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Conditional semi-parametric bounds on pricing kernel dynamics

by

Jalpa Mishra

Thesis supervisor

Piotr Orlowski

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Résumé

Dans cette étude, nous examinons la dynamique des moments conditionnels du noyau de prix obtenus à l'aide des bornes dérivées par Orłowski, Sali et Trojani (2018). Nous calculons les bornes sur les différents moments du noyau de prix en utilisant les données impliquées par les prix des options et la variance conditionnelle du rendement du marché. Nous analysons la dynamique des bornes obtenues et tirons des conclusions sur les propriétés des processus de noyau de prix admissibles. Nous illustrons l'utilité de nos bornes en diagnostiquant le modèle de risque à long terme (LRR) proposé par Bansal et Yaron (2004). Nous examinons le modèle LRR et simulons l'espérance conditionnelle de PK d'ordre supérieur en nous basant sur les paramétrisations impliquées par les calibrations dans Bansal, Kiku, et Yaron (2012).

Abstract

In this study, we examine the dynamics of conditional moments of the pricing kernel obtained with the use of bounds derived by Orłowski, Sali, and Trojani (2018). We compute the bounds on the different PK moments using the data implied by options prices and the conditional variance of the market return. We illustrate the usefulness of our bounds by diagnosing the long-run risk (LRR) model proposed by Bansal and Yaron (2004). We examine the LRR model and simulate conditional expectation of the second, third, and fourth moment of the PK based on the parameterizations implied by the calibrations in Bansal, Kiku, and Yaron (2012).

Keywords: Pricing Kernel, Long-run risk, Asset pricing, Cumulant generating function.

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List of Abbreviations

Abbreviation	Full form
AFD	Arbitrage-free dispersion
CGF	Cumulant generating function
CRSP	Center for Research in security prices
GARCH	Generalized autoregressive conditional heteroskedasticity
GMM	Generalized method of moments
IES	Intertemporal elasticity of substitution
IMRS	Intertemporal marginal rate of substitution
LHS	Left-hand side
LRR	Long-run risk
MLE	Maximum-likelihood estimation
NYSE	New York stock exchange
OTM	Out-of-the-money
PK	Pricing kernel
RHS	Right-hand side
S&P	Standard & Poor
SDF	Stochastic discount factor
USA	United States of America
VAR	Vector Autoregressive

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Chapter 1: Introduction

One of the fundamental problems in finance is determining how future payoffs are discounted to yield the value of an asset. In the absence of arbitrage opportunities, there exists a Stochastic Discount Factor (SDF) or Pricing Kernel (PK) which can be used to value an asset's future payoffs (Hansen and Richard 1987). The PK is not directly observable. Indirect inferences about the properties of the PK are possible due to the relationships between the properties of the PK and observed asset returns. This can be seen in the relationship between the price of a security at time t, p_t, and the payoff of that security at time $t+r$, x_{t+r}, which can be represented by $E[m_{t+r}x_{t+r}|\Omega_t] = p_t$ where the variable m_{t+r}, represents the scaler random discount factor. Ω_t denotes the information set available at time t and $E[\bullet|\Omega_t]$ denotes the conditional expectation. Empirical implications of all asset-pricing models can be characterized through their PKs (Cochrane 2001). The PK determines the risk-return trade-off in financial markets. It contains information about investors' risk preferences and beliefs (Linn et al. 2017; Hansen and Renault 2009).

The validity of a PK is determined by its ability to match the observed asset returns. Many "tests" based solely on necessary conditions have been proposed for asset-pricing models (Hansen and Singleton 1982; Chabi-Yo et al. 2005; Otrok and Ravikumar 2020). For instance, Hansen and Jagannathan (1991) examine what data on asset payoffs tell us about PK volatility. They derive a minimum variance PK bound that characterizes the admissible set of PKs for frictionless asset-pricing models that is consistent with the observed asset returns. The Hansen and Jagannathan volatility bound constitutes a necessary condition that an asset-pricing model must satisfy. An asset-pricing model is found to be consistent with the observed asset returns if the volatility of the PK proposed by the model is greater than the volatility implied by the Hansen and Jagannathan volatility bound. The Hansen and Jagannathan bound also expressible in a Sharpe-ratio version imply that the ratio of the standard deviation of a PK to its mean exceeds the Sharpe ratio attained by any portfolio (Cochrane 2001, Ljungqvist and Sargent 2018). Despite being very useful, such PKs obtained by linear projections may not be explanatory enough to diagnose asset pricing models. This becomes particularly accurate for models whose PK dispersion is generated by non-

Gaussianity in returns or nonlinearities in the kernel. In such instances, higher moments of the PK play a crucial role (Almeida and Garcia 2017; Chabi-Yo et al. 2005).

Against this background, we attempt to explore the relation between moments of returns and dynamics of conditional moments of the PK. Our bounds implied by options prices and the conditional variance of the market return serve as a dynamic test of the consistency of asset-pricing models. The expected outcome of the thesis will serve the interest of the finance profession given the interest in finance in the use of (unconditional) Hansen and Jagannathan bounds which serves as a static test for diagnosing asset-pricing models. The new evidence we provide on what conditional bounds on PK dynamics look like can help guide the design of better asset-pricing models. The thesis will incorporate time variation in conditional means and variances of asset returns in these bounds.

Research on the dynamics of conditional moments of the PK collectively is very limited. However, recent developments in the theory of PK bounds allow for the development of conditional bounds on PK dynamics of different order. These bounds are instrumental because these bounds yield information on expected returns and serve as a diagnostic tool to evaluate the performance of dynamic asset pricing models. These bounds can also be used in developing performance measures for portfolio managers. The thesis analyses the dynamics of conditional moments of the PK obtained with the use of bounds derived by Orłowski, Sali, and Trojani (2018), who develop a theory of arbitrage-free dispersion (AFD) defining the testable restrictions of asset pricing models. They derive a broad class of multivariate arbitrage-free inequalities between unobservable and observable regions of the joint distribution of PKs and asset returns, resultant of the convexity of joint cumulant generating function (CGF) of PKs and asset returns. They demonstrate the arbitrage-free inequalities to be interpretable as arbitrage-free constraints on the multivariate dispersion of PKs and returns and incorporate arbitrage-free dispersion constraints into lower and upper bounds on the CGF of PKs and returns.

Our contributions are threefold. First, we present the bounds on the Sharpe ratio of the market portfolio (as a consequence of bounding the PK's variance) and on different PK moments using the bounds derived by Orłowski, Sali, and Trojani (2018). These bounds will be formed by combining a model for the conditional variance of the market return with data about the risk-neutral moments of the market return, compiled from option data. Second, we analyze the dynamics of the obtained bounds and draw inferences about the properties of admissible PK processes. Third, we illustrate the usefulness of our bounds by diagnosing the long-run risk (LRR) model proposed by Bansal and Yaron (2004). Based on a representative agent with standard Epstein and Zin (1989) preferences in an environment with complete markets, the LRR model of Bansal and Yaron (2004) demonstrates consumption and dividend growth rates as encompassing a long-run predictable component and fluctuating economic uncertainty. We examine the LRR model and simulate conditional expectation of the second, third, and fourth moment of the PK based on the parameterizations implied by the calibrations in Bansal, Kiku, and Yaron (2012). We compare the properties of the PK processes drawn from obtained bounds with those proposed by the LRR model.

The structure of the thesis is as follows. In the next chapter, an overview of the literature is presented. Chapter 3 presents the data sources and description. Chapter 4 describes and discusses the models employed. Chapter 5 presents and interprets the results. Finally, chapter 6 contains concluding remarks.

Chapter 2: Literature Review

This study involves two research streams, namely, PK estimation and bounds on the moments of the PK.

2.1 Estimation approaches for the pricing kernel

In this section, we provide a review of the approaches and techniques proposed in the literature to estimate the PK. The PK is a vital concept in financial economics, and its properties affect the pricing of all financial assets. It describes how investors demand compensation for risk. Campbell et al. (1998) and Cochrane (2001) provide an extensive description of the role of PK in asset pricing. Other related papers include Ross (1978), Harrison and Kreps (1979), Hansen and Richard (1987), and Hansen and Jagannathan (1991).

Seminal papers in PK estimation literature include Aıt -Sahalia and Lo (1998), Aıt-Sahalia and Lo (2000), Jackwerth (2000), Rosenberg and Engle (2002), and Barone-Adesi et al. (2008). Aıt -Sahalia and Lo (1998) derive the option price function by non-parametric kernel regression. Aıt-Sahalia and Lo (2000) non-parametrically estimate the PK projected onto equity return states using options data and historical returns data. Jackwerth (2000) non-parametrically estimates the “risk-aversion function" using options data and historical returns data. Rosenberg and Engle (2002) estimate the PK each month from 1991 to 1995 using current asset prices and a predicted asset payoff density. Barone-Adesi et al. (2008) propose a non-parametric estimation model for the PK wherein they relax the normality assumption in Rosenberg and Engle (2002). Yang (2009) and Grith et al. (2011) demonstrate some modified versions of PK estimation that were initially introduced by Rosenberg and Engle (2002). The above-mentioned studies have attempted to estimate the PK and investigate its properties by estimating the risk-neutral measure using options market data and the physical measure using historical returns on the underlying asset. Conversely, few studies have attempted to estimate the PK using aggregate consumption data. For instance, Hansen and Singleton 1982 and Hansen and Singleton 1983 use maximum-likelihood estimation (MLE) and the generalized method of moments (GMM) to estimate the PK. They assume that the PK is a power function of aggregate USA consumption. Chapman (1997) assumes functions of consumption and its lags as PK state variables and suggests the PK function as an orthogonal polynomial expansion.

Over the past twenty years, several other PK estimation methodologies have been proposed by Bekaert and Liu (2004), Grith et al. (2009), Bakshi et al. (2010), Christoffersen et al. (2012), Barone-Adesi and Dallo (2010), Fengler and Hin (2011), Barone-Adesi et. Al (2020) and Kim (2021).

2.2 Bounds on the moments of the pricing kernel

In this section, we provide an overview of the widely recognized methods proposed in the literature for generating the bounds on the moments of the PK. These bounds provide insights into where current asset-pricing models work and where they do not. This information can then be used accordingly to develop new models that provide a better explanation of the data.

Under the fundamental no-arbitrage condition, Hansen and Jagannathan (1991) derive non-parametric bounds for the mean and standard deviation of the consumption-based PK in terms of the mean and standard deviation of the market portfolio excess returns. Snow (1991) extends their work by developing theoretical bounds on the moments of the PK. The bounds are derived as a function of the moments of observed asset returns. Stutzer (1995) derives an information bound that minimizes the Kullback-Leibler information criterion when evaluated using the risk-neutral measure. Bansal and Lehmann (1997) and Alvarez and Jermann (2005) derive restrictions on entropy, an alternative measure of PK dispersion, based on the equity risk premium. Bansal and Lehmann (1997) propose a non-parametric lower bound on the mean of the logarithm of the reciprocal of the set of strictly positive PKs implicit in asset-pricing models. Alvarez and Jermann (2005) develop a lower bound for the volatility of the permanent component of PKs based on the return properties of long-term zero-coupon bonds, risk-free bonds, and other risky securities.

In the last fifteen years, bounds on the moments of the PK have been studied in various contexts and several other methodologies have been proposed by Martin (2008, 2009 and 2011), Backus, Chernov and Zin (2011), Bakshi and Chabi-Yo (2012), Almeida and Garcia (2012), Ghosh et al. (2017), Orłowski, Sali and Trojani (2018), and Liu (2021). Backus, Chernov and Zin

(2011) propose metrics that summarize the properties of asset pricing models based on the PK's dispersion and dynamics. They apply these metrics to representative agent models with recursive preferences, habits and jumps. Bakshi and Chabi-Yo (2012) provide lower bounds on the variance of the permanent component, the transitory component, and the ratio of the permanent to the transitory component of the PK. Almeida and Garcia (2012) propose non-parametric PK bounds that naturally generalize variance, entropy, and higher-moment bounds. Ghosh et al. (2017) decomposes the PK into an observable component and an unobservable component and establish entropy bounds that limit the admissible regions for the PK and its unobservable component. Bakshi and Chabi-Yo (2019) develop a bound on the entropy of the square of the PK and consider models that pass the lower bound on PK, but fail the lower bound on the square of the PK. Liu (2021) develop a new spectrum of bounds on the PK moments based on the solution of an optimization problem that is complimentary to the approach proposed by Hansen and Jagannathan (1991). Their unifying theory of non-parametric bounds is based on the discrepancy between what an economic agent could achieve if all the assets priced by the PK were tradeable and what she can achieve in the real-world market.

The closest to our research are the bounds developed by Orłowski, Sali, and Trojani (2018), who develop a theory of AFD defining the testable restrictions of asset pricing models. They derive a broad class of multivariate arbitrage-free inequalities between unobservable and observable regions of the joint distribution of PKs and asset returns, resultant of the convexity of joint CGF of PKs and asset returns. They demonstrate the arbitrage-free inequalities to be interpretable as arbitrage-free constraints on the multivariate dispersion of PKs and returns and incorporate arbitrage-free dispersion constraints into lower and upper bounds on the CGF of PKs and returns. This thesis contributes to the literature by analyzing the dynamics of conditional moments of the PK obtained with the use of bounds derived by Orłowski, Sali, and Trojani (2018). Our bounds implied by options prices and the conditional variance of the market return are a generalization to the interpretation of CGF of PK and asset returns.

Chapter 3: Data

This chapter presents the data sources and descriptions. The two main data sources for this study are CRSP and OptionMetrics unless otherwise indicated.

We use data on S&P 500 index options obtained via OptionMetrics to determine the conditional risk-neutral moments of the market return. These options are European. The data span the period from January 1996 to December 2021. The dataset contains strike prices, expiration date, highest closing bid across all exchanges, lowest closing ask across all exchanges, and open interest for all the put and call options on the S&P 500 index, as well as the daily forward price of the underlying S&P 500 index. We compute the price of the options as the average of the option's highest closing bid across all exchanges and lowest closing ask across all exchanges. Further, the data were filtered to exclude options whose open interest was 0, the ratio of ask price to bid price exceeded 5, and the bid price was 0. We extracted from this dataset a sample pertaining to options having one month to maturity. Further, we retain only those options that expire on the third Friday (or Thursday if Friday is an NYSE holiday) of every month from 1996-2021. This process eventually reduced the sample to observations for 305 days. The distribution of the sample days considered in the study per year is presented in Table 1.

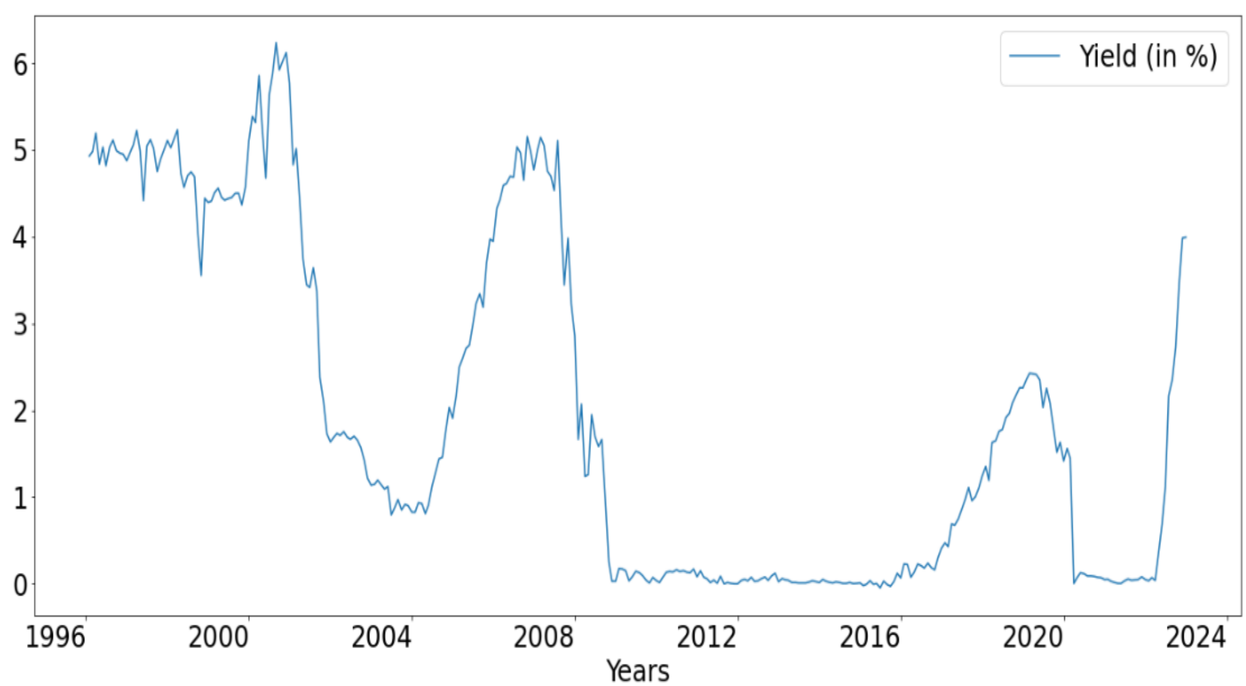
Table 1: Data on the distribution of sample days per year

N	Year	Number of days
1	1996	11
2	1997	12
3	1998	12
4	1999	12
5	2000	11
6	2001	12
7	2002	12
8	2003	11
9	2004	12
10	2005	12
11	2006	12
12	2007	12
13	2008	11
14	2009	12
15	2010	12

16	2011	12
17	2012	12
18	2013	12
19	2014	11
20	2015	10
21	2016	12
22	2017	12
23	2018	12
24	2019	12
25	2020	12
26	2021	11
27	2022	1
	Total	305

To compute the conditional risk-neutral moments of the return, we also source from WRDS CRSP the data on monthly annualized yield calculated from the nominal price of Treasury bills. We use the monthly annualized yield with 30-day maturity, depicted in Figure 1, to convert the option contribution from current to forward prices.

Figure 1: Monthly annualized yield calculated from the nominal price of Treasury bills with 30-day maturity from 1996-2022



To formulate a model for the computation of the conditional second power of the monthly market return, we collect data on daily S&P 500 returns from WRDS CRSP. The data span the period from January 1996 to December 2021. We base our analyses on the daily value-weighted returns which include dividends. The second moment of the market return is computed at time $t + 1 = 30$ days conditional on the information available at time t. The time horizon of the returns is matched to those of options that expire on the third Friday (or Thursday if Friday is an NYSE holiday) of every month from 1996-2021.

Chapter 4: Methodology

This chapter describes the model structures and presents basic assumptions. Section 4.1 presents our main model to find the conditional bounds on the moments of the PK based on the bounds derived by Orłowski, Sali, and Trojani (2018). Our bounds are computed using the data implied by the options prices and the conditional variance of the market return. We employ the model presented in section 4.2 and the mathematical identity presented in section 4.3 to find the inputs of our main model. Section 4.4 describes the Long-Run Risk (LRR) model proposed by Bansal and Yaron (2004) that we examine to illustrate the usefulness of our obtained bounds. Bansal and Yaron (2004) constitute one of the few studies to look at the conditional moments of the PK of higher order in a semi-parametric matter i.e., in a model-free setting.

4.1 Lower bounds on the conditional moments of the pricing kernel

We first summarize the information provided by a well-defined subset of values on the joint CGF of PK and asset return which uniquely characterizes the joint distribution of PK components and returns. In a simplified setting of a single PK, m, and a traded asset return, R, the joint CGF is expressed as

$$\kappa_{MR}(m,r) = \log E[M^m R^r]\,;(m,r) \in \mathbb{R}^2 \tag{1}$$

The set of empirically observable information on κ_{MR} can be summarized by the values of the CGF on the observable set $\mathcal{O}_{\kappa_{MR}} = \{(m,r) \in dom(\kappa_{MR}): m = 0 \text{ or } (m,r) = (1,1)\}$. $\kappa_{MR}(1,1)$ represents the fundamental arbitrage-free asset pricing constraint $\log E[M,R] = \log 1 \Leftrightarrow 0$. $\kappa_{MR}(0,r) = \log E[R^r]$ is directly observed from the statistical observation of the marginal distribution of moments of returns.

A broad class of multivariate arbitrage-free inequalities can be derived between the unobservable and observable regions of the joint CGF of PK and asset returns, attributable to the convexity of the CGFs. We derive the inequalities for the lower bound of the second, third, and fourth moment of the PK by finding the coordinates of convex combinations of the observable regions of the CGF (the derivation is provided in Appendix A). The lower bound derived for the second, third, and fourth moment of the PK is expressed below in eq. (2), eq. (3) and eq. (4) respectively,

$$\log E_t[M_{t+1}^2] \geq -\log E_t[R_{t+1}^2] \tag{2}$$

where $E_t[M_{t+1}^2]$ denotes the conditional expectation of the second power of the PK and $E_t[R_{t+1}^2]$ is the conditional expectation of the second power of the market return.

$$\log E_t[M_{t+1}^3] \geq 3\log E_t^Q[R_{t+1}^{4/3}] - 2\log E_t[R_{t+1}^2] \tag{3}$$

where $E_t[M_{t+1}^3]$ denotes the conditional expectation of the third power of the PK and $E_t^Q[R_{t+1}^{4/3}]$ is the conditional risk-neutral expectation of the $4/3^{rd}$ power of the market return.

$$\log E_t[M_{t+1}^4] \geq 4\log E_t^Q[R_{t+1}^{3/2}] - 3\log E_t[R_{t+1}^2] \tag{4}$$

where $E_t[M_{t+1}^4]$ denotes the conditional expectation of the fourth power of the PK and $E_t^Q[R_{t+1}^{3/2}]$ is the conditional risk-neutral expectation of the $3/2^{nd}$ power of the market return.

Figure 2: Bounds on the unobservable moments of the pricing kernel

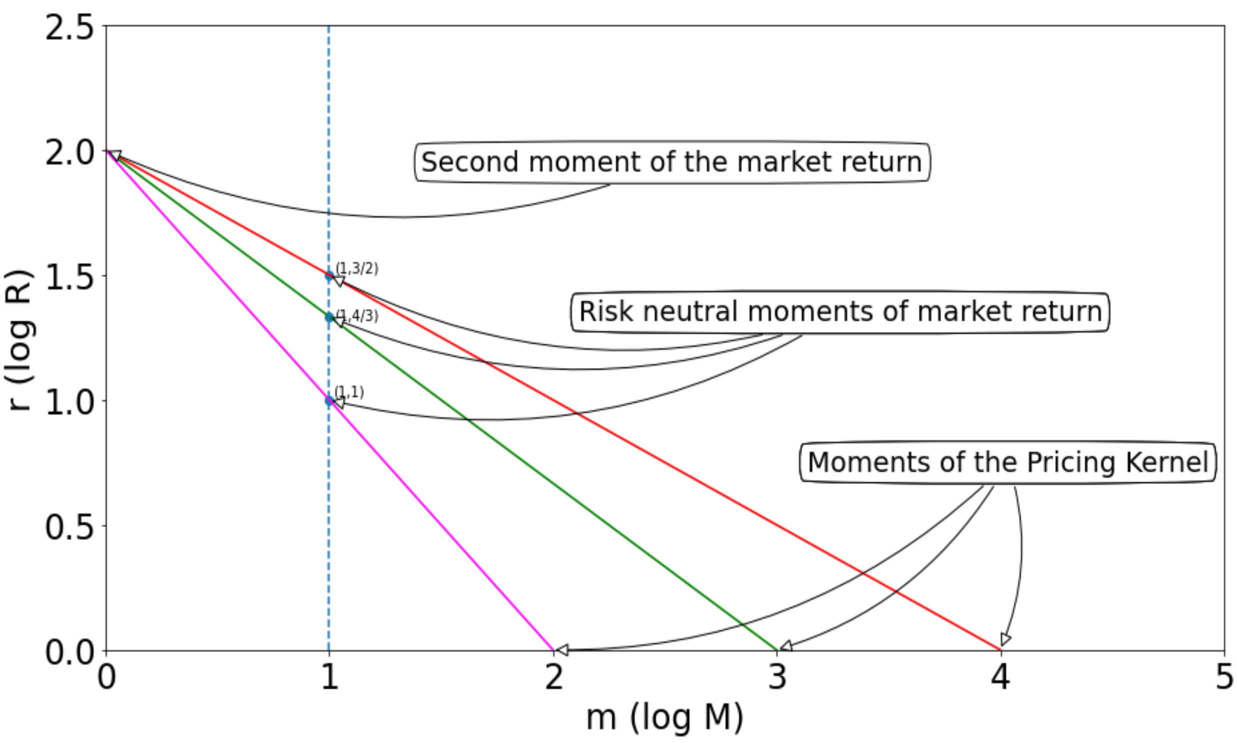


Figure 2 depicts the relationship between moments of the PK, risk-neutral moments of the market return, and the second moment of the market return on the CGF of PK and asset return which form the basis of our inequalities. “Moments of the Pricing Kernel” depicted in the figure constitutes the unobservable region of the CGF of PK and asset return. “Risk-neutral moments of the market return” and “Second moment of the market return” depicted in the figure constitutes the observable regions of the CGF of PK and asset return. Whenever we can assume statistical observability of the return distribution, moments of the market return, expressed as $\kappa_{MR}(0, r)$, can be computed. We compute the conditional expectation of the second power of the market return, $E_t[R_{t+1}^2]$, from a Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model on daily value-weighted returns (including dividends) with the errors assumed to follow a normal distribution (discussed in section 4.2)**.** We compute the conditional risk-neutral moments of the PK, $E_t^Q[R_{t+1}^{4/3}]$ and $E_t^Q[R_{t+1}^{3/2}]$, using a replication strategy based on the approximation of the Carr and Madan (2001) replication identity for a non-linear payoff (discussed in section 4.3). The value of $E_t^Q[R_{t+1}^1] = 1$ is obtained from the fundamental arbitrage-free asset pricing constraint.

4.2 Model for finding the conditional expectation of the second power of the monthly market return

We estimate the conditional second moment of the market return by using a GARCH model which is one of the most widely used methods for modeling and forecasting time-varying volatility. A GARCH model is expressed as

$$r_t \sim GARCH(p, q)$$

$$r_t = \mu + \sigma_t \epsilon_t \tag{5}$$

where μ is the expected value of r_t, σ_t is the standard deviation of r_t in time t, ϵ_t is an error term for time t and $r_t \sim N(0, \sigma_t^2)$.

$$\sigma_t^2 = \omega + \sum_{k=1}^{p} \alpha_k \cdot \varepsilon_{t-k}^2 + \sum_{k=1}^{q} \beta_k \cdot \sigma_{t-k}^2 \tag{6}$$

We calculate the second moment of market return at time $t + 1 = 30$ days conditional on the information available at time t by estimating a GARCH (1,1) model on the logarithm of daily value-weighted returns (including dividends) from 1996 to 2021. The errors have been assumed to follow a normal distribution. We matched the time horizon of monthly returns, R_{t+1}, to those of options that expire on the third Friday (or Thursday if Friday is an NYSE holiday) of every month from 1996-2021. We accordingly compute $E_t[R_{t+1}^2]$ by simulating data from the GARCH (1,1) model based on the model parameters for the exact number of business days between the start and expiry of the options. The number of business days between the start and expiry of the options thereby serves as the number of data points to simulate. We source initial values used when initializing the simulation from the conditional volatility computed from the GARCH (1,1) model.

4.3 Calculation of the conditional risk-neutral moments of the market return

We calculate risk-neutral moments of the market return using a replication strategy based on the approximation of the Carr and Madan (2001) replication identity for a non-linear payoff. We consider a market for forwards and options on a stock index. Investors can trade at times $t \in \{0,1, \dots .. \tau, \dots . T\}$ in forward contracts with fixed maturity T and options with strike prices $K \geq 0$ having the same maturity as the forwards. The forward price of the stock index on trading date t is represented by F_t and accordingly the log return by $r_t = \log(F_t/F_{t-1})$. The market return is expressed as the sum of a single non-linear payoff and a weighted sum of forward price increments.

The replication strategy is formulated using a finite set of options with a discrete set of available strike prices. We compute $E_t^Q[R^{3/2}]$ and $E_t^Q[R^{4/3}]$ from an approximation to the price of the options replicating portfolio for non-linear payoff $\Phi(F_T/F_0) = -\ln(F_T/F_0)$, based on the Carr and Madan (2001) portfolio weights $\Phi"(k)$ for OTM put and call options with moneyness $k = K/F_0$. The approximation of the Carr and Madan (2001) replication identity for a non-linear payoff as expressed below,

$$\Phi(F_T) - \Phi(F_t) - \Phi'(F_t)(F_T - F_t) = \int_0^\infty \Phi"(k) O_t(F_T, K_j; F_t) \mathrm{dk} \tag{7}$$

$$\approx \sum_j w_j \mathrm{O}(F_T, K_j; F_t) \tag{8}$$

where the quantity on the left-hand side (LHS) is the observed price of the options expressed as a non-linear payoff initiated at time t for maturity T. w_j are the weights determined by minimizing the squared integrated replication error and $\mathrm{O}_t(F_T, K_j; F_t)$ is the payoff of an option subject to the option being OTM at time t. The computation can be expressed as

$$w_j \approx \Phi''(k)\mathrm{dk} \tag{9}$$

$$\mathrm{O}_t\left(F_T, K_j; F_t\right) = \begin{cases} \max\left(F_T - K_j, 0\right) \, if \ K_j > F_t \\ \max\left(K_j - F_T, 0\right) \ if \ K_j \leq F_t \end{cases} \tag{10}$$

We use the quantity on the right-hand side (RHS) of eq. (8) to calculate the options contribution for all OTM options that expire on the third Friday (or Thursday if Friday is an NYSE holiday) of every month from 1996 to 2021. We sum up the contributions of all options having different strikes but the same start and expiry dates to compute $E_t^Q[R^{\frac{3}{2}} - 1]$ and $E_t^Q[R^{\frac{4}{3}} - 1]$. We convert the options contribution from current to forward prices by multiplying $E_t^Q[R^{\frac{3}{2}}]$ and $E_t^Q[R^{\frac{4}{3}}]$ with e^{rt} where t is the time period between the start and expiry date of an option and r is the yield calculated from the nominal price of T-bills with 30-day maturity.

4.4 Long-Run Risk (LRR) Model

This section presents the long-run risks (LRR) model developed by Bansal and Yaron (2004). The model features an underlying environment with complete markets and a representative agent with the Epstein and Zin (1989) recursive preferences, which allow for a separation between the elasticity of intertemporal substitution (IES) and risk aversion. Given such preferences, the gross return, $R_{i,t+1}$, satisfies the asset pricing restrictions which is defined as,

$$E_t[\delta^\theta G_{t+1}^{-\frac{\theta}{\psi}} R_{a,t+1}^{-(1-\theta)} R_{i,t+1}] = 1, \tag{11}$$

where $0 < \delta < 1$ reflects the agent's time preferences, G_{t+1} is the aggregate growth rate of consumption, $\psi \geq 0$ is the IES parameter and $R_{a,t+1}$ is the gross return on the asset that delivers aggregate consumption as its dividends every period. The parameter $\theta = \frac{1-\gamma}{1-\frac{1}{\psi}}$ is determined by the magnitude of the elasticity of substitution and the risk aversion, with $\gamma \geq 0$ being the coefficient of risk aversion.

The log of the intertemporal marginal rate of substitution (IMRS) is expressed as,

$$m_{t+1} = \theta \log \delta - \frac{\theta}{\psi} g_{t+1} + (\theta - 1) r_{a,t+1}, \tag{12}$$

where $g_{t+1} = \log(G_{t+1})$ is the consumption growth rate and $r_{a,t+1} = \log(R_{a,t+1})$ is the continuous return on the consumption asset determining the PK and resultantly the risk premium on the market portfolio. The innovation in m_{t+1} is influenced by the innovations in g_{t+1} and $r_{a,t+1}$. g_{t+1} is modeled as containing a small persistent predictable component capturing long-run risks in consumption growth, x_t, which determines the conditional expectation of consumption growth and time-varying volatility of consumption growth, σ_{t+1}, which reflects fluctuating economic uncertainty. Consumption growth and return dynamics can be summarised as,

$$g_{t+1} = \mu + x_t + \sigma_t \eta_{t+1} \tag{13}$$

$$x_{t+1} = \rho x_t + \varphi_e \sigma_t e_{t+1} \tag{14}$$

$$\sigma_{t+1}^2 = \sigma^2 + \nu_1(\sigma_t^2 - \sigma^2) + \sigma_w w_{t+1} \tag{15}$$

$$\eta_{t+1}, e_{t+1}, w_{t+1} \sim N.i.i.d.(0,1),$$

where parameter ρ dictates the persistence of the expected growth rate process, σ^2 is the unconditional mean of time-varying economic uncertainty incorporated in the consumption growth rate and ν_1 governs the persistence of the volatility process. The shocks are assumed to be i.i.d. normal and uncorrelated.

The model is solved using the standard approximations utilized by Campbell and Shiller (1988),

$$r_{a,t+1} = \kappa_0 + \kappa_1 z_{t+1} - z_t + g_{t+1}, \tag{16}$$

where z_t is the log of the price-consumption ratio. κ_0 and κ_1 are approximating constants that depend upon the average level of z (the details can be found in Appendix B).

The approximate solution for the equilibrium price-consumption ratio is determined as,

$$z_t = A_0 + A_1 x_t + A_2 \sigma_t^2 \tag{17}$$

where A_1 is the coefficient for the effects of the expected growth rate x_t on z_t and A_2 measures the sensitivity of price-consumption ratios to volatility fluctuations. The solution coefficients for A_0, A_1 and A_2 are mentioned in Appendix B.

By calculating the unconditional expectation of the exponential of Eq. (3) raised to a power p, we obtain the unconditional expectation of higher power of the PK given by the LRR model (the derivation is provided in Appendix B):

$$E(m_{t+1}^p) = E(e^{p(\theta \log \delta - \frac{\theta}{\psi} g_{t+1} + (\theta-1) r_{a,t+1})})$$

$$= e^{\left(p(\theta \log \delta - \frac{\theta}{\psi}\mu + (\theta-1)(\kappa_0 + (\kappa_1 - 1)\left(A_0 + A_2\sigma^2\right)) + \mu)\right)}.$$

$$e^{\frac{1}{2}p^2(\frac{\theta^2}{\psi^2}(\sigma^2) + (\theta-1)^2(\kappa_1^2 A_1^2 \varphi_e^2 \sigma_t^2 + \kappa_1^2 A_2^2 \sigma_w^2 + \sigma^2) - 2\frac{\theta}{\psi}(\theta-1)\sigma^2)} \tag{18}$$

On similar lines, we obtain the conditional expectation of higher power of the PK given by the LRR model by calculating the conditional expectation of the exponential of Eq. (3) raised to a power p (the derivation is provided in Appendix B):

$$E_t(m_{t+1}^p) = E_t(e^{p(\theta \log \delta - \frac{\theta}{\psi} g_{t+1} + (\theta-1) r_{a,t+1})})$$

$$= e^{\left(p\left(\theta \log \delta - \frac{\theta}{\psi}(\mu + x_t) + \left((\theta-1)\left(\kappa_0 + \kappa_1\left(A_0 + A_1 \rho x_t + A_2\sigma^2\right) - z_t\right) + \mu + x_t\right)\right)\right)}.$$

$$e^{\frac{1}{2}p^2\left(\left(\frac{\theta^2}{\psi^2}(\sigma_t{}^2)+(\theta-1)^2(\kappa_1^2 A_1^2 \varphi_e^2 \sigma_t^2+\kappa_1^2 A_2^2 \sigma_w^2+\sigma_t^2)-2\frac{\theta}{\psi}(\theta-1)\sigma^2\right)\right)} \tag{19}$$

We use Eq. (18) and (19) to simulate unconditional and conditional expectations of the PK of higher order respectively at the monthly frequency. The simulations are based on the monthly parameterizations implied by the calibrations in Bansal, Kiku, and Yaron (2012) listed in Table 2. We set the values of the model variables at time $t = 0$ at their unconditional expectation (the details can be found in Appendix B). Such simulations based on the LRR model help us to illustrate the usefulness of our bounds. We compare the properties of the PK processes drawn from obtained bounds with those proposed by the LRR model.

Table 2 Parameterizations implied by the calibrations in Bansal, Kiku, and Yaron (2012)

Risk-aversion parameter	$\gamma = 8.12$
IES parameter	$\psi = 2.45$
Time discount factor	$\delta = 0.9990$
Volatility process persistence parameter	$\upsilon_i = 0.9981$
Unconditional mean of time-varying economic uncertainty incorporated in the consumption growth rate	$\sigma^2 = 0.00004096$
Persistence of expected growth rate process	$\rho = 0.97530$
Unconditional expectation of consumption growth rate	$\mu = 0.0016$
Approximating constants that depend upon the average level of z	$\kappa_0 = 0.0057$, $\kappa_1 = 0.9993$

Note: For the fundamentals, $\sigma_w = 0.0000035$ and $\varphi_e = 0.0340$. The values of κ_1 and κ_0 are determined endogenously through Eq. (16). The parameter θ is determined by the magnitude of the IES and the risk aversion.

Chapter 5: Empirical Results

This chapter presents our findings. Section 5.1 presents the derivation of our lower bounds on the conditional expectation of the second, third, and fourth moment of the PK computed using the data implied by options prices and the conditional variance of the market return. As discussed in Chapter 4, we obtain our bounds by employing the GARCH model on the daily value-weighted returns including dividends (presented in section 4.2) and mathematical identity based on Carr and Madan (2001) replication identity for a non-linear payoff (presented in section 4.3). Section 5.2 presents the computation of conditional and unconditional expectations of the PK of higher order based on the LRR model proposed by Bansal and Yaron (2004). Given the absence of a model PK time series corresponding directly to our sample period, the conditional expectation of PK of higher order computed from the LRR model serves as an ideal mechanism to illustrate the usefulness of our obtained bounds.

5.1 Deriving the Bounds

We compute the second moment of the market return at time $t + 1 = 30$ days conditional on the information available at time t, $E_t[R_{t+1}^2]$, by performing 1000 simulations based on the model parameters (presented in Table 3) of the GARCH (1,1) model on the logarithm of daily value-weighted returns (including dividends) from 1996 to 2021. Figure 3 depicts the conditional volatilities estimated from the model that serves as the initial values for initializing the simulation. We use the time period (in days) between the start and expiry of the option as the number of data points to simulate. The time horizon of monthly returns, R_{t+1}, have been matched to those of options that expire on the third Friday (or Thursday if Friday is an NYSE holiday) of every month from 1996-2021. Figure 4 plots $E_t[R_{t+1}^2]$ that form the basis of our inequalities to compute our lower bounds.

Table 3: Parameters of the GARCH (1,1) model on the daily value-weighted returns (including dividends) from 1996 to 2021

	Coefficient
μ	0.000711
ω	0.000003
α_1	0.100000

β_1	0.880000

Figure 3: Conditional volatilities estimated from the GARCH (1,1) model on the daily value-weighted returns (including dividends) from 1996 to 2021

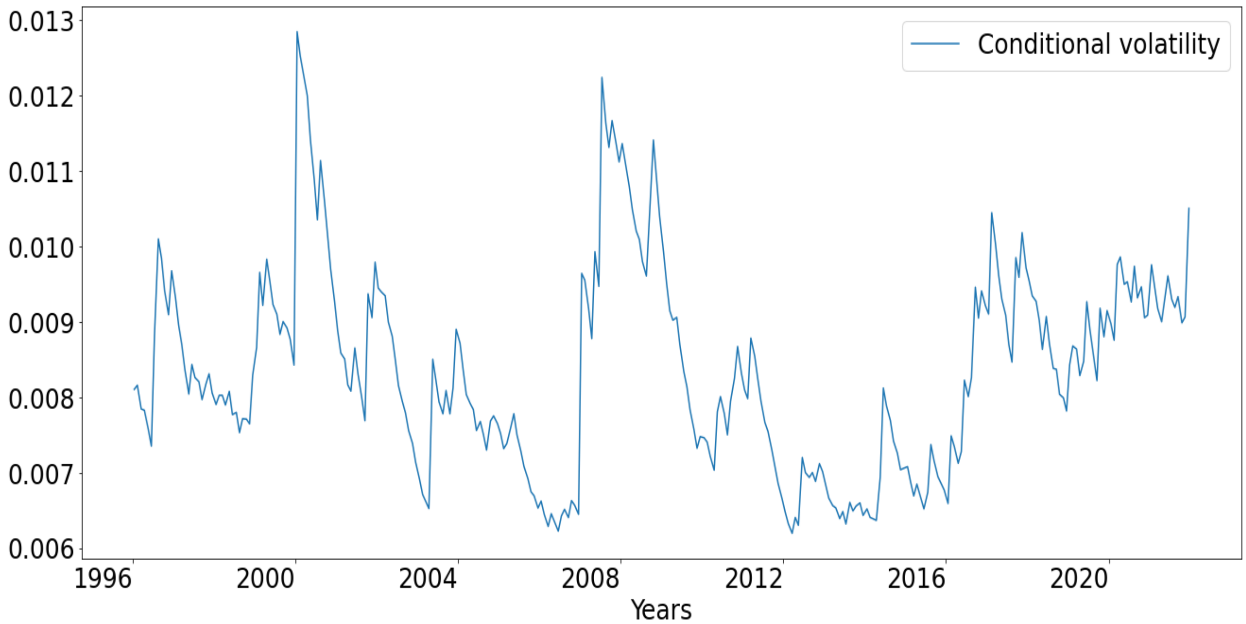


Figure 4: Conditional expectation of the second power of the monthly market return

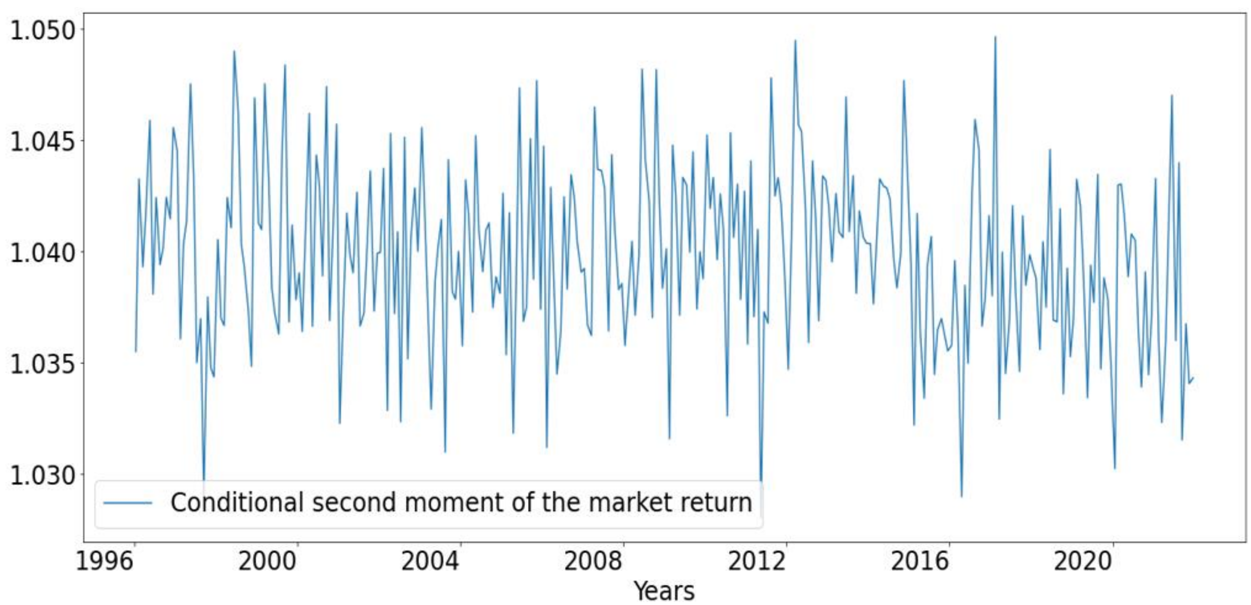


Figure 5 plots the conditional risk-neutral moments (i.e. $E_t^Q\left[R^{\frac{3}{2}}\right].e^{rt}$ and $E_t^Q\left[R^{\frac{4}{3}}\right].e^{rt}$) of the PK computed using the replication strategy based on the approximation of the Carr and Madan (2001) replication identity for a non-linear payoff. The computations are based on the data pertaining to OTM options that expire on the third Friday (or Thursday if Friday is an NYSE holiday) of every month from 1996-2021. Table 4 reports the summary statistics for these conditional risk-neutral moments of the PK.

Figure 5: Conditional risk-neutral moments of the pricing kernel

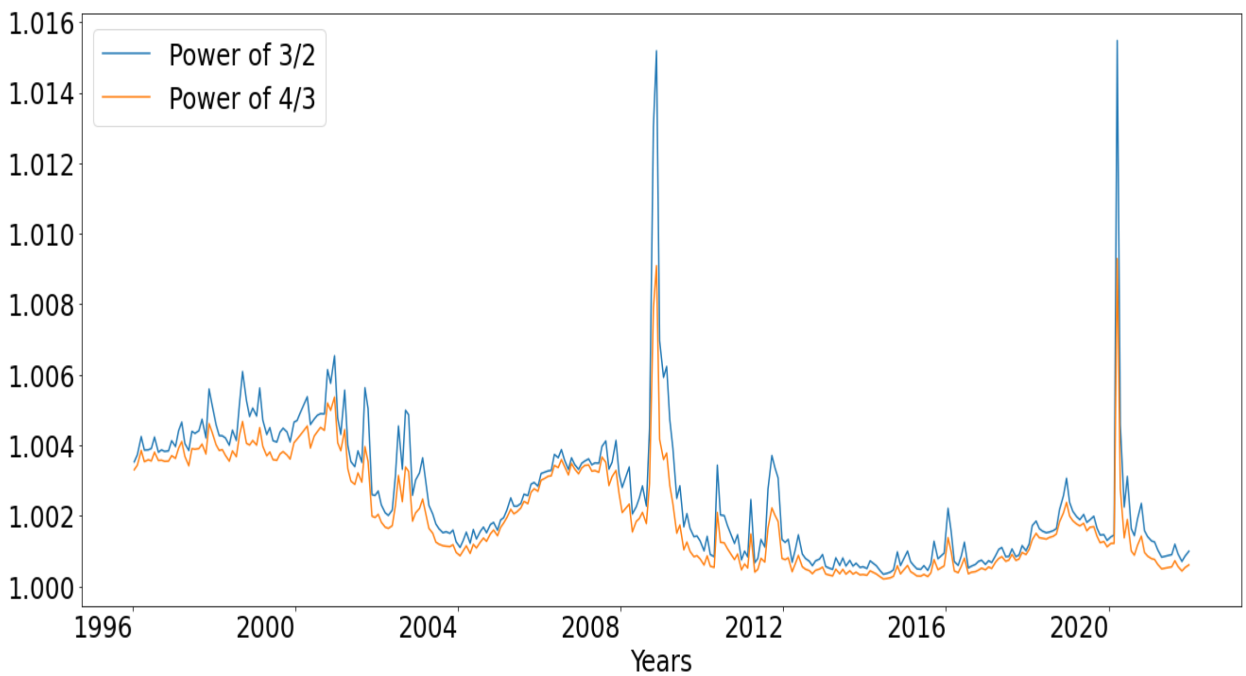


Table 4: Summary statistics for the conditional risk-neutral moments of the pricing kernel

	$E_t^Q\left[R^{\frac{3}{2}}\right].e^{rt}$	$E_t^Q\left[R^{\frac{4}{3}}\right].e^{rt}$
Average	1.002593	1.002047
Standard Deviation	0.001967	0.001520
Skewness	2.417258	1.081918
Kurtosis	11.956496	2.063625
First quintile	1.000900	1.000582

Second quintile	1.001594	1.001235
Third quintile	1.002841	1.002095
Fourth quintile	1.004130	1.003575
Fifth quintile	1.015479	1.009294
Autocorrelation at lag 1	0.757162	0.855751
Autocorrelation at lag 2	0.617296	0.772328
Autocorrelation at lag 3	0.580142	0.747088
Autocorrelation at lag 4	0.523793	0.709741
Autocorrelation at lag 5	0.476716	0.675503
Autocorrelation at lag 6	0.447872	0.651334
Autocorrelation at lag 7	0.436843	0.636946
Autocorrelation at lag 8	0.415839	0.617898
Autocorrelation at lag 9	0.404902	0.605474
Autocorrelation at lag 10	0.406194	0.601256
Autocorrelation at lag 11	0.396983	0.590344
Autocorrelation at lag 12	0.383860	0.575527

Figure 6 depicts the lower bounds on the conditional second, third, and fourth moment of the PK over the period extending from 1996 to 2021. The computation of these bounds is based on the inequalities mentioned in eq. (2), (3), and (4). Table 5 reports the summary statistics for these lower bounds on the conditional moments of the PK.

Figure 6: Lower bounds on the conditional expectation of the second, third, and fourth moment of the pricing kernel

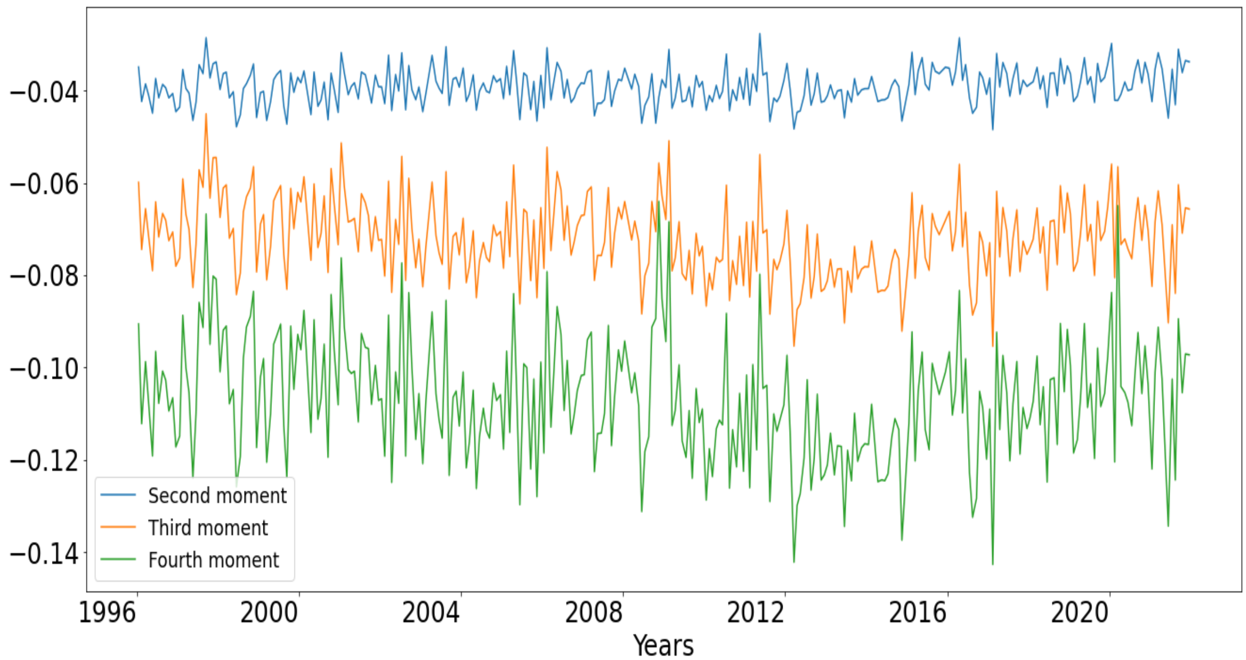


Table 5: Summary statistics for the lower bounds on the moments of the pricing kernel

	$\log E_t[M_{t+1}^2]$	$\log E_t[M_{t+1}^3]$	$\log E_t[M_{t+1}^4]$
Average	-0.039044	-0.071957	-0.106781
Standard Deviation	0.003892	0.008568	0.013276
Skewness	0.094735	0.105487	0.273303
Kurtosis	-0.148131	-0.154462	0.264053
First quintile	-0.042357	-0.079314	-0.118511
Second quintile	-0.040097	-0.074337	-0.110050
Third quintile	-0.038038	-0.070266	-0.104557
Fourth quintile	-0.035993	-0.064818	-0.096084
Fifth quintile	-0.027655	-0.045034	-0.064007
Autocorrelation at lag 1	-0.017897	0.134294	0.149847
Autocorrelation at lag 2	0.107225	0.274864	0.282681
Autocorrelation at lag 3	0.000510	0.151393	0.146167
Autocorrelation at lag 4	-0.024273	0.109862	0.088360
Autocorrelation at lag 5	0.007451	0.125622	0.095126
Autocorrelation at lag 6	0.118282	0.230070	0.198152
Autocorrelation at lag 7	0.017225	0.112484	0.074882
Autocorrelation at lag 8	0.060764	0.172829	0.142867

Autocorrelation at lag 9	-0.015585	0.124585	0.104024
Autocorrelation at lag 10	0.026266	0.149474	0.127786
Autocorrelation at lag 11	-0.040364	0.099276	0.083214
Autocorrelation at lag 12	0.094465	0.204700	0.176817

We obtained several noteworthy findings. Firstly, we observe that the lower bounds on the second, third, and fourth moments of the PK are negative throughout the entire time which can have important implications for risk management and financial decision-making. Moreover, we find that the lower bounds on the second moment of the PK exceed those of the third moment, which in turn exceed those of the fourth moment. This trend persists throughout the duration of our study.

Furthermore, we notice that in periods of economic turbulence (for instance in 2008 and 2020), the third and fourth moments of the PK come very close. We investigate this further by analyzing the relationship between the time series of the lower bounds on the second, third, and fourth moments of the PK. We derive our inferences by testing for causation using Granger's causality test and formulating a Vector Autoregressive (VAR) model.

Granger's causality test helps us to determine the direction of causality of the relationship between two variables. The test findings (p-values) are presented in Table 6 wherein we test the null hypothesis that the coefficients of past values in the regression equation are zero. In the table, the row is the response variable and the column is the predictor series. We infer from the results of the test (at a 10% level of significance) that the lower bound on the second moment of the PK causes the lower bound on the third and the fourth moment of the PK. We also infer that the lower bound on the third moment of the PK causes the lower bound on the second and fourth moment of the PK. Lastly, we infer that the lower bound on the fourth moment of the PK causes the lower bound on the third moment of the PK.

Table 6: P-values of the Granger Causality Test

	Lower bound on the second moment of the PK	Lower bound on the third moment of the PK	Lower bound on the fourth moment of the PK
Lower bound on the second moment of the PK	1.0	0.0804	0.1202

Lower bound on the third moment of the PK	0.0	1.0	0.0460
Lower bound on the fourth moment of the PK	0.0	0.0136	1.0

Further, the VAR model enables us to model each variable as a linear combination of past values of itself and the past values of other variables in the system. We formulate the VAR model of order 1 that gives the least AIC. We present the findings of the VAR (1) model in Table 6.

Table 7: Summary of the regression results derived from the VAR model

Results for the equation of lower bounds on the second moment of the PK		
	Coefficient	Standard error
Constant	-0.039990	0.002320
First lag on the lower bound of the second moment	0.080127	0.116770
First lag on the lower bound of the third moment	-0.045769	0.178508
First lag on the lower bound of the fourth moment	-0.006329	0.103189
Results for the equation of lower bounds on the third moment of the PK		
	Coefficient	Standard error
Constant	-0.080119	0.004714
First lag on the lower bound of the second moment	-1.712284	0.237258
First lag on the lower bound of the third moment	1.479898	0.362700
First lag on the lower bound of the fourth moment	-0.445496	0.209662
Results for the equation of lower bounds on the fourth moment of the PK		
	Coefficient	Standard error
Constant	-0.120314	0.007336
First lag on the lower bound of the second moment	-2.499099	0.369237
First lag on the lower bound of the third moment	1.130831	0.564459

First lag on the lower bound of the fourth moment	0.028067	0.326292

5.2 Diagnosing the Bounds

In this section, we examine the LRR model proposed by Bansal and Yaron (2004). The computation is based on the parameterizations implied by the calibrations in Bansal, Kiku, and Yaron (2012). Table 8 provides a comparison of the properties of the PK processes drawn from the conditional second, third, and fourth moment of the PK based on the LRR model with our data-implied lower bounds. Our results show that the conditional and unconditional expectation of the PK of higher power based on the LRR model satisfies our lower bounds. Given that the LRR model is a reliable method for estimating the second, third, and fourth moment of the PK, it illustrates the usefulness of our bounds. Furthermore, we investigate the correlation between the series of the second, third, and fourth moment of our obtained bounds and those of the second, third, and fourth moment of the PK computed from the LRR model (depicted in Table 9). Our findings reveal a strong correlation between these series of moments.

Table 8: Comparison of the properties of PK processes drawn from obtained bounds with those proposed by the LRR model

	LRR Model			Our obtained lower bounds			LRR Model		
	$\log E[M_{t+1}^2]$	$\log E[M_{t+1}^3]$	$\log E[M_{t+1}^4]$	$\log E_t[M_{t+1}^2]$	$\log E_t[M_{t+1}^3]$	$\log E_t[M_{t+1}^4]$	$\log E_t[M_{t+1}^2]$	$\log E_t[M_{t+1}^3]$	$\log E_t[M_{t+1}^4]$
Average	0.009219	0.035635	0.076589	-0.039044	-0.071957	-0.106781	0.003263	0.023911	0.057237
Standard Deviation	0.008163	0.018366	0.032650	0.003892	0.008568	0.013276	0.001382	0.005043	0.010831
Skewness	-0.177176	-0.177176	-0.177176	0.094735	0.105487	0.273303	-1.357057	-1.452567	-1.396497
Kurtosis	-0.559164	-0.559164	-0.559164	-0.148131	-0.154462	0.264053	1.947250	2.480872	2.403378
First quintile	0.002999	0.021641	0.051711	-0.042357	-0.079314	-0.118511	0.002473	0.020858	0.051025
Second quintile	0.007708	0.032235	0.070545	-0.040097	-0.074337	-0.110050	0.003335	0.024149	0.057065
Third quintile	0.010412	0.038320	0.081363	-0.038038	-0.070266	-0.104557	0.003795	0.025640	0.061121
Fourth quintile	0.018132	0.055690	0.112242	-0.035993	-0.064818	-0.096084	0.004319	0.027563	0.064666
Fifth quintile	0.025159	0.071500	0.140349	-0.027655	-0.045034	-0.064007	0.005697	0.032762	0.076584
Autocorrelation at lag 1	0.994189	0.994189	0.994189	-0.017897	0.134294	0.149847	0.928406	0.943721	0.947714
Autocorrelation at lag 2	0.988652	0.988652	0.988652	0.107225	0.274864	0.282681	0.858181	0.888046	0.895841
Autocorrelation at lag 3	0.983852	0.983852	0.983852	0.000510	0.151393	0.146167	0.799180	0.840247	0.851001
Autocorrelation at lag 4	0.978956	0.978956	0.978956	-0.024273	0.109862	0.088360	0.740149	0.794619	0.808517
Autocorrelation at lag 5	0.974059	0.974059	0.974059	0.007451	0.125622	0.095126	0.683041	0.750267	0.767096
Autocorrelation at lag 6	0.969430	0.969430	0.969430	0.118282	0.230070	0.198152	0.629663	0.708309	0.727769
Autocorrelation at lag 7	0.965240	0.965240	0.965240	0.017225	0.112484	0.074882	0.585147	0.670916	0.692098
Autocorrelation at lag 8	0.960677	0.960677	0.960677	0.060764	0.172829	0.142867	0.546038	0.634921	0.656995
Autocorrelation at lag 9	0.955891	0.955891	0.955891	-0.015585	0.124585	0.104024	0.502243	0.596743	0.620252
Autocorrelation at lag 10	0.951617	0.951617	0.951617	0.026266	0.149474	0.127786	0.463831	0.562067	0.586707
Autocorrelation at lag 11	0.947685	0.947685	0.947685	-0.040364	0.099276	0.083214	0.430175	0.527089	0.552081
Autocorrelation at lag 12	0.943777	0.943777	0.943777	0.094465	0.204700	0.176817	0.387081	0.486718	0.513035

Table 9: Correlation between the series of PK moments

	Obtained lower bounds	LRR model (Conditional)
Correlation between SDF series of power 2 and 3	0.849676	0.990806
Correlation between SDF series of power 3 and 4	0.985529	0.999223
Correlation between SDF series of power 2 and 4	0.810518	0.984704

Chapter 6: Conclusion

The PK is a critical concept in financial economics, and its properties affect the pricing of all financial assets. The PK determines the risk-return trade-off in financial markets and contains information about investors' risk preferences and beliefs. We examine the dynamics of conditional moments of the PK obtained with the use of bounds derived by Orłowski, Sali, and Trojani (2018). We compute the bounds on the different PK moments using the data implied by options prices and the conditional variance of the market return. These bounds are instrumental because these bounds yield information on expected returns and serve as a diagnostic tool to evaluate the performance of dynamic asset pricing models. These bounds can also be used in developing performance measures for portfolio managers. We analyze the dynamics of the obtained bounds and draw inferences about the properties of admissible PK processes. We illustrate the usefulness of our bounds by diagnosing the LRR model proposed by Bansal and Yaron (2004). We examine the LRR model and simulate the conditional expectation of PK of higher order based on the parameterizations implied by the calibrations in Bansal, Kiku, and Yaron (2012).

We demonstrate the following findings that shed light on the properties of the moments of the PK. Firstly, we observed that the lower bounds on the second, third, and fourth moments of the PK are negative throughout the entire period of our study which have important implications for risk management and financial decision-making. Secondly, we found that the lower bounds on the second moment of the PK exceed those of the third moment, which in turn exceed those of the fourth moment. This trend persisted throughout the duration of our study. Thirdly, we found that in periods of economic turbulence, the third and fourth moments of the PK come very close. Fourthly, we infer (at a 10% level of significance) that the lower bound on the second moment of the PK causes the lower bound on the third and the fourth moment of the PK, the lower bound on the third moment of the PK causes the lower bound on the second and fourth moment of the PK and the lower bound on the fourth moment of the PK causes the lower bound on the third moment of the PK. Fifthly, we conclude that the conditional and unconditional expectation of the second, third, and fourth moment of the PK calculated from the LRR model satisfies our lower bounds. Lastly, we conclude that the correlation within the series of lower bounds of our varied PK moments and those of the moments of the PK computed from the LRR model is very high.

In conclusion, our study contributes to the existing literature on the bounds of the moments of the PK and provides valuable insights into its behavior over a significant period. Our findings can aid in the development of more methods for estimating the bounds of the moments of the PK and can have important implications for portfolio management, risk assessment, and financial modeling.

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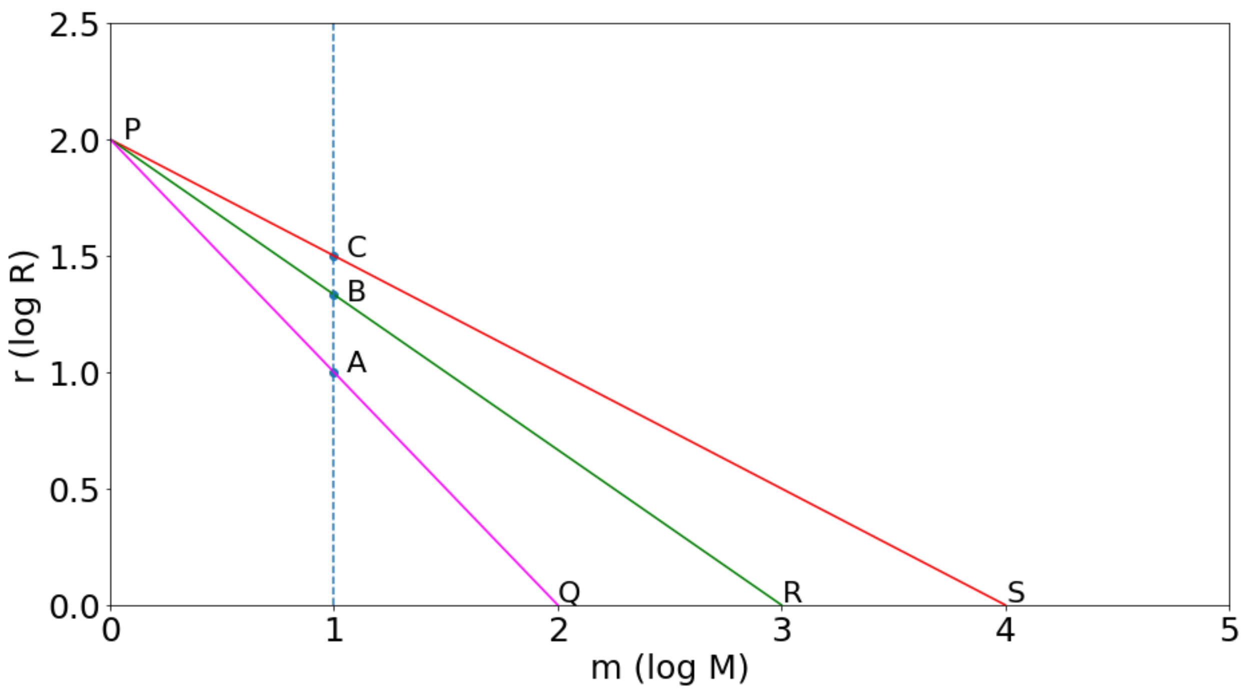
Appendices

Appendix A Inequalities for lower bounds of the moments of the PK

We derive the inequalities for the lower bound of the second, third, and fourth moment of the PK by solving for the coordinates of convex combinations of the observable regions of the CGF of the PK and asset returns. The CGF of the PK and asset returns represented as $\kappa_{MR}(m,r) = \log E[M^m R^r]$ is depicted in Figure 7.

Figure 7: CGF of pricing kernel and asset returns



A.1 Lower bound for the second moment of the PK

Point A depicted in Figure 7 is a convex combination of point P and Point Q having coordinates (0,2) and (2,0) respectively. Assuming the coordinates of point A as $(1, a)$, we solve for a below:

$$\alpha \begin{bmatrix} 0 \\ 2 \end{bmatrix} + (1-\propto) \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ a \end{bmatrix} \tag{A.1}$$

$$\propto = \frac{1}{2}\,,\ 1-\propto = \frac{1}{2} \text{ and } a = 1$$

Plugging in the values represented by points P, Q, and A in Eq (A.1), we get:

$$\frac{1}{2}\log E_t(R_{t+1}^2) + \frac{1}{2}\log E_t(M_{t+1}^2) \geq \log E_t(M_{t+1}R_{t+1}^1)$$

$$\frac{1}{2}\log E_t(R_{t+1}^2) + \frac{1}{2}\log E_t(M_{t+1}^2) \geq 0$$

$$\log \mathrm{E_t}[\mathrm{M_{t+1}^2}] \geq -\log \mathrm{E_t}[\mathrm{R_{t+1}^2}] \tag{A.2}$$

A.2 Lower bound for the third moment of the PK

Point B depicted in Figure 7 is a convex combination of point P and Point R having coordinates $(0,2)$ and $(3,0)$ respectively. Assuming the coordinates of point B as $(1, b)$, we solve for b below:

$$\alpha \begin{bmatrix}0\\2\end{bmatrix} + (1{-}\propto) \begin{bmatrix}3\\0\end{bmatrix} = [\begin{matrix}1\\b\end{matrix}] \tag{A.3}$$

$$\propto = \tfrac{2}{3}\ , 1{-}\propto = \tfrac{1}{3} \text{ and } b = \tfrac{4}{3}$$

Plugging in the values represented by points P, R, and B in Eq (A.3), we get:

$$\frac{2}{3}\log E_t(R_{t+1}^2) + \frac{1}{3}\log E_t(m_{t+1}^3) \geq \log E_t(m_{t+1}R_{t+1}^{4/3})$$

$$\frac{2}{3}\log E_t(R_{t+1}^2) + \frac{1}{3}\log E_t(m_{t+1}^3) \geq \log E_t^Q[R_{t+1}^{4/3}]$$

$$\log E_t[M_{t+1}^3] \geq 3\log E_t^Q[R_{t+1}^{4/3}] - 2\log E_t[R_{t+1}^2] \tag{A.4}$$

A.3 Lower bound for the fourth moment of the PK

Point C depicted in Figure 7 is a convex combination of point P and Point S having coordinates $(0,2)$ and $(4,0)$ respectively. Assuming the coordinates of point C as $(1, c)$, we solve for c below:

$$\alpha \begin{bmatrix}0\\2\end{bmatrix} + (1{-}\propto) \begin{bmatrix}4\\0\end{bmatrix} = [\begin{matrix}1\\c\end{matrix}] \tag{A.5}$$

$$\propto = \tfrac{3}{4}\ , 1{-}\propto = \tfrac{1}{4} \text{ and } c = \tfrac{3}{2}$$

Plugging in the values represented by points P, S, and C in Eq (A.5), we get:

$$\frac{3}{4}\log E_t(R_{t+1}^2) + \frac{1}{4}\log E_t(m_{t+1}^4) \geq \log E_t(m_{t+1}R_{t+1}^{3/2})$$

$$\frac{3}{4}\log E_t(R_{t+1}^2) + \frac{1}{4}\log E_t(m_{t+1}^4) \geq \log E_t^Q[R_{t+1}^{3/2}]$$

$$\log E_t[M_{t+1}^4] \geq 4\log E_t^Q[R_{t+1}^{3/2}] - 3\log E_t[R_{t+1}^2] \tag{A.6}$$

Appendix B Long-run risk model

B.1 Derivation of the conditional expectation of higher power of the PK based on the LRR model

The conditional expectation of higher power of the PK can be solved from the log of IMRS expressed in Eq. (3):

$$\log m_{t+1} = \theta \log \delta - \frac{\theta}{\psi} g_{t+1} + (\theta - 1)\, r_{a,t+1}$$

Taking exponential and raising both sides to a power p, we get:

$$m_{t+1}^{p} = e^{p(\theta \log \delta - \frac{\theta}{\psi} g_{t+1} + (\theta-1) r_{a,t+1})}$$

Taking conditional expectation at time t on both sides, we get:

$$E_t(m_{t+1}^{p}) = E_t(e^{p(\theta \log \delta - \frac{\theta}{\psi} g_{t+1} + (\theta-1) r_{a,t+1})})$$

$$= \; e^{E_t\left(p(\theta \log \delta - \frac{\theta}{\psi} g_{t+1} + (\theta-1) r_{a,t+1})\right) + \frac{1}{2} Var_t\left(p(\theta \log \delta - \frac{\theta}{\psi} g_{t+1} + (\theta-1) r_{a,t+1})\right)}$$

$$= e^{\left(p(\theta \log \delta - \frac{\theta}{\psi} E_t(g_{t+1}) + (\theta-1) E_t\left(r_{a,t+1}\right))\right) + \frac{1}{2} p^2 Var_t\left((\theta \log \delta - \frac{\theta}{\psi} g_{t+1} + (\theta-1) r_{a,t+1})\right)}$$

$$= e^{\left(p\left(\theta \log \delta - \frac{\theta}{\psi} E_t(g_{t+1}) + (\theta-1) E_t\left(r_{a,t+1}\right)\right)\right)} .$$

$$e^{\left(\frac{1}{2} p^2 \left(\left(Var_t\left(-\frac{\theta}{\psi} g_{t+1}\right) + Var_t((\theta-1) r_{a,t+1}) + 2cov(g_{t+1}, r_{a,t+1})\right)\right)\right)} \quad \text{(B.1)}$$

We calculate the conditional expected value of model parameters below,

$$E_t(g_{t+1}) = \; \mu + x_t \quad \text{(B.2)}$$

$$E_t(x_{t+1}) = \; \rho x_t \quad \text{(B.3)}$$

$$E_t(\sigma_{t+1}^2) = \sigma^2 + \nu_1(\sigma_t^2 - \sigma^2) \tag{B.4}$$

$$E_t(z_t) = A_0 + A_1 x_t + A_2\sigma_t^2 \tag{B.5}$$

$$z_{t+1} = A_0 + A_1 x_{t+1} + A_2\sigma_{t+1}^2 \tag{B.6}$$

$$E_t(z_{t+1}) = A_0 + A_1 E_t(x_{t+1}) + A_2 E_t(\sigma_{t+1}^2) \tag{B.7}$$

Substituting Eq. (B.3) for $E_t(x_{t+1})$ and Eq. (B.4) for $E_t(\sigma_{t+1}^2)$ in Eq. (B.7), we obtain

$$E_t(z_{t+1}) = A_0 + A_1\rho x_t + A_2(\sigma^2 + \nu_1(\sigma_t^2 - \sigma^2)) \tag{B.8}$$

$$E_t(r_{a,t+1}) = \kappa_0 + \kappa_1 E_t(z_{t+1}) - z_t + E_t(g_{t+1}) \tag{B.9}$$

Substituting Eq. (B.8) for $E_t(z_{t+1})$ and Eq. (B.2) for $E_t(g_{t+1})$ in Eq. (B.9), we obtain

$$E_t(r_{a,t+1}) = \kappa_0 + \kappa_1(A_0 + A_1\rho x_t + A_2(\sigma^2 + \nu_1(\sigma_t^2 - \sigma^2))) - z_t + \mu + x_t \tag{B.10}$$

Using $Var_t(X) = E_t[(X - E_t(X))^2]$, we calculate the conditional variance of model parameters below

$$Var_t(g_{t+1}) = {\sigma_t}^2 \tag{B.11}$$

$$Var_t(x_{t+1}) = (\varphi_e\sigma_t)^2 \tag{B.12}$$

$$Var_t(\sigma_{t+1}^2) = (\sigma_w)^2 \tag{B.13}$$

$$Var_t(z_t) = 0 \tag{B.14}$$

$$Var_t(r_{a,t+1}) = E_t[((r_{a,t+1}) - E_t(r_{a,t+1}))^2]$$

$$\begin{aligned} Var_t(r_{a,t+1}) = E_t[((&\kappa_0 + \kappa_1 z_{t+1} - z_t + g_{t+1}) \\ &- (\kappa_0 + \kappa_1(A_0 + A_1\rho x_t + A_2(\sigma^2 + \nu_1(\sigma_t^2 - \sigma^2))) - z_t + \mu + x_t))^2] \end{aligned} \tag{B.15}$$

Substituting Eq. (B.6) for z_{t+1} in Eq. (B.15), we obtain

$$Var_t\left(r_{a,t+1}\right) = E_t[(\kappa_1 A_1 x_{t+1} + \kappa_1 A_2 \sigma_{t+1}^2 + g_{t+1} - \kappa_1 A_1 \rho x_t - \kappa_1 A_2 . \sigma^2 - \kappa_1 A_2 \nu_1 (\sigma_t^2 - \sigma^2) - \mu - x_t)^2] \tag{B.16}$$

Substituting value of x_{t+1} from Eq. (14), g_{t+1} from Eq. (13) and the value of σ_{t+1}^2 from Eq. (15), we get:

$$Var_t\left(r_{a,t+1}\right) = \kappa_1^2 A_1^2 \varphi_e^2 \sigma_t^2 + \kappa_1^2 A_2^2 \sigma_w^2 + \sigma_t^2 \tag{B.17}$$

Given $Cov\left(g_{t+1}, r_{a,t+1}\right) = -\frac{\theta}{\psi}(\theta - 1)\sigma^2$, by plugging Eq. (B.2), Eq. (B.10), Eq. (B.11) and Eq. (B.17) into Eq. (B.1), we obtain

$$E_t(m_{t+1}^p) = e^{\left(p\left(\theta \log\delta - \frac{\theta}{\psi}(\mu + x_t) + \left((\theta-1)\left(\kappa_0 + \kappa_1\left(A_0 + A_1 \rho x_t + A_2 \sigma^2\right) - z_t\right) + \mu + x_t\right)\right)\right)}.$$

$$e^{\frac{1}{2}p^2\left(\left(\frac{\theta^2}{\psi^2}(\sigma_t{}^2) + (\theta-1)^2\left(\kappa_1^2 A_1^2 \varphi_e^2 \sigma_t^2 + \kappa_1^2 A_2^2 \sigma_w^2 + \sigma_t^2\right) - 2\frac{\theta}{\psi}(\theta-1)\sigma^2\right)\right)}$$

B.2 Unconditional expectation of LRR model variables

We write below the expressions for the unconditional expectation of the LRR model variables. The values of the model variables at time $t = 0$ have been set equal to their unconditional expectation in the simulation exercise to compute $E_t[R_{t+1}^2]$.

$$E(g_{t+1}) = \mu \tag{B.18}$$

$$0E(x_t) = 0 \tag{B.19}$$

$$E(\sigma_{t+1}^2) = \sigma^2 \tag{B.20}$$

$$E(z_t) = A_0 + A_2 \sigma^2 \tag{B.21}$$

$$E\left(r_{a,t+1}\right) = \kappa_0 + (\kappa_1 - 1)(A_0 + A_2 \sigma^2) + \mu \tag{B.22}$$

B.3 Derivation of the unconditional expectation of higher power of the PK based on the LRR model

The unconditional expectation of higher power of the PK can be solved from the log of IMRS expressed in Eq. (3). By taking the exponential of $\log m_{t+1}$ and raising both sides to a power p, we get:

$$m_{t+1}^{p} = e^{p(\theta \log \delta - \frac{\theta}{\psi} g_{t+1} + (\theta-1) r_{a,t+1})}$$

Taking unconditional expectation at time t on both sides, we get:

$$E(m_{t+1}^{p}) = E(e^{p(\theta \log \delta - \frac{\theta}{\psi} g_{t+1} + (\theta-1) r_{a,t+1})})$$

$$= e^{E\left(p(\theta \log \delta - \frac{\theta}{\psi} g_{t+1} + (\theta-1) r_{a,t+1})\right) + \frac{1}{2} Var\left(p(\theta \log \delta - \frac{\theta}{\psi} g_{t+1} + (\theta-1) r_{a,t+1})\right)}$$

$$= e^{\left(p(\theta \, log \, \delta - \frac{\theta}{\psi} E(g_{t+1}) + (\theta-1) E\left(r_{a,t+1}\right))\right) + \frac{1}{2} p^2 Var\left((\theta \, log \, \delta - \frac{\theta}{\psi} g_{t+1} + (\theta-1) r_{a,t+1})\right)} \tag{B.23}$$

Substituting Eq. (B.18) for $E(g_{t+1})$ and Eq. (B.22) for $E\left(r_{a,t+1}\right)$ in Eq. (B.23), we obtain

$$= e^{\left(p(\theta \, log \, \delta - \frac{\theta}{\psi} \mu + (\theta-1)(\kappa_0 + (\kappa_1 - 1)\left(A_0 + A_2 \sigma^2\right)) + \mu)\right)}.$$

$$e^{\frac{1}{2} p^2 Var\left((\theta \, log \, \delta - \frac{\theta}{\psi} g_{t+1} + (\theta-1) r_{a,t+1})\right)} \tag{B.24}$$

Using $Var(X) = E[(X - E(X))^2]$, we calculate the unconditional variance of model parameters below

$$Var(g_{t+1}) = \sigma^2 \tag{B.25}$$

$$Var(x_{t+1}) = (\varphi_e \sigma_t)^2 \tag{B.26}$$

$$Var(\sigma_{t+1}^2) = (\sigma_w)^2 \tag{B.27}$$

$$Var(z_t) = 0 \tag{B.28}$$

$$Var(r_{a,t+1}) = E[((r_{a,t+1}) - E_t(r_{a,t+1}))^2]$$

$$\begin{aligned} Var(r_{a,t+1}) = \; & E[((\kappa_0 + \kappa_1 z_{t+1} - z_t + g_{t+1}) \\ & - (\kappa_0 + \kappa_1(A_0 + A_2\sigma^2) - (A_0 + A_2\sigma^2) + \mu))^2] \end{aligned} \tag{B.29}$$

Substituting Eq. (B.6) for z_{t+1} in Eq. (B.29), we obtain

$$\begin{aligned} Var(r_{a,t+1}) = \; & E[(\kappa_1 A_1 x_{t+1} + \kappa_1 A_2 \sigma_{t+1}^2 - z_t + g_{t+1} - \kappa_1 A_2 . \sigma^2 + A_0 + A_2\sigma^2 \\ & - \mu)^2] \end{aligned} \tag{B.30}$$

Substituting value of x_{t+1} from Eq. (14), g_{t+1} from Eq. (13) and the value of σ_{t+1}^2 from Eq. (15), we get

$$Var(r_{a,t+1}) = \kappa_1^2 A_1^2 \varphi_e^2 \sigma^2 + \kappa_1^2 A_2^2 \sigma_w^2 + \sigma^2 \tag{B.31}$$

Given $Cov(g_{t+1}, r_{a,t+1}) = -\frac{\theta}{\psi}(\theta - 1)\sigma^2$, by plugging Eq. (B.25) and Eq. (B.30) into Eq. (B.24), we obtain

$$E(m_{t+1}^p) = e^{\left(p(\theta \, log \, \delta - \frac{\theta}{\psi}\mu + (\theta-1)(\kappa_0 + (\kappa_1 - 1)(A_0 + A_2\sigma^2)) + \mu)\right)}.$$

$$e^{\frac{1}{2}p^2(\frac{\theta^2}{\psi^2}(\sigma^2) + (\theta-1)^2(\kappa_1^2 A_1^2 \varphi_e^2 \sigma_t^2 + \kappa_1^2 A_2^2 \sigma_w^2 + \sigma^2) - 2\frac{\theta}{\psi}(\theta-1)\sigma^2)}$$

B.4 Expressions for the components of the conditional and unconditional expectations of higher power of the PK

We write below the expressions for the components of the conditional and unconditional expectations of higher power of the PK in terms of the parameters of the LRR model.

$$A_0 = \frac{1}{1-\kappa_1}[\log \delta + \kappa_0 + \left(1 - \frac{1}{\psi}\right)\mu + \kappa_1 A_2 (1 - \nu_1)\sigma^2 + \frac{\theta}{2}(\kappa_1 A_2 \sigma_w)^2] \tag{B.32}$$

$$A_1 = \frac{1 - \frac{1}{\psi}}{1 - \kappa_1 \rho} \tag{B.33}$$

$$A_2 = \frac{0.5\left[(\theta - \frac{\theta}{\psi})^2 + (\theta A_1 \kappa_1 \varphi_e)^2\right]}{\theta(1 - \kappa_1 \nu_1)} \tag{B.34}$$

$$\kappa_0 = \log(1 + \exp(\bar{z})) - (\kappa_1 \bar{z}) \tag{B.35}$$

$$\kappa_1 = \frac{exp\ (\bar{z})}{1 + exp\ (\bar{z})} \tag{B.36}$$

Appendix C

All the computations have been coded in Python scripts in Jupyter Notebook. Developed by Fernando Pérez and Brian Granger, Jupyter served us as an ideal open-source interactive development environment to solve and analyze our bounds. We performed our computations by using Python libraries namely Pandas, NumPy, Scipy, Matplotlib, Seaborn, Statsmodels, PyTables, NLTK, Scikit-learn, Beautiful Soup, and Linearmodels. We ran the computations on a 64-bit operating system with an Intel® Core™ i5-1135G7 CPU of 2.40GHz and an installed RAM of 16.0 GB.