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Long-run Productivity Risk: Theory and Empirics

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Abstract

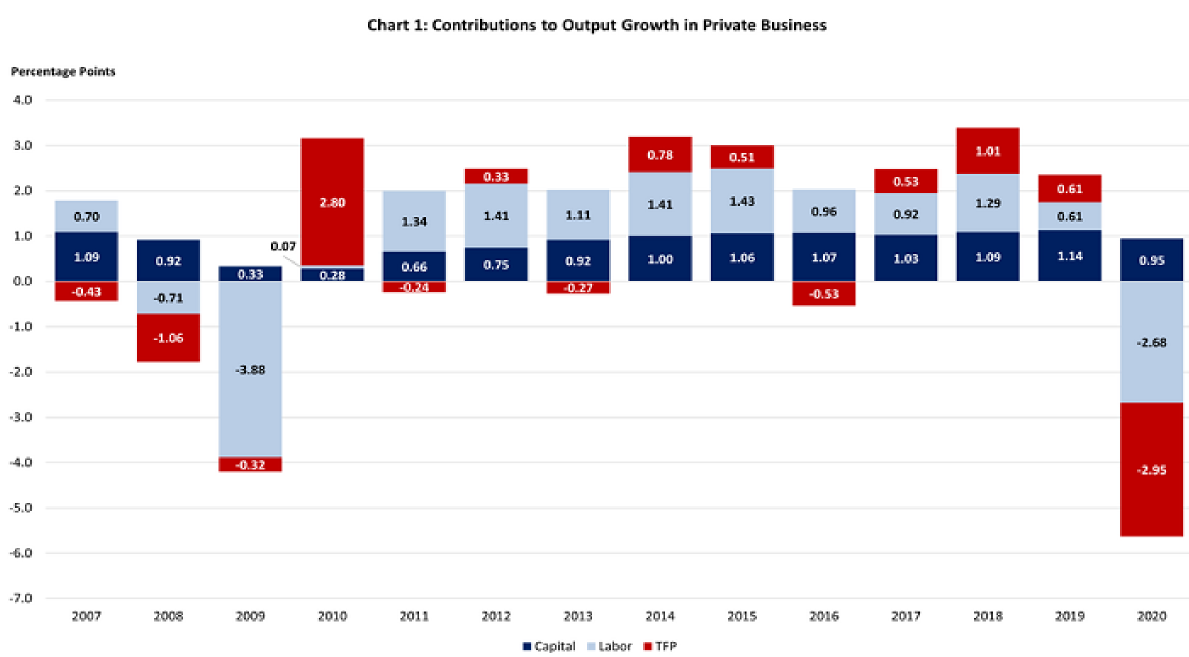
This study examines the effects of the long-run productivity risks as defined in Croce (2014) on macroeconomic variables. First, it examines empirically the predictable component in productivity and the relationship between productivity and news on equity through AR (1), ARMA (1,1) and Bansal and Yaron (2004) models. Then, it considers how the relationship affects theoretical predictions. Since a significant relationship between productivity risk and equity news has been a major concern for the macroeconomic literature, e.g. Rudebusch and Swanson (2012), we introduce Croce (2014)'s theoretical framework to study productivity risk's interaction with asset prices and quantities. We perform a sensitivity analysis of the model concerning various parameters, allowing us to establish that positive short-run and long-run technology shocks affect macroeconomic variables like consumption, investment, Tobin's Q, and labour in different ways. We use Stata for the empirical analysis and Dynare to solve the general-equilibrium real business cycle model. The paper utilizes U.S. data for the analysis (1950-2020).

Keywords: production-based DSGE model, long-run risk, asset pricing, Epstein-Zin preferences

Chapter 1 Introduction

Macroeconomic uncertainty increased during the Covid-19 according to the World Uncertainty Index by IMF. When considering uncertainty with regards to output, it is useful to distinguish between short-run and long-run changes. Given that output growth has high persistence and low volatility, as seen in Rebelo and Jaimovich (2012), low frequency fluctuations, or risks, could be an important aspect for output growth. At the same time, figure 1.1 shows that recent short-term declines in total factor productivity (TFP) and labour inputs have had large effects on the output growth of the private business sector. Sharpe and Tsang (2018) also demonstrate how TFP growth accounts for major declines in labour productivity in Canada, making it even further important in the current economic scenario. Altogether, both short-term and long-term movements in TFP can matter for the macroeconomy, and in turn movements in asset prices and quantities.

Figure 1.1. Contributions to Output Growth



This figure by BLS shows the Contributions to Output Growth in Private Business as of March 2022.

Since the seminal work of Lucas Jr. (1978), research on asset pricing models and their relationship with business cycle measurements through DSGE models has helped link macroeconomics with the finance field. Rudebusch and Swanson (2012) built on this premise to develop a model where the macroeconomic variables, such as consumption and inflation, affect financial variables, namely the term premium of a nominal bond. Their paper demonstrated how technological shocks make up a large source of macroeconomic variability, suggesting the importance of productivity risks. Given the backdrop of falling real GDP due to Covid-19 after-effects, a study of the contribution of technology to output becomes of immense importance.

In this study, we examine how both technology shocks can influence output and asset prices, following the approach of Croce (2014). We find that a positive long-run or short-run technology shock leads to an increase in the excess leveraged returns, even when the share of investment is not a major contributor to total output.

The model uses Epstein-Zin preferences to link TFP shocks and financial variables, such as the excess return. The model incorporates both long-run and short-run risk effects (defined as fluctuations in TFP in the long-run and short-run), which, as shown in Kung and Schmid (2015) for transitory shocks, has long-lasting permanent effects. The model is used to study the co-movements of asset prices and aggregate quantities over the long-run and short-run simultaneously.

We also test the productivity data by traditional models including an AR (1), ARMA (1,1), and the BKY model as seen in Bansal and Yaron (2004) to study productivity risk, defined as the role predictable fluctuations that occur at low frequency play in the long run for productivity growth. The BKY model, here, refers to the Basu et al. (2012) model focusing on the long-run risk arising from consumption smoothing. These models show a definite high persistence in the TFP process but fail to show statistical significance for the relationship between productivity growth and equity news. This is an important by-product to study the productivity risk given the increasing investment in R&D by American companies like Alphabet ($27.57 billion), and Apple ($18.75 billion). Such investments would not be prudent if TFP growth was not directly leading to private business output growth.

Hence, in this study, we re-derive the Croce (2014) model and solve and simulate it in Dynare++. In addition, we consider the model's sensitivity towards alternate benchmark parametric calibrations. We also extend the time frame for the traditional long-run risk testing models like AR (1), and ARMA (1,1) to 1950-2020, to examine the relevance of including data post the 2008-09 global financial crisis. Further, we use two series to study equity value news behaviour, namely, price-dividend ratio and value-weighted return including dividends by CRSP, as compared to Croce (2014) which focuses on a price-dividend measure as the sole proxy for the relation with equity value news.

The remainder of the study is organised as follows. Chapter 2 reviews the literature on long-run risks and productivity analysis and how this study incorporates their findings. Chapter 3 presents the empirical evidence, which is divided into 2 parts: 1) a description of the data and 2) the empirical findings from the different empirical models. In Chapter 4, we introduce Croce (2014)'s main model along with the benchmark calibration and the results for a sensitivity analysis. Chapter 5 concludes this study by summarizing the findings, limitations, and possible extensions.

Chapter 2 Literature Review

Rebelo and Jaimovich (2009) and King et al. (1988) devise a new model that generates aggregate movement and sectoral co-movement in response to shocks in the form of news of an event that will affect output. They find that when shocks are introduced to future values of TFP, consumption, investment, output, hours worked, average labour productivity and capital utilization respond before the realization of the shock. This is in line with the hypothesis that there is a reaction to the news of the shock itself. The paper presents a one-sector model and a two-sector model. These models differ from the model shown by the author of this paper in many aspects. Beginning with the consumer preferences, Rebelo and Jaimovich (2009) and Greenwood et al. (1988) follow King et al. (1988) preferences for special cases of gamma whereas our study uses Epstein-Zin preferences (seen in Epstein and Zin (1989)), as they allow the consideration of future consumption with time-additive preferences along with a more stable stochastic discount factor and a higher IES (i.e., the Intertemporal Elasticity of substitution, which corresponds to the amount of future consumption that the agent is willing to forego for current consumption).

Rudebusch and Swanson (2012) introduce Epstein-Zin preferences in a standard macroeconomic DSGE model to derive bond premiums in line with macro variables like inflation and consumption. Their paper focuses on studying the zero-coupon bond premium from a consumption based DSGE model following technology shocks. The paper models the technology shock in a Cobb-Douglas production function following an exogenous AR (1) process where $|\rho| < 1$ to generate persistent inflation following unanticipated changes. The paper finds that the lower the IES, the lower the consumption volatility which also leads to lower labour volatility. This helps to reduce the variability of hours worked. Also, a positive shock to technology increases consumption as its persistence increases households' perception of future wealth and causes inflation to fall along with nominal bond prices to increase. Although, when they try to incorporate a persistent long-run technological shock like in Croce (2014), they find that their results differ from the data, as the correlation between consumption and inflation becomes positive.

Hirshleifer et al. (2015) introduces extrapolative bias in a production-based economy with Epstein-Zin-Weil (found in Weil (1989) and Epstein and Zin (1989)) preferences to study the behaviour of consumption, investment, output, and financial variables like the equity premium, stock returns, and risk-free rate. Extrapolation bias refers to the concept in behavioural finance that talks about the irrationally higher importance given to the preceding events by people compared to the realised effect of such events. The model also considers a possible break in the expectation build-up in the sense that, if the representative agent has sufficient evidence to believe that the expectation would be changing in the future and there is a break, the model will incorporate this information of a break and will not add higher importance to that recent event. Their model produces a high equity premium even with substantial consumption smoothing, which is intriguing as most of the contemporaneous literature failed to provide a high equity premium due to the low volatility generated in their models. While both these studies set the level of the intertemporal elasticity of substitution at 2, they use a moderately low relative risk aversion coefficient (4), which was more realistic as per the data used by them.

In line with the literature, there is a close to unity persistence seen for the TFP growth risk when they try to integrate the TFP process and leveraged excess returns in Hirshleifer et al. (2015)'s model as also seen in Croce (2014). Hirshleifer et al. (2015)'s model is unique in the way that it produces high volatility for equity returns while deriving low volatility for a risk-free rate, which is in line with our final model results.

Another similarity between the papers cited so far was that all of them use GMM estimators to empirically test the long-run risks models. Schorfheide et al. (2018), however, go beyond to work with a nonlinear Bayesian state-space model to study the predictable component. They use an algorithm called the Markov chain Monte-Carlo (MCMC) method along with measurement error structure which allows them to give consistent and significant results irrespective of the frequency of data. This provides them with a model which is less susceptible to prediction changes for the data. Although, we do not go ahead with this incorporation of measurement error in our model because when Croce (2014) tested his data for results at quarterly and monthly frequencies, he found that except for a decline in the equity premium, all the other moments remained the same and statistically significant; the equity premium still stayed close to 5% as seen in our main model predictions as well. Thus, we do not incorporate this mechanism in our paper.

Considering that it is a ripe time to explore RBC models for equity risk premium given that treasury yields are falling, leverage at hedge funds and life insurance companies is high, while the asset valuations have started to go down from the elevated levels in the past months, a model linking macroeconomic variables like productivity and financial variables like equity excess return and leveraged excess return should help in bolstering interests in the financial markets when financial stability returns.

Chapter 3 Empirical Evidence

3.1 Data

In this study, data on U.S. -based companies is taken from Compustat and CRSP. The data from Compustat is from 1950 to 2020, while the data from CRSP is from 1928-2020. We use this data to build the price-dividend (henceforth, PD) ratio and the value-weighted return which includes dividends (henceforth, vwretd). The merging of the data is based on the GVKEY and Permno IDs in the respective databases due to the reusability of the CUSIP and NCUSIP in the WRDS software. Further, the data are filtered to include only companies listed specifically on U.S. affiliated exchanges by their exchange code specified in the data. Since the data are monthly, we convert to annual frequency by taking a mean of all the variables yearly per company and then averaging per year in total. The dividends taken for the PD-ratio are the annualised mean of individual dividends found on Compustat.

Further, to create the risk-free rate data, we use the 3-month Treasury rate. To convert to real terms, we subtract the current CPI inflation rate. Treasury data for the 3-month risk-free monthly annualized yield is taken from CRSP for the period 1925 to 2020 and is annualized by taking the yearly average. CPI inflation comes from the FRED database starting in 1947 and is measured as the Consumer Price Index for All Urban Consumers: All Items, which is based on the prices of food, housing, transportation, and other expenses by urban consumers. We consider this index as it is the longest consistent data source available representing inflation over the years of interest.

The price-dividend ratio is calculated by converting the PD ratio derived from the above data and converting it to real terms with the inflation rate.

We use TFP data derived from 3 sources: Basu et al. (2006), Bureau of Labour Statistics, and Solow residuals. We use 3 different datasets to account for the difference in the computation of the variables depending on the production function they follow. Solow residuals, in our case, follow the Cobb-Douglas production function while the BLS follows the value-added method of GDP estimation, while Basu et al. (2006) consider the TFP series adjusted for variations in factor utilization of capital and labour and aggregation effects. The data on total factor productivity for

Basu et al. (2006) analysis is from the Federal Reserve Bank of San Francisco for the years 1949-2020. The dataset for the Solow Model is taken from the annual data provided by the Bureau of Economic Analysis starting in 1947 from the NIPA tables (table 1.1.5, 1.1.9, and 5.5.5) and is converted to real terms using the respective price deflators available at the website. The Solow residuals are constructed as follows:

$$lnA_t = lnY_t - \alpha lnK_t - (1-\alpha)lnN_t$$

$$K_t = (1-\delta_K)\, K_{t-1} + I_t$$

$$K_{1929} = \frac{I_{1929}}{\delta_K}$$

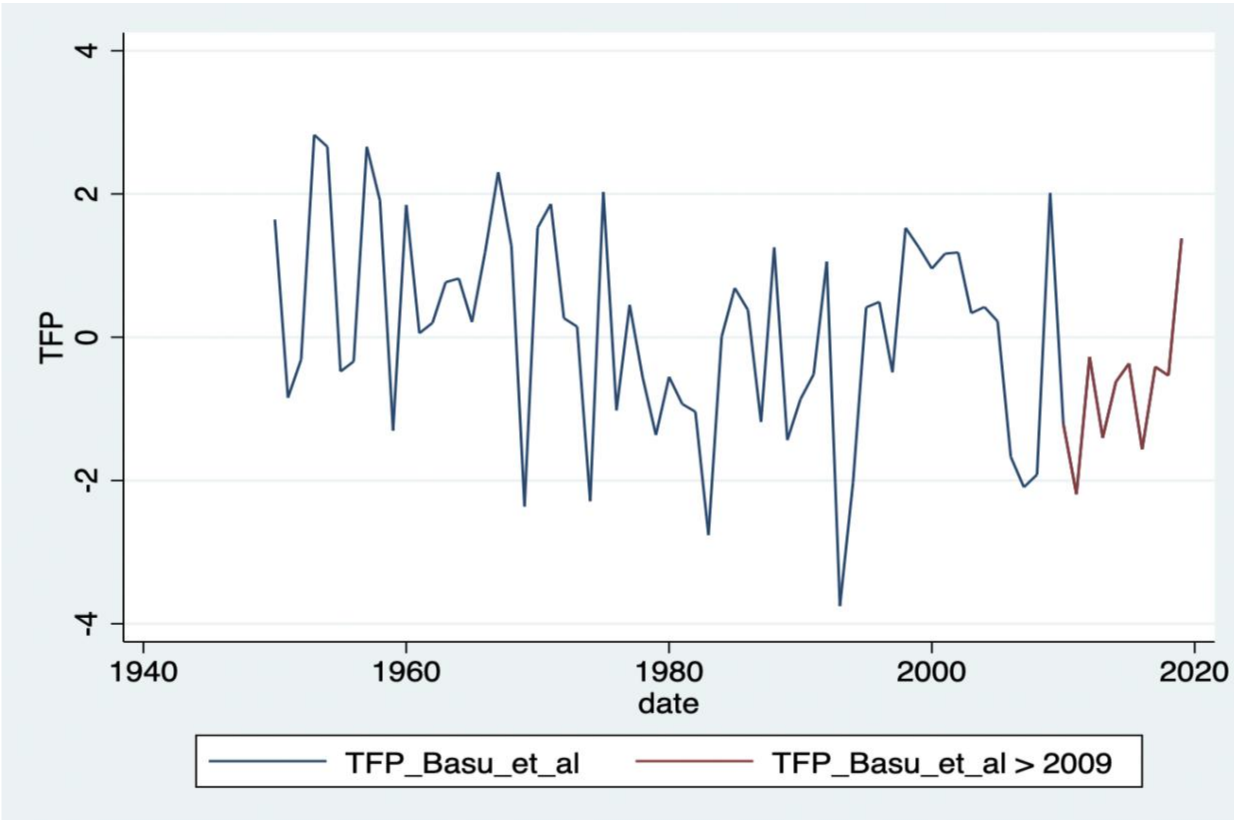
where $\alpha = 0.3$ and $\delta_K = 6\%$

All of this is in line with the data specifications of Croce (2014). The data for BLS is from the BLS website from the Office of Productivity and Technology from 1948-2020. The data for total factor productivity is an index value with base year as 2012.

We demean the data for the Basu et al. (2006) measure since it is already in percent change at an annual rate. We take the first difference of the Solow measure variables calculated from the data. All the resultant data is annual, seasonally adjusted and in real terms except the value-weighted returns. The data is in annual terms to avoid measurement errors coming from seasonality and other measurement problems.

Figure 3.1 shows the demeaned total factor productivity data derived from the Federal Reserve Bank of San Francisco for the years 1950-2020.

Figure 3.1. Long-run Productivity (Basu et al. (2006))

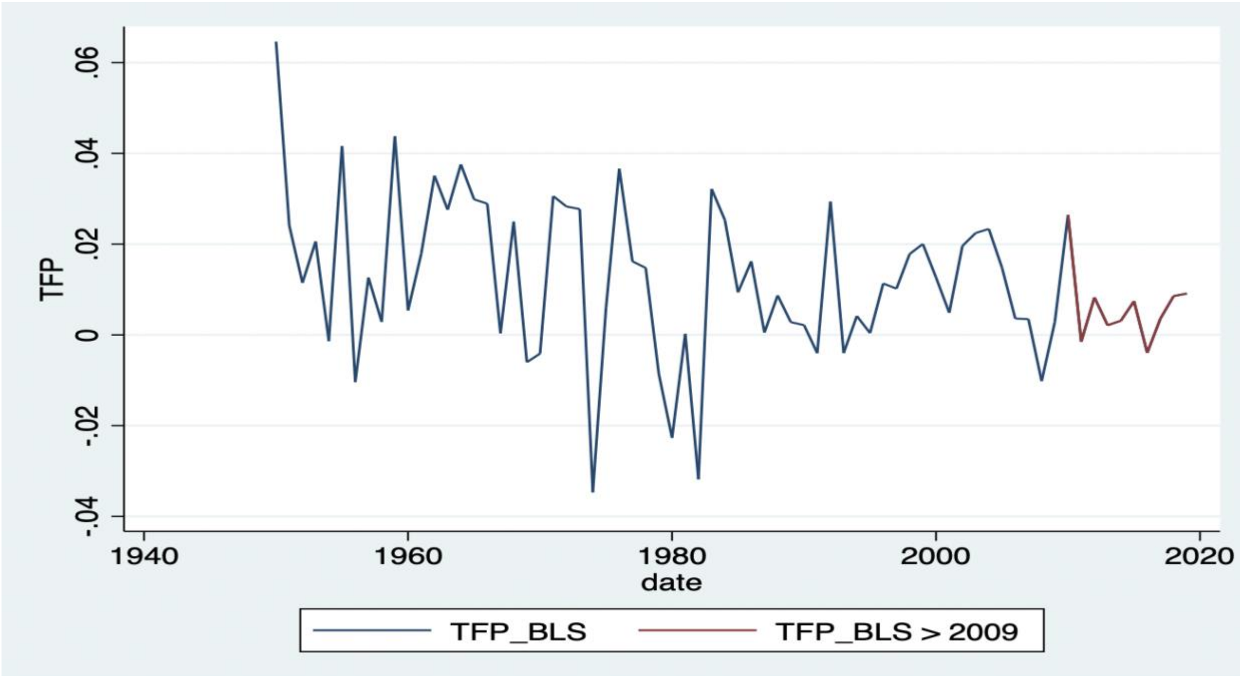


This figure shows the demeaned data of the total factor productivity risk from the Federal Reserve Bank of San Francisco over the years 1950- 2020 extending the dataset studied by Croce (2014).

Moving on, we see the data from the BLS and the Solow residuals from 1950-2020 in Figures 3.2 and 3.3.

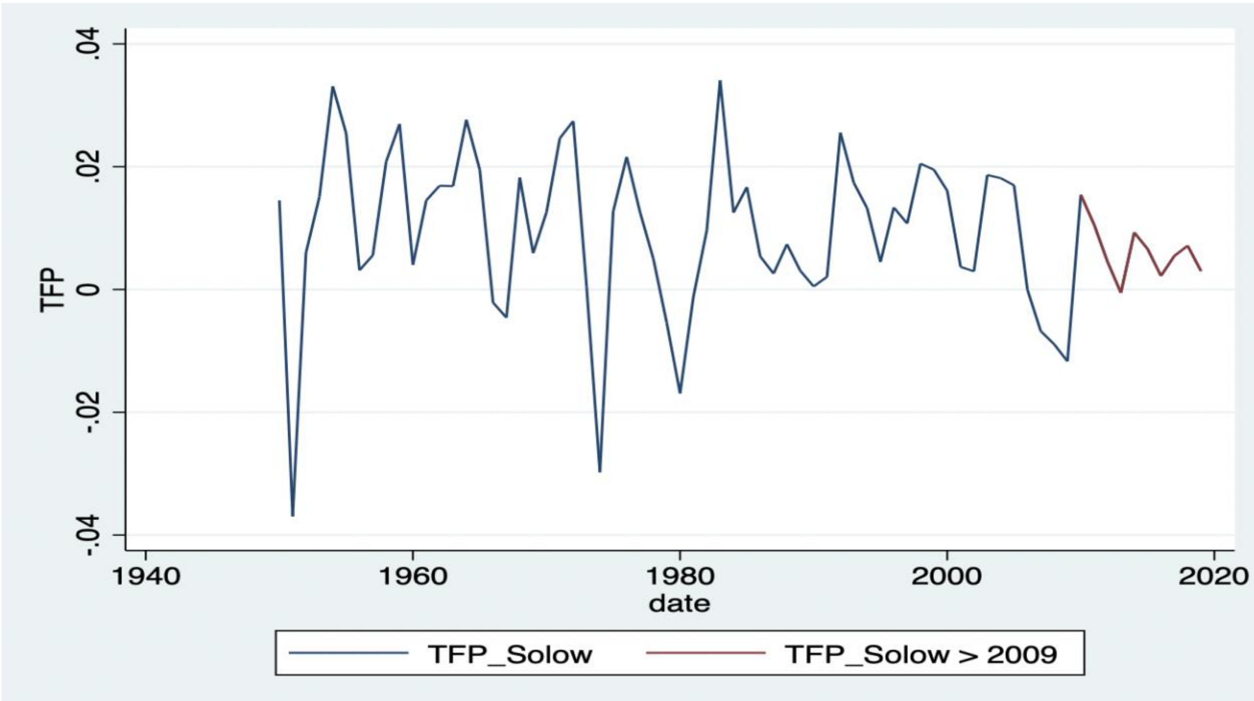
The standard errors used in this study are Eicker–Huber–White robust standard errors rather than Newey-West standard errors to maintain consistency in the data analysis since parts of the BKY model were not compatible with the Newey-West specification.

Figure 3.2. Long-run Productivity (BLS data)



The figure shows the first difference of log of total factor productivity data from the 1950-2020 for Bureau of Labour Statistics residuals over the years 1950- 2020 extending the dataset studied by Croce (2014).

Figure 3.3. Long-run Productivity (Solow residuals data)



The figure shows the first difference of log of total factor productivity data from the 1950-2020 for Solow residuals over the years 1950- 2020 extending the dataset studied by Croce (2014).

3.2 Empirical Findings

Let A_t denote the level of total factor productivity at time t with a_t as the logarithm of A_t and consider the following model:

$$\Delta a_{t+1} = \mu + x_t + \sigma\epsilon_{a,t+1} \quad (1)$$

$$x_t = \rho x_{t-1} + \sigma_x\epsilon_{x,t} \quad (2)$$

where $\epsilon_{a,t+1}$ *and* $\epsilon_{x,t}$ follow white noise processes with mean zero and variance equal to 1.

The x_t refers to the long-run risk component in productivity growth, and $\epsilon_{a,t+1}$ refers to the short-run risk. Note that the lowercase letters refer to the logs of the respective uppercase letters.

We consider three variants of the model, each of which is tested with the three data sources, namely, Basu et al., Solow model, and BLS. In Croce (2014), the analysis is limited to the data until 2008 to avoid the effects of the 2008-2009 recession. However, in this paper, we additionally consider data until 2020 to see whether the results still hold. Also, we use the three TFP series to test the models on equity news. To measure equity news, we use both the PD ratio and value-weighted return including dividends. Following Croce (2014) we begin our analysis with an AR (1) model:

$$\Delta a_{t+1} = \mu(1-\rho) + \rho\Delta a_t + \sigma\epsilon_{(a,t+1)} \quad (3)$$

This is equivalent to imposing $\sigma_x = \rho\sigma$ and $\rho_{xa} = 1$ in equation (1). (See Appendix 7.1 equation 1 for the derivation.)

The second model is ARMA (1,1):

$$\Delta a_{t+1} = \mu(1-\rho) + \rho\Delta a_t - b\epsilon_{a,t} + \epsilon_{(a,t+1)} \quad (4)$$

This is equivalent to imposing $\sigma_x = (\rho - b)\sigma$ and $\rho_{xa} = 1$ in equation (1). (See Appendix 7.1 equation 2 for the derivation.)

The third model is BKY from Bansal and Yaron (2004), the seminal paper on long-run risk:

$$\Delta a_{t+1} = \mu + r_t^f\beta_{rf} + pd_t\beta_{pd} + \epsilon_{a,t+1} \quad (5)$$

Where the estimate $x_t = r_t^f\beta_{rf} + pd_t\beta_{pd}$, and follows an AR (1) process. This is based on the long-run risks model by Bansal et al. (2004) where they model long-run risk by regressing consumption growth on the risk-free rate and PD ratio, as they are linear functions of x_t and σ_t, and can recover the state variables including the short-run risks.

To study the connection between productivity and equity value, we also study the relationship between the productivity risk and equity value news as follows:

$$pd_t = \beta_{0,pd} + \rho_{pd}pd_{t-1} + \epsilon_{pd,t} \quad (6)$$

$$\epsilon_{pd,t} = \beta_0 + \beta_{(\epsilon_a|\epsilon_{pd})}\epsilon_{a,t} + \epsilon_t \quad (7)$$

We first estimate an AR (1) model of the log PD ratio and value weighted returns including dividends (vwretd) and an AR (1) model and then use the residuals of both the estimated models to regress residuals of on the residuals of the log PD ratio or vwretd. The first panel of table 3.2 reports the results of the AR (1) model. The results are divided into 1950-2008 with respect to log PD data and the vwretd data and further extending the study to 1950-2020 for log PD data and the vwretd data.

Apart from the Basu et al. (2006) configuration, the AR (1) model is significant; 10% for the Solow dataset and 5% for the BLS dataset. The R^2 corresponds to the strength of the regression of log PD/vwretd on the residuals of the technological process. The R^2 increases as we increase the horizon of the study although the estimated coefficients are not significant for any of the datasets. The ρ determines the persistence of the shocks and their magnitude based on the restriction imposed on σ_x (see Appendix 7.1 equation 1). It can be noted that on average the estimated persistence of the long-run shocks is 0.45 which is significantly lower than the level seen in Rudebusch and Swanson (2012) at 0.95, as they use a different model to calibrate the parameters leading to different results.

So, in search of a better model, we test the ARMA (1,1) model where the persistence is seen as significant for BLS and Basu et al. (2006) data with an average of 0.9 at 1%. This level of persistence is in line with the findings of Bansal and Yaron (2004). The MA coefficient is significant for all the models at this stage at 1%, bolstered by the higher R^2 for the models. However, there is a high disparity between the level and direction for the MA coefficient across the models, as it is a positive value of 3.237 for the Solow residuals model, while it is negative and close to 1 for the others. The magnitude of these shocks, which is seen as the difference between ρ and b, is then high for the BLS and Basu et al. cases. However, the relationship between productivity news and equity value news, $\beta_{\left(\epsilon_a \middle| \epsilon_{pd}\right)}$, is still not statistically significant. It is interesting to note that the results ρ and b through value-weighted return including dividends for Basu et al. and BLS data are also statistically significant in this model, but the roots are close to unity. Both for the AR (1) model and the ARMA (1,1) model, the roots lie inside the unit circle which means that the series might not be stationary. So, a further test for this is done leading to the result that all the inverse of eigenvalues for the AR (1) and ARMA (1,1) models are inside the unit circle, meaning that the processes are stationary and invertible.

We also considered a specification of the analysis from 1950-2019, stopping before the pandemic. The results do not differ significantly, so they have not been included for brevity.

Thus, we move ahead to the model given by Bansal and Yaron (2004). It is a multivariate approach to modelling the TFP process. The interesting thing to notice here is that the $\frac{\sigma_x}{\sigma}$ is statistically different from zero, which confirms the predictability of the TFP process. Further, it ranges between 5% to 35% of the short-run shocks. Moreover, the persistence of the technological shock is only significant when taken with the log PD specification but at a statistical significance of 1%. The correlation between the long-run risk and the short-run risk, on the other hand, is only significant twice and has different directions of correlation, suggesting that in general there is not a strong correlation.

Hence, in the next chapter, we work on Croce (2014)'s model which incorporates the persistent TFP process and provides a path to study its relationship with equity news while accounting for technological shocks.

Table 3.1. Empirical Analysis of long-run Productivity risk and stock market news

Productivity Sample		Solow 1930-2020 (log pd)	Solow 1930-2020 (vwretd)	Solow 1930-2008 (log pd)	Solow 1930-2008 (vwretd)	BLS 1950-2020 (log pd)	BLS 1950-2020 (vwretd)	BLS 1950-2008 (log pd)	BLS 1950-2020 (vwretd)	Basu et al. 1950-2020 (log pd)	Basu et al. 1950-2020 (vwretd)	Basu et al. 1950-2008 (log pd)	Basu et al. 1950-2008 (vwretd)
AR(1) model	ρ	0.479*	0.479*	0.481*	0.481*	0.395**	0.395**	0.415***	0.415***	0.119	0.119	0.155	0.155
		(0.254)	(0.254)	(0.257)	(0.257)	(0.131)	(0.131)	(0.137)	(0.137)	(0.125)	(0.125)	(0.137)	(0.137)
	R^2	0.00	0.007	0.016	0.008	0.058	0.033	0.052	0.037	0.013	0.024	0.000	0.023
	$\beta_{\varepsilon_a \| \varepsilon_{pd}}$	0.113	0.053	1.796	0.056	-2.682*	0.134	-2.214	0.142	0.017	0.002	0.0004	0.001
		(2.227)	(0.072)	(2.160)	(0.074)	(1.547)	(0.098)	(1.620)	(0.102)	(0.019)	(0.001)	(0.018)	(0.001)
ARMA(1,1) model	ρ	0.269	0.269	0.272	0.272	0.993***	0.993***	0.994***	0.994***	0.888***	0.888***	0.843***	0.843***
		(0.347)	(0.347)	(0.349)	(0.349)	(0.025)	(0.025)	(0.019)	(0.019)	(0.118)	(0.118)	(0.204)	(0.204)
	b	3.237***	3.237***	3.25***	3.25***	-0.902***	-0.902***	-0.910***	-0.910***	-0.788***	-0.788***	-0.727***	-0.727***
		(1.199)	(1.199)	(1.197)	(1.197)	(0.079)	(0.079)	(0.085)	(0.085)	(0.135)	(0.135)	(0.247)	(0.247)
	R^2	0.0001	0.005	0.015	0.006	0.054	0.03	0.049	0.032	0.02	0.043	0.002	0.042
	$\beta_{\varepsilon_a \| \varepsilon_{pd}}$	0.126	0.046	1.738	0.048	-2.650	0.131	-2.175	0.133	0.021	0.002	0.006	0.002
		(2.216)	(0.071)	(2.198)	(0.074)	(1.609)	(0.106)	(1.679)	(0.111)	(0.019)	(0.001)	(0.019)	(0.001)
BKY model	ρ_x	0.862***	0.127	0.922***	-0.007	0.761***	0.406***	0.813***	0.072	0.819***	-0.104	0.814***	0.072
		(0.108)	(0.128)	(0.052)	(0.171)	(0.127)	(0.124)	(0.180)	(0.141)	(0.065)	(0.131)	(0.180)	(0.141)
	σ_x/σ	0.075***	0.147***	0.064***	0.171***	0.175***	0.204***	0.319***	0.353**	0.047***	0.207***	0.133***	0.237***
		(0.002)	(0.002)	(0.002)	(0.002)	(0.001)	(0.001)	(0.001)	(0.001)	(0.084)	(0.087)	(0.096)	(0.097)
	$\rho_{\varepsilon_a,\varepsilon_x}$	-0.314**	-0.043	-0.141	0.002	0.149	-0.033	0.265**	-0.018	0.173	-0.026	0.1287	0.014

The table summarizes the estimation results along with their confidence levels for the TFP process from the Empirical Evidence Chapter. Numbers in parentheses are robust standard errors. Data are annual.

Chapter 4 Main Model

4.1 Model description

We start with a model for a representative agent which Epstein-Zin preferences. Epstein-Zin preferences are used here for their importance in incorporating recursive utility and generating significant risk premia while allowing room for risk aversion movement as noted by Loublier and Pariès (2010).

$$U_t = \left[(1-\delta)\widetilde{C}_t^{1-\frac{1}{\Psi}} + \delta\left(E_t\left[U_{t+1}^{1-\gamma}\right]\right)^{\frac{1-\frac{1}{\Psi}}{1-\gamma}}\right]^{\frac{1}{1-\frac{1}{\Psi}}}$$

Here, $\widetilde{C}_t$ refers to the aggregate consumption bundle at time t. The parameters ψ and γ represent respectively the intertemporal elasticity of substitution (IES), which corresponds to the relation between future consumption and current consumption along with the real interest rate, and relative risk of aversion (RRA) which refers to risk preferences of an agent. The δ is the subjective discount factor which corresponds to the patience of the consumer for future consumption, and the higher the value of δ, higher the value of future consumption as the δ is augmenting the tradeoff for future consumption. The consumption bundle is the aggregate of consumption in general along with the last period TFP and leisure at time t. Given that TFP grows over time, lagged TFP is included in the specification with leisure to be consistent with balanced growth preferences. Thus:

$$\widetilde{C}_t = \left[oC_t^{1-\frac{1}{\xi_l}} + (1-o)(A_{t-1}l_t)^{1-\frac{1}{\xi_l}}\right]^{\frac{1}{1-\frac{1}{\xi_l}}}$$

The ξ_l refers to the consumption elasticity, while o is the weight of consumption versus leisure in preferences. Our model builds in a different direction than many works of literature, like Hirshleifer et al. (2015), as it also includes leisure in the preferences.

The output, Y, of the economy, is as per the Neo-classical production function in terms of capital, K, technology, A, and labour, n:

$$Y_t = K_t^{\alpha}[A_t n_t]^{1-\alpha}$$

The total factor productivity growth rate is $\Delta a_{t+1} \equiv \log(A_{t+1}/A_t)$ and it follows equation (1) described in Chapter 2, Empirical Evidence.

The other constraints of the economy are:

$$Y_t \geq C_t + I_t$$
$$1 \geq n_t + l_t$$
$$K_{t+1} \leq (1 - \delta_k)K_t + I_t - G_t K_t$$

where δ_k is the rate of depreciation of the capital stock, and I_t refers to the investment at time t.

G_t is the convex adjustment cost for the capital stock. It follows:

$$G_t = \frac{I_t}{K_t} - \left[\left(\frac{\alpha_1}{1 - \frac{1}{\xi}}\right)\left(\frac{I_t}{K_t}\right)^{1-\frac{1}{\xi}} + \alpha_0\right]$$

Solving the consumer's problem, we find the resultant stochastic discount factor (S.D.F.) to be:

$$M_{t+1} = \frac{\lambda_{t+1}}{\lambda_t} = \delta\left(\frac{\tilde{C}_{t+1}}{\tilde{C}_t}\right)^{\left(\frac{1}{S_t}\right)-\left(\frac{1}{\psi}\right)}\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{S_t}}\left(\frac{U_{t+1}}{E_t\left(U_{t+1}^{1-\gamma}\right)^{\frac{1}{1-\gamma}}}\right)^{\left(\frac{1}{\psi}\right)-\gamma}$$

For a detailed solution of the main model, refer to Appendix Chapter 7.2, equation 7 of this study.

The M_{t+1} corresponds to the units of consumption that the agent is willing to let go of today for more units of future consumption. The last part of the equation determines the information about the continuation of the agent's value creation. The future utility is very sensitive to long-run news here which can lead to higher volatility in the S.D.F., even for moderate levels of relative risk aversion.

The risk-free rate is $E_t[M_{t+1}]^{-1} = R_t^f$, and the capital gross returns is given as follows:

$$R_{t+1} \equiv \frac{\alpha\left(\frac{Y_{t+1}}{K_{t+1}}\right) + Q_{t+1}(1-\delta_k) + Q_{t+1}\left[G'_{t+1}\left(\frac{I_{t+1}}{K_{t+1}}\right) - G_{t+1}\right]}{Q_t}$$

where Q_t is the marginal rate of transformation between new capital and consumption (See equation 11 in the Appendix 7.2). This return is derived from the consumer's first order conditions, namely equations 5,6, and 8 of Appendix 7.2.

The optimality condition for labour is also derived as the rate of change between consumption and labour being equivalent to the marginal product of labour (see Appendix 7.2 equation (8) for details):

$$\frac{\frac{\partial \tilde{C}_t}{\partial l_t}}{\frac{\partial \tilde{C}_t}{\partial C_t}} = \frac{w_t}{\lambda_t} = \frac{Y_t}{n_t} * (1-\alpha)$$

The benchmark calibration for the model, which replicates the baseline parameter values seen in Croce (2014), is tested through Monte Carlo simulations using Dynare++. The parameters are defined at a monthly rate and then the results are cumulated at the annual value.

The time-invariant mean of productivity, μ, is set to an annual average growth of 1.8% and the annual volatility of its process is set to 1%. This will give us an output growth rate of 3.35%. This is in line with the calibration in Croce (2014). The ρ is set to the annual value of 0.8 which will keep the persistence close to unity in line with the results seen in table 3.2. We have empirical evidence of persistence close to 0.9 in the TFP growth series from the ARMA (1,1) models. The BKY model, however, shows persistence varying with the BLS, Solow residuals, and BLS measures used, from 0.4 to 0.9, averaging at 0.8. Hence, we set the annual value of ρ to 0.8. To keep the volatility of the long-run risk component small, we impose $\sigma_x = 10\% * \sigma$. This is supported by the log PD results in the table 3.2 for σ_x. On the capital side, we set the annual depreciation rate at 6%, which was also the value used to derive the Solow residuals data in Chapter

2. The capital income share, α, is set to 0.34 as seen in Ortu et al. (2013) while the elasticity of the cost adjustment function is set to 7.

On the preferences side, we set ξ_l at 1 to accommodate for the simple Cobb-Douglas function. The RRA and IES are set to 10 and 2, respectively, in line with the literature on this topic, e.g., Mehra and Prescott (1985) and Bansal et al. (2016). The annual subjective discount factor is set at 0.95 while the steady-state labour share is set to 18%. The steady-state labour share value is set at 18% as per Croce (2014).

In the model, we introduce leveraged excess returns with a distinct dividend growth volatility. This is to model the idiosyncratic pay-out shocks in returns which are the firm-specific pay-out decisions, which is another source of volatility in returns.

$$R_{ex}^{LEV} = \phi_{lev}\left(R_t - R_{t-1}^f\right) + \epsilon_t^d$$

So, as per Rauh and Sufi (2012), the ϕ_{lev} is kept at 2 while the annual volatility of the idiosyncratic pay-out shocks, $\sigma_d\sqrt{12} = 6.5\%$, in line with Bansal and Yaron (2004). The error is assumed to be independently and identically distributed and is not priced so that it does not affect the premium value.

We use Dynare++ for the benchmark calibration and the other sensitivity testing calibrations at 3rd order approximation. Whilst the 2nd order approximation captures the non-linearities of the model, allowing the correct identification of standard deviation of shocks that drive our DSGE model, our use of 3rd order approximation allows to incorporate time varying volatility of aggregate productivity in the model as seen in table 3.2 through (σ_x/σ).

4.2 Sensitivity Analysis

This sub-chapter is divided mainly in two parts: the results from the baseline calibration mentioned in the preceding sub-chapter and the predicted parametric value sensitivity of the model seen in Chapter 3, Main Model. We did 100 simulations for a period of 840 months or 70 years. Period is kept at monthly frequency to maintain consistency with the monthly data which had to be annualised to get the Table 3.2. Further, we take the endogenous variables thusly obtained from the Taylor approximation to the decision rule to simulate 100 times over 840 periods to get a unique distribution of each variable, adhered to its unique standard deviation (leveraged excess return has different volatility than the rest of the variables). We then use this distribution to get the expectation, autocorrelation, and volatility values along with their correlations with each other. For the figures 4.1, 4.2, and 4.3, we report the positive impulse to the short-run, $\epsilon_{x,t}$, and long-run, $\epsilon_{a,t+1}$, technological shocks of the specifically calibrated model.

Table 4.1 shows about the predictions of asset prices and aggregate quantities with the baseline calibration and the subsequent columns represent the sensitivity of the model for a specific parameter value. It is followed by the impulse response to the positive persistent technological shock seen in each case.

We see in Table 4.1 that the value of γ is set to 10 in line with Mehra and Prescott (1985) to see that the excess returns desired by the representative agent are high enough to compensate for the risk taken, which in our model is around 5% on average. The model explains the leveraged excess returns well, which are high and positive otherwise but go to a lower level when the ρ is reduced to 0.4 as seen in the statistically significant results for the AR (1) model in Chapter 2, Empirical Evidence. As the agent is not rewarded for the low risk taken by him in this scenario, it bolsters the acceptability of the model that agents receive higher premiums for taking risks in the market and even more for staying long-run in the market. This is partially opposite to what we see in Hirshleifer et al. (2015) where the mean of the leveraged excess returns is positive.

Although, the share of investment stays relatively high with respect to output in this economy, ranging from 24% – 47% with the depreciation rate of capital at 6%, the risk of investment with respect to output is also not high to lose the incentive to save for the future consumption, as it ranges from 2.2 to 4.5 units.

We can see in Table 4.1 that the leveraged excess return and the consumption are not highly correlated at an average of 0.1, which is an important result to notice as this shows that the high equity premium is derived without having high dependence on the consumption by the equity market return. The returns are highest when the correlation becomes negative with a high labour elasticity. This shows the importance of labour, as seen in Rudebusch and Swanson (2012), where the shocks are shown to affect output by way of labour and capital rather than consumption.

Thus overall, the model seems well-suited for studying the long-run productivity risk as the model works well in line with the data for the persistence seen in the data, and it effectively predicts moments of asset prices and quantities while keeping the equity premium at a reasonably high value.

Table 4.1. Predictions of asset prices and quantities

	Baseline Calibration	Calibration 1	Calibration 2	Calibration 3	Calibration 4	Calibration 5
μ	0.0015	**0.0042**	0.0015	0.0015	0.0015	0.0015
ρ	0.9816	0.9816	**0.9265**	0.9816	0.9816	0.9816
ψ (IES)	2.0000	2.0000	2.0000	**0.9000**	2.0000	2.0000
α	0.3400	0.3400	0.3400	0.3400	0.3400	0.3400
ξ	7.0000	7.0000	7.0000	7.0000	7.0000	7.0000
ξ_l	1.0000	1.0000	1.0000	1.0000	**1.2500**	**0.7500**
δ_k	0.0050	0.0050	0.0050	0.0050	0.0050	0.0050
γ	10.0000	10.0000	10.0000	10.0000	10.0000	10.0000
σ	0.0097	0.0097	0.0097	0.0097	0.0097	0.0097
σ(Δy) (%)	6.2461	4.6502	3.5772	5.5143	5.1177	5.4706
σ(Δc)/σ(Δy)	0.8548	0.8282	0.7007	0.8711	0.7948	0.8123
σ(Δi)/σ(Δy)	2.9526	2.3834	2.1946	2.9525	2.8584	4.5248
E[I|Y]%	24.4800	41.2500	46.7600	23.6400	28.4700	18.1000
ρ(Δc,Δi)	0.8834	0.8870	0.5262	0.8549	0.8204	0.7907
ρ(Δc,r_lev)	0.1188	0.0651	0.1662	-0.0309	0.0034	-0.0881
E[r_lev] (%)	4.7455	5.2282	0.8345	6.2797	4.8288	7.0390
σ(r_lev) (%)	8.4180	7.6173	6.6318	8.8033	7.7368	10.0148
σ(q)	0.8545	0.3985	0.2022	0.7571	0.4754	0.5715
ACF[r_lev]	-0.0840	-0.1827	-0.2512	-0.0868	-0.1201	-0.1338
ACF[rf]	0.7382	0.6714	0.1441	0.7808	0.6709	0.3744
ACF[q]	0.9432	0.9007	0.8606	0.9569	0.9123	0.8192
ACF[Δc]	0.7147	0.6268	0.2020	0.7053	0.5485	0.4327

The table shows the Predictions for sensitivity analysis-based quantities and prices for the model derived in this Chapter. The benchmark monthly calibrations have been specified in the top part with the resultant sensitivity to respective factors have been mentioned in the below area. The highlighted part refers to the parameter value which the predictions are sensitive to in that calibration. Further, $\tau = 1.5$, $\epsilon = 100$, periods = 840 months which are time aggregated for 100 simulations. δ_{k} is the depreciation rate of capital, α is the capital share, σ is the short-run productivity risk component, γ is the RRA value of representative agent, ψ is the IES in the economy, ρ refers to the persistence of the long-run productivity risk, ξ_{l} is the labour elasticity, ξ is the elasticity of cost function, and μ is the long-run time invariant TFP value.

Moving to the model sensitivity to labour elasticity, it is interesting to note that when the value for labour elasticity is changed to greater than or less than unity, the volatility of consumption,

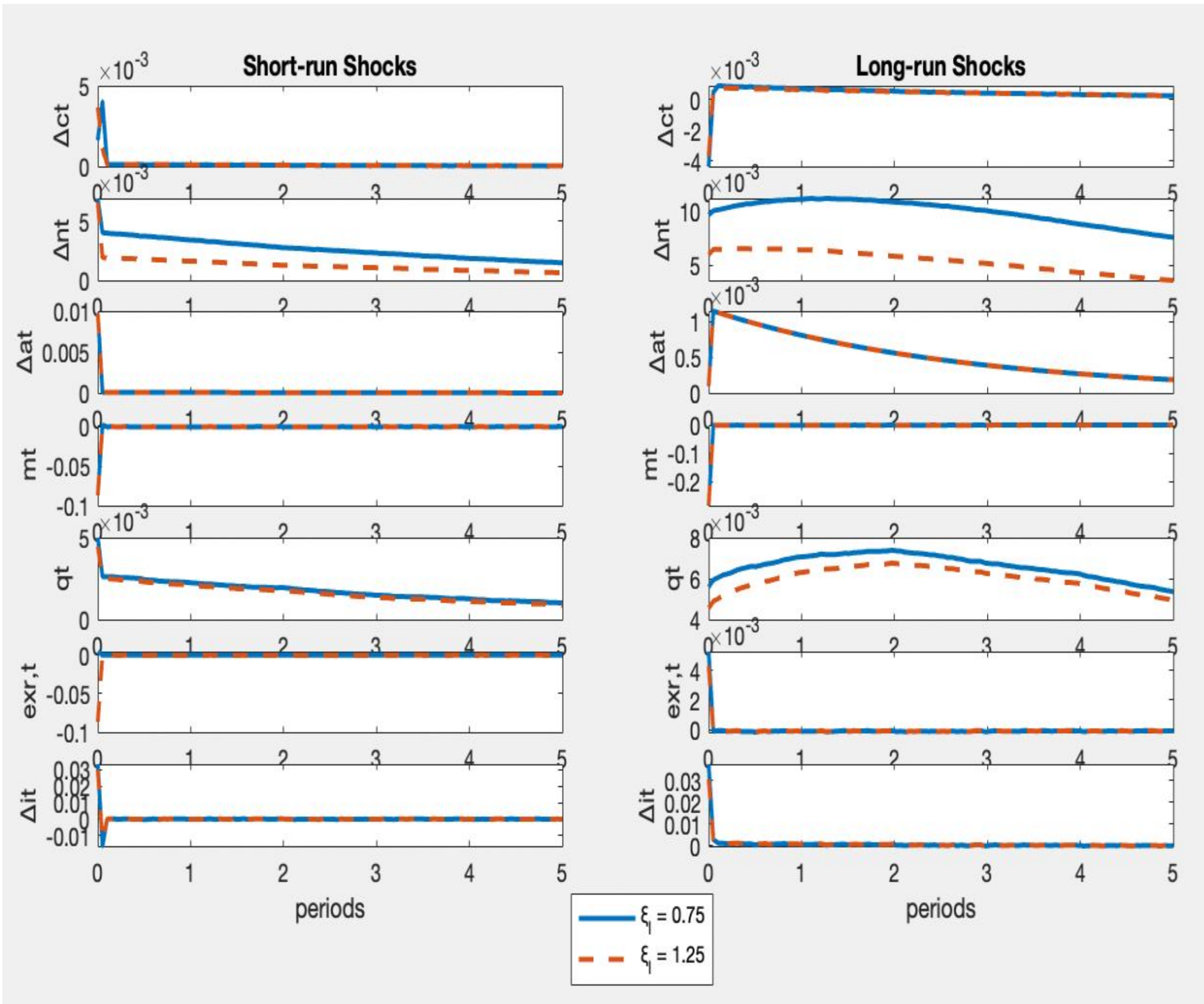
$\sigma_{\Delta c}$, and output, $\sigma_{\Delta y}$, does not change significantly as seen for calibration 4 and 5 in Table 4.1 from the base calibration. The change in the value of labour elasticity corresponds to the substitution between labour and leisure, where the higher the value of labour elasticity corresponds to a higher share of leisure for the representative agent. In Figure 4.1 we see that the behaviour for variables differs in response to whether we are talking about a long-run shock or a short-run shock, but the value of the labour elasticity does not matter for individual variables for the same type of shock. However, this parameter has a substantial impact on the investment in the economy, as the volatility of investment with respect to output, $(\sigma_{\Delta i}/\sigma_{\Delta y})$ almost doubles, while the contribution of investment to output, E[I/Y]%, falls significantly, from 29% when $\xi_l = 1.25$ compared to 18% when $\xi_l = 0.7$ as seen in Table 4.1 calibration 4 and 5. Despite this, the behaviour of investment stays the same for both levels of labour elasticity following short-run and long-run shocks.

On the other hand, the labour elasticity is shown to have a greater effect on consumption as the impulse response to the positive short-run shock leads to a short increase in consumption before going back to its steady-state value. A positive long-run shock, on the other hand, leads to a persistent fall in consumption.

The q_t in figure 4.1 refers to Tobin's Q, which is seen to persistently fall in response to a positive short-run shock. But the model also shows that this is a good time to invest after the initial fall, as the excess return from the lower Tobin's Q can be realized before the economy adjusts itself to the new behaviour of the firm's market value.

In the case of a positive long-run shock, however, we see that the investment and excess return fall before reverting to the normalized value in the long-run as Tobin's Q reverts to the long-run value over several periods. Although the consumption and leveraged excess returns are negatively correlated when $\xi_l = 1.25$, the impulse response follows the same pattern in both the long-run and short-run positive risk scenario, showing that it does not affect the economy indirectly by way of consumption and capital but by output as mentioned in Rebelo and Jaimovich (2009) which showed productivity change directly affecting output rather than through labour or capital.

Figure 4.1. Role of Labour Elasticity



The results are from of positive long-run and short-run shocks from the Table 4.1; calibration 4 and 5 for sensitivity of the model to $\xi_l = 1.25$ and $\xi_l = 0.75$. The results are from monthly calibrations. $\Delta n_{\%}$ is the first difference log of labour, $\Delta c_{\%}$ is the first difference log of aggregate consumption, $\Delta a_{\%}$ is the first difference log of TFP, $\Delta i_{\%}$ is the first difference log of investment, $m_{\%}$ is the stochastic discount factor, $q_{\%}$ is the Tobin's Q value, and $ex_{r,t}$ is the excess returns in the stock market.

Figure 4.2 exhibits how the transmission of positive technological shocks is sensitive to IES values as seen in Bansal et al. (2016). So far, we have calibrated the IES value at 2 to demonstrate moderate consumption substitutability. We see that the variables are unaffected by the change in the parameter value, that is, there is not a significant sensitivity towards the level of IES.

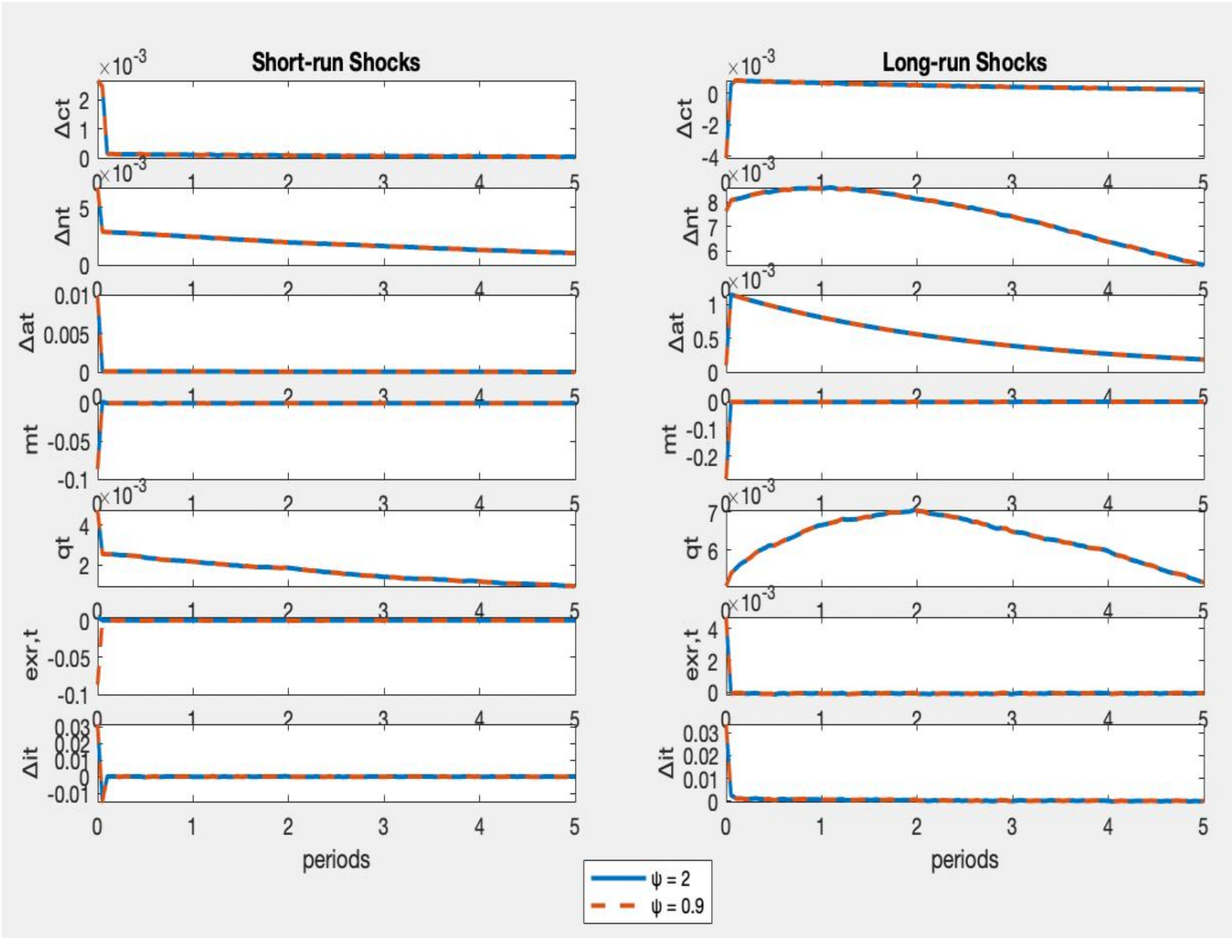
We see a significant persistence in consumption and Tobin's Q when $\psi = 0.9$. The increase in investment allows the representative agent to gain from a temporary increase in capital and aids in consumption smoothing. We see a strong income effect in the positive long-run shock as there is an increase in consumption while a simultaneous drop in investment, meaning that the agent is more motivated to consume today, which makes the excess returns in this case lower as the opportunity cost of consumption prompts lower investment. An interesting thing to note here is that the excess returns materialize following a short-run shock at time 1 when $\psi = 0.9$ and converge to the response to a positive short-run shock when $\psi = 2$. This behaviour of excess returns is consistent with the close to 1 correlation it has with the consumption as the behaviour of consumption also does not respond to the short-run shock when $\psi = 2$.

Further, we see that the effect on Tobin's Q is long-term as the value reverts to its mean over several periods, showing high persistence (as seen in table 4.1 where we find the ACF value of Q is 0.96). The positive short-run shock materializes only in period 1.

Usually, the long-run risk models are sensitive to the IES value, as it is an integral parameter, but we find the impulse response remains similar even when it goes slightly lower than unity. Yet, the risk-free rate and leveraged excess returns increase without a substantial increase in the accompanying volatility showing a lesser sensitivity towards the opportunity cost of future consumption. Households invest less and value current consumption more.

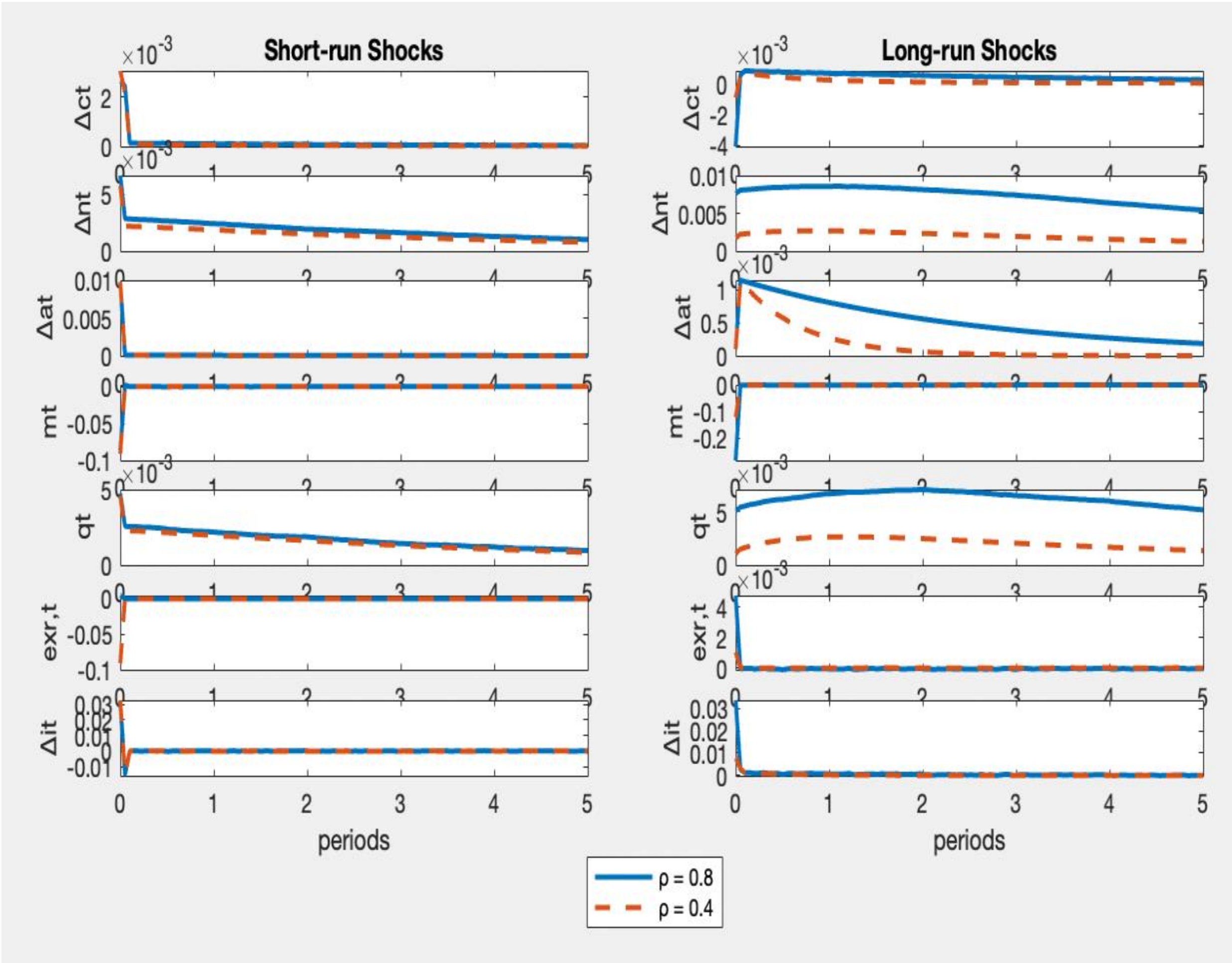
We study the model's sensitivity when $\rho = 0.4$ as we found in some cases from the AR (1) specification of Chapter 2. We can see in figure 4.3 that the response to the positive long-run shock is fairly different in this case. Although when the positive long-run shock materializes we see the income effect being dominant as consumption increases and investment falls, its effect on TFP and Tobin's Q is very sensitive to the predictability of the long-run component. When $\rho = 0.4$, the firm's market value converges to its long-run level faster compared to when the persistence is higher, as the agent is increasing its consumption more.

Figure 4.2. The role of IES



The results are from of positive long-run and short-run shocks from the Table 4.1; base calibration and calibration 3 for sensitivity of the model to $\psi = 2$ and $\psi = 0.9$. The results are from monthly calibrations. Δn_t is the first difference log of labour, Δc_t is the first difference log of aggregate consumption, Δa_t is the first difference log of TFP, Δi_t is the first difference log of investment, m_t is the stochastic discount factor, q_t is the Tobin's q value, and exr, t is the excess returns in the stock market.

Figure 4.3. The role of the persistence of the long-run component in productivity



The results are from of positive long-run and short-run shocks from the Table 4.1; base calibration and calibration 2 for sensitivity of the model to $\rho = 0.8$ and $\rho = 0.4$. The results are from monthly calibrations. Δn_t is the first difference log of labour, Δc_t is the first difference log of aggregate consumption, Δa_t is the first difference log of TFP, Δi_t is the first difference log of investment, m_t is the stochastic discount factor, q_t is the Tobin's q value, and exr, t is the excess returns in the stock market.

Thus, the model turns out to be more sensitive to the labour elasticity and the aggregate productivity persistence than the IES of the agent. Our study fortified the need for a model which accurately models productivity risk and equity news while providing a model that showed the dynamics of labour elasticity in relation to the TFP process seen in Sharpe and Tsang (2018).

Chapter 5 Conclusion

We started our estimation of long-run risks with the traditional time series models available for the evaluation of the TFP processes for example the AR (1) model, ARMA (1,1) model, and the model of Bansal and Yaron (2004). Although it provided evidence of high persistence in the long-run and short-run risk in productivity, it failed to account for the presence of the relation between the long-run component of productivity, and the equity news, proxied by price-dividend ratio and value-weighted returns including dividends series.

Considering the current state of financial stability that the economy is experiencing, a model to exploit the benefits from the unelevated levels of financial asset valuation certainly helps in wealth creation of agents. A model to gauge the impact of the next technological shock (or revolution) will help in taking the first-mover advantage. So, we then derived Croce (2014)'s model which can generate valid theoretical predictions for prices and quantities of macroeconomic and financial variables simultaneously. Although we see the persistence of the long-run component in productivity in the literature and our empirical evidence is on similar lines, the model derived in this paper was ahead in its approach to mask both short-run and long-run component behaviour which differs substantially from each other when it materializes as seen in the results of the sensitivity analysis conducted.

Nonetheless, there are limitations to this study arising from the data being available only from 1950 and with certain variables having a more restrictive definition than others like the CPI used in this paper (restricted to only Urban consumers in the USA). We strived to arrive at an analysis which is more consistent and extends the work of Croce (2014) using more recent data. The theoretical model in this study can be extended to incorporate the extrapolative bias discussed in Hirshleifer et al. (2015) with the difference arising from the presence of long-run and short-run shocks. Another extension can be on the lines of a model where more variables are endogenous, like cashflow variables, to better evaluate the effect of equity news on productivity.

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Chapter 7 Appendix

7.1 Empirical Findings derivation

This chapter explains how the AR (1) and ARMA (1,1) models have been formulated in this study in the Chapter 2 Empirical Evidence. We define the transformed total factor productivity, Δa_t:

$$\Delta a_{t+1} = \mu + x_t + \sigma\epsilon_{a,t+1}$$

$$x_t = \rho x_{t-1} + \sigma_x \epsilon_{x,t}$$

$$\therefore x_{t-1} = \Delta a_t - \mu - \sigma\epsilon_{a,t}$$

$$\Delta a_{t+1} = \mu(1-\rho) + \rho\Delta a_t - \sigma\rho\epsilon_{a,t} + \sigma_x\epsilon_{x,t} + \sigma\epsilon_{a,t+1}$$

Imposing $\sigma_x = \rho\sigma$ and $\rho_{xa} = 1$ as the correlation between the long-run risk and TFP allows us to easily model the long-run risks in the model. We have:

$$\Delta a_{t+1} = \mu(1-\rho) + \rho\Delta a_t + \sigma\left(\epsilon_{a,t+1} - \rho\epsilon_{a,t} + \rho\epsilon_{x,t}\right)$$

Since $\rho_{xa} = 1$, the last two terms negate each other, leaving:

$$\Delta a_{t+1} = \mu(1-\rho) + \rho\Delta a_t + \sigma\epsilon_{a,t+1} \quad (1)$$

which is the AR (1) process equation if the $\epsilon_{a,t+1}$ follows a standard deviation of 1and σ is the magnitude of change of one standard deviation in error.

Using again,

$$\Delta a_{t+1} = \mu(1-\rho) + \rho\Delta a_t - \sigma\rho\epsilon_{a,t} + \sigma_x\epsilon_{x,t} + \sigma\epsilon_{a,t+1}$$

Imposing $\rho_{xa} = 1$ and $\sigma_x = (\rho - b)\sigma$, we have

$$\Delta a_{t+1} = \mu(1-\rho) + \rho\Delta a_t + \sigma\left(\epsilon_{a,t+1} - \rho\epsilon_{a,t}\right) + (\rho\sigma - b\sigma)\epsilon_{x,t}$$

$$\Delta a_{t+1} = \mu(1-\rho) + \rho\Delta a_t + \sigma(-b\epsilon_{x,t} + \epsilon_{a,t+1})$$

Since $\rho_{xa} = 1, \epsilon_{x,t} = \epsilon_{a,t}$

$$\therefore \Delta a_{t+1} = \mu(1-\rho) + \rho\Delta a_t + \sigma\left(-b\epsilon_{a,t} + \epsilon_{a,t+1}\right) \quad (2)$$

Which is the ARMA (1,1) process equation if the error term follows a standard deviation of 1, and σ is the magnitude of change of one standard deviation in error.

7.2 Main Model derivation

The purpose of this Appendix is to supplement the model results discussed in the Chapter 3 of this study. This model is the main model developed in Croce (2014).

The agent follows Epstein-Zin preferences for the Consumption bundle, $\widetilde{C}_t$.

$$\mathrm{U_t} = \left[(1-\delta)\widetilde{\mathrm{C}}_{\mathrm{t}}^{1-\frac{1}{\Psi}} + \delta\left(E_t\left[U_{t+1}^{1-\gamma}\right]\right)^{\frac{1-\frac{1}{\Psi}}{1-\gamma}}\right]^{\frac{1}{1-\frac{1}{\Psi}}} \quad (3)$$

Aggregate Consumption is derived from Consumption and leisure as follows:

$$\widetilde{\mathrm{C}}_{\mathrm{t}} = \left[\mathrm{oC_t}^{1-\frac{1}{\xi_l}} + (1-\mathrm{o})(\mathrm{A_{t-1}l_t})^{1-\frac{1}{\xi_l}}\right]^{\frac{1}{1-\frac{1}{\xi_l}}} \quad (4)$$

Output is defined as follows:

$$\mathrm{Y_t} = \mathrm{K_t^{\alpha}}[\mathrm{A_t n_t}]^{1-\alpha}$$

Constraints:

$$\mathrm{Y_t} \geq \mathrm{C_t} + \mathrm{I_t}$$

$$1 \geq \mathrm{n_t} + \mathrm{l_t}$$

$$\mathrm{K_{t+1}} \leq (1-\delta_k)\mathrm{K_t} + \mathrm{I_t} - \mathrm{G_t K_t}$$

$$\mathrm{G_t} = \frac{\mathrm{I_t}}{\mathrm{K_t}} - \left[\left(\frac{\alpha_1}{1-\frac{1}{\xi}}\right)\left(\frac{\mathrm{I_t}}{\mathrm{K_t}}\right)^{1-\frac{1}{\xi}} + \alpha_0\right]$$

So, we develop a Lagrangian as follows:

$$\mathcal{L}_{\tilde{U}_t, C_t, I_t, n_t, K_{t+1}, l_t, \lambda_t, w_t, v_t} = \tilde{U}_{t-1} - \sum_{t=0}^{\infty} \mu_t \left[\tilde{U}_t - (1-\delta) \left\{ \left(oC_t^{1-\frac{1}{\xi_l}} + (1-o)(A_{t-1} l_t)^{1-\frac{1}{\xi_l}} \right)^{\frac{1}{1-\frac{1}{\xi_l}}} \right\}^{\rho} - \delta \left(E_t \tilde{U}_{t+1}^{\frac{1-\gamma}{\rho}} \right)^{\frac{\rho}{1-\gamma}} \right] - \sum_{t=0}^{\infty} \lambda_t [C_t + I_t - K_t^{\alpha} [A_t n_t]^{1-\alpha}] - \sum_{t=0}^{\infty} w_t [n_t - l_t - 1] - \sum_{t=0}^{\infty} v_t \left[K_{t+1} - (1-\ \delta_k) K_t - I_t + I_t - \left(\frac{\alpha_1}{1-\frac{1}{\xi}} \right) I_t^{1-\frac{1}{\xi}} K_t^{1-1+\frac{1}{\xi}} - K_t \alpha_0 \right]$$

We get the following FOCs:

$$\frac{\partial \mathcal{L}}{\partial \tilde{U}_t}: \mu_t = \ \mu_{t+1}\, \delta E_t \left(\tilde{U}_{t+1}^{\frac{1-\gamma}{\rho}} \right)^{\frac{\rho-1+\gamma}{1-\gamma}} \tilde{U}_t^{\frac{1-\gamma-\rho}{\rho}} \qquad (5)$$

$$\frac{\partial \mathcal{L}}{\partial C_t}: \lambda_t = (1-\delta)\mu_t \rho \tilde{C}_t^{\rho-1} \tilde{C}_t^{\frac{1}{S_t}} o C_t^{-\frac{1}{S_t}} \qquad (6)$$

Combining (1) and (2) to eliminate μ_t:

$$\frac{\lambda_t}{(1-\delta)\rho \tilde{C}_t^{\rho-1} \tilde{C}_t^{\frac{1}{S_t}} oC_t^{-\frac{1}{S_t}}} = \frac{\delta \lambda_{t-1}}{(1-\delta)\rho \tilde{C}_{t-1}^{\rho-1} \tilde{C}_{t-1}^{\frac{1}{S_t}} oC_{t-1}^{-\frac{1}{S_t}}} * E_t \left(\tilde{U}_{t+1}^{\frac{1-\gamma}{\rho}} \right)^{\frac{\rho-1+\gamma}{1-\gamma}} \left(\tilde{U}_t^{\frac{1-\gamma-\rho}{\rho}} \right)$$

Going 1 period ahead in time:

$$\frac{\lambda_{t+1}}{\lambda_t} = \delta \frac{\tilde{C}_{t+1}^{\rho-1} \tilde{C}_{t+1}^{\left(\frac{1}{S_t}\right)} C_t^{-\frac{1}{S_t}}}{\tilde{C}_t^{\rho-1} \tilde{C}_t^{\frac{1}{S_t}} C_t^{-\frac{1}{S_t}}} E_t \left(\tilde{U}_{t+1}^{\frac{1-\gamma}{\rho}} \right)^{\frac{\rho-1+\gamma}{1-\gamma}} \left(\tilde{U}_{t+1}^{\frac{1-\gamma-\rho}{\rho}} \right) \qquad \text{given that } \ \tilde{U}_{t+1} = U_{t+1}^{\rho}$$

Recall that $\rho = \left(1 - \left(\frac{1}{\psi}\right)\right)$, so that $(\rho - 1) = \ -\frac{1}{\psi}$. Then:

$$\frac{\lambda_{t+1}}{\lambda_t} = \delta \left(\frac{\tilde{C}_{t+1}}{C_t} \right)^{-\frac{1}{\psi}} \left(\frac{\tilde{C}_{t+1}}{C_t} \right)^{\frac{1}{S_t}} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{S_t}} E_t \left(U_{t+1}^{1-\gamma} \right)^{\frac{\left(-\frac{1}{\psi}\right)+\gamma}{1-\gamma}} \left(U_{t+1}^{-\gamma+\left(\frac{1}{\psi}\right)} \right)$$

$$m_{t+1} = \frac{\lambda_{t+1}}{\lambda_t} = \delta\left(\frac{\widetilde{C}_{t+1}}{\widetilde{C}_t}\right)^{\left(\frac{1}{S_t}\right)-\left(\frac{1}{\psi}\right)}\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{S_t}}\left(\frac{U_{t+1}}{E_t\left(U_{t+1}^{1-\gamma}\right)^{\frac{1}{1-\gamma}}}\right)^{\left(\frac{1}{\psi}\right)-\gamma} \quad (7)$$

$$\frac{\partial \mathcal{L}}{\partial I_t}: \lambda_t = -\alpha_1\left(\frac{K_t}{I_t}\right)^{\frac{1}{\xi}}$$

$$\frac{\partial \mathcal{L}}{\partial n_t}: w_t = \lambda_t\left(\frac{K_t}{n_t}\right)^{\alpha} A_t^{1-\alpha}(1-\alpha)$$

Multiplying $\frac{n_t}{n_t}$ on RHS we get,

$$\frac{w_t}{\lambda_t} = \left(\frac{K_t}{n_t}\right)^{\alpha} A_t^{1-\alpha}(1-\alpha) * \left(\frac{n_t}{n_t}\right)$$

$$\frac{\frac{\partial \widetilde{C}_t}{\partial l_t}}{\frac{\partial \widetilde{C}_t}{\partial C_t}} = \frac{w_t}{\lambda_t} = \frac{Y_t}{n_t} * (1-\alpha) \quad (8)$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}}: E_t\beta\lambda_{t+1}\alpha K_{t+1}^{\alpha-1}[A_{t+1}n_{t+1}]^{1-\alpha} + E_t\beta v_{t+1}\left[(1-\delta_k) + \left(\frac{\alpha_1}{1-\left(\frac{1}{\xi}\right)}\right) I_{t+1}^{1-\left(\frac{1}{\xi}\right)} K_{t+1}^{\left(\frac{1}{\xi}\right)-1}\left(\frac{1}{\xi}\right) + \alpha_0\right] = v_t$$

Assuming $\frac{v_t}{\lambda_t} = q_t$, multiplying $\frac{1}{\lambda_t}$ on both sides,

Also, we know:

$$\frac{v_{t+1}}{\lambda_t} = \frac{v_{t+1}}{\lambda_{t+1}} * \frac{\lambda_{t+1}}{\lambda_t} = q_{t+1} * m_{t+1}$$

It is also given that $E_t[m_{t+1}]^{-1} = R_t^f$.

Substituting this information in the above equation we get:

$$R_{t+1} = \frac{1}{q_t}\left[\alpha K_{t+1}^{\alpha-1}[A_{t+1}n_{t+1}]^{1-\alpha} + q_{t+1}\left[1-\delta_k + \alpha_0 + \left(\frac{\alpha_1}{\xi-1}\right)\left(\frac{I_{t+1}}{K_{t+1}}\right)^{1-\frac{1}{\xi}}\left(\frac{1}{\xi}\right)\right]\right] \quad (9)$$

$$\frac{\partial G_t}{\partial\left(\frac{I_t}{K_t}\right)}: 1 - \alpha_1\left(\frac{I_{t+1}}{K_{t+1}}\right)^{-\frac{1}{\xi}} = G'_{t+1}$$

$$G'_{t+1}\left(\frac{I_{t+1}}{K_{t+1}}\right) - G_{t+1} = \alpha_0 + \frac{\alpha_1^{\frac{1}{\xi}}}{1-\frac{1}{\xi}}\left(\frac{I_{t+1}}{K_{t+1}}\right)^{1-\frac{1}{\xi}} \quad (10)$$

Putting equation (10) in (9), we get:

$$R_{t+1} \equiv \frac{\alpha\left(\frac{Y_{t+1}}{K_{t+1}}\right)+Q_{t+1}(1-\delta_k)+Q_{t+1}\left[G'_{t+1}\left(\frac{I_{t+1}}{K_{t+1}}\right)-G_{t+1}\right]}{Q_t} \qquad (11)$$