \\
&+ (1-\delta) \left(q_1(n^6) + q_2^*(n^6) + q_3^*(n^6) \right) + \left(q_1(n^i) + q_2^*(n^i) + q_3^*(n^i) \right) - \overline{S} \Big]
\end{aligned}
$$

$$
\begin{aligned}
&+ m_1(n^0)\left(q_1(n^0) - k_1^0\right) \\
&+ \sum_{i=1,2} \pi(n^i) m_1(n^i)\left(q_1(n^i) - (1-\mu)k_1^0 - I_1(n^0)\right) \\
&+ \sum_{i=3,4} \pi(n^i) m_1(n^i)\left[q_1(n^i) - (1-\mu)^2 k_1^0 - (1-\mu)I_1(n^0) - I_1(n^1)\right] \\
&+ \sum_{i=5,6} \pi(n^i) m_1(n^i)\left[q_1(n^i) - (1-\mu)^2 k_1^0 - (1-\mu)I_1(n^0) - I_1(n^2)\right] \\
&+ \sum_{i=7,8} \pi(n^i) m_1(n^i)\left[q_1(n^i) - (1-\mu)^3 k_1^0 - (1-\mu)^2 I_1(n^0) - (1-\mu)I_1(n^1) - I_1(n^3)\right] \\
&+ \sum_{i=9,10} \pi(n^i) m_1(n^i)\left[q_1(n^i) - (1-\mu)^3 k_1^0 - (1-\mu)^2 I_1(n^0) - (1-\mu)I_1(n^1) - I_1(n^4)\right] \\
&+ \sum_{i=11,12} \pi(n^i) m_1(n^i)\left[q_1(n^i) - (1-\mu)^3 k_1^0 - (1-\mu)^2 I_1(n^0) - (1-\mu)I_1(n^2) - I_1(n^5)\right] \\
&+ \sum_{i=13,14} \pi(n^i) m_1(n^i)\left[q_1(n^i) - (1-\mu)^3 k_1^0 - (1-\mu)^2 I_1(n^0) - (1-\mu)I_1(n^2) - I_1(n^6)\right] \\
&+ \sum_{i=7,8} \frac{1}{2}\pi(n^i) v_1\left[(1-\mu)^3 k_1(n^0) + (1-\mu)^2 I_1(n^0) + (1-\mu)I_1(n^1) + I_1(n^3)\right]^2 \\
&+ \sum_{i=9,10} \frac{1}{2}\pi(n^i) v_1\left[(1-\mu)^3 k_1(n^0) + (1-\mu)^2 I_1(n^0) + (1-\mu)I_1(n^1) + I_1(n^4)\right]^2 \\
&+ \sum_{i=11,12} \frac{1}{2}\pi(n^i) v_1\left[(1-\mu)^3 k_1(n^0) + (1-\mu)^2 I_1(n^0) + (1-\mu)I_1(n^2) + I_1(n^5)\right]^2 \\
&+ \sum_{i=13,14} \frac{1}{2}\pi(n^i) v_1\left[(1-\mu)^3 k_1(n^0) + (1-\mu)^2 I_1(n^0) + (1-\mu)I_1(n^2) + I_1(n^6)\right]^2
\end{aligned}
$$

where $m_1^c(n^i)$ and $m_1(n^i)$ are Lagrange multipliers associated with the coupled constraint (2.22) and capacity constraint (2.23) at each node, respectively, taking into account the expanded forms for $K_j(n^t)$ (2.26) and $S(n^t)$ (2.27).
The linear complementarity conditions for the above problem are of the following form:

$$\frac{\partial L_1}{\partial I_1(n_i)} \leq 0,\ I_1(n_i) \geq 0,\ I_1(n_i).\frac{\partial L_1}{\partial I_1(n_i)} = 0, \quad i = 0, ..., 6, \tag{2.28}$$

$$\frac{\partial L_1}{\partial q_1(n_i)} \leq 0,\ q_1(n_i) \geq 0,\ q_1(n_i).\frac{\partial L_1}{\partial q_1(n_i)} = 0, \quad i = 0, ..., 14, \tag{2.29}$$

$$\frac{\partial L_1}{\partial m_1^c(n_i)} \leq 0,\ m_1^c(n_i) \geq 0,\ m_1^c(n_i).\frac{\partial L_1}{\partial m_1^c(n_i)} = 0, \quad i = 0, ..., 14, \tag{2.30}$$

$$\frac{\partial L_1}{\partial m_1(n_i)} \leq 0,\ m_1(n_i) \geq 0,\ m_1(n_i).\frac{\partial L_1}{\partial m_1(n_i)} = 0, \quad i = 0, ..., 14. \tag{2.31}$$

Similarly, we can obtain the linear complementarity conditions for other two players. Imposing the normalized equilibrium condition

$$m_i^c(n^j) = \frac{m_1^c(n^j) \times r_1}{r_j}, \tag{2.32}$$

leads to a system of 126 equations and 126 unknowns.[5]

[5]Condition (2.32) is used to omit the set of conditions (2.30) for players 2 and 3 and their associated Lagrange multipliers ($m_2^c(n^j)$ and $m_3^c(n^j)$ where $j = 0, ..., 14$). Hence, we have a reduction in the size of the system (from 156 to 126). This elimination is necessary, as skipping it results in a nonsingular matrix in LCP.

Chapter 3

Incentive Equilibrium Strategies in Dynamic Games Played Over Event Trees[1]

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abstract

We characterize incentive equilibrium strategies and their credibility conditions for the class of linear-quadratic dynamic games played over event trees. In such games, the transition from one node to another is nature's decision and cannot be influenced by players' actions. Assuming that two players wish to optimize their joint payoff over a given planning horizon, we show that this outcome can be achieved as an incentive equilibrium, and hence ensures that cooperation will continue from one node onward. A simple example illustrates these strategies and the credibility conditions.

Keywords: Dynamic games, Incentive equilibria, Event tree, Cooperation, Linear-state dynamic games, Linear-quadratic dynamic games.

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[1]This paper is under second round review in Automatica journal.

3.1 Introduction

A main issue in cooperative dynamic games is how to sustain cooperation over time, that is, how to ensure that each player will indeed implement her part of the agreement as time goes by. The breakdown of long-term agreements before their maturity has been empirically observed. Schematically, a breakdown will occur either if all the parties agree at an intermediate instant of time to replace the initial agreement by a new one for the remaining periods, or if one of the players finds it (individually) rational to deviate, that is, to switch to her noncooperative strategy from that time onward [17]. The literature in (state-space) dynamic games suggests mainly three approaches to sustain cooperation over time.

Time consistency: A cooperative agreement is time consistent at an initial date and state if, at any intermediate instant of time, the cooperative payoff-to-go of each player dominates, at least weakly, her noncooperative payoff-to-go; see, e.g., [24]. Note that a time-consistent payment schedule always exists, and that the cooperative and noncooperative payoffs-to-go are compared along the cooperative state trajectory, which implicitly assumes that the players have so far played cooperatively. A stronger concept is agreeability, which requires the cooperative payoff-to-go to dominate the noncooperative payoff-to-go along *any* state trajectory; see, e.g., [29]. For a survey of time consistency, see [33].

Cooperative equilibrium: If the cooperative solution is an equilibrium, then it is self-supported, and the durability of the agreement issue is emptied. To endow the cooperative solution with an equilibrium property, one approach is to use trigger strategies that credibly and effectively punish any player deviating from the agreement; see, e.g., [21], [22] and [23].

Incentive equilibrium: Trigger strategies may embody large discontinuities, i.e., a slight deviation from an agreed-upon path triggers harsh retaliation, generating a very different path from the agreed-upon one. An alternative approach, which will be followed here, is to use incentive strategies that are continuous in the information. An incentive equilibrium has the property that when both players implement their incentive strategies, the cooperative outcome is realized as an equilibrium. Therefore, no player should be tempted to deviate from the agreement during the course of the game, provided that the incentive strategies are credible. An incentive strategy is credible if it is better for a player who has been cheated to use her strategy than to stick to the coordinated solution. Ehtamo and Hämäläinen in [36] and [38] used linear incentive strategies in a dynamic resource game and demonstrated that such strategies are credible when deviations are not too large.

The concept of incentive strategies has of course been around for a long time in dynamic games (and economics), but it was often understood and used in a leader-follower (or principal-agent) sense. The idea is that the leader designs an incentive to induce the follower to reply in a certain way, which is often meant to be (only) in the leader's best interest, but may also be in the best collective interest (see the early contributions by [79] and [80]). In such a case, the incentive is one sided. Here, we focus on two-sided incentive strategies, with the aim of implementing the joint optimization solution.

The objective of this paper is to characterize incentive equilibrium strategies and outcomes for the class of dynamic games played over event trees (DGET). In these games, the transition from one node to another is nature's decision and cannot be influenced by the players' actions. For a detailed description of DGET, see [6]. We focus on linear-quadratic dynamic games, a popular class in applications because it admits closed-form solutions (see, e.g., the books by Engwerda [39] and Haurie et al. [6]). Martín-Herrán and Zaccour ([41] and [42]) characterized incentive strategies and their credibility for linear-state and linear-quadratic dynamic games (LQDG), but in a deterministic setting.

The rest of the paper is organized as follows. In section 3.2, we briefly recall the ingredients of DGPET and derive the coordinated solution. In Section 3.3, we define the incentive equilibrium strategies and provide a numerical illustration in Section 3.4. We briefly conclude in Section 3.5.

3.2 Linear-quadratic DGPET

Let $\mathcal{T} = \{0,\ 1,\ \ldots,\ T\}$ be the set periods, and denote by $(\xi\,(t) : t \in \mathcal{T})$ the exogenous stochastic process represented by an event tree, with a root node n^0 in period 0 and a set of nodes $\mathcal{N}^t$ in period $t = 0,\ 1,\ \ldots,\ T$. Each node $n^t \in \mathcal{N}^t$ represents a possible sample value of the history h^t of the $\xi\,(.)$ process up to time t. Let $a(n^t) \in \mathcal{N}^{t-1}$ be the unique predecessor of node $n^t \in \mathcal{N}^t$ for $t = 0,\ 1,\ \ldots,\ T$, and denote by $S(n^t) \in \mathcal{N}^{t+1}$ the set of all possible direct successors of node $n^t \in \mathcal{N}^t$ for $t = 0,\ 1,\ \ldots,\ T-1$. We call *scenario* any path from node n^0 to a terminal node n^T. Each scenario has a probability, and the probabilities of all scenarios sum up to 1. We denote by π^{n^t} the probability of passing through node n^t, which corresponds to the sum of the probabilities of all scenarios that contain this node. In particular, $\pi^{n^0} = 1$, and π^{n^T} is equal to the probability of the single scenario that terminates in (leaf) node $n^T \in \mathcal{N}^T$. Also, $\sum_{n^t \in \mathcal{N}^t} \pi^{n^t} = 1, \forall t$.

Denote by $u_i(n^t) \in U_i^{n^t} \subseteq \mathbb{R}^{m_i^{n^t}}$ the decision variable of player i at node n^t, where $U_i^{n^t}$ is the control set, and $m_i^{n^t}$ is the dimension of the decision variable for player i, $i = 1, 2$. Let $\mathbf{u}(n^t)$ denote the vector of decision variables for both players at node n^t, i.e., $\mathbf{u}(n^t) = (u_1(n^t), u_2(n^t))$. Let $X \subseteq \mathbb{R}^q$, with q being given positive integer, be a state set. Denote by $x(n^t)$ the state vector at node n^t. An admissible S-adapted strategy for player i (where S stands for sample, as in the terminology of [6]), is a vector $\mathbf{u}_i = \{u_i(n^t) : n^t \in \mathcal{N}^t, t = 0, ..., T-1\}$, that is, a plan of actions adapted to the history of the random process represented by the event tree.

Assuming a linear-quadratic game structure,[2] the optimization problem of player i is as follows:

$$\begin{aligned}
\max_{\mathbf{u}_i} V_i(x, \mathbf{u}) = \max_{\mathbf{u}_i} \sum_{t=0}^{T-1} \sum_{n^t \in \mathcal{N}^t} \pi^{n^t} \Big(\frac{1}{2} x'(n^t) Q_i(n^t) x(n^t) \\
+ p_i'(n^t) x(n^t) + \frac{1}{2} \sum_{j=1}^{2} u_j'(n^t) R_{ij}(n^t) u_j(n^t) \Big) \qquad (3.1) \\
+ \sum_{n^T \in \mathcal{N}^T} \pi^{n^T} \Big(\frac{1}{2} x'(n^T) Q_i(n^T) x(n^T) + p_i'(n^T) x(n^T) \Big),
\end{aligned}$$

subject to

$$x(n^t) = A(a(n^t)) x(a(n^t)) + \sum_{j=1}^{2} B_j(a(n^t)) u_j(a(n^t)), \qquad (3.2)$$

$$x(n^0) = x_0,$$

where $Q_i(n^t) \in \mathbb{R}^{q \times q}$, $R_{ij}(n^t) \in \mathbb{R}^{m_j^{n^t} \times m_j^{n^t}}$, $p_i(n^t) \in \mathbb{R}^q$, $A(n^t) \in \mathbb{R}^{q \times q}$ and $B_j(n^t) \in \mathbb{R}^{q \times m_j^{n^t}}$ for all $n^t \in \mathcal{N}^t, t \in T$.

Assumption 1: The matrices $Q_i(n^t)$ are symmetric and $R_{ii}(n^t)$ is negative definite. Additionally, the matrices$R_{ij}(n^t),\ i \neq j$ are such that $R_{ii}(n^t) + R_{ji}(n^t)$ are negative definite as well.

By Assumption 1, the objective function in (3.1) will be strictly concave in the control variables.

3.2.1 Cooperative solution

Suppose that the two players agree to cooperate and maximize their joint payoff, that is, $\max_{\mathbf{u}_i} \sum_{i=1}^{2} V_i(x, \mathbf{u})$ subject to (3.2). The Lagrangian associated with the joint optimization

[2]For the linear state game, see the Appendix B.

problem is given by:

$$\mathcal{L}^C = \sum_{t=0}^{T-1} \sum_{n^t \in \mathcal{N}^t} \pi^{n^t} \Bigg(\frac{1}{2} x'(n^t)\big(Q_1(n^t) + Q_2(n^t)\big)x(n^t) + \big(p_1(n^t) + p_2(n^t)\big)' x(n^t)$$

$$+ \frac{1}{2} \begin{bmatrix} u_1(n^t) \\ u_2(n^t) \end{bmatrix}' \begin{bmatrix} R_{11}(n^t) + R_{21}(n^t) & 0 \\ 0 & R_{12}(n^t) + R_{22}(n^t) \end{bmatrix} \begin{bmatrix} u_1(n^t) \\ u_2(n^t) \end{bmatrix}$$

$$+ \sum_{n^T \in \mathcal{N}^T} \pi^{n^T} \Big(\frac{1}{2} x'(n^T)\big(Q_1(n^T) + Q_2(n^T)\big)x(n^T) + \big(p_1(n^T) + p_2(n^T)\big)' x(n^T)\Big) + \big(\lambda^C(n^0)\big)'(x(n^0) - x_0)$$

$$+ \sum_{t=1}^{T} \sum_{n^t \in \mathcal{N}^t} \pi^{n^t} \big(\lambda^C(n^t)\big)' \Big(x(n^t) - A(a(n^t))x(a(n^t)) - \sum_{j=1}^{2} B_j(a(n^t))u_j(a(n^t)) \Big). \quad (3.3)$$

This is a standard dynamic optimization problem with the following optimality conditions:

$$\frac{\partial \mathcal{L}^C}{\partial u_i^{n^t}} = \pi^{n^t} u_i'(n^t) \sum_{j=1}^{2} R_{ji}(n^t) + \lambda^C(S(n^t)) B_i^{n^t} = 0,$$

$$\Rightarrow u_i^C(n^t) = -\frac{1}{\pi^{n^t}} \Bigg(\sum_{j=1}^{2} R_{ji}(n^t) \Bigg)^{-1} B_i^{n^t} \lambda^C(S(n^t)). \quad (3.4)$$

$$\lambda^C(n^t) = \frac{\partial \mathcal{L}^C}{\partial x^{n^t}} = \pi^{n^t} \sum_{j=1}^{2} (Q_j^{n^t} x^{n^t} + p_j^{n^t}) + A^{n^t} \lambda^C(S(n^t)) - \pi^{n^t} \lambda^C(n^t),$$

$$\lambda^C(n^T) = \frac{\partial \mathcal{L}^C}{\partial x^{n^T}} = \pi^{n^T} \sum_{j=1}^{2} (Q_j^{n^T} x^{n^T} + p_j^{n^T}), \quad (3.5)$$

where $\lambda^C(S(n^t)) = \sum_{\nu \in S(n^t)} \pi^\nu \lambda^C(\nu)$. Let us define k^{n^t} and α^{n^t} recursively as follows:

$$k^{n^t} = \frac{1}{1 + \pi^{n^t}} \Big(\pi^{n^t} \sum_{j=1}^{2} Q_j + A^{n^t} k^\nu (I + \sum_{j=1}^{2} S_j^{n^t} k^\nu)^{-1} A^{n^t} \Big),$$

$$\alpha^{n^t} = \frac{1}{1 + \pi^{n^t}} \Big(\pi^{n^t} \sum_{j=1}^{2} p_j + A^{n^t} \big(\alpha^\nu - k^\nu (I + \sum_{j=1}^{2} S_j^{n^t} k^\nu)^{-1} \sum_{j=1}^{2} S_j^{n^t} \alpha^\nu \big) \Big), \quad (3.6)$$

where

$$S_i^{n^t} = \frac{1}{\pi^{n^t}} B_i(n^t) (\sum_{j=1}^{2} R_{ij}^{n^t})^{-1} B_i'(n^t), \quad (3.7)$$

and k^ν and α^ν are, respectively, the values of k^{n^t} and α^{n^t} at the successor nodes of n^t.

Assumption 2: The set of all matrices $(I + \sum_{j=1}^{2} S_j^{n^t} k^\nu)$, $\forall n^t$, $t = 0, ..., T-1$ are invertible.[3]

Proposition 1. *Under Assumptions 1 and 2, the cooperative (joint optimization) solution is given by*

$$u_i^C(n^t) = -\frac{1}{\pi^{n^t}}\Big(\sum_{j=1}^{2} R_{ji}(n^t)\Big)^{-1} B_i^{n^t}\Big(k^\nu(I+\sum_{j=1}^{2} S_j^{n^t}k^\nu)^{-1}(A^{n^t}x^C(n^t)-\sum_{j=1}^{2} S_j^{n^t}\alpha^\nu)+\alpha^\nu\Big), \tag{3.8}$$

where x^C is the associated state trajectory determined by

$$x^C(\nu) = (I+\sum_{j=1}^{2} S_j^{n^t}k^\nu)^{-1}(A^{n^t}x^C(n^t)-\sum_{j=1}^{2} S_j^{n^t}\alpha^\nu). \tag{3.9}$$

Proof. See Appendix A. □

Remark 3. *Assumption 1 guarantees an interior solution (control variable as a function of costate variable). Further, using Assumption 2, we obtain the cooperative solution (control variable as a function of state variable). Thus, assumptions (1) and (2) are sufficient to guarantee the existence of a cooperative solution.*

Denote by $\mathbf{u}_i^C = \{u_i^C(n^t) : n^t \in \mathcal{N}^t, t = 0, ..., T-1\}$ the cooperative solution.

3.3 *S*-Adapted incentive equilibria

As mentioned in the introduction, our aim is to design incentive equilibrium strategies to support the cooperative (or coordinated) solution $\mathbf{u}^C(n^t) = (u_1^C(n^t), u_2^C(n^t)) \in U_1^{n^t} \times U_2^{n^t}$. Denote by

$$\Psi_i = \{\psi_i | \psi_i : U_j \to U_i\}, \quad i, j = 1, 2; \;\; i \neq j,$$

the set of admissible strategies over the event tree.

Definition 5. *A strategy $\psi_i \in \Psi_i, i = 1, 2$ is an incentive equilibrium at $\mathbf{u}^C$, if*

$$\begin{aligned}
&V_1(u_1^C, u_2^C) \geq V_1(u_1, \psi_2(u_1)), \quad \forall u_1 \in U_1,\\
&V_2(u_1^C, u_2^C) \geq V_2(\psi_1(u_2), u_2), \quad \forall u_2 \in U_2,\\
&\psi_1(u_2^C(n^t)) = u_1^C(n^t),\\
&\psi_2(u_1^C(n^t)) = u_2^C(n^t), \quad n^t \in \mathcal{N}^t, t = 0, \dots, T-1.
\end{aligned}$$

[3]The satisfaction of this assumption clearly depends on the parameter values.

The above definition states that if a player implements her part of the agreement, then the best response of the other player is to do the same. In this sense, each player's incentive strategy represents a threat to implement a different control than the optimal one if the other player deviates from her optimal strategy. To determine these incentive strategies, we need to solve two optimal control problems, in each of which one player assumes that the other player is using her incentive strategy. The optimization problem of Player i is given by (3.1)-(3.2) with the additional constraint stating that Player j is using her incentive strategy, that is,

$$u_j(n^t) = \psi_j(u_i(n^t)), \quad i, j = 1, 2\ , i \neq j.$$

To show how it works, let us introduce the following corresponding Lagrangian to Player i's optimization problem, $i, j = 1, 2, i \neq j$:

$$\begin{aligned}
&\mathcal{L}_i(\lambda_i, x, u_i) = \frac{1}{2}\Big\{x'(n^0)Q_i^{n^0}x(n^0) + 2p_i'^{n^0}x(n^0) \\
&+ u_i'(n^0)R_{ii}^{n^0}u_i(n^0) + \psi_j'(u_i(n^0))R_{ij}^{n^0}\psi_j(u_i(n^0))\Big\} \\
&+ \sum_{t=1}^{T-1}\sum_{n^t\in\mathcal{N}^t}\frac{\pi^{n^t}}{2}\Big\{x'(n^t)Q_i^{n^t}x(n^t) + 2p_i'^{n^t}x(n^t) \\
&+ u_i'(n^t)R_{ii}^{n^t}u_i(n^t) + \psi_j'(u_i(n^t))R_{ij}^{n^t}\psi_j(u_i(n^t))\Big\} \\
&+ \sum_{n^T\in\mathcal{N}^T}\frac{\pi^{n^T}}{2}\Big\{x'(n^T)Q_i^{n^T}x(n^T) + 2p_i'^{n^T}x(n^T)\Big\} \\
&+ \lambda_i(n^0)(x_0 - x(n^0)) + \sum_{t=1}^{T}\sum_{n^t\in\mathcal{N}^t}\pi^{n^t}\lambda_i(n^t)\Big\{A^{a(n^t)}x(a(n^t)) \\
&+ B_i^{a(n^t)}u_i(a(n^t)) + B_j^{a(n^t)}\psi_j(u_i(a(n^t))) - x(n^t)\Big\},
\end{aligned}$$

where $\lambda_i(\cdot)$ represents the vector of Lagrange multipliers. The first-order optimality conditions are

$$\frac{\partial\mathcal{L}_i}{\partial u_i^{n^t}} = \pi^{n^t}(u_i'(n^t)R_{ii}^{n^t} + \psi_j'(u_i(n^t))\frac{\partial\psi_j}{\partial u_i}R_{ij}^{n^t}) + \sum_{\nu\in S(n^t)}\pi^\nu\lambda_i^\nu(B_i^{n^t} + B_j^{n^t}\frac{\partial\psi_j}{\partial u_i}) = 0, \quad (3.10)$$

$$\Rightarrow u_i(n^t) = -\frac{1}{\pi^{n^t}}(R_{ii}^{n^t})^{-1}(B_i^{n^t} + B_j^{n^t}\frac{\partial\psi_j}{\partial u_i})\lambda_i(S) - (R_{ii}^{n^t})^{-1}\psi_j'(u_i(n^t))\frac{\partial\psi_j}{\partial u_i}R_{ij}^{n^t},$$

$$\lambda_i(n^t) = \frac{\partial\mathcal{L}_i}{\partial x^{n^t}} = \pi^{n^t}(Q_i^{n^t}x(n^t) + p_i^{n^t}) + A^{n^t}\lambda_i(S) - \pi^{n^t}\lambda_i(n^t),$$

$$\lambda_i(n^T) = \pi^{n^T}(Q_i^{n^T}x(n^T) + p_i^{n^T}), \quad (3.11)$$

where $\lambda_i(S) = \sum_{\nu\in S(n^t)}\pi^\nu\lambda_i(\nu)$. The proposition below states the conditions to be satisfied by incentive strategies.

Proposition 2. *To be an incentive equilibrium at $\mathbf{u}^C$, a pair of strategies $(\psi_1, \psi_2) \in \Psi_1 \times \Psi_2$ must satisfy the following conditions:*

$$
\begin{aligned}
&(R_{1i}(n^t) + R_{2i}(n^t))^{-1} B_i^{n^t} (k^\nu x^C(n^t) + \alpha^\nu) \\
&= R_{ii}^{-1}(n^t)(B_i^{n^t} + B_j^{n^t} \frac{\partial \psi_j}{\partial u_i})(k_i^\nu x^C(\nu) + \alpha_i^\nu) \\
&+ \pi^{n^t} R_{ii}^{-1} R_{ij}^{n^t} \frac{\partial \psi_j}{\partial u_i} \psi_j'(u_i(n^t)), \quad i, j = 1, 2, \ i \neq j,
\end{aligned}
$$

with all functions evaluated at (u_1^C, u_2^C), and where $k_i^{n^t}$ and $\alpha_i^{n^t}$ are recursively defined by

$$
\begin{aligned}
k_i^{n^t} &= \frac{1}{1+\pi^{n^t}} \Big(\pi^{n^t} Q_i^{n^t} + A^{n^t} k_i^\nu (I + \sum_{j=1}^{2} (S_j^{n^t} + l_j^{n^t}) k_i^\nu)^{-1} A^{n^t} \Big), \qquad (3.12) \\
\alpha_i^{n^t} &= \frac{1}{1+\pi^{n^t}} \Bigg(\pi^{n^t} p_i^{n^t} + A^{n^t} \Big\{ \alpha_i^\nu \\
&- k_i^\nu \big(I + \sum_{j=1}^{2} (S_j^{n^t} + l_j^{n^t}) k_i^\nu \big)^{-1} \sum_{j=1}^{2} \big((S_j^{n^t} + l_j^{n^t}) \alpha_i^\nu + m_i^{n^t} \big) \Big\} \Bigg),
\end{aligned}
$$

where

$$
\begin{aligned}
S_i^{n^t} &= \frac{1}{\pi^{n^t}} B_i^{n^t} (R_{ii}^{n^t})^{-1} B_i^{n^t}, \qquad (3.13) \\
l_i^{n^t} &= \frac{1}{\pi^{n^t}} B_i^{n^t} (R_{ii}^{n^t})^{-1} B'^{n^t}_j \frac{\partial \psi_j}{\partial u_i}, \\
m_i^{n^t} &= B_i^{n^t} (R_{ii}^{n^t})^{-1} \psi'_j(u_i(n^t)) \frac{\partial \psi_j}{\partial u_i} R_{ij}^{n^t}, \quad i, j = 1, 2, \ i \neq j.
\end{aligned}
$$

Proof. See Appendix A. □

One important concern with incentive strategies is their credibility. These strategies are said to be credible if it is in each player's best interest to implement her incentive strategy upon detecting a deviation by the other player from the agreed-upon solution. Otherwise, the threat is not believable, and a player can freely cheat on the agreement without being punished. A formal definition of credibility follows.

Definition 6. *The incentive equilibrium strategy $(\psi_i^{n^t} \in \Psi_i, \ \forall i)$ is credible at $\mathbf{u}^C \in U_1 \times U_2$*

if the following inequalities are satisfied:

$$V_1(\psi_1(u_2(n^t)), u_2(n^t)) \geq V_1(u_1^C(n^t), u_2(n^t)), \quad \forall u_2 \in U_2, \forall n^t,$$
$$V_2(u_1(n^t), \psi_2(u_1(n^t))) \geq V_2(u_1(n^t), u_2^C(n^t)), \quad \forall u_1 \in U_1, \forall n^t. \tag{3.14}$$

Note that the above definition characterizes the credibility of the equilibrium strategies for any possible deviation in the set $U_1 \times U_2$. To give additional insight, let us retain the following functional forms, where game parameters may vary in different nodes:

$$L_i^{n^t}(x(n^t), \mathbf{u}(n^t)) = \frac{1}{2}\Big\{r_i^{n^t} u_i(n^t)^2 - d_i^{n^t} x(n^t)^2 - c_i^{n^t} x(n^t)\Big\},$$
$$\Phi_i^{n^T} = -\frac{1}{2}\Big\{d_i^{n^T} x(n^T)^2 + c_i^{n^T} x(n^T)\Big\},$$
$$x(n^t) = \sum_{j=1}^{2} g_j^{a(x^{n^t})} u_j(a(x^{n^t})) + k^{a(x^{n^t})} x(a(x^{n^t})); \quad x(n^0) = x_0, \tag{3.15}$$

where $L_i^{n^t}$ and $\Phi_i^{n^T}$ are the instantaneous payoff for player i at node $n^t \in \mathcal{N}^t, t = 0, ..., T-1$ and terminal node n^T, respectively. In this linear-quadratic game, Player i's optimal payoff under cooperation is given by

$$V_i(\mathbf{u}^C) = \sum_{t=0}^{T-1} \sum_{n^t \in \mathcal{N}^t} \frac{\pi^{n^t}}{2}\Big\{r_i^{n^t} u_i^C(n^t)^2 - d_i^{n^t} x^C(n^t)^2$$
$$- c_i^{n^t} x^C(n^t)\Big\} - \sum_{n^T \in \mathcal{N}^T} \frac{\pi^{n^T}}{2}\Big\{d_i^{n^T} x^C(n^T)^2 + c_i^{n^T} x^C(n^T)\Big\}, \tag{3.16}$$

where $u_i^C(n^t)$ and $x^C(n^t)$ are given by (3.8) and (3.9), respectively.

Proposition 3. *Consider the game defined by (3.15). Denote by* $(\mathbf{u}^C)$ *its cooperative solution. The incentive equilibrium strategy* $(\psi_i \in \Psi_i)$ *at* $u_i^C(n^t)$ *for* $i = 1, 2$, *is credible in*

$U_1 \times U_2$ *if the following conditions hold:*

$$
\begin{aligned}
&\frac{1}{2} r_1^{n^0} \big(u_1^C(n^0)^2 - \psi_1(u_2(n^0))^2\big) \\
&+ \sum_{t=1}^{T-1} \sum_{n^t} \frac{\pi^{n^t}}{2} \Big(r_1^{n^t} \big(u_1^C(n^t)^2 - \psi_1(u_2(n^t))^2\big) \\
&- d_1^{n^t} [x\big(u_1^C(n^t), u_2(n^t)\big)^2 - x\big(\psi_1(u_2(n^t)), u_2(n^t)\big)^2] \\
&- c_1^{n^t} [x\big(u_1^C(n^t), u_2(n^t)\big) - x\big(\psi_1(u_2(n^t)), u_2(n^t)\big)] \Big) \\
&- \sum_{n^T} \frac{\pi^{n^T}}{2} \Big(d_1^{n^T} [x\big(u_1^C(n^T), u_2(n^T)\big)^2 - x\big(\psi_1(u_2(n^T)), u_2(n^T)\big)^2] \\
&+ c_1^{n^T} [x\big(u_1^C(n^T), u_2(n^T)\big) - x\big(\psi_1(u_2(n^T)), u_2(n^T)\big)] \Big) \leq 0, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall u_2 \in U_2,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2} r_2^{n^0} \big(u_2^C(n^0)^2 - \psi_2(u_1(n^0))^2\big) \\
&+ \sum_{t=1}^{T-1} \sum_{n^t} \frac{\pi^{n^t}}{2} \Big(r_2^{n^t} \big(u_2^C(n^t)^2 - \psi_2(u_1(n^t))^2\big) \\
&- d_2^{n^t} [x\big(u_1(n^t), u_2^C(n^t)\big)^2 - x\big(u_1(u^t), \psi_2(u_1(n^t))\big)^2] \\
&- c_2^{n^t} [x\big(u_1(n^t), u_2^C(n^t)\big) - x\big(u_1(n^t), \psi_2(u_1(n^t))\big)] \Big) \\
&- \sum_{n^T} \frac{\pi^{n^T}}{2} \Big(d_2^{n^T} [x\big(u_1(n^T), u_2^C(n^T)\big)^2 - x\big(u_1(n^T), \psi_2(u_1(n^T))\big)^2] \\
&+ c_2^{n^T} [x\big(u_1(n^T), u_2^C(n^T)\big) - x\big(u_1(n^T), \psi_2(u_1(n^T))\big)] \Big) \leq 0, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall u_1 \in U_1,
\end{aligned}
$$

where $x^C(n^t)$ is the cooperative state variable defined in (3.9).

Proof. It suffices to compute the expressions of the different payoffs in the inequalities (3.14) taking into account the expression of Player i's payoff along a given decision established in (3.16). $\square$

Up to now, we have not assumed any particular functional form for the incentive strategies. For illustrative purpose, let us consider the following simple linear form:

$$\psi_i(u_j(n^t)) = u_i^C\left(n^t\right) + b_i\left(n^t\right)\left(u_j^C\left(n^t\right) - u_j(n^t)\right), \qquad i,j = 1,2\ , i \neq j, \tag{3.17}$$

where $b_i\left(n^t\right)$ is the penalty coefficient to be determined optimally.

Remark 4. *Although the adoption of linear strategies is standard in LQDG, this does not exclude the possibility of using nonlinear strategies. In some instances, such nonlinear strategies can be attractive, as they lead to (i) higher payoffs for at least one player (see, e.g., [81] in the context of the design of incentive games); (ii) a Pareto solution under some conditions (see [82] for an example in environmental economics), or credible incentive strategies in linear-state deterministic differential games (see [41]).*

Assuming an interior solution, solving the necessary optimality conditions given in (3.10) and (3.11) yields the values of the incentive control and costate variables, that is, $u_i^I(n^t)$ and $\lambda_i^I(n^t)$, on which we impose the equality $u_i^I(n^t) = u_i^C(n^t)$. This implies that $u_i^C(n^t)$ must satisfy its associated condition given in (3.10), and moreover we have

$$\psi_i(u_j^C(n^t)) = u_i^C(n^t), \quad \text{for } i = 1,2,$$

which simplifies the arguments in condition (3.10). Additionally, $\mathbf{u}^C$ satisfies the condition in (3.4) that characterizes the cooperative solution. Using equations (3.4) and the simplified (3.10), one may establish the necessary conditions that must be satisfied by the incentive equilibrium strategies. In the following section, these necessary conditions will be derived for LQDG.

3.4 Numerical illustration

The credibility conditions involve overlong expressions to be amenable to a qualitative analysis. To visualize the set of credible incentive strategies, we shall resort to a simple numerical example. The event tree is depicted in Figure 1, and the parameter values are given in Table 1, with the last line specifying the variation of each parameter value with respect to its value at the preceding node. In other words, each parameter increases (decreases) in its upper (lower) child node according to its variation level. Note that in a k-level binary tree, each of the two conditions defined in the above proposition contains $2^{k-1} - 1$ variables; therefore, three in this example. Again, we assume that the incentive strategies are linear,

Table 3.1: Parameter values

Parameters	r_1	r_2	c_1	c_2	d_1	d_2	g_1	g_2	k
Value at n^0	-5	-6.5	0.7	1	0.3	0.5	1	1.5	-0.3
Variation	± 0.1	± 0.15	± 0.03	± 0.08	± 0.01	± 0.05	± 0.04	± 0.09	± 0.1

Table 3.2: Control and penalty values

	Controls		Penalty parameters	
Node	u_1	u_2	b_1	b_2
n^0	0.0512	0.0591	0.9317	1.0731
n^1	0.0293	0.0338	0.9189	1.0874
n^2	0.0381	0.0441	0.9486	1.0553

with their expressions being given in (3.17). Table 2 provides the optimal control values as well as the penalty terms at the different nodes. We note that in this example the product of the penalty terms at each node is equal to one, i.e., $b_1(n^t) \times b_2(n^t) = 1$ for all n^t. Also, each of the credibility conditions defined in Proposition 3 corresponds to the area inside a polyhedron, as shown in Figures 2 and 3. Lower and upper bounds for the decision variables may be found based on the drawn polyhedrons.

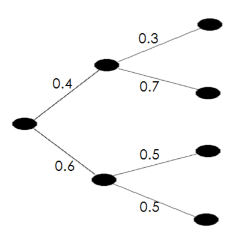

Figure 3.1: Event tree

3.5 Conclusion

We determined incentive equilibrium strategies for linear-quadratic dynamic games played over event trees, and characterized the conditions under which these strategies are credible. We illustrated the implementation of such equilibria on a simple example, where we obtained non-empty regions for credibility. Two extensions of this work are worth considering. First, the results were obtained under the assumption of linear incentive strategies. As mentioned

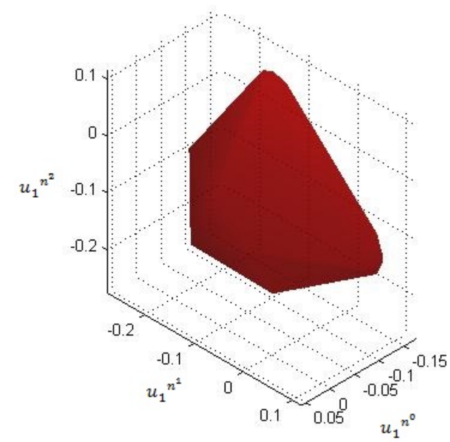

Figure 3.2: Credibility conditions for Player 1.

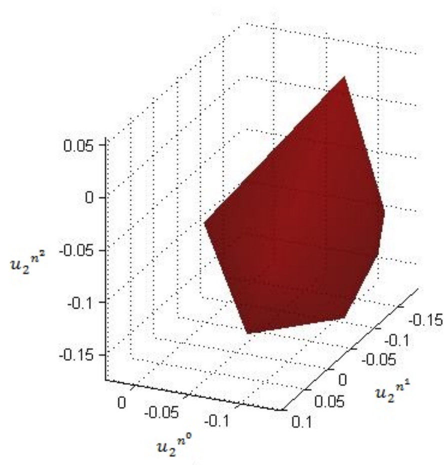

Figure 3.3: Credibility conditions for Player 2.

in Remark 2, using other forms is clearly possible, and it would be of interest to see the impact of having non-linear strategies on the credibility regions. Second, extending the formalism of incentive strategies to more than two players is a challenging and relevant research question.

3.6 Appendix A

3.6.1 Proof of Proposition 1

Due to strict concavity of the objective function, we have the unique relation given by (3.4) for all n^t. If $S_i^{n^t}$ is defined by (3.7),

$$\begin{gathered} x^C(\nu) = A^{n^t} x^C(n^t) - \sum_{i=1}^{2} S_i^{n^t} \lambda^C(S(n^t)); \\ x^C(n^0) = x_0;\ \nu \in S(n^t);\ n^t \in \mathcal{N}^t;\ t = 1, ..., T \end{gathered}$$

Let us suppose that the costate variables are linear in the state (see [39]), that is,

$$\lambda^C(n^t) = k^{n^t} x^C(n^t) + \alpha^{n^t}; \quad n^t \in \mathcal{N}^t; \forall t$$

which leads to

$$x^C(\nu) = A^{n^t} x^C(n^t) - \sum_{\nu \in S(n^t)} \pi^\nu \sum_{i=1}^{2} S_i^{n^t} k^\nu x^C(\nu) - \sum_{\nu \in S(n^t)} \pi^\nu \sum_{i=1}^{2} S_i^{n^t} \alpha^\nu.$$

The right-hand side of the above equation contains the expected value of the terms evaluated at the successor nodes $\nu \in S(n^t)$. We know that

$$\begin{aligned} & x^C(\nu_1) = x^C(\nu_2); \quad \forall \nu_1, \nu_2 \in S(n^t), \\ & \sum_{\nu \in S(n^t)} \pi^\nu x^C(\nu) = x^C(\nu). \end{aligned}$$

Since the matrix $(I + \sum_{i=1}^{2} S_i^{n^t} k^\nu)$ is assumed to be invertible, we have

$$x^C(\nu) = (I + \sum_{i=1}^{2} S_i^{n^t} k^\nu)^{-1} (A^{n^t} x^C(n^t) - \sum_{i=1}^{2} S_i^{n^t} \alpha^\nu), \tag{3.18}$$

$$\begin{aligned} \lambda^C(\nu) &= k^\nu x^C(\nu) + \alpha^\nu \qquad (3.19) \\ &= k^\nu (I + \sum_{i=1}^{2} S_i^{n^t} k^\nu)^{-1} (A^{n^t} x^C(n^t) - \sum_{i=1}^{2} S_i^{n^t} \alpha^\nu) + \alpha^\nu. \end{aligned}$$

From the optimality conditions given in (3.5), we have

$$-\lambda^C(n^t) = \frac{\partial \mathcal{L}^C}{\partial x^{n^t}} = \pi^{n^t} \sum_{j=1}^{2} (Q_j^{n^t} x^{n^t} + p_j^{n^t}) + A^{n^t} \lambda^C(S(n^t)) - \pi^{n^t} \lambda^C(n^t).$$

Substituting (3.19) in the above equation yields the following equation:

$$\begin{aligned}
&(1+\pi^{n^t})(k^{n^t} x^C(n^t) + \alpha^{n^t}) \\
&= \pi^{n^t}\Big((Q_1^{n^t} + Q_2^{n^t}) x^C(n^t) + (p_1^{n^t} + p_2^{n^t})\Big) \\
&+ A^{n^t}\Big(k^\nu (1 + \sum_{i=1}^{2} S_i^{n^t} k^\nu)^{-1} (A^{n^t} x^C(n^t) - \sum_{i=1}^{2} S_i^{n^t} \alpha^\nu) + \alpha^\nu\Big) \\
&= \Big(\pi^{n^t}(Q_1^{n^t} + Q_2^{n^t}) + A^{n^t} k^\nu (1 + \sum_{i=1}^{2} S_i^{n^t} k^\nu)^{-1} A^{n^t}\Big) x^C(n^t) \\
&+ \pi^{n^t}(p_1^{n^t} + p_2^{n^t}) + A^{n^t}\Big(\alpha^\nu - k^\nu (1 + \sum_{i=1}^{2} S_i^{n^t} k^\nu)^{-1} \sum_{i=1}^{2} S_i^{n^t} \alpha^\nu\Big).
\end{aligned}$$

Collecting the coefficients of $x^C(n^t)$, the relations in (3.6) follow. The remaining statements follow from using the terminal conditions.

3.6.2 Proof of Proposition 2

Using the optimality conditions defined in (3.10)

$$\begin{aligned}
x^C(\nu) &= A^{n^t} x^C(n^t) \\
&\quad - B_1^{n^t}\Big\{\frac{1}{\pi^{n^t}}(R_{11}^{n^t})^{-1}(B_1^{n^t} + B_2^{n^t}\frac{\partial \psi_2}{\partial u_1})\lambda_1(S(n^t)) \\
&\quad - (R_{11}^{n^t})^{-1}\psi_2'(u_1(n^t))\frac{\partial \psi_2}{\partial u_1}R_{12}^{n^t}\Big\} \\
&\quad - B_2^{n^t}\Big\{\frac{1}{\pi^{n^t}}(R_{22}^{n^t})^{-1}(B_2^{n^t} + B_1^{n^t}\frac{\partial \psi_1}{\partial u_2})\lambda_2(S(n^t)) \\
&\quad - (R_{22}^{n^t})^{-1}\psi_1'(u_2(n^t))\frac{\partial \psi_1}{\partial u_2}R_{21}^{n^t}\Big\}.
\end{aligned}$$

Using the definitions in (3.13), we can simplify the above relation as follows:

$$x^C(\nu) = A^{n^t} x^C(n^t) - \sum_{i=1}^{2}\Big((S_i^{n^t} + l_i^{n^t})\lambda_i(S(n^t)) + m_i^{n^t}\Big).$$

Now define

$$\lambda_i(n^t) = k_i^{n^t} x^C(n^t) + \alpha_i^{n^t}; \quad n^t \in \mathcal{N}^t; \forall t,$$

which leads to

$$\begin{aligned}
x^C(\nu) &= A^{n^t} x^C(n^t) - \sum_{\nu\in S(n^t)} \pi^\nu \sum_{i=1}^{2}(S_i^{n^t} + l_i^{n^t})k_i^\nu x^C(\nu) \\
&\quad - \sum_{\nu\in S(n^t)} \pi^\nu \sum_{i=1}^{2}\Big((S_i^{n^t} + l_i^{n^t})\alpha_i^\nu + m_i^{n^t}\Big).
\end{aligned}$$

Since $x^C(\nu_1) = x^C(\nu_2); \quad \forall \nu_1, \nu_2 \in S(n^t)$ and $\sum_{\nu\in S(n^t)} \pi^\nu x^C(\nu) = x^C(\nu)$, and the matrix $(I + \sum_{i=1}^{2}(S_i^{n^t} + l_i^{n^t})k^\nu)$ is assumed to be invertible, we have

$$x^C(\nu) = (I + \sum_{i=1}^{2}(S_i^{n^t} + l_i^{n^t})k^\nu)^{-1}(A^{n^t} x^C(n^t) - \sum_{i=1}^{2}(S_i^{n^t} + l_i^{n^t})\alpha^\nu + m_i^{n^t}),$$

and

$$
\begin{aligned}
\lambda_i(\nu) &= k_i^\nu x^C(\nu) + \alpha_i^\nu \\
&= k_i^\nu (I + \sum_{i=1}^{2} (S_i^{n^t} + l_i^{n^t}) k^\nu)^{-1} \left(A^{n^t} x^C(n^t) - \sum_{i=1}^{2} \Big((S_i^{n^t} + l_i^{n^t}) \alpha_i^\nu + m_i^{n^t} \Big) \right) + \alpha_i^\nu.
\end{aligned} \tag{3.20}
$$

From the optimality conditions given in (3.10), we have

$$
-\lambda_i(n^t) = \frac{\partial \mathcal{L}_i}{\partial x^{n^t}} = \pi^{n^t} \sum_{i=1}^{2} (Q_i^{n^t} x(n^t) + p_i^{n^t}) + A^{n^t} \lambda_i(S(n^t)) - \pi^{n^t} \lambda_i(n^t).
$$

Substituting (3.20) in the above equation, we obtain the following equation:

$$
\begin{aligned}
&(1 + \pi^{n^t})(k_i^{n^t} x^C(n^t) + \alpha_i^{n^t}) = \pi^{n^t} \Big(Q_i^{n^t} x(n^t) + p_i^{n^t} \Big) \\
&+ A^{n^t} \left\{ k_i^\nu (I + \sum_{i=1}^{2} (S_i^{n^t} + l_i^{n^t}) k_i^\nu)^{-1} \left(A^{n^t} x^C(n^t) - \sum_{i=1}^{2} \Big((S_i^{n^t} + l_i^{n^t}) \alpha_i^\nu + m_i^{n^t} \Big) \right) + \alpha_i^\nu \right\} \\
&= \Big(\pi^{n^t} Q_i^{n^t} + A^{n^t} k_i^\nu (I + \sum_{i=1}^{2} (S_i^{n^t} + l_i^{n^t}) k_i^\nu)^{-1} A^{n^t} \Big) x^C(n^t) \\
&+ \pi^{n^t} p_i^{n^t} + A^{n^t} \Big\{ \alpha_i^\nu - k_i^\nu (I + \sum_{i=1}^{2} (S_i^{n^t} + l_i^{n^t}) k_i^\nu)^{-1} \sum_{i=1}^{2} \Big((S_i^{n^t} + l_i^{n^t}) \alpha_i^\nu + m_i^{n^t} \Big) \Big\}.
\end{aligned}
$$

Collecting the coefficients of $x^C(n^t)$ leads to the relations in (3.12). The remaining statements directly follow from using the terminal conditions.

3.7 Appendix B

3.7.1 Linear-state DGPET

In a linear-state dynamic game the payoff functions and the state dynamics are polynomial of degree one in the state variables and there are no cross terms between the state and the control variables. Consider the game defined as follows,

$$V_i\left(\mathbf{u}, x^0\right) = \max_{\mathbf{u}_i} \sum_{t=0}^{T-1} \sum_{n^t \in \mathcal{N}^t} \pi^{n^t} L_i^{n^t}\left(x\left(n^t\right), \mathbf{u}(n^t)\right) + \sum_{n^T \in \mathcal{N}^T} \pi^{n^T} \Phi_i^{n^T}\left(x\left(n^T\right)\right), \quad i = 1, 2, \tag{3.21}$$

subject to:

$$x\left(n^t\right) = f^{a\left(n^t\right)}\left(x\left(a\left(n^t\right)\right), \mathbf{u}(a\left(n^t\right))\right), \qquad x\left(n_0\right) = x^0 \tag{3.22}$$
$$\mathbf{u}(a\left(n^t\right)) \in U^{a\left(n^t\right)}, \qquad n^t \in \mathcal{N}^t, t = 1, \ldots, T.$$

where $L_i^{n^t}(x(n^t), \mathbf{u}^{n^t})$, $\Phi_i(x^{n^T})$ and $f^{n^t}(x(n^t), \mathbf{u}^{n^t})$ satisfy the property of linear-state games. In particular, this implies that

$$\frac{\partial^2 \mathcal{L}_i}{\partial u_j^{n^t} \partial x^{n^t}} = 0, \quad i = 1, 2,\ j = 1, 2,\ t = 0, ..., T-1,\ n^t \in \mathcal{N}^t, \tag{3.23}$$
$$\frac{\partial^2 \mathcal{L}_i}{\partial {x^{n^t}}^2} = 0, \quad i = 1, 2,\ t = 0, ..., T,\ n^t \in \mathcal{N}^t, \tag{3.24}$$

which shows that the optimality conditions

$$\frac{\partial \mathcal{L}_i}{\partial u_i^{n^t}}(\lambda_i, x, \mathbf{u}) = 0, \qquad t = 0, 1, ..., T-1,\ n^t \in \mathcal{N}^t,$$

are independent of the state, and that the costate equations

$$\lambda_i(n^t) = \frac{\partial \mathcal{L}_i}{\partial x(n^t)}, \qquad t = 0, 1, ..., T,\ n^t \in \mathcal{N}^t,$$

do not include the state variables. In turn, this implies that the costate and control trajectories may be computed independently from the initial state, which yields the well-known result that an open-loop equilibrium is Markov perfect for this class of games. In this context, it is easy to show that $\lambda^C(n^t) = \sum_{i=1}^2 \lambda_i(n^t)$.[4][5] Optimality conditions for the cooperative game implies that for each player $i = 1, 2$ at $\mathbf{u}^C(n^t),\ \forall n^t,\ t = 0, ..., T-1$ we have

$$\frac{\partial \mathcal{L}^C}{\partial u_k(n^t)} = \pi^{n^t}\left(\partial_{u_i} L_1^{n^t} + \partial_{u_i} L_2^{n^t}\right) + \partial_{u_i} f^{n^t} \sum_{\nu \in \mathcal{S}(n^t)} \pi^{\nu} \lambda^C(\nu) = 0,$$

which together with $\lambda^C(n^t) = \sum_{i=1}^2 \lambda_i(n^t),\ \forall n^t$ imply,

$$\pi^{n^t} \partial_{u_1} L_1^{n^t} + \partial_{u_1} f^{n^t} \sum_{\nu \in \mathcal{S}(n^t)} \pi^{\nu} \lambda_1(\nu) = -\pi^{n^t} \partial_{u_1} L_2^{n^t} - \partial_{u_1} f^{n^t} \sum_{\nu \in \mathcal{S}(n^t)} \pi^{\nu} \lambda_2(\nu), \quad (3.25)$$

$$\pi^{n^t} \partial_{u_2} L_2^{n^t} + \partial_{u_2} f^{n^t} \sum_{\nu \in \mathcal{S}(n^t)} \pi^{\nu} \lambda_2(\nu) = -\pi^{n^t} \partial_{u_2} L_1^{n^t} - \partial_{u_2} f^{n^t} \sum_{\nu \in \mathcal{S}(n^t)} \pi^{\nu} \lambda_1(\nu). \quad (3.26)$$

The proposition below states the conditions to be satisfied by incentive strategies.

Proposition 4. *To be an incentive equilibrium at* $\mathbf{u}^C$*, a pair of strategies* $(\psi_1, \psi_2) \in \Psi_1 \times \Psi_1$ *must satisfy the following conditions:*[6]

$$\frac{\partial \psi_1}{\partial u_2}(u_2^C(n^t)) \times \frac{\partial \psi_2}{\partial u_1}(u_1^C(n^t)) = 1, \quad (3.27)$$

and

$$\psi_1'(u_2^C) = -\frac{\pi^{n^t} \partial_{u_2} L_2^{n^t} + \partial_{u_2} f^{n^t} \sum_{\nu \in \mathcal{S}(n^t)} \pi^{\nu} \lambda_2(\nu)}{\pi^{n^t} \partial_{u_1} L_2^{n^t} + \partial_{u_1} f^{n^t} \sum_{\nu \in \mathcal{S}(n^t)} \pi^{\nu} \lambda_2(\nu)}, \quad (3.28)$$

$$\psi_2'(u_1^C) = -\frac{\pi^{n^t} \partial_{u_1} L_1^{n^t} + \partial_{u_1} f^{n^t} \sum_{\nu \in \mathcal{S}(n^t)} \pi^{\nu} \lambda_1(\nu)}{\pi^{n^t} \partial_{u_2} L_1^{n^t} + \partial_{u_2} f^{n^t} \sum_{\nu \in \mathcal{S}(n^t)} \pi^{\nu} \lambda_1(\nu)}, \quad (3.29)$$

where

$$\pi^{n^t} \partial_{u_j} L_i^{n^t} + \partial_{u_j} f^{n^t} \sum_{\nu \in \mathcal{S}(n^t)} \pi^{\nu} \lambda_i(\nu), \quad i, j = 1, 2,\ i \neq j,$$

are assumed to be nonzero.

Proof. In a linear-state game, optimality conditions for the non-cooperative game can be

[4]The proof is straightforward simply because of (3.23), (3.24) and the condition $u_i^I = u_i^C, \forall i$.

[5]This condition directs us to,

$$\sum_{\nu \in \mathcal{S}(n^t)} \pi^{\nu} \lambda^C(\nu) = \sum_{\nu \in \mathcal{S}(n^t)} \pi^{\nu} \sum_{i=1}^{2} \lambda_i(\nu).$$

[6]All functions evaluated at (u_1^C, u_2^C).

simplified to[7]

$$
\begin{aligned}
&[\pi^{n^t}\partial u_i^{n^t} L_i^{n^t} + \partial u_i^{n^t} f^{n^t} \sum_{\nu\in S(n^t)} \pi^\nu \lambda_i(\nu)] \\
&+\frac{\partial \psi_j}{\partial u_i}[\pi^{n^t}\partial u_j^{n^t} L_i^{n^t} + \partial u_j^{n^t} f^{n^t} \sum_{\nu\in S(n^t)} \pi^\nu \lambda_i(\nu)] = 0, \quad i,j=1,2,\ i\neq j.
\end{aligned} \tag{3.30}
$$

Replacing the equations (3.25) and (3.26) in (3.30) for $i = 2,\ j = 1$, we get,

$$
[\pi^{n^t}\partial u_2^{n^t} L_1^{n^t} + \partial u_2^{n^t} f^{n^t} \sum_{\nu\in S(n^t)} \pi^\nu \lambda_1(\nu)] + \frac{\partial \psi_1}{\partial u_2}[\pi^{n^t}\partial u_1^{n^t} L_1^{n^t} + \partial u_1^{n^t} f^{n^t} \sum_{\nu\in S(n^t)} \pi^\nu \lambda_1(\nu)] = 0. \tag{3.31}
$$

Besides, from (3.30) for $i = 1,\ j = 2$, we have,

$$
[\pi^{n^t}\partial u_1^{n^t} L_1^{n^t} + \partial u_1^{n^t} f^{n^t} \sum_{\nu\in S(n^t)} \pi^\nu \lambda_1(\nu)] = -\frac{\partial \psi_2}{\partial u_1}[\pi^{n^t}\partial u_2^{n^t} L_1^{n^t} + \partial u_2^{n^t} f^{n^t} \sum_{\nu\in S(n^t)} \pi^\nu \lambda_1(\nu)]. \tag{3.32}
$$

Substituting the right-hand side of the last equation in (3.31) and arranging the terms, we obtain

$$
[\pi^{n^t}\partial u_2^{n^t} L_1^{n^t} + \partial u_2^{n^t} f^{n^t} \sum_{\nu\in S(n^t)} \pi^\nu \lambda_1(\nu)][1 - \frac{\partial \psi_1}{\partial u_2} \times \frac{\partial \psi_2}{\partial u_1}] = 0
$$

Since $[\pi^{n^t}\partial u_2^{n^t} L_1^{n^t} + \partial u_2^{n^t} f^{n^t} \sum_{\nu\in S(n^t)} \pi^\nu \lambda_1(\nu)]$ is assumed to be nonzero, we have condition (3.27). Moreover, (3.28) and (3.29) are direct results of (3.30) for $i = 2,\ j = 1$ and $i = 1,\ j = 2$ respectively assuming $\pi^{n^t}\partial u_j^{n^t} L_i^{n^t} + \partial u_j^{n^t} f^{n^t} \sum_{\nu\in S(n^t)} \pi^\nu \lambda_i(\nu)$ is nonzero for $i,j = 1,2, i \neq j$. □

To get more insight into the results, let us assume the following specific functional forms for the 2-player linear-state game under consideration:

$$
\begin{aligned}
L_i^{n^t}(x(n^t), \mathbf{u}(n^t)) &= \frac{1}{2} r_i^{n^t} u_i(n^t)^2 - d_i^{n^t} x(n^t), \\
\Phi_i^{n^T} &= -d_i^{n^T} x(n^T), \\
x^{n^t} &= \sum_{i=1}^{2} g_i^{a(n^t)} u_i(a(n^t)) + k^{a(n^t)} x(a(n^t)); \quad x(n^0) = x_0.
\end{aligned} \tag{3.33}
$$

Note that in the above formulation, the parameters vary in different nodes. Denoting by $u_i^C(n^t)$ the cooperative strategy of player i at node n^t, the optimality conditions for the

[7]To keep the notation simple, the arguments of all functions are omitted.

cooperative and non-cooperative games can be rewritten as follows:

$$\frac{\partial \mathcal{L}^C}{\partial u_i^{n^t}} = \pi^{n^t} r_i^{n^t} u_i^C(n^t) + g_i^{n^t} \sum_{\nu \in \mathcal{S}(n^t)} \pi^\nu \lambda^C(\nu) = 0 \Rightarrow u_i^C(n^t) = -\frac{g_i^{n^t}}{\pi^{n^t} r_i^{n^t}} \sum_{\nu \in \mathcal{S}(n^t)} \pi^\nu \lambda^C(\nu),$$

$$\frac{\partial \mathcal{L}_i}{\partial u_i^{n^t}} = \pi^{n^t} r_i^{n^t} u_i(n^t) + \left(g_i^{n^t} + g_j^{n^t} \frac{\partial \psi_j(\mathbf{u}_i(n^t))}{\partial u_i} \right) \left(\sum_{\nu \in \mathcal{S}(n^t)} \pi^\nu \lambda_i(\nu) \right) = 0,$$

$$\Rightarrow u_i(n^t) = -\frac{\sum_{\nu \in \mathcal{S}(n^t)} \pi^\nu \lambda_i(\nu)}{\pi^{n^t} r_i^{n^t}} \left(g_i^{n^t} + g_j^{n^t} \frac{\partial \psi_j(\mathbf{u}_i(n^t))}{\partial u_i} \right); \quad i, j = 1, 2, \ i \neq j.$$

The associated cooperative costate variables are given by

$$\lambda^C(n^t) = -\frac{1}{1+\pi^{n^t}} \left(\pi^{n^t} \sum_{i=1}^{2} d_i^{n^t} - k^{n^t} \sum_{\nu \in \mathcal{S}(n^t)} \pi^\nu \lambda^C(\nu) \right), \ t = 0, ..., T-1,$$

$$\lambda^C(n^T) = -\frac{\pi^{n^T} \sum_{i=1}^{2} d_i^{n^T}}{1+\pi^{n^T}},$$

and their non-cooperative counterparts by

$$\lambda_i(n^t) = -\frac{1}{1+\pi^{n^t}} \left(\pi^{n^t} d_i^{n^t} - k^{n^t} \sum_{\nu \in \mathcal{S}(n^t)} \pi^\nu \lambda_i(\nu) \right), \ t = 0, ..., T-1,$$

$$\lambda_i(n^T) = -\frac{\pi^{n^T} d_i^{n^T}}{1+\pi^{n^T}}.$$

We collect the results for the cooperative case in the following proposition.

Proposition 5. *If the players optimize their joint payoffs, then the optimal control is constant and given by*

$$u_i^C(n^t) = -\frac{g_i^{n^t}}{\pi^{n^t} r_i^{n^t}} \sum_{\nu \in \mathcal{S}(n^t)} \pi^\nu \lambda^C(\nu),$$

where the costate variables are obtained by solving recursively the following equations:

$$\lambda^C(n^t) = -\frac{1}{1+\pi^{n^t}} \left(\pi^{n^t} \sum_{i=1}^{2} d_i^{n^t} - k^{n^t} \sum_{\nu \in \mathcal{S}(n^t)} \pi^\nu \lambda^C(\nu) \right), \ t = 0, ..., T-1,$$

$$\lambda^C(n^T) = -\frac{\pi^{n^T} \sum_{i=1}^{2} d_i^{n^T}}{1+\pi^{n^T}}.$$

The cooperative state trajectory $x^C(n^t)$ is given by

$$x^C(n^t) = \sum_{i=1}^{2} g_i^{a(n^t)} u_i^C(a(n^t)) + k^{a(n^t)} x^C(a(n^t)), \tag{3.34}$$

and Player i's optimal payoff by

$$V_i(\mathbf{u}^C) = \sum_{t=0}^{T-1} \sum_{n^t \in \mathcal{N}^t} \pi^{n^t} \left(\frac{1}{2} r_i u_i^C(n^t)^2 - d_i^{n^t} x^C(n^t) \right) - \sum_{n^T \in \mathcal{N}^T} \pi^{n^T} d_i^{n^T} x^C(n^T). \tag{3.35}$$

To fully characterize incentive strategies and the credibility conditions, we need to assume a certain functional form for these strategies. Consider the linear strategies given by

$$\psi_1(u_2(n^t)) = u_1^C(n^t) + b_1^{n^t}(u_2(n^t) - u_2^C(n^t)), \tag{3.36}$$

$$\psi_2(u_1(n^t)) = u_2^C(n^t) + b_2^{n^t}(u_1(n^t) - u_1^C(n^t)). \tag{3.37}$$

The node-varying parameter $b_i^{n^t}$ represents the penalty that player i imposes on the other player deviation from cooperation at node n^t. Of course, the idea is to have no deviation so that the penalty becomes immaterial. Note that under the linearity assumption of the incentive strategies, it is easy to verify that the conditions in (3.28)-(3.29) and (3.27) become

$$\psi_1'(u_2) = b_1^{n^t}, \quad \psi_2'(u_1) = b_2^{n^t}, \quad b_1^{n^t} \times b_2^{n^t} = 1.$$

The following proposition characterizes the conditions under which these incentive strategies are credible.

Proposition 6. *Consider the game defined by (3.33) and denote by $(\mathbf{u}^C)$ its cooperative solution. The incentive equilibrium strategy $(\psi_i \in \Psi_i)$ at $u_i^C(n^t)$ for $i = 1, 2$, is credible in*

Table 3.3: Parameter values

Node	r_1	r_2	d_1	d_2	g_1	g_2	k
n^0	-5	-6.5	0.7	1	1	1.5	-0.3
Variation	± 0.1	± 0.15	± 0.03	± 0.08	± 0.04	± 0.09	± 0.1

$U_1 \times U_2$ if the following conditions hold:

$$
\begin{aligned}
&\frac{1}{2} r_1^{n^0} \left(u_1^C(n^0)^2 - \psi_1(u_2(n^0))^2\right) + \sum_{t=1}^{T-1} \sum_{n^t} \pi^{n^t} \Big(\frac{1}{2} r_1^{n^t} \left(u_1^C(n^t)^2 - \psi_1(u_2(n^t))^2\right) \\
&- d_1^{n^t} [g_1^{a(n^t)} (u_1^C(a(n^t)) - \psi_1(u_2(a(n^t)))) + k^{a(n^t)} \big(x_1(a(n^t)) - x_2(a(n^t))\big)] \Big) \\
&- \sum_{n^T} \pi^{n^T} d_1^{n^T} [g_1^{a(n^T)} (u_1^C(a(n^T) - \psi_1(u_2(a(n^T)))) \\
&+ k^{a(n^T)} \big(x_1(a(n^T)) - x_2(a(n^T))\big)] \leq 0, \ \forall u_2 \in U_2,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2} r_2^{n^0} \left(u_2^C(n^0)^2 - \psi_2(u_1(n^0))^2\right) + \sum_{t=1}^{T-1} \sum_{n^t} \pi^{n^t} \Big(\frac{1}{2} r_2^{n^t} \left(u_2^C(n^t)^2 - \psi_2(u_1(n^t))^2\right) \\
&- d_2^{n^t} [g_2^{a(n^t)} (u_2^C(a(n^t)) - \psi_2(u_1(a(n^t)))) + k^{a(n^t)} \big(x_3(a(n^t)) - x_4(a(n^t))\big)] \Big) \\
&- \sum_{n^T} \pi^{n^T} d_2^{n^T} [g_2^{a(n^T)} (u_2^C(a(n^T)) - \psi_2(u_1(a(n^T)))) \\
&+ k^{a(n^T)} \big(x_3(a(n^T)) - x_4(a(n^T))\big)] \leq 0, \ \forall u_1 \in U_1.
\end{aligned}
$$

where $x_1(n^t)$, $x_2(n^t)$, $x_3(n^t)$, and $x_4(n^t)$ are state variables defined by (3.34) at $(u_1^C(n^t), u_2(n^t))$, $(\psi_1^{n^t}(u_2), u_2(n^t))$, $(u_1(n^t), u_2^C(n^t))$, and $(u_1(n^t), \psi_2^{n^t}(u_1))$ respectively.

Proof. It suffices to compute the expressions of the different payoffs in the inequalities (3.14) taking into account the expression of player i's payoff in (3.35). □

3.7.2 Numerical illustration

The event tree is depicted in Figure 3.1, and the parameter values are given in Table 3.3.

The values of the control variables and the penalties at the different nodes are given in Table 3.4. Observe that $b_1(n^t) \times b_2(n^t) = 1$ for all n^t.

The credibility conditions defined in Proposition 6 correspond to the polyhedron regions, which are not necessarily with flat faces and straight edges, shown in Figures 3.4 and 3.5. This representation holds true for any set of parameters in the linear-state game with

Table 3.4: Control and penalty values

	Controls		Penalty parameters	
Node	u_1	u_2	b_1	b_2
n^0	0.0535	0.0617	0.9202	1.0843
n^1	0.0303	0.0351	0.9205	1.0886
n^2	0.0394	0.0456	0.9546	1.0504

linear-incentive strategies, that is, the credibility conditions correspond to the area inside a polyhedron. Further, for any set of parameters, we may find the lower and upper bounds for the decision variables for which the polyhedrons are drawn.

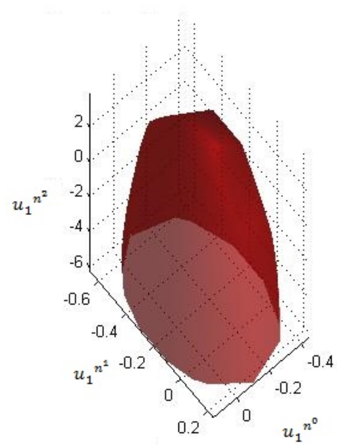

Figure 3.4: Credibility conditions for Player 1.

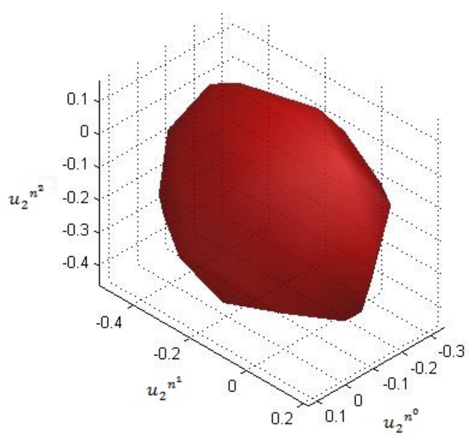

Figure 3.5: Credibility conditions for Player 2.

Chapter 4

Cost-Revenue Sharing in a Closed Loop Supply Chain Played Over Event Trees

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abstract

This article deals with a game of closed-loop supply chain (CLSC) with a single manufacturer and a single retailer played over an event tree, i.e., a tree where the transition from one node to another is nature's decision and cannot be influenced by the players' actions. We characterize and compare strategies and outcomes in two non-cooperative scenarios played à la Stackelberg, where the retailer acts as leader and the manufacturer as follower. In the benchmark scenario, we assume that there is no collaboration between the two players and the manufacturer pays the cost of green activity (GA) efforts which aims at increasing the product return. In the second scenario, namely cost-revenue sharing (CRS) program, the retailer contributes in the cost of GA efforts incurred by the manufacturer and the latter transfers a part of his revenues to the former in return. In both scenarios, the event tree is defined based on the fluctuations in the demand law parameters. Furthermore, a numerical illustrative example in environmental economics is presented for more elaboration.

Keywords: Closed-loop supply chain, Cost-revenue sharing, Dynamic games, Event tree, Return rate.

4.1 Introduction

In a Closed-Loop Supply Chain (CLSC), forward and reverse activities are combined into a unique system to increase economic, environmental, and social performance [52]. In particular, reverse activities include collecting back previously sold products when they reach their end of life. The interest in doing so lies in the cost reduction that results from producing by means of used components instead of only with new materials. It is then not surprising to see that the manufacturer is the agent who is most interested in closing the loop and appropriating the returns' residual value, while other members of the supply chain are excluded from the benefits [54]. At the same time, as retailers are close to the customers, they can be highly effective in creating awareness about the environmental benefit of recycling, and therefore play a key role in a CLSC [51], [55]. Intuitively, the retailer will participate in closing the loop only if she gets some of the savings that the manufacturer realizes when producing with material extracted from used products [56]. One straightforward way of sharing this benefit is in reducing the wholesale price. On the other hand, the retailer may find it optimal to pay part of the manufacturer's cost incurred to increase the return of products by consumers at the end of their useful life. This reasoning implies that there is room for a two-way incentive scheme in a CLSC, i.e., sharing both revenues and (some) costs. This is the line of thought pursued in this paper.

In a revenue sharing contract (RSC), the retailer pays the manufacturer a percentage of the total revenues. The rationale for a RSC is the mitigation of the double-marginalization effect, i.e., a RSC leads to a lower price and higher demand than in a standard wholesale price contract, see, e.g., [47] and [46]. In a Reverse Revenue Sharing Contract (RRSC), it is typically the manufacturer who transfers part of her revenues to the retailer. For examples of RSC and RRSC, see [44], [45], [48], [49], [50] and [51].

In this article, assuming that the data of the problem is stochastic, we want to check whether the manufacturer should rely on her own or financially involve the retailer in the remanufacturing process through implementing a RRSC. Also, we investigate under which conditions RRSC is a Pareto improving solution with respect to the traditional wholesale pricing model. Furthermore, we run a sensitivity analysis to assess the impact of main model's parameters on the results. In order to answer the above questions, we develop a dynamic game of CLSC played over an uncontrolled event tree, that is, a game where the transition from one node to another is nature's decision and cannot be influenced by the players' actions, and design a RRSC that adapts to the realization of the stochastic demand at each node of the tree. The dynamic feature of the game stems naturally from the fact that the purchase and return of the product take place at different moments in time.

We assume that the manufacturer can influence the return of used products by conducting some "green" activities (GA) such as advertising and communications campaigns about the recycling policies, logistics services, monetary and symbolic incentives, employees-training programs, etc.

We characterize and compare strategies and outcomes in two non-cooperative scenarios played à la Stackelberg, where the retailer acts as leader and the manufacturer as follower. In the first scenario, which plays the role of a benchmark, the retailer does not participate financially in the GA program and the manufacturer does not offer any discount on the wholesale price. In the second scenario, the two members of the supply chain implement a cost-revenue sharing contract. To account for the stochastic process evolving over time, we use the formalism of dynamic games played over event trees, which was introduced in Zaccour [1] and Haurie et al. [2].

The rest of the paper is organized as follows. In Section 4.2, we state the dynamic game model played over the event tree followed by analytical solutions of the benchmark (non-collaborative) and cost-revenue sharing (CRS) scenarios. In Section 4.4, we discuss an illustrative example in environmental economics. Section 4.5 contains the conclusion.

4.2 The model

The CLSC is formed of one (re)manufacturer (player M) who sells her product through a retailer (player R). To account for the return by (some) consumers of previously purchased products at the end of their useful life, we naturally retain a dynamic model. The planning horizon is finite and let $\mathcal{T} = \{0, 1, \ldots, T\}$ be the set of periods. We suppose that the demand is random and described by a stochastic process defined by an event tree. We denote the root node by n^0 in period 0 and consider a set of nodes $\mathcal{N}^t$ in period $t = 1, \ldots, T$. Let $a(n_l^t) \in \mathcal{N}^{t-1}$ be the unique predecessor of node $n_l^t \in \mathcal{N}^t$ for $t = 1, \ldots, T$, and denote by $S(n_l^t) \in \mathcal{N}^{t+1}$ the set of all possible direct successors of node $n_l^t \in \mathcal{N}^t$ for $t = 0, 1, \ldots, T-1$. In what follows, the dependence on a node $n_l^t \in \mathcal{N}^t$ in period $t = 1, \ldots, T$ is shown as a superscript for parameters and as an argument for variables. We denote by $\pi^{n_l^t}$ the probability of passing through node n_l^t. In particular, we have $\pi^{n_0} = 1$ and $\pi^{n_l^T}$ is equal to the probability of the single scenario that terminates in (leaf) node $n_l^T \in \mathcal{N}^T$. Also, $\sum_{n_l^t \in \mathcal{N}^t} \pi^{n_l^t} = 1, \forall t.$[1]

Denote by $p(n_l^t)$ the price-to-consumer chosen by the retailer in node $n_l^t \in \mathcal{N}^t$, $t = 1, \ldots, T$. At each node, the demand $Q(n_l^t)$ is a decreasing function of

[1] See [6] for an introduction to this class of games and [83] for the extension to the case where the players face coupling constraints.

price. Following a long tradition in economics, we retain the following linear form:[2]

$$Q\left(n_l^t\right) = \alpha^{n_l^t} - \beta^{n_l^t} p(n_l^t), \tag{4.1}$$

where $\alpha^{n_l^t} > 0$ is the market potential, and $\beta^{n_l^t} > 0$ represents consumer's sensitivity to price at node $n_l^t \in \mathcal{N}^t$ for $t = 0, 1, \ldots, T$. Note that both parameters' values are node-dependent. To have nonnegative demand and positive margin for the retailer, the condition $w < p(n_l^t) \leq \frac{\alpha^{n_l^t}}{\beta^{n_l^t}}$ must be satisfied at all nodes where w is the fixed wholesale price.

The manufacturer can use either new materials or old materials extracted from the returned (past-sold) products in the production process. Assuming away inventories in production and in consumer's basement, we shall refer to the number of product units that come back as *returns* and denote them by $r\left(n_l^t\right)$, at node $n_l^t \in \mathcal{N}^t$ in period $t = 1, \ldots, T$. In the CLSC literature, the rate (or quantity) of returned products has been modeled following essentially three approaches. A first group of authors assumed that the return rate is exogenous (see, e.g., [84], [85], [86], [87], [88]). The second stream also adopted a passive approach, but modeled the return rate as a random variable, e.g., an independent Poisson (see, e.g., [89], [90], [91]). The third group of studies considered an active approach, with the return rate being a function of players' strategies (see, e.g., [55], [92]). In this paper, we follow this active approach. More specifically, we suppose that the life duration of a product is one period. At the end of this period, some used products will be returned to the manufacturer for recycling, while some others will end-up in the landfield. The quantity of returned units at node n_l^t depend therefore on demand $Q(a(n_l^t))$, and on the investment $G(n_l^t)$ in the GA program at that node, that is,

$$r(n_l^t) = f^{n^t}(Q(a(n_l^t)), G(n_l^t)), \quad r(n^0) = r^0. \tag{4.2}$$

To solve the model, we retain the following form for the above state dynamics:

$$r(n_l^t) = G(n_l^t)Q(a(n_l^t)) - \delta r(a(n_l^t)), \quad r(n^0) = r_0, \tag{4.3}$$

where δ is a decay rate. This specification assumes that the GA program targets only the customers who bought the product in antecedent node. This amounts at assuming that products bought more than one period before have either already been returned to the manufacturer or disposed of in the environment. This is clearly a simplifying assumption as nothing precludes a consumer to wait a number of periods before returning a used product.

[2]To be more rigorous, we should write the demand function as $Q(p(n^t))$, but to simplify the notation, we write it as $Q(n^t)$.

The decay term $\delta r(a(n^t))$ is capturing the idea, in albeit a crude manner, that a product cannot be recycled infinitely many times.

Remark 5. *To account for the fact that the node $a(n^0)$ cannot materialize, we impose $\alpha^{a(n^0)} = \beta^{a(n^0)} = r(a(n^0)) = 0$.*

Denote by $d(G(n^t))$ the manufacturer's investment cost in the GA program. We suppose that $d(G(n_l^t))$ is convex increasing, with $d(0) = 0$, that is, there is no fixed cost. For simplicity, we shall assume in the sequel that this cost is quadratic and given by

$$d(G(n_l^t)) = \frac{k(G(n_l^t))^2}{2}, \tag{4.4}$$

where $k > 0$ is a scaling parameter.

The rationale for returns can be purely environmental. For instance, to avoid having used products being dispersed in the environment, the government may request the manufacturer to collect back these products and store them in an appropriate site. As mentioned before, here we additionally suppose that there is a cost advantage in recycling, namely, that producing with used parts is cheaper that manufacturing with exclusively new material. Denote by $c(r(n_l^t))$ the unit production cost. As using old material is intuitively subject to marginal decreasing return, we assume $c(r(n_l^t))$ to be convex decreasing. For mathematical tractability, we follow [50] and adopt the following linear functional form:

$$c(r(n_l^t)) = c_0 - c_r r(n_l^t), \tag{4.5}$$

where $c_0 > 0$ is the unit production cost when using only new material, and $c_r > 0$ represents the marginal reduction in cost due to returns. We assume that the parameters are such that the unit cost is positive at all nodes. Instead of formally imposing this constraint, we shall check ex-post for its fulfillment.

Remark 6. *The returns $r(n_l^t)$ constitute a measure of environmental performance at node n_l^t, and the term $c_r r(n_l^t)$ gives the associated economic gain.*

We shall characterize and compare the equilibrium solutions in two scenarios.

Benchmark: The two members of the supply chain do not share the cost of the GA program, nor the revenues. The game is played non-cooperatively, that is, the manufacturer and the retailer maximize their individual profits given by

$$\max_{G(n_l^t)>0} J_M(G,p,r) = \sum_{t=0}^{T}\sum_{n_l^t\in\mathcal{N}^t} \pi^{n_l^t}\Big((w-c(r(n_l^t)))Q(n_l^t)-d(G(n_l^t))\Big), \tag{4.6}$$

$$\max_{p(n^t)\geq 0} J_R(p) = \sum_{t=0}^{T}\sum_{n_l^t\in\mathcal{N}^t} \pi^{n_l^t}\left(p(n_l^t)-w\right)Q(n_l^t), \tag{4.7}$$

where w is the fixed wholesale price and the returns are given by (4.2).

Cost-Revenue Sharing (CRS): In this case, the retailer pays a share $B\left(n_l^t\right)$, $0 \leq B\left(n_l^t\right) \leq 1$, of the manufacturer's GA cost, and the manufacturer discounts the wholesale price by an amount $I(r(n_l^t))$ to compensate the retailer for sharing the cost of the GA program. Therefore, the wholesale price in this scenario is given by $w - I(r(n_l^t))$. Note that the reduction in the wholesale price depends on the returns; it is a way of incentivizing the retailer to contribute at a higher rate in the cost of GA program, whose objective is to increase the returns. The manufacturer's net margin at node $n_l^t\in\mathcal{N}^t,\ t=0,...,T$ is therefore $w-c(r(n_l^t))-I(r(n_l^t))$. As noted before, the non-cooperative game is played à la Stackelberg with the retailer as the leader and the manufacturer as the follower, that is, the retailer first decides the support rate and the price to consumer, and next the manufacturer the investment in the GA program.

Following [46], [44] and [93], we retain the following form for the incentive function:

$$I(r(n_l^t)) = \phi(w-c(r(n_l^t))) = \phi\left(w-c_0+c_r r(n_l^t)\right), \tag{4.8}$$

where $\phi\in[0,1]$ is the sharing parameter and stands for the percentage of manufacturer's profit margin transferred to the retailer. In the benchmark scenario, $\phi=0$ as well as $B\left(n_l^t\right)=0$ for all $n_l^t\in\mathcal{N}^t,\ t=0,...,T$.

In this cost-revenue sharing scenario, the optimization problems are as follows

$$\max_{G(n_l^t)>0} J_M(G,p,B,r)$$
$$= \sum_{t=0}^{T} \sum_{n_l^t \in \mathcal{N}^t} \pi^{n_l^t} \Big(\big(w - c(r(n_l^t)) - I(r(n_l^t))\big) Q(n_l^t) - (1 - B(n_l^t)) d(G(n_l^t)) \Big), \qquad (4.9)$$

$$\max_{\substack{p(n_l^t)>0 \\ B(n_l^t)>0}} J_R(G,p,B,r)$$
$$= \sum_{t=0}^{T} \sum_{n_l^t \in \mathcal{N}^t} \pi^{n_l^t} \Big(\big(p(n_l^t) - w + I(r(n_l^t))\big) Q(n_l^t) - B(n_l^t) d(G(n_l^t)) \Big), \qquad (4.10)$$

subject to the returns dynamics in (4.2). Notice that in this scenario, the two players are strategically linked, as their payoffs depend on both players' actions and on the returns.

Remark 7. *The reason for having a fixed wholesale price w, which may appear counter intuitive, is to have a meaningful comparison between the two scenarios. If w were a decision variable, then the manufacturer can manipulate its level in both scenarios rendering the incentive $I(r(n_l^t))$ meaningless.*

To wrap up, we have defined a two-player dynamic game played over an event tree, with one decision variable for the manufacturer ($G(n_l^t) \geq 0$) and two decision variables for the retailer ($p(n_l^t) \geq 0$ and $0 \leq B(n_l^t) \leq 1$). In the following section, we characterize the equilibrium solutions in the two scenarios and draw some comparisons.

4.3 Equilibrium solutions

In this section, we characterize equilibrium strategies and outcomes for both scenarios. First, we solve for the benchmark scenario in which the retailer does not participate in the green efforts undertaken by the manufacturer and the manufacturer does not discount the wholesale price to the retailer, i.e., $B(n_l^t) = I(r(n_l^t)) = 0$, $\forall n_l^t$. Second, we solve for the cost-revenue sharing (CRS) scenario in which the retailer, as the leader, sets first her support rate and the price, and next the manufacturer, as the follower, decides about her GA efforts. Next, we compare the two equilibrium results.

4.3.1 Benchmark scenario

In this scenario, the retailer's optimization problem is independent of the manufacturer's control variable G and of the state variable r. Consequently, the retailer optimizes, at each node, a static problem without any regard of what the manufacturer is doing. The implication of this structure is that it does not matter if the game is played simultaneously à la Nash or sequentially à la Stackelberg, the result would be the same.

Introduce the manufacturer's Hamiltonian

$$\mathcal{H}_M = \sum_{t=0}^{T} \sum_{n_l^t \in \mathcal{N}^t} \pi^{n_l^t} \Big\{ \left(\alpha^{n_l^t} - \beta^{n_l^t} p(n_l^t)\right) \left(w - c_0 + c_r r(n_l^t)\right) - \frac{k}{2}(G(n_l^t))^2$$
$$+ \lambda(n_l^t) \left(G(n_l^t) \left(\alpha^{a(n_l^t)} - \beta^{a(n_l^t)} p(a(n_l^t))\right) - \delta r(a(n_l^t)) \right) \Big\},$$

where $\lambda = \left(\lambda\left(n_l^t\right)\right)_{n_l^t \in \mathcal{N}^t, t=1,\ldots,T}$, is the costate variable appended by the manufacturer to the state variable r. The value at node n_l^t corresponds to the shadow price or marginal value of the returns at that node. The following proposition characterizes the equilibrium strategies (superscripted with $\sim$) in this scenario.

Proposition 1. *Assuming an interior solution, the equilibrium GA and price values at node at node $n_l^t \in \mathcal{N}^t$for $t = 0,\ 1,\ \ldots,\ T$ are as follows:*

$$\tilde{G}(n_l^t) = \frac{(\alpha^{a(n_l^t)} - \beta^{a(n_l^t)} w)}{2k} \left(\frac{\pi^{n_l^t} c_r}{2} (\alpha^{n_l^t} - \beta^{n_l^t} w) - \delta \sum_{\nu \in S(n_l^t)} \pi^{\nu} \tilde{\lambda}(\nu) \right), \tag{4.11}$$

$$\tilde{p}(n_l^t) = \frac{\alpha^{n_l^t} + \beta^{n_l^t} w}{2\beta^{n_l^t}}, \tag{4.12}$$

where $\tilde{\lambda}\left(\cdot\right)$ is governed by the following difference equation:

$$\tilde{\lambda}(n_l^t) = \pi^{n_l^t} c_r \Big(\frac{\alpha^{n_l^t} - \beta^{n_l^t} w}{2} \Big) - \delta \sum_{\nu \in S(n_l^t)} \pi^{\nu} \tilde{\lambda}(\nu)$$
$$\tilde{\lambda}(n_l^{T+1}) = 0$$

Proof. See Appendix. □

The results deserve the two following comments:

1. As expected, the retail price is increasing in the market potential $\alpha^{n_l^t}$ and decreasing in consumer sensitivity to price given by $\beta^{n_l^t}$. Note that $p(n_l^t)$ is increasing in the wholesale price w, a result that has a strategic complementarity flavor, that is, increasing the value of one strategic variable leads to an increase in the other one. Of course, the interpretation here is stretched a bit as w is given and is not a decision variable.

2. The equilibrium GA value is determined by the familiar rule of marginal cost equals marginal revenue. Indeed, observing that

$$\tilde{Q}\left(n_l^t\right) = \frac{\alpha^{n_l^t} - \beta^{n_l^t} w}{2}, \tag{4.13}$$

it is then easy to see from the Appendix that

$$k\tilde{G}(n_l^t) = \tilde{\lambda}(n_l^t)\tilde{Q}\left(a\left(n_l^t\right)\right).$$

The left-hand side is $d'\left(\tilde{G}\left(n_l^t\right)\right)$, while the right-hand side gives the marginal revenue, which is the product of demand at antecedent node multiplied by the shadow price of returns.

4.3.2 Cost-revenue sharing scenario

In this scenario, at node $n_l^t \in \mathcal{N}^t$ for $t = 0,\ 1,\ \ldots,\ T$, the manufacturer transfers a part of her revenues $\phi\left(w - c_0 + c_r r(n_l^t)\right)$, where ϕ is a given parameter, and the retailer pays a percentage $B(n_l^t)$ of manufacturer's GA cost, where $B(n_l^t)$ is a strategic variable. The implication is that now the returns influence the retailer's payoff and therefore become a relevant variable.

Denote by μ_r is the retailer's costate variable appended to the returns dynamics. Recall that λ is the costate variable appended by the manufacturer to the state dynamics, and let μ_λ be the costate variable appended by the retailer to the state equation describing the evolution of λ, which becomes an additional state variable for her (see Appendix for details). The following proposition characterizes the equilibrium strategies in the leader-follower dynamic game.

Proposition 2. *Assuming an interior solution, the Stackelberg equilibrium values at node*

$n_l^t \in \mathcal{N}^t$*for* $t = 0,\ 1,\ \ldots,\ T$ *are given by*

$$G(n_l^t) = \frac{(1+\phi)}{2k}\left(\alpha^{a(n_l^t)} - \beta^{a(n_l^t)}p(a(n_l^t))\right)\Big\{\pi^{n_l^t}c_r(\alpha^{n_l^t} - \beta^{n_l^t}p(n_l^t)) - \frac{\delta}{1-\phi}\sum_{\nu\in S(n_l^t)}\pi^{\nu}\lambda(\nu)\Big\}, \tag{4.14}$$

$$p(n_l^t) = \frac{k\pi^{n_l^t}\left(\alpha^{n_l^t} + \beta^{n_l^t}\left(\phi\left(c_0 - c_r r(n_l^t)\right) + (1-\phi)\left(w - \pi^{n_l^t}c_r\mu_\lambda(n_l^t)\right)\right)\right) - \alpha^{n_l^t}\beta^{n_l^t}\Gamma}{\beta^{n_l^t}\left(2k\pi^{n_l^t} - \beta^{n_l^t}\Gamma\right)}, \tag{4.15}$$

$$B(n_l^t) = \frac{3\phi - 1}{\phi + 1}, \tag{4.16}$$

where

$$\Gamma = \sum_{\nu\in S(n_l^t)}\pi^{\nu}\left(\frac{1}{2}\lambda(\nu) + \mu_r(\nu)\right)^2,$$

and the costate and state variables are governed by the following difference equations:

$$\mu_r(n_l^t) = \pi^{n_l^t}c_r\phi(\alpha^{n_l^t} - \beta^{n_l^t}p(n_l^t)) - \delta\sum_{\nu\in S(n_l^t)}\pi^{\nu}\mu_r(\nu), \tag{4.17}$$

$$\mu_r(n_l^{T+1}) = 0,$$

$$\mu_\lambda(n_l^t) = \frac{(1+\phi)\,\pi^{n_l^t}Q^2(a(n_l^t))}{4k\,(1-\phi)^2}\left(2\mu_\lambda(n_l^t)\,(1-\phi)) - \lambda(n_l^t)\,(3\phi - 1)\right) + \pi^{n_l^t}\pi^{a(n_l^t)}\mu_r(a(n_l^t)), \tag{4.18}$$

$$\mu_\lambda(n_l^{T+1}) = 0,$$

$$\lambda(n_l^t) = \pi^{n_l^t}c_r(1-\phi)(\alpha^{n_l^t} - \beta^{n_l^t}p(n_l^t)) - \delta\sum_{\nu\in S(n_l^t)}\pi^{\nu}\lambda(\nu), \tag{4.19}$$

$$\lambda(n_l^{T+1}) = 0,$$

$$r(n_l^t) = \frac{(1+\phi)\,\lambda(n_l^t)Q^2(a(n_l^t))}{2k\,(1-\phi)} - \delta r(a(n_l^t)), \tag{4.20}$$
$$r(n^0) = r_0.$$

Proof. See Appendix. □

In order to find the exact values of the variables at each node, we need to solve (4.14), (4.15), (4.17), (4.18), (4.19) and (4.20) as a set of simultaneous equations because of their interdependency owing to the recursive form of equations over the event tree. Clearly, retailer's support rate, B, can be specified independently by (4.16).

The main result is that the support provided by the retailer to the manufacturer is constant over nodes, and only depends on the sharing-revenue parameter ϕ. For this support to be interior, that is, $0 < B(n_l^t) < 1$ rate, we must have

$$\frac{1}{3} < \phi < 1.$$

We shall assume from now on that the parameter ϕ satisfies the above bounds. Further, this support rate in increasing in ϕ. Indeed,

$$\frac{dB(n_l^t)}{d\phi} = \frac{3\,(\phi+1) - (3\phi-1)}{(\phi+1)^2} = \frac{4}{(\phi+1)^2} > 0.$$

It is not easy to derive conditions on the parameter values that guarantee positivity of the price. We shall check in the numerical examples that the price is indeed interior.

To interpret $G(n_l^t)$, we have from (4.19) that

$$\lambda(n_l^t) = \frac{\partial \mathcal{H}_M}{\partial r(n_l^t)} = \pi^{n_l^t} c_r (1-\phi)(\alpha^{n_l^t} - \beta^{n_l^t} p(n_l^t)) - \delta \sum_{\nu \in S(n_l^t)} \pi^{\nu} \lambda(\nu).$$

Using the above expression, we can rewrite the expression of $G(n_l^t)$ as

$$kG(n_l^t) = \frac{(1+\phi)}{2(1-\phi)} Q\left(a\left(n_l^t\right)\right)\lambda(n_l^t).$$

Using $B(n_l^t) = \frac{3\phi-1}{\phi+1}$, we get

$$kG(n_l^t)(1 - B(n_l^t)) = \lambda(n_l^t) Q\left(a\left(n_l^t\right)\right). \tag{4.21}$$

As in the previous scenario, the equilibrium GA effort is determined by the familiar rule of marginal cost equals marginal revenue. Observe that the marginal cost now includes the term $(1 - B(n_l^t))$, which is the share of the cost taken by the manufacturer.

4.4 Illustrative examples

As no more insight can be obtained analytically, especially in terms of comparing strategies and outcomes in the two scenarios, we shall run some numerical simulations. We increase the realism and complexity of the problem by assuming a two part planning horizon. In the first part, $t = 0, ..., \tau$, we have the regular binary event tree representing the uncertainty in the parameters while in the second part, $t = \tau + 1, ..., T$, there is no uncertainty involved and hence there exists 2^τ scenarios, prolonging from each of the nodes at $t = \tau$. In other words, the parameters in the second period are fixed and equal to the their levels at $t = \tau$. We assumed that we have equal probabilities for each of the two branches stemmed from node n^t, $t = 0, ..., \tau - 1$. The event tree is depicted in Figure (4.1) and table (4.1) summarizes some useful information in this structure.

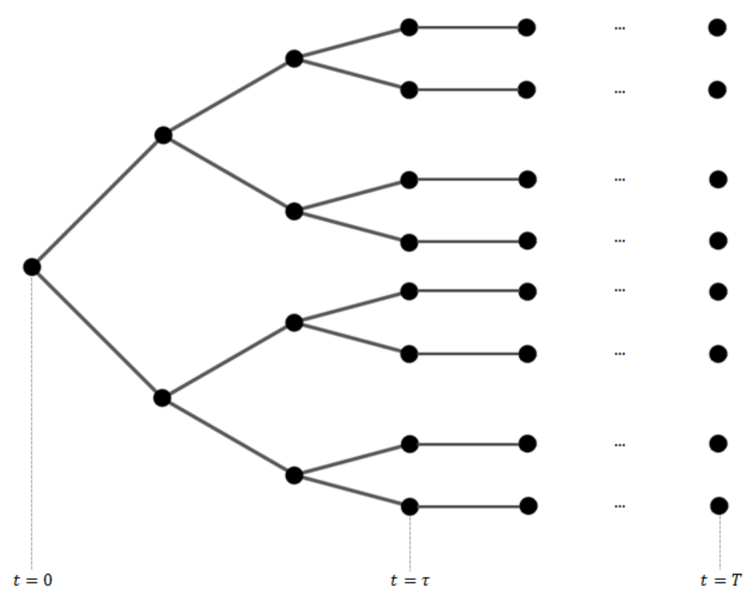

Figure 4.1: General event tree

Although we can solve for any finite event tree, we provide the results for only two planning horizons.[3] More specifically, we shall discuss in details the results for a short planning horizon ($T = 7$) and show some results for a longer planning horizon ($T = 18$). To start, let us consider an eight-period planning horizon ($\tau = 3$, $T = 7$ and $t = 0, \ldots, 7$).

[3] Other results can be obtained from the author upon request.

Table 4.1: Important numbers of the extended tree

Number of stochastic nodes	$2^{\tau+1}-1$
Number of nodes in the last stochastic stage = Total number of scenarios	2^{τ}
Number of deterministic nodes	$2^{\tau}\times(T-\tau)$
Total number of nodes	$2^{\tau}\times(T-\tau+2)-1$

In this case, there are in total eight scenarios emanating from the initial (root) node and terminating in one of the node $n_l^T \in \mathcal{N}^T$.

The model has 8 parameters, namely:

$$\begin{aligned} \text{Stochastic demand parameters} &: \alpha, \beta \\ \text{Cost parameters} &: c^0, c_r, k \\ \text{Wholesale price} &: w \\ \text{Sharing parameter} &: \phi \\ \text{Decay parameter} &: \delta \end{aligned}$$

In the sequel, we shall fix once and for all the values of three parameters, namely, c^0, k and δ, whose impact on the results is expected to be purely quantitative, without much qualitative insight.[4] More precisely, we normalize the two cost parameters at one ($c^0 = k = 1$) and set the decay parameter $\delta = 0.2$.

The stochastic demand parameters evolve as follows along the tree:

$$\begin{aligned} \alpha^{n_l^t} &= (1 \pm r_\alpha)\cdot \alpha\left(a\left(n_l^t\right)\right), \quad \alpha^0 = 3, \quad r_\alpha = 0.1, \quad t = 1,2,3, \\ \beta^{n_l^t} &= (1 \pm r_\beta)\cdot \beta\left(a\left(n_l^t\right)\right), \quad \beta^0 = 1, \quad r_\beta = 0.05, \ \ t = 1,2,3. \end{aligned}$$

The positive variation rate is applied to upward pointing successor nodes, and the negative rate to downward pointing successor nodes, for $t = 0,\ldots,3$. For $t = 4,\ldots,7$, the values of α and β stay fixed and equal to their associated levels at $t = 4$. In the following simulations, we shall consider the event tree made of 8 periods and the demand parameters are as given in Table (4.2).

In the base case, we fix three parameters $c_r = 0.5$, $w = 2$, and $\phi = 0.6$ and solve both benchmark and the CRS scenarios. The results are given in Table (4.3). For the next three sets of simulations, the aim is to investigate the impact of different values of ϕ, w, and c_r on

[4]See Appendix C for the related simulations.

Table 4.2: Demand parameters over the tree

Parameters	α	β
Value at root	3	1
Variation rate	0.1	0.05

the expected payoffs separately. More precisely, in the second run we investigate the impact of changing ϕ on the expected payoffs and similarly in the third and forth run we aim at studying the impact of w and c_r, respectively. In each simulation, the idea is to take two values below the base case and two others above for the parameter under study while the other two parameters are kept fixed and set at their levels in the base case. Thus, we have $\phi \in \{0.4, 0.5, 0.6, 0.7, 0.8\}$, $w \in \{1.5, 1.75, 2.0, 2.25, 2.5\}$, and $c_r \in \{0.3, 0.4, 0.5, 0.6, 0.7\}$. The results are given in Tables (4.4), (4.5), and (4.6) respectively.

For each set of simulations, we draw two different comparisons. First, we analyze the impact of changing the parameter under study on the payoffs in the CRS scenario (sensitivity analysis). Second, we check whether for any level of the parameter under study, we reach to a Pareto improving result for which the expected payoff of each player should be compared with his associated benchmark payoff. Clearly, the benchmark results for the second case (impact of ϕ) is the same as the benchmark in the first run (base case). For the other two simulations the benchmark results are given in two bottom rows of Tables (4.5) and (4.6).

Sensitivity analysis (payoff wise comparisons) leads to the following immediate observations based on the simulations results:

1. In the benchmark scenario, the parameter c_r does not affect retailer's payoff which is expected as it does not appear in his objective function.

2. In the benchmark scenario, higher c_r results in higher payoff for the manufacturer.

3. In the CRS scenario, higher c_r results in higher payoff for both players.

4. In both scenarios, benchmark and CRS, lower w leads to higher payoff for the retailer.

5. In the benchmark scenario, the manufacturer has his highest payoff for $w = 2$.

6. In the CRS scenario,the manufacturer has higher payoff for higher levels of w.

7. In the CRS scenario, higher sharing parameter, ϕ results in higher payoff for the retailer.

8. In the CRS scenario, the manufacturer has higher payoff for lower levels of ϕ.

Table 4.3: Base case

Scenario	CRS	Benchmark
parameters	$c_r = 0.5, w = 2, \phi = 0.6, c_0 = 1, k = 1, d = 0.2$	
Manufacturer	2.6240	4.0217
Retailer	5.2850	2.1017

Table 4.4: Impact of ϕ on the expected payoff for each player

parameters	$c_r = 0.5, w = 2, c_0 = 1, k = 1, d = 0.2$				
ϕ	0.4	0.5	0.6	0.7	0.8
Manufacturer	3.4070	3.0566	2.6240	2.1068	1.5010
Retailer	4.0452	4.6412	5.2850	5.9796	6.7291

Table 4.5: Impact of w on the expected payoff for each player

parameters	$c_r = 0.5, \phi = 0.6, c_0 = 1, k = 1, d = 0.2$				
w	1.5	1.75	2	2.25	2.5
CRS					
Manufacturer	1.5204	2.1122	2.6240	3.0560	3.4083
Retailer	6.6813	5.9615	5.2850	4.6512	4.0599
Benchmark					
Manufacturer	3.0940	3.7981	4.0217	3.7582	3.0023
Retailer	4.6017	3.2267	2.1017	1.2267	0.6017

Table 4.6: Impact of c_r on the expected payoff for each player

parameters	$w = 2, \phi = 0.6, c_0 = 1, k = 1, d = 0.2$				
c_r	0.3	0.4	0.5	0.6	0.7
CRS					
Manufacturer	2.5823	2.6003	2.6240	2.6542	2.6919
Retailer	5.2439	5.2616	5.2850	5.3145	5.3510
Benchmark					
Manufacturer	4.0078	4.0139	4.0217	4.0313	4.0427
Retailer	2.1017	2.1017	2.1017	2.1017	2.1017

Looking closer to the players' payoff function at each node in the benchmark scenario, we have,

$$\frac{\partial J_M^B(n_l^t)}{\partial c_r} = r(n_l^t)Q(n_l^t) \geq 0,$$
$$\frac{\partial J_R^B(n_l^t)}{\partial c_r} = 0.$$

Similarly, for the CRS scenario, we have,

$$\frac{\partial J_M^{CRS}(n_l^t)}{\partial c_r} = (1-\phi)r(n_l^t)Q(n_l^t) \geq 0,$$
$$\frac{\partial J_R^{CRS}(n_l^t)}{\partial c_r} = \phi r(n_l^t)Q(n_l^t) \geq 0.$$

Above derivations analytically confirm players' preferences with respect to c_r, i.e., the retailer is indifferent about changes of c_r in the benchmark scenario while the manufacturer benefits from higher c_r as his marginal profit with respect to c_r is non-negative at each node. Similarly, marginal increase of profit for both players in the CRS scenario show that higher c_r results in higher payoff for them. Note that marginal increase of the manufacturer's profit due to a unit of increment in c_r is higher in the benchmark scenario compare to CRS scenario. This result is expected as the manufacturer is transferring a part of her revenues (due to the cost saving effect of the return volume) to the retailer under CRS contract.

Now, considering the outcomes of the sensitivity analysis, we go one step further and use above result to run the next set of simulations. The objective is to find Pareto improving solutions. As both players' perspectives regarding c_r are in line and hence setting c_r at a higher level would be appealing for the manufacturer and the retailer, from now on we fix c_r to its higher level, i.e., 0.7. On the contrary, since the ideal level of w and ϕ for the manufacturer is not the same as those of the retailer, for two different levels of w namely 1.5 and 2.5, we run a set of simulations for various levels of the sharing parameter, ϕ. The results are shown in Tables (4.7) and (4.8).

Table 4.7: Impact of ϕ on the expected payoff for each player for low w

parameters	$w = 1.5, c_r = 0.7, c_0 = 1, k = 1, d = 0.2$							
Scenario	CRS							Benchmark
ϕ	0.34	0.4	0.5	0.6	0.7	0.8	0.9	N/A
Manufacturer	2.3586	2.2010	1.9184	1.6085	1.2679	0.8917	0.4729	3.1844
Retailer	5.7565	5.9806	6.3717	6.7877	7.2332	7.7143	8.2405	4.6017

The results show that when w is fixed at a low level, i.e., $w = 1.5$, the players do not

Table 4.8: Impact of ϕ on the expected payoff for each player for high w

parameters	$\mathbf{w = 2.5}, c_r = 0.7, c_0 = 1, k = 1, d = 0.2$							
Scenario	CRS							Benchmark
ϕ	**0.34**	**0.4**	**0.5**	**0.6**	0.7	0.8	0.9	N/A
Manufacturer	**4.0368**	**4.0092**	**3.8296**	**3.4582**	2.9102	2.1676	1.2104	3.0052
Retailer	**2.1538**	**2.5410**	**3.2709**	**4.0989**	5.0479	6.1299	7.3725	0.6017
$B(n^t)$	0.0149	0.1428	0.3333	0.5	0.6470	0.7778	0.8947	N/A

find it optimal to commit to CRS contract while for the higher wholesale price, $w = 2.5$, we have appealing results for both players if the sharing parameter is not very high. The bold entries in table (4.8) show these Pareto improving solutions. To put it differently, both players find it in their interest to abandon the benchmark scenario in which they play independently and commit to the CRS contract for high wholesale price and low sharing parameter which intuitively makes sense. When the wholesale price is high enough, the retailer finds it worthwhile to financially support the manufacturer's GA efforts to receive discount on the wholesale price. Clearly, the higher her support is, the more the discount will be. On the other hand, the manufacturer commits to CRS contract when the discount is not so high, i.e., the manufacturer does not find it rational to condone a huge share of his profit even when he is charging the retailer a high wholesale price and the latter is paying a significant portion of GA efforts' cost.

4.4.1 Short vs. long planning horizon

In this section, we are interested in studying the impact of the length of the planning horizon on values of the variables in the model. We keep the previously used formalism and run the simulation for two cases, namely, short and long planning horizon. First, we solve the problem with τ and T as in the previous part, i.e., $\tau = 3$ and $T = 7$ and then we extend the planning horizon by keeping τ equal to its previous level, 3, and increasing the length of the second period from 4 to 15, i.e., we set $T = 18$. The demand parameters are as given in Table (4.2). We fix $c_0 = 0,\ k = 1,\ \delta = 0.2$ as before and we set $c_r = 0.7,\ w = 2$, and $\phi = 0.7$. Full results of the two scenarios, benchmark and CRS over short and long planning horizons, are given in Tables (4.9), (4.10), (4.11) and (4.12) of appendix B respectively.

The first look at the results shows that regardless of the length of the planning horizon, the price is always lower in the CRS scenario compared to the benchmark scenario which leads to higher demand in the CRS scenario. We always have significantly higher return rate in the CRS scenario which is expected. In order to visualize the difference between the level of the variables in different scenarios, we use the top scenario of the complete event tree, i.e., the scenario which starts at root node and terminates at node n^{127}. As shown in the graphs of Figure (4.2), we have lower price in CRS scenario at each node compared to the price at the same node in benchmark scenario. This relation is reverse for G, Q, and r, i.e., level of G, Q and r at each node is higher in CRS scenario compared to the same node in benchmark scenario. The results are qualitatively the same for any other complete scenario over the event tree. The graphs are drawn for the top scenario over the long planning horizon. All the variables follow the same trend over short planning horizon as well.

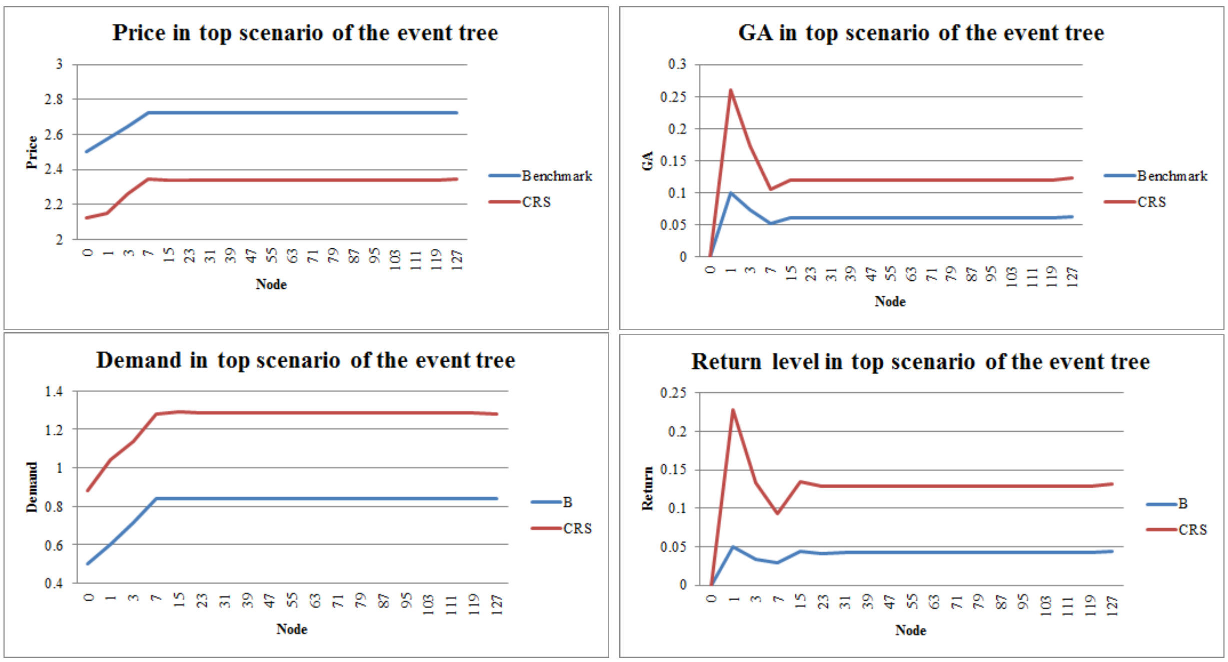

Figure 4.2: p, G, Q and r in the top scenario of the event tree

Getting closer to the end of planning horizon, under CRS contract regardless of the length of the planning horizon, we have lower demand in the last stage which is due to setting higher prices. Note that in short planning horizon we have lower demand (higher price) in stage $t = 7$ (the last stage) compared to the demand (price) in the same stage in long planning horizon (an intermediate stage).

4.5 Conclusion

This paper is the first attempt to show how to implement CRS contract in a closed loop supply chain in the class of games played over an event tree. We introduced a game with two players, one manufacturer and one retailer, played over an event tree which is specified by the fluctuations in demand law parameters. We have characterized the equilibrium strategies in two different structures, namely benchmark and CRS scenarios with a dynamic return volume level that serves to decrease the unit production cost. In the benchmark scenario in which the players play simultaneously, we characterized the price and GA efforts as the control variables of the retailer and the manufacturer respectively. In CRS scenario, the retailer acts as the leader and sets the retail price and the level of his contribution in GA efforts executed by the manufacturer and then the latter as the follower sets his GA efforts level.

In the first scenario, we came up with strategies that are state independent, which is a result of the linear structure of the game while in the CRS scenario, we obtained time dependent optimal price and GA efforts and constant support rate. In the CRS scenario, we always observed higher demand level due to the lower price set by the retailer. Furthermore, the return volume turned to be higher in the CRS scenario. We could also show that under a specific parameter setting, the CRS scenarios leads to a Pareto-improving solution and hence is appealing to both players.

In any modeling effort, some restrictive assumptions are required for mathematical tractability. Thus, we believe that some extensions in terms of modeling are worth conducting. We assumed that cost saving is the main purpose of the supply chain. It has been shown in the literature that the customers who return a product, usually purchase a new one [51]. One interesting extension would be to assume demand enhancing as the other purpose of closing the chain by letting the return volume to influence the demand level. In that case, current return volume will also appear in the demand function.

Furthermore, we assumed that a product can be reused in the manufacturing process for infinite number of times without any limitation. Another crucial assumption is that the returned products are remanufactured and sold as new product and the customers do not distinguish between the two versions of the product, i.e., there is no secondary market. Investigating the results without these simplifying assumptions would be interesting though challenging.

The other avenue for the research is to investigate the effect of offering some monetary incentives to the costumers to return the product. Differently, a penalty can be added per non-returned product. Clearly, the latter needs major modeling modifications in order to induce customers to buy the product at the first place.

Finally, competition might be another interesting factor to investigate. One may consider a situation with more than one CLSC whose decisions influence each other. In addition to demand level, they may compete in the collection of end-of-use products.

4.6 Appendix A

4.6.1 Proof of Proposition 1

The retailer's optimization problem is given by

$$\max_{p(n_l^t)\geq 0} J_R\left(p\right) = Q(n^0)\Big(p(n^0) - w\Big) + \sum_{t=1}^{T} \sum_{n_l^t \in \mathcal{N}^t} \pi\left(n_l^t\right)\Big\{Q(n_l^t)\Big(p(n_l^t) - w\Big)\Big\},$$

which is independent of the manufacturer's decision variable G and of the state variable r. Assuming an interior solution, the first-order optimality condition at node n_l^t reads

$$\frac{dJ_R}{dp(n_l^t)} = \pi^{n_l^t}\Big(\alpha^{n_l^t} - 2\beta^{n_l^t} p(n_l^t) + \beta^{n_l^t} w\Big) = 0,$$

which yields

$$p(n_l^t) = \frac{\alpha^{n_l^t} + \beta^{n_l^t} w}{2\beta^{n_l^t}}, \quad \forall n_l^t \in \mathcal{N}^t, t = 0, ..., T.$$

Clearly, $p(n_l^t) > 0$ and as $J_R\left(p\right)$ is concave in $p(n_l^t)$, we have an interior maximum.

Introduce the manufacturer's Hamiltonian

$$\begin{aligned}
\mathcal{H}_M &= (\alpha^{n^0} - \beta^{n^0} p(n^0))(w - c(r(n^0))) - \frac{k}{2}(G(n^0))^2 \\
&+ \sum_{t=1}^{T} \sum_{n_l^t \in \mathcal{N}^t} \pi^{n_l^t}\Big\{\big(\alpha^{n_l^t} - \beta^{n_l^t} p(n_l^t)\big)\Big(w - \big(c_0 - c_r r(n_l^t)\big)\Big) - \frac{k}{2}(G(n_l^t))^2 \\
&+ \lambda(n_l^t)\Big(G(n_l^t)\Big(\alpha^{a(n_l^t)} - \beta^{a(n_l^t)} p(a(n_l^t))\Big) - \delta r(a(n_l^t))\Big)\Big\},
\end{aligned} \tag{4.22}$$

where $\lambda\left(\cdot\right)$ is the vector of costate variables.

Differentiating $\mathcal{H}_M$ with respect to $G(n_l^t)$ and equating to zero, we get

$$\frac{\partial \mathcal{H}_M}{\partial G(n_l^t)} = \pi^{n_l^t}\Big(\lambda(n_l^t)\Big(\alpha^{a(n_l^t)} - \beta^{a(n_l^t)} p(a(n_l^t))\Big) - kG(n_l^t)\Big) = 0,$$

$$\Leftrightarrow G(n_l^t) = \frac{\lambda(n_l^t)\Big(\alpha^{a(n_l^t)} - \beta^{a(n_l^t)} p(a(n_l^t))\Big)}{k}, \quad \forall n_l^t, t = 1, ..., T. \tag{4.23}$$

The costate variables are derived using the following system of equations,

$$\lambda(n_l^t) = \frac{\partial \mathcal{H}_M}{\partial r(n_l^t)} = \pi^{n_l^t} c_r \Big(\frac{\alpha^{n_l^t} - \beta^{n_l^t} w}{2} \Big) - \delta \sum_{\nu \in S(n_l^t)} \pi^{\nu} \lambda(\nu), \quad \forall n_l^t \in \mathcal{N}^t, 0 = 1, ..., T-1 \quad (4.24)$$

$$\lambda(n_l^T) = \pi^{n_l^T} c_r \Big(\alpha^{n_l^T} - \beta^{n_l^T} p(n_l^T) \Big) = \pi^{n_l^T} c_r \Big(\frac{\alpha^{n_l^T} - \beta^{n_l^T} w}{2} \Big)$$

The transversality condition is $\lambda(v) = 0, \forall v \in S(n^t)$, meaning that returns after period T have no value to the manufacturer.

Substituting for (4.24) in $G(n_l^t)$ leads to the results in the proposition.

4.6.2 Proof of Proposition 2

To compute a Stackelberg equilibrium, we first determine the reaction function of the manufacturer to the announcement of the retailer of a retail price $p(n_l^t)$ and a support rate $B(n_l^t)$, $\forall n_l^t \in \mathcal{N}^t, t = 1, ..., T$. The manufacturer's Hamiltonian is as follows:

$$\begin{aligned} \mathcal{H}_M = \sum_{t=0}^{T} \sum_{n_l^t \in \mathcal{N}^t} \pi^{n_l^t} \Big\{ & \Big(\alpha^{n_l^t} - \beta^{n_l^t} p(n_l^t) \Big) (1 - \phi) \left(w - c_0 + c_r r(n_l^t) \right) \\ & - \frac{k}{2} \left(1 - B(n_l^t) \right) (G(n_l^t))^2 + \lambda(n_l^t) \left(G(n_l^t) Q(a(n_l^t)) - \delta r(a(n_l^t)) \right) \Big\}, \end{aligned}$$

where $\lambda(\cdot)$ is the costate variable appended by the manufacturer to the state dynamics in (4.3). Maximizing with respect to $G(n_l^t)$, we get

$$\frac{\partial \mathcal{H}_M}{\partial G(n_l^t)} = \pi^{n_l^t} \Big\{ -k(1 - B(n_l^t)) G(n_l^t) + \lambda(n_l^t) Q(a(n_l^t)) \Big\} = 0,$$

which yields

$$G(n_l^t) = \lambda(n_l^t) \frac{Q(a(n_l^t))}{k(1 - B(n_l^t))}, \quad \forall n_l^t \in \mathcal{N}^t, 0 = 1, ..., T.$$

Note that under the assumptions of an interior solution and positive demand, $\lambda(n_l^t)$ must be positive to have a positive $G(n_l^t)$. The conditions with respect to $\lambda(n_l^t)$ are given by

$$\begin{aligned} \lambda(n_l^t) &= \frac{\partial \mathcal{H}_M}{\partial r(n_l^t)} = \pi^{n_l^t} c_r (1 - \phi)(\alpha^{n_l^t} - \beta^{n_l^t} p(n_l^t)) - \delta \sum_{\nu \in S(n_l^t)} \pi^{\nu} \lambda(\nu), \quad \forall n_l^t \in \mathcal{N}^t, 0 = 1, ..., T, \\ \lambda(n_l^{T+1}) &= 0. \end{aligned}$$

This costate variable will play the role of an additional state variable in the retailer's problem.

Substituting for $G(n_l^t)$ in the state dynamics, we get

$$r(n_l^t) = \lambda(n_l^t)\frac{Q^2(a(n_l^t))}{k(1-B(n_l^t))} - \delta r(a(n_l^t)), \quad r(n^0) = r_0. \tag{4.25}$$

The retailer's Hamiltonian is given by

$$\begin{aligned}\mathcal{H}_R = \sum_{t=0}^{T}\sum_{n_l^t\in\mathcal{N}^t} \pi^{n_l^t}\Big\{ & \left(\alpha^{n_l^t} - \beta^{n_l^t}p(n_l^t)\right)\left(p(n_l^t) - w + \phi\left(w - c_0 + c_r r(n_l^t)\right)\right) \\ & - \frac{\lambda^2(n_l^t)B(n_l^t)Q^2(a(n_l^t))}{2k(1-B(n_l^t))^2} + \mu_r(n_l^t)\left(\frac{\lambda(n_l^t)Q^2(a(n_l^t))}{k(1-B(n_l^t))} - \delta r(a(n_l^t))\right) \\ & + \mu_\lambda(n_l^t)\left(\pi^{n_l^t}c_r(1-\phi)\left(\alpha^{n_l^t} - \beta^{n_l^t}p(n_l^t)\right) - \sum_{\nu\in S(n_l^t)}\pi^\nu\lambda(\nu)\right)\Big\}\end{aligned}$$

where μ_r and μ_λ are the associated costate variables to the state variables r and λ, respectively.

Assuming an interior solution, the first-order optimality conditions for all $n_l^t \in \mathcal{N}^t, t = 0, ..., T$ are given by

$$\begin{aligned}\frac{\partial\mathcal{H}_R}{\partial p(n_l^t)} = {} & \pi^{n_l^t}\left(\alpha^{n_l^t} - \beta^{n_l^t}\left(2p(n_l^t) - w + \phi\left(w - c_0 + c_r r(n_l^t)\right) + \pi^{n_l^t}c_r(1-\phi)\mu_\lambda(n_l^t)\right)\right) \\ & + \frac{\beta^{n_l^t}(\alpha^{n_l^t} - \beta^{n_l^t}p(n_l^t))}{k}\sum_{\nu\in S(n_l^t)}\pi^\nu\left(\frac{\lambda^2(\nu)B(\nu)}{(1-B(\nu))^2} - \frac{2\mu_r(\nu)\lambda(\nu)}{(1-B(\nu))}\right) = 0,\end{aligned} \tag{4.26}$$

$$\frac{\partial\mathcal{H}_R}{\partial B(n_l^t)} = \pi^{n_l^t}\left(-\frac{\lambda^2(n_l^t)Q^2(a(n_l^t))(1+B(n_l^t))}{2k(1-B(n_l^t))^3} + \frac{\mu_r(n_l^t)\lambda(n_l^t)Q^2(a(n_l^t))}{k(1-B(n_l^t))^2}\right) = 0, \tag{4.27}$$

$$\mu_r(n_l^t) = \frac{\partial\mathcal{H}_R}{\partial r(n_l^t)} = \pi^{n_l^t}c_r\phi(\alpha^{n_l^t} - \beta^{n_l^t}p(n_l^t)) - \delta\sum_{\nu\in S(n_l^t)}\pi^\nu\mu_r(\nu), \tag{4.28}$$

$$\mu_r(n_l^{T+1}) = 0, \tag{4.29}$$

$$\mu_\lambda(n_l^t) = \frac{\partial \mathcal{H}_R}{\partial \lambda(n_l^t)} = \frac{\pi^{n_l^t} Q^2(a(n_l^t))}{k(1-B(n_l^t))^2}\left(\mu_\lambda(n_l^t)(1-B(n_l^t)) - \lambda(n_l^t)B(n_l^t)\right)$$
$$+ \pi^{n_l^t}\pi^{a(n_l^t)}\mu_r(a(n_l^t)), \tag{4.30}$$
$$\mu_\lambda(n_l^{T+1}) = 0, \tag{4.31}$$

$$\lambda(n_l^t) = \frac{\partial \mathcal{H}_M}{\partial r(n_l^t)} = \pi^{n_l^t} c_r(1-\phi)(\alpha^{n_l^t} - \beta^{n_l^t} p(n_l^t)) - \delta \sum_{\nu \in S(n_l^t)} \pi^\nu \lambda(\nu), \tag{4.32}$$
$$\lambda(n_l^{T+1}) = 0. \tag{4.33}$$

We first prove that

$$B(n_l^t) = \frac{3\phi - 1}{\phi + 1}, \text{ for } n_l^t \in \mathcal{N}^t, 0 = 1, ..., T,$$

Straightforward manipulations allow to simplify the expression of $B(n_l^t)$ to

$$B(n_l^t) = \frac{2\mu_r(n_l^t) - \lambda(n_l^t)}{2\mu_r(n_l^t) + \lambda(n_l^t)}. \tag{4.34}$$

Let us define $S^m(n_l^t)$ as the set of m^{th} level children of node n_l^t.[5] Clearly, in the case of a binary tree, we have $|S^m(n_l^t)| = 2^m$. Let us assume that n_l^t is a node at stage k of the event tree, $0 \le k \le T$. A direct observation is that members of $S^{T-k}(n_l^t)$ are 2^{T-k} nodes from the last stage, T.[6] We start by computing values of the multipliers $\mu_r(n_l^t)$ and $\lambda(n_l^t)$ not in a

[5] $S^1(n_l^t) = S(n_l^t)$ is the set of the children of n_l^t and $S^2(n_l^t)$ is the set of the grand children of n_l^t.
[6] Number of nodes at stage T is equal to 2^T.

recursive form[7] as follows:

$$\begin{aligned}
\mu_r(n_l^t) &= \pi^{n_l^t}\phi c_r Q(n_l^t) - \delta \sum_{\nu_1 \in S(n^t)} (\pi^{\nu_1})^2 \phi c_r Q(\nu_1) \\
&+ \delta^2 \sum_{\nu_2 \in S^2(n_l^t)} \sum_{\nu_1 \in S(n_l^t)} \pi^{\nu_1}(\pi^{\nu_2})^2 \phi c_r Q(\nu_2) \\
&- \delta^3 \sum_{\nu_3 \in S^3(n_l^t)} \sum_{\nu_2 \in S^2(n_l^t)} \sum_{\nu_1 \in S(n_l^t)} \pi^{\nu_1}\pi^{\nu_2}(\pi^{\nu_3})^2 \phi c_r Q(\nu_3) \\
&+ \delta^4 \sum_{\nu_4 \in S^4(n_l^t)} \sum_{\nu_3 \in S^3(n_l^t)} \sum_{\nu_2 \in S^2(n_l^t)} \sum_{\nu_1 \in S(n_l^t)} \pi^{\nu_1}\pi^{\nu_2}\pi^{\nu_3}(\pi^{\nu_4})^2 \phi c_r Q(\nu_4) \\
&+ ... + (-1)^k \delta^k \sum_{\nu_1 \in S(n_l^t)} ... \sum_{\nu_k \in S^{T-k}(n_l^t)} \pi^{\nu_1} ... (\pi^{\nu_k})^2 \phi c_r Q(\nu_k) \\
&= \phi c_r \Big\{ \pi^{n_l^t} Q(n_l^t) - \delta \sum_{\nu_1 \in S(n_l^t)} (\pi^{\nu_1})^2 Q(\nu_1) \\
&+ ... + (-1)^k \delta^k \sum_{\nu_1 \in S(n_l^t)} ... \sum_{\nu_k \in S^{T-k}(n_l^t)} \pi^{\nu_1} ... (\pi^{\nu_k})^2 Q(\nu_k) \Big\},
\end{aligned}$$

and,

$$\begin{aligned}
\lambda(n_l^t) &= \pi^{n_l^t}(1-\phi) c_r Q(n_l^t) - \delta \sum_{\nu_1 \in S(n_l^t)} (\pi^{\nu_1})^2 (1-\phi) c_r Q(\nu_1) \\
&+ \delta^2 \sum_{\nu_2 \in S^2(n_l^t)} \sum_{\nu_1 \in S(n_l^t)} \pi^{\nu_1}(\pi^{\nu_2})^2 (1-\phi) c_r Q(\nu_2) \\
&+ ... + (-1)^k \delta^k \sum_{\nu_1 \in S(n_l^t)} ... \sum_{\nu_k in S^{T-k}(n_l^t)} \pi^{\nu_1} ... (\pi^{\nu_k})^2 (1-\phi) c_r Q(\nu_k) \\
&= (1-\phi) c_r \Big\{ \pi^{n_l^t} Q(n_l^t) - \delta \sum_{\nu_1 \in S(n_l^t)} (\pi^{\nu_1})^2 Q(\nu_1) \\
&+ ... + (-1)^k \delta^k \sum_{\nu_1 \in S(n_l^t)} ... \sum_{\nu_k \in S^{T-k}(n_l^t)} \pi^{\nu_1} ... (\pi^{\nu_k})^2 Q(\nu_k) \Big\}
\end{aligned}$$

Now, let us compute the numerator and denominator of the expression of $B(n^t)$:

[7] There is no $\mu_r(.)$ or $\lambda(.)$ at the right-hand side of the equations.

$$
\begin{aligned}
2\mu_r(n_l^t) - \lambda(n_l^t) &= 2\phi c_r \Big\{ \pi^{n^t} Q(n^t) - \delta \sum_{\nu_1 \in S(n^t)} (\pi^{\nu_1})^2 Q(\nu_1) \\
&+ ... + (-1)^k \delta^k \sum_{\nu_1 \in S(n^t)} ... \sum_{\nu_k in S^{T-k}(n^t)} \pi^{\nu_1} ... (\pi^{\nu_k})^2 Q(\nu_k) \Big\} \\
&- (1-\phi) c_r \Big\{ \pi^{n^t} Q(n^t) - \delta \sum_{\nu_1 \in S(n^t)} (\pi^{\nu_1})^2 Q(\nu_1) \\
&+ ... + (-1)^k \delta^k \sum_{\nu_1 \in S(n^t)} ... \sum_{\nu_k in S^{T-k}(n^t)} \pi^{\nu_1} ... (\pi^{\nu_k})^2 Q(\nu_k) \Big\} \\
&= c_r \Big(2\phi - (1-\phi) \Big) \Big\{ \pi^{n^t} Q(n^t) - \delta \sum_{\nu_1 \in S(n^t)} (\pi^{\nu_1})^2 Q(\nu_1) \\
&+ ... + (-1)^k \delta^k \sum_{\nu_1 \in S(n^t)} ... \sum_{\nu_k in S^{T-k}(n^t)} \pi^{\nu_1} ... (\pi^{\nu_k})^2 Q(\nu_k) \Big\}
\end{aligned}
$$

$$
\begin{aligned}
2\mu_r(n_l^t) + \lambda(n_l^t) &= 2\phi c_r \Big\{ \pi^{n_l^t} Q(n_l^t) - \delta \sum_{\nu_1 \in S(n_l^t)} (\pi^{\nu_1})^2 Q(\nu_1) \\
&+ ... + (-1)^k \delta^k \sum_{\nu_1 \in S(n_l^t)} ... \sum_{\nu_k \in S^{T-k}(n_l^t)} \pi^{\nu_1} ... (\pi^{\nu_k})^2 Q(\nu_k) \Big\} \\
&+ (1-\phi) c_r \Big\{ \pi^{n_l^t} Q(n_l^t) - \delta \sum_{\nu_1 \in S(n_l^t)} (\pi^{\nu_1})^2 Q(\nu_1) \\
&+ ... + (-1)^k \delta^k \sum_{\nu_1 \in S(n_l^t)} ... \sum_{\nu_k \in S^{T-k}(n_l^t)} \pi^{\nu_1} ... (\pi^{\nu_k})^2 Q(\nu_k) \Big\} \\
&= c_r \Big(2\phi + (1-\phi) \Big) \Big\{ \pi^{n_l^t} Q(n_l^t) - \delta \sum_{\nu_1 \in S(n_l^t)} (\pi^{\nu_1})^2 Q(\nu_1) \\
&+ ... + (-1)^k \delta^k \sum_{\nu_1 \in S(n_l^t)} ... \sum_{\nu_k \in S^{T-k}(n_l^t)} \pi^{\nu_1} ... (\pi^{\nu_k})^2 Q(\nu_k) \Big\}
\end{aligned}
$$

Simplifying for the common factors in the fraction, we end up having

$$
B(n_l^t) = \frac{2\mu_r(n_l^t) - \lambda(n_l^t)}{2\mu_r(n_l^t) + \lambda(n_l^t)} = \frac{2\phi - (1-\phi)}{2\phi + (1-\phi)} = \frac{3\phi - 1}{\phi + 1}.
$$

Substituting for $B(n_l^t)$ in the expressions of the control variables and costate and state variables, lead to the expressions in the Proposition.

4.7 Appendix B

4.7.1 Numerical results: short planning horizon

Table 4.9: Benchmark scenario: short planning horizon

Description	Node	p	G	α	β	Q	r
root node	0	2.5	0	3	1	0.5	0
	1	2.5714	0.0999	3.3	1.05	0.60003	0.04995
end of stage 1	2	2.4211	0.0666	2.7	0.95	0.399955	0.0333
	3	2.6463	0.073	3.63	1.1025	0.712454	0.033812
	4	2.4887	0.0499	2.97	0.9975	0.487522	0.019951
	5	2.4887	0.0333	2.97	0.9975	0.487522	0.006659
end of stage 2	6	2.3463	0.0213	2.43	0.9025	0.312464	0.001859
last stochastic stage	7	2.7247	0.051	3.993	1.1576	0.838887	0.029573
	8	2.5596	0.0356	3.267	1.0474	0.586075	0.018601
	9	2.5596	0.0244	3.267	1.0474	0.586075	0.007905
	10	2.4104	0.0162	2.673	0.9476	0.388905	0.003908
	11	2.5596	0.0244	3.267	1.0474	0.586075	0.010564
	12	2.4104	0.0162	2.673	0.9476	0.388905	0.006566
	13	2.4104	0.0104	2.673	0.9476	0.388905	0.002878
last stochastic node	14	2.2754	0.0063	2.187	0.8574	0.236072	0.001597
	15	2.7247	0.0601	3.993	1.1576	0.838887	0.044503
	16	2.5596	0.0293	3.267	1.0474	0.586075	0.013452
	17	2.5596	0.0293	3.267	1.0474	0.586075	0.015591
	18	2.4104	0.0129	2.673	0.9476	0.388905	0.004235
	19	2.5596	0.0293	3.267	1.0474	0.586075	0.015059
	20	2.4104	0.0129	2.673	0.9476	0.388905	0.003704
	21	2.4104	0.0129	2.673	0.9476	0.388905	0.004441
end of stage 4	22	2.2754	0.0048	2.187	0.8574	0.236072	0.000814
	23	2.7247	0.0601	3.993	1.1576	0.838887	0.041517
	24	2.5596	0.0293	3.267	1.0474	0.586075	0.014482
	25	2.5596	0.0293	3.267	1.0474	0.586075	0.014054
	26	2.4104	0.0129	2.673	0.9476	0.388905	0.00417
	27	2.5596	0.0293	3.267	1.0474	0.586075	0.01416
	28	2.4104	0.0129	2.673	0.9476	0.388905	0.004276
	29	2.4104	0.0129	2.673	0.9476	0.388905	0.004129
end of stage 5	30	2.2754	0.0048	2.187	0.8574	0.236072	0.00097
	31	2.7247	0.0601	3.993	1.1576	0.838887	0.042114
	32	2.5596	0.0293	3.267	1.0474	0.586075	0.014276
	33	2.5596	0.0293	3.267	1.0474	0.586075	0.014361
	34	2.4104	0.0129	2.673	0.9476	0.388905	0.004183
	35	2.5596	0.0293	3.267	1.0474	0.586075	0.01434
	36	2.4104	0.0129	2.673	0.9476	0.388905	0.004162
	37	2.4104	0.0129	2.673	0.9476	0.388905	0.004191
end of stage 6	38	2.2754	0.0048	2.187	0.8574	0.236072	0.000939
	39	2.7247	0.0601	3.993	1.1576	0.838887	0.041994
	40	2.5596	0.0293	3.267	1.0474	0.586075	0.014317
	41	2.5596	0.0293	3.267	1.0474	0.586075	0.0143
	42	2.4104	0.0129	2.673	0.9476	0.388905	0.00418
	43	2.5596	0.0293	3.267	1.0474	0.586075	0.014304

Continued on next page

Table 4.9 – continued from previous page

Description	Node	p	G	α	β	Q	r
	44	2.4104	0.0129	2.673	0.9476	0.388905	0.004185
	45	2.4104	0.0129	2.673	0.9476	0.388905	0.004179
end of stage 7	46	2.2754	0.0048	2.187	0.8574	0.236072	0.000945

Table 4.10: CRS scenario: short planning horizon

Description	Node	p	G	B	Q	r
root node	0	2.1201	0	0	0.8799	0
	1	2.1512	0.259746	0.647019	1.04124	0.2286
end of stage 1	2	2.0223	0.194458	0.647059	0.778815	0.1711
	3	2.2591	0.172221	0.646917	1.139342	0.1336
	4	2.1147	0.130103	0.647059	0.860587	0.0897
	5	2.127	0.095911	0.646772	0.848318	0.0405
end of stage 2	6	1.9901	0.07169	0.646931	0.633935	0.0216
last stochastic stage	7	2.3461	0.10556	0.647059	1.277155	0.0935
	8	2.1913	0.080381	0.647059	0.971832	0.0648
	9	2.1989	0.060198	0.646891	0.963872	0.0338
	10	2.0542	0.045353	0.647059	0.72644	0.0211
	11	2.1968	0.059425	0.647395	0.966072	0.0424
	12	2.052	0.044834	0.646168	0.728525	0.03
	13	2.0552	0.033345	0.646388	0.725492	0.0168
last stochastic node	14	1.9222	0.024787	0.647059	0.538906	0.0114
	15	2.3361	0.119414	0.647059	1.288731	0.1338
	16	2.1941	0.06832	0.647226	0.9689	0.0535
	17	2.1928	0.067905	0.646558	0.970261	0.0587
	18	2.0535	0.038283	0.647059	0.727103	0.0236
	19	2.1931	0.068011	0.647727	0.969947	0.0572
	20	2.0539	0.038393	0.647059	0.726724	0.022
	21	2.0534	0.038306	0.647727	0.727198	0.0244
end of stage 4	22	1.9228	0.021071	0.647059	0.538391	0.0091
	23	2.3375	0.120367	0.646681	1.28711	0.1283
	24	2.1936	0.068114	0.647226	0.969423	0.0554
	25	2.1938	0.068209	0.647226	0.969214	0.0545
	26	2.0537	0.038318	0.647059	0.726914	0.0232
	27	2.1938	0.068187	0.647226	0.969214	0.0547
	28	2.0536	0.038298	0.647059	0.727009	0.0235
	29	2.0537	0.038323	0.647059	0.726914	0.023
end of stage 5	30	1.9227	0.021051	0.647059	0.538477	0.0095
	31	2.3371	0.120216	0.646681	1.287573	0.129
	32	2.1936	0.06815	0.647226	0.969423	0.055
	33	2.1935	0.068136	0.647226	0.969528	0.0552
	34	2.0536	0.038308	0.647059	0.727009	0.0232
	35	2.1935	0.068136	0.647226	0.969528	0.0551
	36	2.0536	0.038313	0.647059	0.727009	0.0232
	37	2.0536	0.038308	0.647059	0.727009	0.0233
end of stage 6	38	1.9227	0.021054	0.647059	0.538477	0.0094
	39	2.3421	0.122706	0.647429	1.281785	0.1322
	40	2.1957	0.069702	0.646732	0.967224	0.0566
	41	2.1957	0.069709	0.646732	0.967224	0.0566
	42	2.0545	0.039295	0.646623	0.726156	0.0239
	43	2.1957	0.069709	0.646732	0.967224	0.0566
	44	2.0545	0.039295	0.646623	0.726156	0.0239
	45	2.0545	0.039295	0.646623	0.726156	0.0239
end of stage 7	46	1.923	0.021566	0.64794	0.53822	0.0097

4.7.2 Numerical results: long planning horizon

Table 4.11: Benchmark scenario: long planning horizon

Description	Node	p	G	α	β	Q	r
root node	0	2.5	0	3	1	0.5	0
	1	2.5714	0.0999	3.3	1.05	0.60003	0.04995
end of stage 1	2	2.4211	0.0666	2.7	0.95	0.399955	0.0333
	3	2.6463	0.073	3.63	1.1025	0.712454	0.033812
	4	2.4887	0.0499	2.97	0.9975	0.487522	0.019951
	5	2.4887	0.0333	2.97	0.9975	0.487522	0.006659
end of stage 2	6	2.3463	0.0213	2.43	0.9025	0.312464	0.001859
last stochastic stage	7	2.7247	0.051	3.993	1.1576	0.838887	0.029573
	8	2.5596	0.0356	3.267	1.0474	0.586075	0.018601
	9	2.5596	0.0244	3.267	1.0474	0.586075	0.007905
	10	2.4104	0.0162	2.673	0.9476	0.388905	0.003908
	11	2.5596	0.0244	3.267	1.0474	0.586075	0.010564
	12	2.4104	0.0162	2.673	0.9476	0.388905	0.006566
	13	2.4104	0.0104	2.673	0.9476	0.388905	0.002878
last stochastic node	14	2.2754	0.0063	2.187	0.8574	0.236072	0.001597
	15	2.7247	0.0601	3.993	1.1576	0.838887	0.044503
	16	2.5596	0.0293	3.267	1.0474	0.586075	0.013452
	17	2.5596	0.0293	3.267	1.0474	0.586075	0.015591
	18	2.4104	0.0129	2.673	0.9476	0.388905	0.004235
	19	2.5596	0.0293	3.267	1.0474	0.586075	0.015059
	20	2.4104	0.0129	2.673	0.9476	0.388905	0.003704
	21	2.4104	0.0129	2.673	0.9476	0.388905	0.004441
end of stage 4	22	2.2754	0.0048	2.187	0.8574	0.236072	0.000814
	23	2.7247	0.0601	3.993	1.1576	0.838887	0.041517
	24	2.5596	0.0293	3.267	1.0474	0.586075	0.014482
	25	2.5596	0.0293	3.267	1.0474	0.586075	0.014054
	26	2.4104	0.0129	2.673	0.9476	0.388905	0.00417
	27	2.5596	0.0293	3.267	1.0474	0.586075	0.01416
	28	2.4104	0.0129	2.673	0.9476	0.388905	0.004276
	29	2.4104	0.0129	2.673	0.9476	0.388905	0.004129
end of stage 5	30	2.2754	0.0048	2.187	0.8574	0.236072	0.00097
	31	2.7247	0.0601	3.993	1.1576	0.838887	0.042114
	32	2.5596	0.0293	3.267	1.0474	0.586075	0.014276
	33	2.5596	0.0293	3.267	1.0474	0.586075	0.014361
	34	2.4104	0.0129	2.673	0.9476	0.388905	0.004183
	35	2.5596	0.0293	3.267	1.0474	0.586075	0.01434
	36	2.4104	0.0129	2.673	0.9476	0.388905	0.004162
	37	2.4104	0.0129	2.673	0.9476	0.388905	0.004191
end of stage 6	38	2.2754	0.0048	2.187	0.8574	0.236072	0.000939
	39	2.7247	0.0601	3.993	1.1576	0.838887	0.041994
	40	2.5596	0.0293	3.267	1.0474	0.586075	0.014317
	41	2.5596	0.0293	3.267	1.0474	0.586075	0.0143
	42	2.4104	0.0129	2.673	0.9476	0.388905	0.00418
	43	2.5596	0.0293	3.267	1.0474	0.586075	0.014304
	44	2.4104	0.0129	2.673	0.9476	0.388905	0.004185
	45	2.4104	0.0129	2.673	0.9476	0.388905	0.004179
end of stage 7	46	2.2754	0.0048	2.187	0.8574	0.236072	0.000945
	47	2.7247	0.0601	3.993	1.1576	0.838887	0.042018

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Table 4.11 – continued from previous page

Description	Node	p	G	α	β	Q	r
	48	2.5596	0.0293	3.267	1.0474	0.586075	0.014309
	49	2.5596	0.0293	3.267	1.0474	0.586075	0.014312
	50	2.4104	0.0129	2.673	0.9476	0.388905	0.004181
	51	2.5596	0.0293	3.267	1.0474	0.586075	0.014311
	52	2.4104	0.0129	2.673	0.9476	0.388905	0.00418
	53	2.4104	0.0129	2.673	0.9476	0.388905	0.004181
end of stage 8	54	2.2754	0.0048	2.187	0.8574	0.236072	0.000944
	55	2.7247	0.0601	3.993	1.1576	0.838887	0.042013
	56	2.5596	0.0293	3.267	1.0474	0.586075	0.01431
	57	2.5596	0.0293	3.267	1.0474	0.586075	0.01431
	58	2.4104	0.0129	2.673	0.9476	0.388905	0.004181
	59	2.5596	0.0293	3.267	1.0474	0.586075	0.01431
	60	2.4104	0.0129	2.673	0.9476	0.388905	0.004181
	61	2.4104	0.0129	2.673	0.9476	0.388905	0.004181
end of stage 9	62	2.2754	0.0048	2.187	0.8574	0.236072	0.000944
	63	2.7247	0.0601	3.993	1.1576	0.838887	0.042014
	64	2.5596	0.0293	3.267	1.0474	0.586075	0.01431
	65	2.5596	0.0293	3.267	1.0474	0.586075	0.01431
	66	2.4104	0.0129	2.673	0.9476	0.388905	0.004181
	67	2.5596	0.0293	3.267	1.0474	0.586075	0.01431
	68	2.4104	0.0129	2.673	0.9476	0.388905	0.004181
	69	2.4104	0.0129	2.673	0.9476	0.388905	0.004181
end of stage 10	70	2.2754	0.0048	2.187	0.8574	0.236072	0.000944
	71	2.7247	0.0601	3.993	1.1576	0.838887	0.042014
	72	2.5596	0.0293	3.267	1.0474	0.586075	0.01431
	73	2.5596	0.0293	3.267	1.0474	0.586075	0.01431
	74	2.4104	0.0129	2.673	0.9476	0.388905	0.004181
	75	2.5596	0.0293	3.267	1.0474	0.586075	0.01431
	76	2.4104	0.0129	2.673	0.9476	0.388905	0.004181
	77	2.4104	0.0129	2.673	0.9476	0.388905	0.004181
end of stage 11	78	2.2754	0.0048	2.187	0.8574	0.236072	0.000944
	79	2.7247	0.0601	3.993	1.1576	0.838887	0.042014
	80	2.5596	0.0293	3.267	1.0474	0.586075	0.01431
	81	2.5596	0.0293	3.267	1.0474	0.586075	0.01431
	82	2.4104	0.0129	2.673	0.9476	0.388905	0.004181
	83	2.5596	0.0293	3.267	1.0474	0.586075	0.01431
	84	2.4104	0.0129	2.673	0.9476	0.388905	0.004181
	85	2.4104	0.0129	2.673	0.9476	0.388905	0.004181
end of stage 12	86	2.2754	0.0048	2.187	0.8574	0.236072	0.000944
	87	2.7247	0.0601	3.993	1.1576	0.838887	0.042014
	88	2.5596	0.0293	3.267	1.0474	0.586075	0.01431
	89	2.5596	0.0293	3.267	1.0474	0.586075	0.01431
	90	2.4104	0.0129	2.673	0.9476	0.388905	0.004181
	91	2.5596	0.0293	3.267	1.0474	0.586075	0.01431
	92	2.4104	0.0129	2.673	0.9476	0.388905	0.004181
	93	2.4104	0.0129	2.673	0.9476	0.388905	0.004181
end of stage 13	94	2.2754	0.0048	2.187	0.8574	0.236072	0.000944
	95	2.7247	0.0601	3.993	1.1576	0.838887	0.042014
	96	2.5596	0.0293	3.267	1.0474	0.586075	0.01431
	97	2.5596	0.0293	3.267	1.0474	0.586075	0.01431
	98	2.4104	0.0129	2.673	0.9476	0.388905	0.004181
	99	2.5596	0.0293	3.267	1.0474	0.586075	0.01431
	100	2.4104	0.0129	2.673	0.9476	0.388905	0.004181
	101	2.4104	0.0129	2.673	0.9476	0.388905	0.004181
end of stage 14	102	2.2754	0.0048	2.187	0.8574	0.236072	0.000944

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Table 4.11 – continued from previous page

Description	Node	p	G	α	β	Q	r
	103	2.7247	0.0601	3.993	1.1576	0.838887	0.042014
	104	2.5596	0.0293	3.267	1.0474	0.586075	0.01431
	105	2.5596	0.0293	3.267	1.0474	0.586075	0.01431
	106	2.4104	0.0129	2.673	0.9476	0.388905	0.004181
	107	2.5596	0.0293	3.267	1.0474	0.586075	0.01431
	108	2.4104	0.0129	2.673	0.9476	0.388905	0.004181
	109	2.4104	0.0129	2.673	0.9476	0.388905	0.004181
end of stage 15	110	2.2754	0.0048	2.187	0.8574	0.236072	0.000944
	111	2.7247	0.0601	3.993	1.1576	0.838887	0.042014
	112	2.5596	0.0293	3.267	1.0474	0.586075	0.01431
	113	2.5596	0.0293	3.267	1.0474	0.586075	0.01431
	114	2.4104	0.0129	2.673	0.9476	0.388905	0.004181
	115	2.5596	0.0293	3.267	1.0474	0.586075	0.01431
	116	2.4104	0.0129	2.673	0.9476	0.388905	0.004181
	117	2.4104	0.0129	2.673	0.9476	0.388905	0.004181
end of stage 16	118	2.2754	0.0048	2.187	0.8574	0.236072	0.000944
	119	2.7247	0.06	3.993	1.1576	0.838887	0.04193
	120	2.5596	0.0293	3.267	1.0474	0.586075	0.01431
	121	2.5596	0.0293	3.267	1.0474	0.586075	0.01431
	122	2.4104	0.0129	2.673	0.9476	0.388905	0.004181
	123	2.5596	0.0293	3.267	1.0474	0.586075	0.01431
	124	2.4104	0.0129	2.673	0.9476	0.388905	0.004181
	125	2.4104	0.0129	2.673	0.9476	0.388905	0.004181
end of stage 17	126	2.2754	0.0048	2.187	0.8574	0.236072	0.000944
	127	2.7247	0.0616	3.993	1.1576	0.838887	0.043289
	128	2.5596	0.0301	3.267	1.0474	0.586075	0.014779
	129	2.5596	0.0301	3.267	1.0474	0.586075	0.014779
	130	2.4104	0.0132	2.673	0.9476	0.388905	0.004297
	131	2.5596	0.0301	3.267	1.0474	0.586075	0.014779
	132	2.4104	0.0132	2.673	0.9476	0.388905	0.004297
	133	2.4104	0.0132	2.673	0.9476	0.388905	0.004297
end of stage 18	134	2.2754	0.0049	2.187	0.8574	0.236072	0.000968

Table 4.12: CRS scenario: long planning horizon

Description	Node	p	G	B	Q	r
root node	0	2.1201	0	0	0.8799	0
	1	2.1512	0.259746	0.647019	1.04124	0.2286
end of stage 1	2	2.0223	0.194458	0.647059	0.778815	0.1711
	3	2.2591	0.172221	0.646917	1.139342	0.1336
	4	2.1147	0.130103	0.647059	0.860587	0.0897
	5	2.127	0.095911	0.646772	0.848318	0.0405
end of stage 2	6	1.9901	0.07169	0.646931	0.633935	0.0216
last stochastic stage	7	2.3461	0.10556	0.647059	1.277155	0.0935
	8	2.1913	0.080381	0.647059	0.971832	0.0648
	9	2.1989	0.060198	0.646891	0.963872	0.0338
	10	2.0542	0.045353	0.647059	0.72644	0.0211
	11	2.1968	0.059425	0.647395	0.966072	0.0424
	12	2.052	0.044834	0.646168	0.728525	0.03
	13	2.0552	0.033345	0.646388	0.725492	0.0168
last stochastic node	14	1.9222	0.024787	0.647059	0.538906	0.0114
	15	2.3361	0.119414	0.647059	1.288731	0.1338
	16	2.1941	0.06832	0.647226	0.9689	0.0535
	17	2.1928	0.067905	0.646558	0.970261	0.0587

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Table 4.12 – continued from previous page

Description	Node	p	G	B	Q	r
	18	2.0535	0.038283	0.647059	0.727103	0.0236
	19	2.1931	0.068011	0.647727	0.969947	0.0572
	20	2.0539	0.038393	0.647059	0.726724	0.022
	21	2.0534	0.038306	0.647727	0.727198	0.0244
end of stage 4	22	1.9228	0.021071	0.647059	0.538391	0.0091
	23	2.3375	0.120367	0.646681	1.28711	0.1283
	24	2.1936	0.068114	0.647226	0.969423	0.0554
	25	2.1938	0.068209	0.647226	0.969214	0.0545
	26	2.0537	0.038318	0.647059	0.726914	0.0232
	27	2.1938	0.068187	0.647226	0.969214	0.0547
	28	2.0536	0.038298	0.647059	0.727009	0.0235
	29	2.0537	0.038323	0.647059	0.726914	0.023
end of stage 5	30	1.9227	0.021051	0.647059	0.538477	0.0095
	31	2.3373	0.120216	0.646681	1.287342	0.1291
	32	2.1937	0.06815	0.647226	0.969319	0.055
	33	2.1936	0.068136	0.647226	0.969423	0.0552
	34	2.0536	0.038308	0.647059	0.727009	0.0232
	35	2.1937	0.068136	0.647226	0.969319	0.0551
	36	2.0537	0.038313	0.647059	0.726914	0.0232
	37	2.0536	0.038308	0.647059	0.727009	0.0233
end of stage 6	38	1.9227	0.021054	0.647059	0.538477	0.0094
	39	2.3373	0.120238	0.646681	1.287342	0.129
	40	2.1937	0.068143	0.647226	0.969319	0.0551
	41	2.1937	0.06815	0.647226	0.969319	0.0551
	42	2.0536	0.038313	0.647059	0.727009	0.0232
	43	2.1937	0.068143	0.647226	0.969319	0.0551
	44	2.0536	0.038308	0.647059	0.727009	0.0232
	45	2.0536	0.038313	0.647059	0.727009	0.0232
end of stage 7	46	1.9227	0.021054	0.647059	0.538477	0.0094
	47	2.3373	0.120238	0.646681	1.287342	0.129
	48	2.1937	0.068143	0.647226	0.969319	0.0551
	49	2.1937	0.068143	0.647226	0.969319	0.0551
	50	2.0536	0.038313	0.647059	0.727009	0.0232
	51	2.1937	0.068143	0.647226	0.969319	0.0551
	52	2.0536	0.038313	0.647059	0.727009	0.0232
	53	2.0536	0.038313	0.647059	0.727009	0.0232
end of stage 8	54	1.9227	0.021054	0.647059	0.538477	0.0094
	55	2.3373	0.120238	0.646681	1.287342	0.129
	56	2.1937	0.068143	0.647226	0.969319	0.0551
	57	2.1937	0.068143	0.647226	0.969319	0.0551
	58	2.0536	0.038313	0.647059	0.727009	0.0232
	59	2.1937	0.068143	0.647226	0.969319	0.0551
	60	2.0536	0.038313	0.647059	0.727009	0.0232
	61	2.0536	0.038313	0.647059	0.727009	0.0232
end of stage 9	62	1.9227	0.021054	0.647059	0.538477	0.0094
	63	2.3373	0.120238	0.646681	1.287342	0.129
	64	2.1937	0.068143	0.647226	0.969319	0.0551
	65	2.1937	0.068143	0.647226	0.969319	0.0551
	66	2.0536	0.038313	0.647059	0.727009	0.0232
	67	2.1937	0.068143	0.647226	0.969319	0.0551
	68	2.0536	0.038313	0.647059	0.727009	0.0232
	69	2.0536	0.038313	0.647059	0.727009	0.0232
end of stage 10	70	1.9227	0.021054	0.647059	0.538477	0.0094
	71	2.3373	0.120238	0.646681	1.287342	0.129
	72	2.1937	0.068143	0.647226	0.969319	0.0551

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Table 4.12 – continued from previous page

Description	Node	p	G	B	Q	r
	73	2.1937	0.068143	0.647226	0.969319	0.0551
	74	2.0536	0.038313	0.647059	0.727009	0.0232
	75	2.1937	0.068143	0.647226	0.969319	0.0551
	76	2.0536	0.038313	0.647059	0.727009	0.0232
	77	2.0536	0.038313	0.647059	0.727009	0.0232
end of stage 11	78	1.9227	0.021054	0.647059	0.538477	0.0094
	79	2.3373	0.120238	0.646681	1.287342	0.129
	80	2.1937	0.068143	0.647226	0.969319	0.0551
	81	2.1937	0.068143	0.647226	0.969319	0.0551
	82	2.0536	0.038313	0.647059	0.727009	0.0232
	83	2.1937	0.068143	0.647226	0.969319	0.0551
	84	2.0536	0.038313	0.647059	0.727009	0.0232
	85	2.0536	0.038313	0.647059	0.727009	0.0232
end of stage 12	86	1.9227	0.021054	0.647059	0.538477	0.0094
	87	2.3373	0.120238	0.646681	1.287342	0.129
	88	2.1937	0.068143	0.647226	0.969319	0.0551
	89	2.1937	0.068143	0.647226	0.969319	0.0551
	90	2.0536	0.038313	0.647059	0.727009	0.0232
	91	2.1937	0.068143	0.647226	0.969319	0.0551
	92	2.0536	0.038313	0.647059	0.727009	0.0232
	93	2.0536	0.038313	0.647059	0.727009	0.0232
end of stage 13	94	1.9227	0.021054	0.647059	0.538477	0.0094
	95	2.3373	0.120238	0.646681	1.287342	0.129
	96	2.1937	0.068143	0.647226	0.969319	0.0551
	97	2.1937	0.068143	0.647226	0.969319	0.0551
	98	2.0536	0.038313	0.647059	0.727009	0.0232
	99	2.1937	0.068143	0.647226	0.969319	0.0551
	100	2.0536	0.038313	0.647059	0.727009	0.0232
	101	2.0536	0.038313	0.647059	0.727009	0.0232
end of stage 14	102	1.9227	0.021054	0.647059	0.538477	0.0094
	103	2.3373	0.120238	0.646681	1.287342	0.129
	104	2.1937	0.068143	0.647226	0.969319	0.0551
	105	2.1937	0.068143	0.647226	0.969319	0.0551
	106	2.0536	0.038313	0.647059	0.727009	0.0232
	107	2.1937	0.068143	0.647226	0.969319	0.0551
	108	2.0536	0.038313	0.647059	0.727009	0.0232
	109	2.0536	0.038313	0.647059	0.727009	0.0232
end of stage 15	110	1.9227	0.021054	0.647059	0.538477	0.0094
	111	2.3373	0.120238	0.646681	1.287342	0.129
	112	2.1937	0.068143	0.647226	0.969319	0.0551
	113	2.1937	0.068143	0.647226	0.969319	0.0551
	114	2.0536	0.038313	0.647059	0.727009	0.0232
	115	2.1937	0.068143	0.647226	0.969319	0.0551
	116	2.0536	0.038313	0.647059	0.727009	0.0232
	117	2.0536	0.038313	0.647059	0.727009	0.0232
end of stage 16	118	1.9227	0.021054	0.647059	0.538477	0.0094
	119	2.3371	0.120238	0.646681	1.287573	0.1289
	120	2.1936	0.068143	0.647226	0.969423	0.0551
	121	2.1936	0.068143	0.647226	0.969423	0.0551
	122	2.0536	0.038313	0.647059	0.727009	0.0232
	123	2.1936	0.068143	0.647226	0.969423	0.0551
	124	2.0536	0.038313	0.647059	0.727009	0.0232
	125	2.0536	0.038313	0.647059	0.727009	0.0232
end of stage 17	126	1.9227	0.021054	0.647059	0.538477	0.0094
	127	2.3421	0.122706	0.647429	1.281785	0.1322

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Table 4.12 – continued from previous page

Description	Node	p	G	B	Q	r
	128	2.1957	0.069702	0.646732	0.967224	0.0566
	129	2.1957	0.069702	0.646732	0.967224	0.0566
	130	2.0545	0.039295	0.646623	0.726156	0.0239
	131	2.1957	0.069702	0.646732	0.967224	0.0566
	132	2.0545	0.039295	0.646623	0.726156	0.0239
	133	2.0545	0.039295	0.646623	0.726156	0.0239
end of stage 18	134	1.923	0.021566	0.64794	0.53822	0.0097

4.8 Appendix C

4.8.1 Effect of k, c_0 and δ

Table (4.13) presents the expected payoff for each player over the short planning horizon for different values of the sharing parameter, ϕ.

Table 4.13: Expected payoff for each player for different sharing parameters

parameters	$w = 2, c_r = 0.7, c_0 = 1, k = 1, \delta = 0.2$							
Scenario	CRS							Benchmark
ϕ	0.9	0.8	0.7	0.6	0.5	0.4	0.34	N/A
Manufacturer	0.852858	1.57178	2.180285	2.6919	3.114845	3.454346	3.619371	4.04252808
Retailer	7.798791	6.895269	6.085199	5.35098	4.681337	4.068695	3.726233	2.10173035

Tables (4.14)-(4.16) show how changing parameter k, c_0, δ affect the expected payoff for the players in different scenarios. Comparing the numerical results in these tables with those of table (4.13) provides more insight about the model. Increasing k, does not change the expected payoff for the retailer in the benchmark scenario as expected while reduces the manufacturer's expected payoff. This change leads to a slightly lower expected payoffs for both players in the CRS scenario. Increasing c_0 is not in manufacturer's interest as he faces significantly lower expected payoff in both scenarios. Although the retailer is not affected in the benchmark scenario, he is highly penalized for higher c_0 in the CRS scenario. Decreasing δ leads to a negligible increase in both players' expected payoff.

Table 4.14: Effect of k on the expected payoff for each player for different ϕ

Scenario	CRS		Benchmark
parameters	$w = 2, c_r = 0.7, c_0 = 1, \mathbf{k = 3}, \delta = 0.2$		
ϕ	0.7	0.4	N/A
Manufacturer	2.082615	3.390384	4.014169
Retailer	5.944443	4.036965	2.10173

Table 4.15: Effect of c_0 on the expected payoff for each player for different ϕ

Scenario	CRS		Benchmark
parameters	$w = 2, c_r = 0.7, \mathbf{c_0 = 2}, k = 1, \delta = 0.2$		
ϕ	0.7	0.4	N/A
Manufacturer	0.011484	0.018462	0.042759
Retailer	2.12916	2.115438	2.10173

Table 4.16: Effect of δ on the expected payoff for each player for different ϕ

Scenario	CRS		Benchmark
parameters	$w = 2, c_r = 0.7, c_0 = 1, k = 1, \delta = \mathbf{0.1}$		
ϕ	0.7	0.4	N/A
Manufacturer	2.195456	3.46373	4.047214
Retailer	6.108875	4.073544	2.10173

Chapter 5

Conclusion

In this thesis, composed of three essays, dynamic games are used to study the strategic interactions between independent agents where the data of the problem are stochastic. We take into account the uncertainty involved in the problem by means of a class of dynamic games with the particularity of being played over the event trees, that is, trees where the transition from one node to another is nature's decision and cannot be influenced by the players' actions. In other words, the agents' expectation about the uncertainty involved in the problem is given beforehand as an event tree. The main interest of this class of games lies in its wide range of application in economics and management sciences, where it is quite natural to assume that some of the problem's data are stochastic.

In the first essay entitled "*$\mathcal{S}$-Adapted Equilibria in Games Played Over Event Trees with Coupled Constraints*", we extend the framework of dynamic games played over event trees (DGPET) to the setting where the players face coupled constraints. This essay deals with a policy coordination model where a supranational agent has to induce a set of countries competing on an oligopolistic market to achieve a common global constraint. A game of multiple players (countries) producing a homogeneous good is considered while there is an uncertain fluctuation in the price of a commonly used nonrenewable resource. At the same time, the players must keep the total pollutant emissions less than a certain level. The concept of normalized equilibrium is used to solve the problem. Existence and uniqueness conditions for this equilibrium are provided, as well as a stochastic-control formulation of the game and a maximum principle. The problem is also solved through introducing the penalty tax rates for the violation of the coupled constraint.

This essay gives directions for the future research. The main challenge for future developments is to compute the more conceptually appealing feedback equilibria for this class of games. Also, one may seek to compute the S-adapted normalized equilibrium as solution to an extended variational inequality, for which we have shown the existence and uniqueness conditions.

The second essay of this thesis addresses sustainability in cooperative dynamic games, that is, how to ensure that each player will indeed implement her part of the agreement as time goes by. In the second essay entitled "*Incentive Equilibrium Strategies in Dynamic Games Played over Event Trees*", we characterize incentive equilibrium strategies and outcomes for the class of dynamic games played over event trees. We show that the coordinated solution that optimizes the joint payoff can be achieved as an incentive equilibrium. We determined incentive equilibrium strategies for two popular classes of dynamic games in applications namely, linear-state and linear-quadratic games, and characterize the conditions under which these strategies are credible. We illustrate the implementation of such equilibria on a simple example, where we obtained non-empty regions for credibility.

Two extensions of this essay are worth considering. First, the results have been obtained under the assumption of linear incentive strategies. Using other forms is clearly possible and it would be of interest to see the impact of having non-linear strategies on the credibility regions. Second, extending the formalism of incentive strategies to more than two players is a challenging and relevant research question.

Continuing the labor of investigating the games played over the event trees, in the third part of this thesis, we develop a dynamic game of Closed-Loop Supply Chain (CLSC) played over uncontrolled event trees through introducing a cost-revenue sharing (CRS) program along with a reverse revenue sharing contract (RRSC). In the third essay of this thesis, entitled "*Cost-Revenue Sharing in a Closed Loop Supply Chain Played over Event Trees*", we add the flavor of uncertainty to the game of a CLSC consisting of a manufacturer and a retailer by assuming that the parameters of the model are not fixed over time and vary based on a predetermined event tree. As producing with used parts is more efficient than producing with exclusively new material, the manufacturer invests in different green activities (GA) to encourage consumers to bring back their used products at the end of their useful life. Two scenarios are analyzed and compared, namely, a scenario where the retailer is not involved in GA, and a second where the retailer pays part of the GA efforts' cost. In return, the manufacturer reduces the wholesale price by an amount that depends on the return of used products. Both games are played non-cooperatively à la Stackelberg, with the retailer acting as leader and the manufacturer as follower. Also, in both games, we assume that the demand is stochastic.

In this essay, some extensions in terms of modeling are worth conducting. One interesting extension would be to assume demand enhancing as the other purpose of closing the chain (in addition to cost saving) by letting the return volume to influence the demand level. Besides, relaxing the simplifying assumption on reusability of the products for infinite number of times and assuming the existence of a secondary market would be of great interest. The other avenue for the research is to investigate competition in this setting. One may consider a situation with more than one CLSC whose decisions influence each other. In that setting, in addition to demand level, they may compete in the collection of end-of-use products.

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