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Mortgage and Mortgage-Backed Securities Valuation

par

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Abstract

Mortgage Backed-Securities are exposed to prepayment risk, both from prepayment and default occurring in the underlying pool of mortgages. The probability of such events is modelled using intensity functions: prepayment is mostly driven by the current level of interest rates relative to the mortgage contractual rate; default is primarily driven by the ratio of the mortgage outstanding balance relative to the housing value. Interest rates are assumed to follow an Ornstein-Uhlenbeck process, as in Vasicek. Housing prices are modeled using a geometric Brownian motion, correlated with the interest rate process. In order to capture a wide range of possible states for the interest rate and housing price variables over the long life of a mortgage, a Markov chain is used, based on two grids, one for the interest rates and one for the housing prices. To ensure housing prices do not fall outside the bounds of the grid in spite of the housing price long-term upward drift, only the stochastic part of the housing price dynamics gets deployed on the grid. A mortgage is then valued using dynamic programming. The drift part of the housing price process is calculated off-line and added at each time step to the diffusion part. The state variable dynamics parameters are estimated using maximum likelihood. We fit the prepayment intensity functions parameters to the observed prepayment rates by minimizing squared differences. The model has been stress-tested and turns out to be robust even in extreme scenarios such as negative interest rates.

Keywords : Mortgage, MBS, valuation, prepayment, default, Vasicek, geometric Brownian motion, interest rates, housing prices, Markov chain.

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1 Literature review

About two households out of three own their home in Canada or the US.[1] A key factor behind so high a homeownership rate lies in relatively low mortgage rates. Banks can offer affordable rates to mortgagors, in part thanks to the existence of a secondary market. On such a market, banks can quickly offload and turn into cash the mortgages they issue. Mortgages are split, aggregated and packaged into tradeable securities, called mortgage-backed securities ("MBS"). Investors purchasing MBS need accurate risk-sensitive models to price what they buy.

In practice, valuation models available to investors are either very simplified, using a deterministic one-size-fit-all linear prepayment model;[2] or highly complex and linked to the swaption market,[3] which may or may not be relevant to investors who do not use swaptions to hedge their exposure to mortgage prepayment.

It may be useful to explore alternative valuation models that faithfully reflect the random nature of interest rates, housing prices, prepayment and default events; incorporate stable estimates of the parameters governing the probability distribution thereof; and remain relatively simple numerically.

Extensive literature has been dedicated to valuing MBS. An early focus has been on the risk of early termination. At first, the emphasis was on the risk of prepayment. A single state variable, the interest rate, was considered (for example in Schwartz and Torous (1989) or Stanton (1995)). Then, the risk of default was also incorporated, and a second state

[1] For Canada, see Statistics Canada, at https://www150.statcan.gc.ca/n1/pub/11-402-x/2011000/chap/fam/fam-eng.htm. For the US, see U.S. Census Bureau, Homeownership Rate for the United States [RHORUSQ156N], retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/RHORUSQ156N, June 2, 2019.

[2] For a description of the PSA Prepayment Model developed by the Public Securities Association for the US in 1985 and still widely in use, see https://en.wikipedia.org/wiki/PSA_prepayment_model. For a Canadian equivalent tailored to National Housing Act ("NHA") MBS, see a description of the Linear Liquidation Model at https://iiac.ca/wp-content/uploads/IIAC-MBS-Committee-NHA-MBS-Linear-Liquidation-Model-v-1.1.pdf.

[3] For a description of the LIBOR Market Model, used for example by Bloomberg in their MBS valuation functions, see Brigo and Mercurio (2007).

variable, the house price, was introduced (see Schwartz and Torous (1992) or Stanton *et al.* (2005)).

In such papers, the interest rate is modeled using a Cox-Ingersoll-Ross mean-reverting square-root process, and, where applicable, housing prices are assumed to follow a geometric Brownian motion correlated with the interest-rate Brownian.

For the prepayment or default function, reduced-form models have been developed, as in Schwartz and Torous (1992): mortgagors are assumed to be rational and to exercise their prepayment or default option when it is optimal. Thus, they prepay their mortgage when current rates fall below their mortgage contractual rate; and they default on their mortgage when the mortgaged house value falls below the value of the mortgage liability. The risk of prepayment is captured by a hazard function whereby the prepayment intensity is a direct function of the gap between the mortgage contract rate and the current mortgage market rate. Similarly, the risk of default is captured by a hazard function whereby the default intensity is direct function of a mortgage outstanding balance to the value of the mortgaged house. In reduced-form models, prepayment and default function parameters are provided exogenously based on empirical observations (Schwartz and Torous (1992)).

Conversely, structural models have been developed by for example Stanton (1995). Stanton notes that the mortgagors' default is not necessarily rational, as they do not exercise their prepayment or default option optimally. Friction or transaction costs are modeled and aim at capturing different, and suboptimal, observed behaviors among different mortgagors. Other stylized facts, such as seasoning patterns, are emphasized and modeled (Stanton).

Various numerical approaches have been implemented, from solving a partial differential equation (or a couple of partial differential equations) to Monte Carlo simulations (as described in Hull (2012)) to combining a binomial tree with a Monte Carlo simulation (as described in Veronesi (2010)).

More recently, alternative interest rate models have been applied to valuing MBS, especially the LIBOR-market model (LLM). The MBS is modeled as a tradable callable bond, the prepayment option of which could be hedged using swaptions, and the aim is to capture the volatility skew observed on the swaption market. See for example Karpishan *et al.* (2010) or chapter 31 in Hull (2012).

We used a two-state-variable (interest rate and housing price) model. The interest rate and the housing prices respectively follow a single-factor Ornstein-Uhlenbeck process, as in Vasicek (1977), and a Geometric Brownian motion correlated therewith. We modeled the risk of prepayment and default using two intensity functions, which express the rational behavior of option holders, however combined with background prepayment and default rates, which translate non-rational observed behaviour. Also, as we adopt the perspective of a MBS pool on the US Agency TBA market for 15-year mortgages, where pools are fungible, mortgagor-specific transaction costs can be ignored, as the individual differences can reasonably be assumed to average out from the law of larger numbers. The burnout effect is naturally expressed by the fact that the aggregate principal of the pool is reduced by earlier prepayments, so that the absolute prepayment amount at the pool level decreases as time passes. The burnout effect may also be indirectly expressed by the lower intensity in the default function as the mortgage balance decreases (see equation (27)), so that prepayment is cross-expressed by both the prepayment and the default function.

We implemented the valuation using dynamic programming and a two-dimensional grid, similar to the Markov-chain approach applied by Duan and Simonato (2001) to the valuation of American equity options.

We estimated the parameters in the interest-rate and housing price dynamics using log-likelihood and we estimated the parameters in the prepayment function using a least squared difference approach.

2 Mortgage and Mortgage-Backed Securities (MBS) Valuation Methodology

2.1 No prepayment

2.1.1 One mortgage

The monthly payment p of a mortgage with a face value F, a maturity of T months, and a monthly interest rate (coupon) C satisfies:

$$F = p \sum_{t=1}^{T} \left(\frac{1}{1+C} \right)^t .$$

But $\sum_{t=1}^{T} \left(\frac{1}{(1+C)} \right)^t = \frac{1}{1+C} \sum_{t=1}^{T} \left(\frac{1}{1+C} \right)^{t-1}$, where we recognize a geometric series. Hence,

$$F = p \frac{1}{1+C} \frac{1 - \left(\frac{1}{1+C}\right)^T}{1 - \frac{1}{1+C}} = p \frac{1 - \left(\frac{1}{1+C}\right)^T}{C}. \tag{1}$$

Therefore, the monthly payment is

$$p = F \frac{C}{1 - \left(\frac{1}{1+C}\right)^T}. \tag{2}$$

Lemma 1. *At month t, the balance is*

$$B_t = F(1+C)^t \left(1 - \frac{(1+C)^T - (1+C)^{T-t}}{(1+C)^T - 1} \right). \tag{3}$$

Proof. Initially, $B_0 = F$. Then, $\forall t \in \{1, T\}$, the balance at t, noted B_t, equals the balance at $t-1$, less the payment made at t, net of the interest amount accrued over one period on the $t-1$-balance. In equation:

$$B_t = B_{t-1} - (p - CB_{t-1}) \tag{4}$$

$$= B_{t-1}(1+C) - p.$$

Using recursion over time yields

$$\begin{aligned} B_t &= F(1+C)^t - p\sum_{u=0}^{t-1}(1+C)^u \\ &= F(1+C)^t + \frac{FC}{1-\left(\frac{1}{1+C}\right)^T}\frac{1-(1+C)^t}{C} \\ &= F(1+C)^t\left(1 - \frac{(1+C)^T - (1+C)^{T-t}}{(1+C)^T - 1}\right). \end{aligned}$$

□

2.1.2 Pool of mortgages

Exact calculations Consider a pool of M mortgages. Assume they are all issued at the same issuance date, which matches the pool origination date, and share the same maturity date. The initial face value is

$$F = \sum_{m=1}^{M} F^{(m)}.$$

The equations for the face value as a function of C, the total fixed payment, and the unpaid balances at any time t are simple sums of the right-hand terms in equations (1), (2) and (3) respectively. Thus,

$$F = \sum_{m=1}^{M} p^{(m)} \frac{1-\left(\frac{1}{1+C^{(m)}}\right)^T}{C^{(m)}} \tag{5}$$

$$P = \sum_{m=1}^{M} p^{(m)} = \sum_{m=1}^{M} F^{(m)} \frac{C^{(m)}}{1-\left(\frac{1}{1+C^{(m)}}\right)^T} \tag{6}$$

$$B_t = \sum_{m}^{M} B_t^{(m)} = \sum_{m=1}^{M} F^{(m)}(1+C^{(m)})^t\left(1 - \frac{(1+C^{(m)})^T - (1+C^{(m)})^{T-t}}{(1+C^{(m)})^T - 1}\right). \tag{7}$$

In practice, for Fannie Mae Single-Family MBS, coupon information available to investors used to be limited to the pool average coupon, and some percentiles. Since January 1, 2013, individual loan-level information has been available.[4] On the to-be-announced (TBA) market, such information is however pointless, as MBS security holders do not know which pool, from a cohort of thousands, will be assigned to their security. Hence the need to work with cohort averages.

[4]See Fannie Mae PoolTalk © FAQ's, retrieved from http:www.fanniemae.com/resources/file/mbs/pdf/mbsfaqs.pdf on April 26, 2018.

Average-based calculations Define the original-balance-weighted average of the mortgage coupons:

$$\bar{C} = \sum_{m=1}^{M} \frac{F^{(m)}}{F} C^{(m)}. \tag{8}$$

Then, similar to equation (1), $\bar{P}$, an approximation to the pool aggregate fixed payment, satisfies

$$F = \bar{P} \sum_{t=1}^{T} \left(\frac{1}{1+\bar{C}} \right)^t .$$

Therefore,

$$\bar{P} = F \, \frac{\bar{C}}{1 - \left(\frac{1}{1+\bar{C}} \right)^T} . \tag{9}$$

As a corollary, similar to equation (3),

$$\bar{B}_t = F(1+\bar{C})^t \left(1 - \frac{(1+\bar{C})^T - (1+\bar{C})^{T-t}}{(1+\bar{C})^T - 1} \right) . \tag{10}$$

Remark The total monthly payment P, from equation (6) is different from a fixed monthly payment calculated at the pool level using $\bar{C}$ and a maturity of $T \geq max_m(T^{(m)})$, as is done in equation (9):

$$\bar{P} = \frac{F\bar{C}}{1 - \left(\frac{1}{1+\bar{C}} \right)^T} = \frac{\sum_{m=1}^{M} F^{(m)} \bar{C}^{(m)}}{1 - \left(\frac{1}{1+\sum_{m'=1}^{M} \frac{F^{(m')}}{F} \bar{C}^{(m')}} \right)^T} .$$

In the case where all coupon rates $C^{(m)}$ and all maturities $T^{(m)}$ are the same, then $C^{(m)} = \bar{C}$, $T^{(m)} = T$ and

$$\bar{P} = \sum_{m=1}^{M} \frac{F^{(m)} C}{1 - \left(\frac{1}{1+C \sum_{m'=1}^{M} \frac{F^{(m')}}{F}} \right)^T} = \sum_{m=1}^{M} \frac{F^{(m)} C}{1 - \left(\frac{1}{1+C} \right)^T} = P.$$

Likewise, $\bar{B}_t \neq B_t$.

Nevertheless, $\bar{C}$, $\bar{P}$, and $\bar{B}_t$ are reasonable approximations for C, P, and B_t when projecting the cash flows of a pool underlying an MBS, since pooled mortgages share similar terms and coupons.

2.1.3 Mortgage-backed security (MBS)

Principal payments are passed through integrally to MBS holders. F and B_t, as defined for a pool in equations (5) and (7), or the approximation of the latter, $\bar{B}_t$, as described in equation (10), also apply to the related MBS.

Interest payments are passed through at a reduced rate $\check{C} < \bar{C}$, net of servicing fees (paid to the mortgage servicer) and a credit risk compensation (paid to the guarantor Agency). Based on $\bar{C}$ as defined in equation (8), a reasonable projection for the MBS payment in month t is then, from equation (4):

$$\begin{aligned}\check{P}_t &= \bar{B}_{t-1}(1+\check{C}) - \bar{B}_t \\ &= \bar{B}_{t-1}(1+\check{C}-\bar{C}+\bar{C}) - \bar{B}_t \\ &= \bar{B}_{t-1}(1+\bar{C}) - \bar{B}_t - \bar{B}_{t-1}(\bar{C}-\check{C}) \\ &= \bar{P} - \bar{B}_{t-1}(\bar{C}-\check{C}).\end{aligned}$$

Conclusion A bank who lends money to mortgagors will show them the coupon C, the payment p and the resulting mortgage balance amortizing schedule made up of B_t. Such numbers may be "customer-friendly", but they are not the whole story. Where does this coupon C come from?

2.2 Prepayment

Consider again a single mortgage.

2.2.1 Mortgage valuation

A mortgage is exposed to interest rate risk, default risk, and prepayment risk throughout its life.

Risk-free rate The risk-free rate r_t satisfies

$$dr_t = \alpha\left(\beta - r_t\right)dt + \sigma dW_t^{\mathbb{P},r}, \tag{11}$$

where

- β is the interest rate long-term average;
- α is the speed of mean-reversion;
- $W^{\mathbb{P},r}$ is a Brownian motion under the physical probability measure $\mathbb{P}$.

Housing price The housing price satisfies

$$dH_t = \mu_{H,t}H_t dt + \sigma_H H_t dW_t^{\mathbb{P},H}, \tag{12}$$

where

- H_t is the housing price at time t, which can be proxied by some housing index;
- μ_H is the housing price log-return drift;
- σ_H is the housing price log-return volatility.

Moreover,

$$\text{Corr}\left(W_t^{\mathbb{P},r}, W_t^{\mathbb{P},H}\right) = \rho \;\forall t > 0.$$

Therefore, we can write $W^{\mathbb{P},H} = \rho W_t^{\mathbb{P},r} + \sqrt{1-\rho^2} W_t^{\mathbb{P},\perp}$, where $W^{\mathbb{P},r}$ and $W^{\mathbb{P},\perp}$ are two independent Brownian motions. The state variables r_t and H_t generate a filtration $\mathbb{F}$, to which is associated a σ-algebra $(\mathcal{F}_t)_{t\geq 0}$.

Risk-neutral world In a risk-neutral world,

$$W_t^{\mathbb{Q},r} = W_t^{\mathbb{P},r} + \int_0^T \gamma_s^r ds$$

and

$$W_t^{\mathbb{Q},H} = W_t^{\mathbb{P},H} + \int_0^T \gamma_s^H ds.$$

If the Novikov condition is satisfied, there is a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that

- $$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \gamma_s^r dW_s^{\mathbb{P},r} - \int_0^T \gamma_s^\perp dW_s^{\mathbb{P},\perp} - \frac{1}{2}\int_0^T (\gamma_s^r)^2 + (\gamma_s^\perp)^2 ds\right)$$
- $W^{\mathbb{Q},r}$ and $W^{\mathbb{Q},\perp}$ are two independent $\mathbb{Q}$-Brownian motions.

Now, with respect to the Brownian associated with housing prices,

$$\gamma_t^H = \frac{\mu_H - r_t}{\sigma_H}. \tag{13}$$

Replacing r_t with its definition (see equation (30) in section 3.1.1 further down), and using Jensen inequality and Fubini's theorem,[5] it would be possible to show that

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \left(\gamma_s^H ds\right)^2\right)\right] < \infty, \tag{14}$$

so that the Novikov condition is satisfied.

Default time In continuous time, let $\Lambda_t^{(\eta)}$ be the default intensity at time t. The default time is

$$\tau_\eta = \inf\{t > 0 : \int_0^t \Lambda_s^{(\eta)} ds > E_1\}$$

where E_1 is an exponential random variable of expectation 1 independent of $W^{\mathbb{P},r}$ and $W^{\mathbb{P},\perp}$. In that case, the conditional survival probability with respect to default is

$$\mathbb{P}\left(\tau_\eta > T \,\middle|\, \mathcal{G}_t\right) = \mathbb{E}^{\mathbb{P}}\left[\exp\left(-\int_t^T \Lambda_s^{(\eta)} ds\right) \,\middle|\, \mathcal{G}_t\right] \delta_{\tau_\eta > t}, \tag{15}$$

where

[5] See for example https://math.stackexchange.com/questions/133691/can-i-apply-the-girsanov-theorem-to-an-ornstein-uhlenbeck-process.

- δ is an indicator function;

- $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t^{(\eta)} \vee \mathcal{H}_t^{(\pi)}$, with $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^{(\eta)} \vee \mathbb{H}^{(\pi)}$. $\mathcal{H}_t^{(\eta)}$ is a σ-algebra associated with the filtration $\mathbb{H}^{(\eta)}$ generated by the indicator of default. $\mathcal{H}_t^{(\pi)}$ is a σ-algebra associated with the filtration $\mathbb{H}^{(\pi)}$ generated by the indicator of prepayment (see below). $\mathcal{F}_t$ is a σ-algebra associated with $\mathbb{F}$, the filtration generated by the state variables r_t and H_t as already defined. The objective probability measure $\mathbb{P}$ is thus defined on the filtered probability space $\{\Omega, \mathbb{P}, \mathbb{G}, \mathcal{G}_{t\geq 0}\}$.

Prepayment time Similarly, let $\Lambda_t^{(\pi)}$ be the prepayment intensity at time t. The prepayment time is

$$\tau_\pi = \inf\{t > 0 : \int_0^t \Lambda_s^{(\pi)} ds > E_2\},$$

where E_2 is an exponential random variable of expectation 1 independent of $W^{\mathbb{P},r}$, $W^{\mathbb{P},\perp}$, and E_1.
The conditional survival probability with respect to prepayment

$$\mathbb{P}\left(\tau_\pi > T \,\middle|\, \mathcal{G}_t\right) = \mathbb{E}^{\mathbb{P}}\left[\exp\left(-\int_t^T \Lambda_s^{(\pi)} ds\right) \,\middle|\, \mathcal{G}_t\right] \delta_{\tau_\pi > t}. \quad (16)$$

Risk-free discount factor Let the risk-free discount factor be

$$D_{t,T}^{(RF)} = \exp\left(-\int_t^T r_s \, ds\right)$$

Mortgage value Let $t_i^* = i\Delta_t$, with Δ_t set to one month. A mortgage at time t_i^* is worth

$$\begin{aligned} V_{t_i^*} = \mathbb{E}^{\mathbb{Q}}\Bigg[\sum_{j=i+1}^{N} D_{t_i^*,t_j^*}^{(RF)} \Bigg\{ & \underbrace{p\delta_{\tau_\eta > t_j^*}\delta_{\tau_\pi > t_j^*}}_{\text{regular payment}} + \underbrace{\kappa B_{t_{j-1}^*}\delta_{t_{j-1}^* < \tau_\eta \leq t_j^*}\delta_{\tau_\pi > t_j^*}}_{\text{payment on default}} \\ & + \underbrace{B_{t_{j-1}^*}(1+C)\delta_{t_{j-1}^* < \tau_\pi \leq t_j^*}\delta_{\tau_\eta > t_j^*}}_{\text{prepayment}} \Bigg\} \,\Bigg|\, \mathcal{G}_{t_i^*}\Bigg] \delta_{\tau_\eta > t_i^*}\delta_{\tau_\pi > t_i^*}, \end{aligned} \quad (17)$$

where N is the total number of monthly payments stipulated in the mortgage agreement, κ is the recovery rate, comprised between 0 and 1, in the event of a default, and i is a time index for such payments.[6]

[6]Equation (17) makes implicitly two assumptions. First, it assigns a probability of nil to the case when default and prepayment occur simultaneously, as both events can reasonably be considered mutually exclusive. Second, it assumes that a mortgage can only be prepaid in full, never in part. Arguably, curtailments have a much milder financial impact that full prepayment to a lender and can reasonably be ignored. Moreover, our model could still handle partial prepayments: it would suffice to subdivide a given mortgage into smaller balances and apply equation (17) to such balances separately.

Using the tower property of conditional expectations and recursion,

$$
\begin{aligned}
V_{t_i^*} &= \mathbb{E}^{\mathbb{Q}}\Big[D^{(RF)}_{t_i^*,t_{i+1}^*}\Big\{p\delta_{\tau_\eta>t_{i+1}^*}\delta_{\tau_\pi>t_{i+1}^*}+\kappa B_{t_i^*}\delta_{t_i^*<\tau_\eta\le t_{i+1}^*}\delta_{\tau_\pi>t_{i+1}^*} \\
&\quad +B_{t_i^*}(1+C)\delta_{t_i^*<\tau_\pi\le t_{i+1}^*}\delta_{\tau_\eta>t_i^*}\Big\}\,\Big|\,\mathcal{G}_{t_i^*}\Big]\,\delta_{\tau_\eta>t_i^*}\delta_{\tau_\pi>t_i^*} \\
&\quad +\mathbb{E}^{\mathbb{Q}}\Bigg[D^{(RF)}_{t_i^*,t_{i+1}^*}\mathbb{E}^{\mathbb{Q}}\Bigg[\sum_{j=i+2}^{N}D^{(RF)}_{t_{i+1}^*,t_j^*}\Big\{p\delta_{\tau_\eta>t_j^*}\delta_{\tau_\pi>t_j^*}+\kappa B_{t_{j-1}^*}\delta_{t_{j-1}^*<\tau_\eta\le t_j^*}\delta_{\tau_\pi>t_j^*} \\
&\quad +B_{t_{j-1}^*}(1+C)\delta_{t_{j-1}^*<\tau_\pi\le t_j^*}\delta_{\tau_\eta>t_j^*}\Big\}\,\Big|\,\mathcal{G}_{t_{i+1}^*}\Bigg]\,\delta_{\tau_\eta>t_{i+1}^*}\delta_{\tau_\pi>t_{i+1}^*}\,\Bigg|\,\mathcal{G}_{t_i^*}\Bigg]\,\delta_{\tau_\eta>t_i^*}\delta_{\tau_\pi>t_i^*} \\
&= \mathbb{E}^{\mathbb{Q}}\Big[D^{(RF)}_{t_i^*,t_{i+1}^*}\Big\{p\delta_{\tau_\eta>t_{i+1}^*}\delta_{\tau_\pi>t_{i+1}^*}+\kappa B_{t_i^*}\delta_{t_i^*<\tau_\eta\le t_{i+1}^*}\delta_{\tau_\pi>t_{i+1}^*} \\
&\quad +B_{t_i^*}(1+C)\delta_{t_i^*<\tau_\pi\le t_{i+1}^*}\delta_{\tau_\eta>t_i^*}\Big\}\,\Big|\,\mathcal{G}_{t_i^*}\Big]\,\delta_{\tau_\eta>t_i^*}\delta_{\tau_\pi>t_i^*} \\
&\quad +\mathbb{E}^{\mathbb{Q}}\Big[D^{(RF)}_{t_i^*,t_{i+1}^*}V_{t_{i+1}^*}\,\Big|\,\mathcal{G}_{t_i^*}\Big]\,\delta_{\tau_\eta>t_i^*}\delta_{\tau_\pi>t_i^*} \\
&= \mathbb{E}^{\mathbb{Q}}\Big[D^{(RF)}_{t_i^*,t_{i+1}^*}\Big\{\Big(p+V_{t_{i+1}^*}\Big)\delta_{\tau_\eta>t_{i+1}^*}\delta_{\tau_\pi>t_{i+1}^*}+\kappa B_{t_i^*}\delta_{t_i^*<\tau_\eta\le t_{i+1}^*}\delta_{\tau_\pi>t_{i+1}^*} \\
&\quad +B_{t_i^*}(1+C)\delta_{t_i^*<\tau_\pi\le t_{i+1}^*}\delta_{\tau_\eta>t_i^*}\,\Big|\,\mathcal{G}_{t_i^*}\Big]\,\delta_{\tau_\eta>t_i^*}\delta_{\tau_\pi>t_i^*}.
\end{aligned}
\tag{18}
$$

Further,

$$
\begin{aligned}
&\frac{\frac{d\mathbb{Q}}{d\mathbb{P}}\big|_{\mathcal{F}_{t_{i+1}^*}}}{\frac{d\mathbb{Q}}{d\mathbb{P}}\big|_{\mathcal{F}_{t_i^*}}}D^{(RF)}_{t_i^*,t_{i+1}^*} \\
&= \exp\left(-\int_{t_i^*}^{t_{i+1}^*}\gamma_s^r dW_s^{\mathbb{P},r}-\int_{t_i^*}^{t_{i+1}^*}\gamma_s^\perp dW_s^{\mathbb{P},\perp}-\frac{1}{2}\int_{t_i^*}^{t_{i+1}^*}(\gamma^r)^2+(\gamma^\perp)^2ds-\int_{t_i^*}^{t_{i+1}^*}r_s ds\right)
\end{aligned}
\tag{19}
$$

Time discretization Let $t_i=\frac{t_i^*}{\Delta_t}$, so that $t_i=i$ for $i\in\{0,\frac{T}{\Delta_t}\}$. In discrete time, equations (11) and (21) respectively become

$$
r_{t_{i+1}}=r_{t_i}+\alpha\left(\beta-r_{t_i}\right)\Delta_t+\sigma\left(W^{\mathbb{P},r}_{t_{i+1}^*}-W^{\mathbb{P},r}_{t_i^*}\right) \tag{20}
$$

and

$$
H_{t_{i+1}}=H_{t_i}+\mu_H H_{t_i}\Delta_t+\sigma_H H_{t_i}\left(W^{\mathbb{P},H}_{t_{i+1}^*}-W^{\mathbb{P},r}_{t_i^*}\right). \tag{21}
$$

Then,

$$
D^{(RF)}_{t_i,t_{i+1}}=\exp\left(-\int_{t_i^*}^{t_{i+1}^*}r_s\,ds\right)\simeq\frac{1}{\exp\left(r_{t_i}\Delta_t\right)}\simeq\frac{1}{1+r_{t_i}\Delta_t}.
$$

Similarly, discretizing equation (19) yields

$$\begin{aligned}
&\frac{\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_{t_{i+1}}}}{\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_{t_i}}} D^{(RF)}_{t_i,t_{i+1}} \\
&\simeq \frac{1}{1+\gamma^r_{t_i}\left(\underbrace{W^{\mathbb{P},r}_{t^*_{i+1}} - W^{\mathbb{P},r}_{t^*_i}}_{\simeq 0}\right) + \gamma^{\perp}_{t_i}\left(\underbrace{W^{\mathbb{P},\perp}_{t^*_{i+1}} - W^{\mathbb{P},\perp}_{t^*_i}}_{\simeq 0}\right) + r_{t_i}\Delta_t + \frac{1}{2}\left\{(\gamma^r_{t_i})^2 + (\gamma^{\perp}_{t_i})^2\right\}\Delta_t} \\
&\simeq \frac{1}{1 + r_{t_i}\Delta_t + \underbrace{\frac{1}{2}\left\{(\gamma^r_{t_i})^2 + (\gamma^{\perp}_{t_i})^2\right\}\Delta_t}_{\text{spread}}}.
\end{aligned}$$

Assume γ^r and $\gamma^{\perp}$ are constant. We are left with the following risky discount factor under $\mathbb{P}$:

$$D_{t_i,t_{i+1}} = \frac{1}{1 + r_{t_i}\Delta_t + \frac{1}{2}\left\{(\gamma^r)^2 + (\gamma^{\perp})^2\right\}\Delta_t} = \frac{1}{1 + (r_{t_i} + S)\Delta_t} \tag{22}$$

where S is a constant risky spread.

Also, going back to the prepayment probabilities in equation (18), from discretizing equations (15) and (16) respectively, we get

$$\mathbb{E}^{\mathbb{P}}\left[\delta_{\tau_\eta \geq t^*_{i+1}} \,\middle|\, \mathcal{G}_{t^*_i}\right] = \mathbb{P}\left(\tau_\eta > t^*_{i+1} \,\middle|\, \mathcal{G}_{t^*_i}\right) \simeq \mathbb{E}^{\mathbb{P}}\left[\exp\left(-\Lambda^{(\eta)}_{t^*_i}\Delta_t\right) \,\middle|\, \mathcal{G}_{t^*_i}\right]\delta_{\tau_\eta > t^*_i} \tag{23}$$

and

$$\mathbb{E}^{\mathbb{P}}\left[\delta_{\tau_\pi \geq t^*_{i+1}} \,\middle|\, \mathcal{G}_{t^*_i}\right] = \mathbb{P}\left(\tau_\pi > t^*_{i+1} \,\middle|\, \mathcal{G}_{t^*_i}\right) \simeq \mathbb{E}^{\mathbb{P}}\left[\exp\left(-\Lambda^{(\pi)}_{t^*_i}\Delta_t\right) \,\middle|\, \mathcal{G}_{t^*_i}\right]\delta_{\tau_\pi > t^*_i}. \tag{24}$$

As a result, in a discrete time version of equation (18), a mortgage at time $i\Delta_t$ is worth

$$\begin{aligned}
V_{i\Delta_t} = \mathbb{E}^{\mathbb{P}}\Big[D_{(i\Delta_t,(i+1)\Delta_t)}\Big\{&\left(p + V_{(i+1)\Delta_t}\right)\exp\left(-\Lambda^{(\eta)}_{i\Delta_t}\Delta_t\right)\exp\left(-\Lambda^{(\pi)}_{i\Delta_t}\Delta_t\right) \\
&+\kappa B_{i\Delta_t}\left(1 - \exp\left(-\Lambda^{(\eta)}_{i\Delta_t}\Delta_t\right)\right)\exp\left(-\Lambda^{(\pi)}_{i\Delta_t}\Delta_t\right) \\
&+B_{i\Delta_t}(1+C)\left(1 - \exp\left(-\Lambda^{(\pi)}_{i\Delta_t}\Delta_t\right)\right)\exp\left(-\Lambda^{(\eta)}_{i\Delta_t}\Delta_t\right)\Big\} \,\Big|\, \mathcal{G}_{i\Delta_t}\Big]\delta_{\tau_\eta > i\Delta_t}\delta_{\tau_\pi > i\Delta_t}.
\end{aligned} \tag{25}$$

Relationship between prepayment and default Recall that prepayment and default are assumed to be mutually exclusive. The mortgage termination time is the minimum between two stopping times, τ_η and τ_π, and is thus itself a stopping time. Default is mostly driven by the gap between the mortgage unpaid balance and the house value: the greater the gap, the greater the probability of default, as US mortgagors may walk away from both their house and mortgage, thus getting rid of a liability which is larger than their asset. Prepayment is mostly driven by the gap between the mortgage coupon C and current mortgage rate

C_t, which makes currently issued mortgages trade at par. We can therefore model default and prepayment as two separate functions.

Such a model handles the cases when mortgagors may both want to default and prepay. This is reflected in positive probabilities for both default and prepayment. The relative magnitude of the default intensity rate, a function of the ratio mortgage unpaid balance to house price, versus the prepayment intensity rate, a function of the difference between the mortgage coupon and the current mortgage rate,[7] will automatically make default probability greater than prepayment probability.[8]

2.2.2 Determining the fair coupon at inception

In equation (25), p, C, and B_t are given to the investors, and V_{t_i} does not need to equal F. But to a mortgage issuer, at $t_i = 0$, i.e. on the issuance date, the fair coupon needs to be determined by setting V_0 equal to F and solving for p (or C). Then, using equations (2) and (3), C (or p) and $B_s, \forall\, s \in \{1, T\}$, can be found.

2.3 Default intensity function

Temporarily ignore the possibility of prepayment, and focus on default.

Recall equation (21). Housing price log-returns are modelled as following a geometric Brownian motion:

$$H_{t_{i+1}} = H_{t_i} \exp\left(\left(\mu_{H,t} - \frac{\sigma_H^2}{2}\right)\Delta t + \sigma_H \left(W^{(\mathbb{P},H)}_{t^*_{i+1}} - W^{(\mathbb{P},H)}_{t^*_i}\right)\right). \tag{26}$$

The higher the ratio of B_{t_i} — the mortgage yet unpaid balance when mortgagors may debate whether to default at time t_{i+1} — to the house value H_{t_i}, the higher the mortgagors debt relative to the backing asset, and the more likely a default. A time t_0, set $H_{t_0} = \frac{B_{t_0}}{\upsilon}$, where υ is the average loan-to-value ratio at origination, as observed on the market at the time of the mortgage issuance.[9] The one-period default intensity in equation (23) can thus be specified as

$$\Lambda^{(\eta)}_{t_i} = a_\eta + b_\eta \left(\min\left[\frac{B_{t_i}}{H_{t_i}}, 1\right]\right)^{c_\eta}, \tag{27}$$

where

- $a_\eta + b_\eta$ is a positive constant background default rate, to be estimated. Its interpretation is that a default may occur for other reasons than a high unpaid balance relative to the house value. Such events are captured by a_η;

[7] For details about such functions, see sections 2.3 and 2.4 further below.

[8] This is different from Schwartz and Torous (1992), who model the interaction between default and prepayment using two functions by part, both conditioned on housing prices and mortgage rates, and set the probability of default (prepayment) to zero when prepayment (default) dominates default (prepayment). See Section 1 for more details.

[9] Such a ratio typically hovers around 0.8.

- b_η, a positive constant, may have a magnifying or reducing effect, whether it is smaller or greater than 1;
- c_η is a positive constant ≥ 1, since Λ^η is expected to be increasing and convex in $\frac{B_{t_{i-1}}}{H_{t_i}}$;
- the minimum function ensures the default rate accelerates as c_η increases.
- The interpretation of the floor set to 1 is that, when the mortgage balance is smaller than the house value, the mortgagor has no economic incentive to default. The only source of default is then the constant background rate $a_\eta + b_\eta$, described further above.

Remark Going back to equation (23), observe how, the greater the ratio of B_{t_i} to H_{t_i}, the greater $\Lambda_{t_i}^{(\eta)}$, and so the smaller the survival probability $\exp\left(-\Lambda_{t_i}^{(\eta)}\right)$ and the greater the default rate $1 - \exp\left(-\Lambda_{t_i}^{(\eta)}\right)$ and the probability of defaulting at t_{i+1}.

2.4 Prepayment intensity function

Ignore temporarily the possibility of default. At any time t_j,[10] mortgagors may compare the holding value of their mortgage, noted $V_{t_i}^{(h)}$, and the prepayment or exercise value $V_{t_i}^{(e)}$. They want to minimize their liability, so that, the wider the gap between $V_{t_i}^{(h)}$ and $V_{t_i}^{(e)}$, the greater the incentive to prepay.

The prepayment intensity in equation (24) is defined as

$$\Lambda_{t_i}^{(\pi)} = \left(a_\pi + b_\pi \left(\left[V_{t_i}^{(h)} - V_{t_i}^{(e)}\right]^+\right)^{c_\pi}\right), \tag{28}$$

with a_π, b_π, and c_π, respectively similar to a_η, b_η, and c_η as already defined in equation (27). Taking the positive part of the expression ensures that the intensity remains positive.

If the mortgagors hold to their mortgage, they owe p, the payment scheduled for t_j, plus the present value of their current mortgage remaining payments, that is $V_{t_{j+1}}$, from equation (17), multiplied by the one-month discounting factor $D_{t_j,t_{j+1}}$. If they refinance, they only owe $CB_{t_{j-1}}$,[11] the interest owing for the period just ended in t_j, plus the expected present value of a new mortgage, $V_{t_j}^{(C_{t_j})}$, the coupon of which, C_{t_j}, makes said present value equal to par, i.e. $B_{t_{j-1}}$, which is the opening balance for period t_j.[12] We thus have that

$$\begin{aligned} V_{t_j}^{(h)} &= \mathbb{E}^{\mathbb{P}}\left[V_{t_{j+1}} \,\middle|\, \mathcal{G}_{t_j}\right] D_{t_j,t_{j+1}} + p, \\ V_{t_j}^{(e)} &= B_{t_{j-1}}\left(1 + C\right), \end{aligned}$$

[10]We assume here that default or prepayment can occur at the end of a period only.

[11]Recall that no prepayment penalty applies to mortgages underlying Fannie Mae conventional MBS.

[12]In equation (25), $B_{t_{j-1}}$ is substituted to V_{t_j} on the left-hand side, and the currently observed mortgage rate C_{t_j}, as well as a corresponding fixed payment p_{t_j} are substituted to C and p, while every remaining future balance $B_{t_{j-1}}$ in the sum gets updated accordingly, so that the equality is satisfied.

so that, equation (28) becomes

$$\Lambda_{t_j}^{(\pi)} = a_\pi + b_\pi \left(\left[\mathbb{E}^{\mathbb{P}} \left[V_{t_{j+1}} \,\middle|\, \mathcal{G}_{t_j} \right] D_{t_j,t_{j+1}} + p - B_{t_{j-1}}(1+C) \right]^+ \right)^{c_\pi}, \tag{29}$$

where $V_{t_{j+1}}$ is obtained from equation (25).[13]

3 Numerical implementation

3.1 State variable dynamics and probability calculations

Such a valuation model can be implemented using a time sequence of two-dimensional grids, with one axis for the housing price diffusion term and the other axis for the interest rate diffusion term. The information from both housing prices and interest rates feeds into the default and prepayment functions, as defined by equations (23) and (24). Using a Markov chain, and moving backwards through such a sequence of grids from the mortgage maturity date to the valuation date, we can thus calculate the default probability, the prepayment probability and the continuation probability at each node. The value of the mortgage at each node is the probability-weighted average of such three values, as described in more detail in equations (36) further below. Moving back from time step to time step to the valuation date ultimately yields the mortgage present value, weighted by the different path probabilities, and discounted accordingly.

3.1.1 The interest rate dynamics

The solution to the Ornstein-Uhlenbeck equation (11) is

$$r_t = \beta + e^{-\alpha t} (r_0 - \beta) + \sigma \int_0^t e^{-\alpha(t-s)} dW_s^{(\mathbb{Q})} \tag{30}$$

Writing the Ornstein-Uhlenbeck process as an autoregressive process of order 1, or AR(1), yields

$$r_{t_{j+1}} = e^{-\alpha \Delta_t} r_{t_j} + \beta \left(1 - e^{-\alpha \Delta_t}\right) + \sigma \sqrt{\frac{1 - e^{-2\alpha \Delta_t}}{2\alpha}} Z_{t_{j+1}}, \tag{31}$$

where $\{Z_{t_j}\}_{j \in \mathbb{N}}$ are i.i.d standard normal random variables, and $\Delta_t = \frac{1}{12}$.

Define the diffusion term

$$x_{t_{j+1}} = \sigma \sqrt{\frac{1 - e^{-2\alpha \Delta_t}}{2\alpha}} Z_{t_{j+1}} = r_{t_{j+1}} - \left(e^{-\alpha \Delta_t} r_{t_j} + \beta \left(1 - e^{-\alpha \Delta_t}\right)\right),$$

which will become handy in calculating transition probabilities further down.

[13]For a detailed description of the recursive implementation of the mortgage valuation, moving backwards from $V_{t_{j+1}}$ to $\mathbb{E}^{\mathbb{P}} \left[V_{t_{j+1}} \,\middle|\, \mathcal{G}_{t_j} \right]$ and V_{t_j}, see Section 3.

We now build a grid for interest rate values. We center the grid around β, the distribution long-term reverting mean. We set $r^{(N)}$, the upper bound of the grid, to the maximum level observed in US short-term interest rate empirical time series over the last 30 years, i.e. 10%, which was reached by LIBOR 1-month in March 1989. We then observe that a 33 basis point rate decrease would save a typical $150,000-indebted Fannie Mae 15-year mortgagor $500 in annual interest payments, a difference that could trigger prepayment. So we determine that physical increments should be no more than 33 basis points, and we set such increments, noted Δ_x, to 0.33% / 5. This yields a grid of equally spaced interest rates $r^{(n)}$, ranging from 10% to $\beta - (10\% - \beta)$.[14]

Finally, $\forall n \in \{2, N-1\}$, we partition the vector into cells $(D_n, D_{n+1}]$, with $D_n = \frac{r^{(n-1)}+r^{(n)}}{2}$, so that every interest rate level $r^{(n)}$ is the midpoint in a vector cell. Further, set $D_1 = -\infty$ and $D_{N+1} = \infty$.

3.1.2 The housing price dynamics

Turning to housing price dynamics, discretizing equation (26) gives

$$H_{t_{j+1}} = H_{t_j} \exp\left(\left(\mu_{H,t_j} - \frac{\sigma_H^2}{2}\right)\Delta_t + \sigma_H\sqrt{\Delta_t}\, Z_{t_{j+1}}^{(H)}\right) \tag{32}$$

where $\mu_{H,t_j} = \mu^H - \gamma_H\sigma_H = r_t$, with μ^H as defined in equation (26) and γ_H as defined in equation (13).

A challenge is to build a grid that is wide enough to cover every reasonable state of H_{t_j} over up to 15 years, subject to computer memory constraints.[15] A solution, suggested by Duan and Simonato [5] for equity prices S, is to define an adjusted asset price $\tilde{S}_t$ rid of the drift term such that

$$\tilde{S}_{t_j} = S_{t_j}\exp\left(-\mu^{(S)} j\Delta_t\right) = S_0 \exp\left(\sigma_S\sqrt{j\Delta_t}\sum_{j=1}^{J} Z_{t_j}^{(S)}\right), \tag{33}$$

where S_0 is the asset price at the asset valuation date and $\mu^{(S)}$ is the drift parameter for S.

We cannot exactly replicate such a scheme since, in equation (32), the drift term $\mu_{t_j}^{(\mathbb{P},H)}$ is not a constant, but a stochastic r_t. We can however build, in the same spirit, a grid for the housing dynamic diffusion part.[16]

Let M, the number of elements in the housing price vector, be any odd positive integer, only subject to computer memory constraints. We build the grid of housing prices in two

[14] N, the number of grid points is thus a function of β. We estimate β to be 0.0174 in section 4.1. As a result, N equals 250.

[15] Using MATLAB on a Windows Intel i5-7003U system at 2.60 GHz with 16 GB memory, we could implement, with a monthly discretization, a 250-by-180 node interest rate grid and a 49-by-180 node housing price grid.

[16] Denoted h_t, as defined further down in section 3.1.3.

steps: first, a temporary grid, where H_0 coincides with a grid node; second, the final grid is obtained from slightly shifting the temporary grid, so that H_0 falls exactly in the middle of the grid middle node.

Let's label the temporary grid nodes $\bar{C}$. And let the middle node $\bar{C}^{(\frac{M+1}{2})} = H_0$. Further, set the grid upper and lower bounds to $\bar{C}^{(\frac{M+1}{2})} + \sqrt{\frac{M-1}{2}}\sigma_H\sqrt{\Delta_t}\sqrt{t_J}$ and $\bar{C}^{(\frac{M+1}{2})} - \sqrt{\frac{M-1}{2}}\sigma_H\sqrt{\Delta_t}\sqrt{t_J}$ respectively, where $t_J = T$, the mortgage fixed maturity date.[17] Subdivide the intervall thus bounded into $M-1$ subintervalls of equal size[18]

$$\Delta_C \approx 2\frac{\sqrt{\frac{M-1}{2}}\sigma_H\sqrt{\Delta_t}\sqrt{t_J}}{M-1}.$$

Then, $\forall m \in \{2, M-1\}$, partition the vector into cells $(C^{(m)}, C^{(m+1)}]$, with $C^{(m)} = \frac{\bar{C}^{(m-1)}+\bar{C}^{(m)}}{2}$, so that, in particular, $H_0 = \frac{C^{(\frac{M}{2})}+C^{(\frac{M}{2}+1)}}{2}$ as planned. Further, set $C^{(1)} = -\infty$ and $C^{(M+1)} = \infty$.

Remark Rather than simply centering the housing diffusion term tree around H_0, it would make sense to center such a tree around H_0 plus β, the interest rate long-term average parameter, or a weighted average of such variables. However, a grid simply centered around H_0 performs adequately, including in very high or very low interest rate environments, as evidenced in our stress test.[19]

3.1.3 The bivariate transition probabilities

Let $p_{(mn,uv)}$ be, $\forall\, m, \forall\, u \in \{1, M\}$ and $\forall\, n, \forall v \in \{1, N\}$, the joint transition probability under the probability measure $\mathbb{P}$ that both

- the housing reference price move from state $H^{(m)}$, at any time t_j, to state $H^{(u)}$ at time t_{j+1}, and
- the interest rate move from state $r^{(n)}$, at any time t_j, to state $r^{(v)}$ at time t_{j+1}.

[17] Setting $M = 101$, for example, thus yields upper and lower bounds 7.91σ apart from the middle node since T is fixed and equal to 15 years, regardless of the remaining term-to-maturity at the valuation date.

[18] See Duan and Simonato [5] for such a choice of Δ_h, which helps ensure convergence.

[19] See section 5 for stress-tests.

In equation,

$$
\begin{aligned}
p_{(mn,uv)} &= \mathbb{P}\left(C_u < H_{t_{j+1}} \leq C_{u+1} \,\text{and}\, D_v < r_{t_{j+1}} \leq D_{v+1} \;\middle|\; H_{t_j} = H^{(m)} \,\text{and}\, r_{t_j} = r^{(n)}\right) \\
&= \mathbb{P}\Bigg(C_u < H_{t_j} \exp\left(\left(\mu_{H,t_j} - \frac{\sigma^2}{2}\right)\Delta_t + \sigma_H \sqrt{\Delta_t} Z_{t_{j+1}}\right) \leq C_{u+1} \\
&\quad \text{and}\, D_v < r_{t_{j+1}} \leq D_{v+1} \;\Big|\; H_{t_j} = H^{(m)} \,\text{and}\, r_{t_j} = r^{(n)}\Bigg) \\
&= \mathbb{P}\Bigg(\ln\left(\frac{C_u}{H^{(m)}}\right) < \left(\mu_{H,t_j} - \frac{\sigma^2}{2}\right)\Delta_t + \sigma_H \sqrt{\Delta_t} Z_{t_{j+1}} \leq \ln\left(\frac{C_{u+1}}{H^{(m)}}\right) \\
&\quad \text{and}\, D_v - \left(e^{-\alpha\Delta_t} r^{(n)} + \beta\left(1 - e^{-\alpha\Delta_t}\right)\right) < r_{t_{j+1}} - \left(e^{-\alpha\Delta_t} r^{(n)} + \beta\left(1 - e^{-\alpha\Delta_t}\right)\right) \\
&\quad \leq D_{v+1} - \left(e^{-\alpha\Delta_t} r^{(n)} + \beta\left(1 - e^{-\alpha\Delta_t}\right)\right) \;\Big|\; H_{t_j} = H^{(m)} \,\text{and}\, r_{t_j} = r^{(n)}\Bigg) \\
&= \mathbb{P}\Bigg(\ln\left(\frac{C_u}{H^{(m)}}\right) - \left(r^{(n)} + \gamma_H \sigma_H - \frac{\sigma^2}{2}\right)\Delta_t < \sigma_H \sqrt{\Delta_t} Z_{t_{j+1}} \\
&\quad \leq \ln\left(\frac{C_{u+1}}{H^{(m)}}\right) - \left(r^{(n)} + \gamma_H \sigma_H - \frac{\sigma^2}{2}\right)\Delta_t \\
&\quad \text{and}\, D_v - \left(e^{-\alpha\Delta_t} r^{(n)} + \beta\left(1 - e^{-\alpha\Delta_t}\right)\right) < r_{t_{j+1}} - \left(e^{-\alpha\Delta_t} r^{(n)} + \beta\left(1 - e^{-\alpha\Delta_t}\right)\right) \\
&\quad \leq D_{v+1} - \left(e^{-\alpha\Delta_t} r^{(n)} + \beta\left(1 - e^{-\alpha\Delta_t}\right)\right) \;\Big|\; H_{t_j} = H^{(m)} \,\text{and}\, r_{t_j} = r^{(n)}\Bigg).
\end{aligned}
\tag{34}
$$

Define

$$c_u^{(m)} = \ln\left(\frac{C_u}{H^{(m)}}\right) - \left(r^{(n)} + \gamma_H \sigma_H - \frac{\sigma^2}{2}\right)\Delta_t$$

and

$$d_v^{(n)} = D_v - \left(e^{-\alpha\Delta_t} r^{(n)} + \beta\left(1 - e^{-\alpha\Delta_t}\right)\right).$$

Also define the housing price dynamics diffusion term

$$h_{t_{j+1}} = \sigma_H \sqrt{\Delta_t}\, Z_{t_{j+1}}^{(H)} = \sigma_H \sqrt{\Delta_t}\left(\rho Z_{t_{j+1}} + \sqrt{1-\rho^2} Z_{t_{j+1}}^{(\perp)}\right)$$

with $\mathrm{Corr}\left(Z_{t_{j+1}}, Z_{t_{j+1}}^{(\perp)}\right) = 0$, and $\{Z_{t_j}^{(\perp)}\}_{j\in\mathbb{N}}$ $i.i.d.$ $N(0,1)$.

Then, substituting $c_u^{(m)}$, $d_v^{(n)}$, $h_{t_{j+1}}$ and $x_{t_{j+1}}$ into equation (34) yields

$$\begin{aligned}
p_{(mn,uv)} &= \mathbb{P}\left(d_v^{(n)} < x_{t_{j+1}} \leq d_{v+1}^{(n)} \,\text{and}\, c_u^{(m)} < h_{t_{j+1}} \leq c_{u+1}^{(m)}\right) \\
&= \mathbb{P}\left(\frac{d_v^{(n)}}{\sigma\sqrt{\frac{1-e^{-2\alpha\Delta_t}}{2\alpha}}} < Z_{t_{j+1}} \leq \frac{d_{v+1}^{(n)}}{\sigma\sqrt{\frac{1-e^{-2\alpha\Delta_t}}{2\alpha}}}\right. \\
&\quad \left.\text{and}\, \frac{c_u^{(m)}}{\sigma_H\sqrt{\Delta_t}} < \rho Z_{t_{j+1}} + \sqrt{1-\rho^2} Z_{t_{j+1}}^{(\perp)} \leq \frac{c_{u+1}^{(m)}}{\sigma_H\sqrt{\Delta_t}}\right) \\
&= \int_{\frac{d_v^{(n)}}{\sigma\sqrt{\frac{1-e^{-2\alpha\Delta_t}}{2\alpha}}}}^{\frac{d_{v+1}^{(n)}}{\sigma\sqrt{\frac{1-e^{-2\alpha\Delta_t}}{2\alpha}}}} \left(\int_{\frac{\frac{c_u^{(m)}}{\sigma_H\sqrt{\Delta_t}}-\rho z}{\sqrt{1-\rho^2}}}^{\frac{\frac{c_{u+1}^{(m)}}{\sigma_H\sqrt{\Delta_t}}-\rho z}{\sqrt{1-\rho^2}}} \phi(z^{(\perp)})\mathrm{d}z^{(\perp)}\right)\phi(z)\mathrm{d}z \\
&= \int_{\frac{d_v^{(n)}}{\sigma\sqrt{\frac{1-e^{-2\alpha\Delta_t}}{2\alpha}}}}^{\frac{d_{v+1}^{(n)}}{\sigma\sqrt{\frac{1-e^{-2\alpha\Delta_t}}{2\alpha}}}} \left\{\Phi\left(\frac{\frac{c_{u+1}^{(m)}}{\sigma_H\sqrt{\Delta_t}}-\rho z}{\sqrt{1-\rho^2}}\right) - \Phi\left(\frac{\frac{c_u^{(m)}}{\sigma_H\sqrt{\Delta_t}}-\rho z}{\sqrt{1-\rho^2}}\right)\right\}\phi(z)\mathrm{d}z,
\end{aligned} \tag{35}$$

where ϕ and Φ are respectively probability density and cumulative distribution functions of a standard normal random variable. We implement such a double integral using MATLAB *mvncdf* function.

3.2 Mortgage valuation using dynamic programming

First, default intensity values and one-step discount factors can be calculated off-line.

For all $m \in \{1, M\}$, for all $j \in \{1, J\}$, and for any $n \in \{1, N\}$, the (mn)-node default intensity $\Lambda_{t_j}^{(\eta)(m)}$ can be calculated using equation (27). $H_{t_j}^{(m)}$ in the equation can be recovered from the stationary grid at each time step by combining $H^{(m)}$, β and t_j as follows:[20]

$$H_{t_j}^{(m)} \approx H_0 \exp\left(\left(\beta - \frac{\sigma^2}{2}\right) j\Delta_t + \left(\left(m - \frac{M+1}{2}\right)\Delta_h\right)\right).$$

For all $n \in \{1, N\}$, for all $j \in \{1, J\}$, and for any $m \in \{1, M\}$, the (mn)-node discount factor $D_{t_i,t_{i+1}}^{(n)} = \frac{1}{1+(r^{(n)}+S)\Delta_t}$, from equation (22).

Then the backward algorithm can be run, starting from the maturity date.

Consider $V_{t_{J-1}}^{(mn)}$, the mortgage value at time $J-1$ on node (mn), where $H_{t_{J-1}} = H^{(m)}$

[20] The long term average parameter for the interest rate dynamic is used as a reasonable approximation to recover the housing prices from the stochastic interest rate r_t.

and $r_{t_{J-1}} = r^{(n)}$. Using equation (25) with $i = J-1$ yields

$$\begin{aligned} V_{t_{J-1}} = \mathbb{E}^{\mathbb{P}} \Big[D_{t_{J-1},t_J} \Big\{ (p + V_{t_J}) \exp\left(-\Lambda^{(\eta)}_{t_{J-1}} \Delta_t\right) \exp\left(-\Lambda^{(\pi)}_{t_{J-1}} \Delta_t\right) \\ + \kappa B_{t_{J-1}} \left(1 - \exp\left(-\Lambda^{(\eta)}_{t_{J-1}} \Delta_t\right)\right) \exp\left(-\Lambda^{(\pi)}_{t_{J-1}} \Delta_t\right) \\ + B_{t_{J-1}} (1+C) \left(1 - \exp\left(-\Lambda^{(\pi)}_{t_{J-1}} \Delta_t\right)\right) \exp\left(-\Lambda^{(\eta)}_{t_{J-1}} \Delta_t\right) \Big\} \,\Big|\, \mathcal{G}_{t_{J-1}} \Big] \delta_{\tau_\eta > t_i} \delta_{\tau_\pi > t_i}. \end{aligned} \tag{36}$$

At maturity time t_J, there is no future payment left, so $V_{t_J} = 0$. Also, immediately after t_{J-1}, prepayment becomes irrelevant, so that

$$\mathbb{E}^{\mathbb{P}} \left[\exp\left(-\Lambda^{(\pi)}_{t_{J-1}} \Delta_t\right) \right] = 1.$$

Moreover, knowing the mortgage is still alive at t_{J-1}, $\delta_{\tau_\eta > t_{J-1}} = 1$ and $\delta_{\tau_\pi > t_{J-1}} = 1$. Equation (36) thus simplifies to

$$V^{(mn)}_{t_{J-1}} = D^{(n)}_{t_{J-1},t_J} \left[p \exp\left(-\Lambda^{(\eta)(m)}_{t_{J-1}} \Delta_t\right) + \kappa B_{t_{J-1}} \left(1 - \exp\left(-\Lambda^{(\eta)(m)}_{t_{J-1}} \Delta_t\right)\right)\right].$$

Set $V^{(uv)}_{t_{J-1}} = V^{(mn)}_{t_{J-1}}$, and move one time step back, to t_{J-2}, where, for any of the M times N pairs (mn),

$$\begin{aligned} V^{(mn)}_{t_{J-2}} &= \mathbb{E}^{\mathbb{P}} \big[D_{t_{J-2,J-1}} \big\{ p \delta_{\tau_\eta > t_{J-1}} \delta_{\tau_\pi > t_{J-1}} + \kappa B_{t_{J-2}} \delta_{t_{J-2} < \tau_\eta \le t_{J-1}} \delta_{\tau_\pi > t_{J-1}} \\ &\quad + B_{t_{J-2}} (1+C) \delta_{t_{J-2} < \tau_\pi \le t_{J-1}} \delta_{\tau_\eta > t_{J-1}} + V_{t_{J-1}} \delta_{\tau_\eta > t_{J-1}} \delta_{\tau_\pi > t_{J-1}} \big\} \,\big|\, \mathcal{G}_{t_{J-2}} \big] \delta_{\tau_\eta > t_{J-2}} \delta_{\tau_\pi > t_{J-2}} \\ &= D^{(n)}_{t_{J-2,J-1}} \left[\exp\left(-\Lambda^{(\eta)(m)}_{t_{J-2}} \Delta_t\right) \exp\left(-\Lambda^{(\pi)(n)}_{t_{J-2}} \Delta_t\right) \left(p + \sum_{u=1}^{M} \sum_{v=1}^{N} p_{(mn,uv)} V^{(uv)}_{t_{J-1}} \right) \right. \\ &\quad + \kappa B_{t_{J-2}} \left(1 - \exp\left(-\Lambda^{(\eta)(m)}_{t_{J-2}} \Delta_t\right)\right) \exp\left(-\Lambda^{(\pi)(n)}_{t_{J-2}} \Delta_t\right) \\ &\quad \left. + B_{t_{J-2}} (1+C) \left(1 - \exp(-\Lambda^{(\pi)(n)}_{t_{J-2}} \Delta_t\right) \exp\left(-\Lambda^{(\eta)(m)}_{t_{J-2}} \Delta_t\right) \right], \end{aligned} \tag{37}$$

where $\Lambda^{(\pi)(n)}_{t_{J-2}}$ is the prepayment intensity $\Lambda^{(\pi)}_{t_{J-2}}$, as defined by equation (29), associated with interest rate $r^{(n)}$ and continuation value $\sum_{u=1}^{M} \sum_{v=1}^{N} p_{(mn,uv)} V^{(uv)}_{t_{J-1}}$. From equation (29), $\forall j \in \{1, J\}$,

$$\begin{aligned} \Lambda^{(\pi)(n)}_{t_j} &= a_\pi + b_\pi \left(\max \left[\mathbb{E}^{\mathbb{P}} \left[V_{t_{j+1}} \,\middle|\, \mathcal{G}_{t_j} \right] D^{(n)}_{t_j,t_{j+1}} + p - B_{t_{j-1}} (1+C), 0 \right] \right)^{c_\pi} \\ &= a_\pi + b_\pi \left(\max \left[\sum_{u=1}^{M} \sum_{v=1}^{N} p_{(mn,uv)} V^{(uv)}_{t_{j+1}} D^{(n)}_{t_j,t_{j+1}} + p - B_{t_{j-1}} (1+C), 0 \right] \right)^{c_\pi}. \end{aligned}$$

Moving further backwards until time t_i yields the mortgage value V_{t_i}, as in equation (25), as at the valuation date.

4 Parameter estimation

4.1 Housing Price and Interest Rate Dynamics Parameter Estimation

Recall that a time step Δ_t is equal to one month. Let $k = t_k$, the index for such time steps, and define $\hat{h}_k = \ln(H_k)$. Recall the discretization for the interest dynamics from equation (31) and apply a similar discretization to equation (26) with respect to the housing price log-returns. We thus have

$$\hat{h}_k = \hat{h}_{k-1} + \left(\mu_{H,k-1} - \frac{\sigma_H^2}{2}\right)\Delta_t + \sigma_H \sqrt{\Delta_t}\, Z_k^{(H)}$$

and

$$r_k = e^{-\alpha\Delta_t} r_{k-1} + \beta\left(1 - e^{-\alpha\Delta_t}\right) + \sigma\sqrt{\frac{1 - e^{-2\alpha\Delta_t}}{2\alpha}} Z_k$$

where $\{Z_k^{(H)}\}_{k\in\mathbb{N}}$ and $\{Z_k\}_{k\in\mathbb{N}}$ are i.i.d standard normal random variable, with

$$Corr(Z_k^{(H)}, Z_k) = \rho.$$

Let Σ be the variance-covariance matrix for $\hat{h}_k$ and r_k. Denote Σ^{-1} its inverse, and $\left|\,\Sigma\,\right|$, its determinant. We then have

$$\Sigma \;= \begin{bmatrix} \sigma_H^2 \Delta_t & \rho\sigma_H\sqrt{\Delta_t}\,\sigma\sqrt{\frac{1-e^{-2\alpha\Delta_t}}{2\alpha}} \\ \rho\sigma_H\sqrt{\Delta_t}\,\sigma\sqrt{\frac{1-e^{-2\alpha\Delta_t}}{2\alpha}} & \sigma^2\left(\frac{1-e^{-2\alpha\Delta_t}}{2\alpha}\right) \end{bmatrix},$$

$$\left|\,\Sigma\,\right| \;= \left(1-\rho^2\right)\sigma_H^2\Delta_t\,\sigma^2\left(\frac{1 - e^{-2\alpha\Delta_t}}{2\alpha}\right),$$

and

$$\Sigma^{-1} = \frac{1}{\left|\,\Sigma\,\right|}\begin{bmatrix} \sigma^2\left(\frac{1-e^{-2\alpha\Delta_t}}{2\alpha}\right) & -\rho\sigma_H\sqrt{\Delta_t}\,\sigma\sqrt{\frac{1-e^{-2\alpha\Delta_t}}{2\alpha}} \\ -\rho\sigma_H\sqrt{\Delta_t}\,\sigma\sqrt{\frac{1-e^{-2\alpha\Delta_t}}{2\alpha}} & \sigma_H^2\Delta_t \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{(1-\rho^2)\sigma_H^2\Delta_t} & \frac{-\rho\sqrt{2\alpha}}{(1-\rho^2)\sigma_H\sqrt{\Delta_t}\,\sigma\sqrt{1-e^{-2\alpha\Delta_t}}} \\ \frac{-\rho\sqrt{2\alpha}}{(1-\rho^2)\sigma_H\sqrt{\Delta_t}\,\sigma\sqrt{1-e^{-2\alpha\Delta_t}}} & \frac{2\alpha}{(1-\rho^2)\sigma^2\left(1-e^{-2\alpha\Delta_t}\right)} \end{bmatrix}.$$

The joint probability density function at time $k\Delta_t$

$$
\begin{aligned}
&f(h_k, r_k) \\
&= \frac{1}{\sqrt{2\pi \,|\, \Sigma \,|}} \exp -\frac{1}{2}\left\{ \begin{bmatrix} \hat{h}_k - \hat{h}_{k-1} - (r_{k-1} + \gamma_H \sigma_H)\Delta_t + \frac{\sigma_H^2}{2}\Delta_t \\ r_k - e^{-\alpha\Delta_t} r_{k-1} - \beta\left(1 - e^{-\alpha\Delta_t}\right) \end{bmatrix}' \Sigma^{-1} \begin{bmatrix} \hat{h}_k - \hat{h}_{k-1} - (r_{k-1} + \gamma_H \sigma_H)\Delta_t + \frac{\sigma_H^2}{2}\Delta_t \\ r_k - e^{-\alpha\Delta_t} r_{k-1} - \beta\left(1 - e^{-\alpha\Delta_t}\right) \end{bmatrix} \right\} \\
&= \frac{1}{\sqrt{2\pi \,|\, \Sigma \,|}} \exp -\frac{1}{2}\left\{ \frac{1}{(1-\rho^2)\,\sigma_H^2 \Delta_t} \left(\hat{h}_k - \hat{h}_{k-1} - (r_{k-1} + \gamma_H \sigma_H)\Delta_t + \frac{\sigma_H^2}{2}\Delta_t\right)^2 \right. \\
&- \frac{2\rho\sqrt{2\alpha}}{(1-\rho^2)\,\sigma_H \sqrt{\Delta_t}\, \sigma\sqrt{1 - e^{-2\alpha\Delta_t}}} \left(\hat{h}_k - \hat{h}_{k-1} - (r_{k-1} + \gamma_H \sigma_H)\Delta_t + \frac{\sigma_H^2}{2}\Delta_t\right) \\
&\left(r_k - e^{-\alpha\Delta_t} r_{k-1} - \beta\left(1 - e^{-\alpha\Delta_t}\right)\right) \\
&\left. + \frac{2\alpha}{(1-\rho^2)\,\sigma^2\left(1 - e^{-2\alpha\Delta_t}\right)} \left(r_k - e^{-\alpha\Delta_t} r_{k-1} - \beta\left(1 - e^{-\alpha\Delta_t}\right)\right)^2 \right\}.
\end{aligned}
$$

The log-likelihood function

$$
\begin{aligned}
L(\gamma_H, \sigma_H, \alpha, \beta, \sigma, \rho) &= \ln\left(\prod_{k=1}^{N}\left(f(\hat{h}_k, r_k \,\middle|\, \hat{h}_{k-1}, r_{k-1})\right)\right) = \sum_{k=1}^{N}\left(\ln\left(f(\hat{h}_k, r_k \,\middle|\, \hat{h}_{k-1}, r_{k-1})\right)\right) \\
&= -N\ln\left(\sqrt{2\pi}\right) - N\ln\left(\sqrt{1-\rho^2}\sigma_H\sqrt{\Delta_t}\,\sigma\sqrt{\frac{1 - e^{-2\alpha\Delta_t}}{2\alpha}}\right) \\
&- \frac{1}{2}\sum_{k=1}^{N}\left\{ \frac{1}{(1-\rho^2)\,\sigma_H^2\Delta_t}\left(\hat{h}_k - \hat{h}_{k-1} - (r_{k-1} + \gamma_H \sigma_H)\Delta_t + \frac{\sigma_H^2}{2}\Delta_t\right)^2 \right. \\
&- \frac{2\rho\sqrt{2\alpha}}{(1-\rho^2)\,\sigma_H\sqrt{\Delta_t}\,\sigma\sqrt{1 - e^{-2\alpha\Delta_t}}}\left(\hat{h}_k - \hat{h}_{k-1} - (r_{k-1} + \gamma_H \sigma_H)\Delta_t + \frac{\sigma_H^2}{2}\Delta_t\right) \\
&\left(r_k - e^{-\alpha\Delta_t} r_{k-1} - \beta\left(1 - e^{-\alpha\Delta_t}\right)\right) \\
&\left. + \frac{2\alpha}{(1-\rho^2)\,\sigma^2\left(1 - e^{-2\alpha\Delta_t}\right)}\left(r_k - e^{-\alpha\Delta_t} r_{k-1} - \beta\left(1 - e^{-\alpha\Delta_t}\right)\right)^2 \right\}.
\end{aligned}
$$

Maximizing the expression above is equivalent to minimizing its negative, without the constant term $N\ln\left(\sqrt{2\pi}\right)$, that is

$$
\begin{aligned}
&g(\gamma_H, \sigma_H, \alpha, \beta, \sigma, \rho) = N\ln\left(\sqrt{1-\rho^2}\sigma_H\sqrt{\Delta_t}\,\sigma\sqrt{\frac{1 - e^{-2\alpha\Delta_t}}{2\alpha}}\right) \\
&+ \sum_{k=1}^{N}\left\{ \frac{1}{2\,(1-\rho^2)\,\sigma_H^2\Delta_t}\left(\hat{h}_k - \hat{h}_{k-1} - (r_{k-1} + \gamma_H \sigma_H)\Delta_t + \frac{\sigma_H^2}{2}\Delta_t\right)^2 \right. \\
&- \frac{\rho\sqrt{2\alpha}}{(1-\rho^2)\,\sigma_H\sqrt{\Delta_t}\,\sigma\sqrt{1 - e^{-2\alpha\Delta_t}}} \\
&\left(\hat{h}_k - \hat{h}_{k-1} - (r_{k-1} + \gamma_H \sigma_H)\Delta_t + \frac{\sigma_H^2}{2}\Delta_t\right)\left(r_k - e^{-\alpha\Delta_t} r_{k-1} - \beta\left(1 - e^{-\alpha\Delta_t}\right)\right) \\
&\left. + \frac{\alpha}{(1-\rho^2)\,\sigma^2\left(1 - e^{-2\alpha\Delta_t}\right)}\left(r_k - e^{-\alpha\Delta_t} r_{k-1} - \beta\left(1 - e^{-\alpha\Delta_t}\right)\right)^2 \right.
\end{aligned}
$$

Table 1: Housing log-return and interest-rate process parameter estimates

Parameter	Estimation	+/- Error
σ_H	0.0226	0.0016
β	0.0174	0.0586
α	0.0745	0.1292
σ	0.0104	0.0007
ρ	0.1999	0.0973

The parameters for the housing log-return and interest rate processes described by equations (32) and (31) respectively have been estimated by minimizing numerically the negative of the log-likelihood, as given by equation (4.1). The errors have been calculated using Proposition B.5.1 in Rémillard [7]. Fisher information matrix has been estimated using the Hessian, as returned by MATLAB fminunc function. The data used for the estimates are the monthly Case-Shiller U.S. National Home Price Index [CSUSHPINSA] and the monthly 1-Month London Interbank Offered Rate (LIBOR), based on U.S. Dollar [USD1MTD156N] between January 1987 and March 2018 inclusively, both retrieved from FRED, Federal Reserve Bank of St. Louis. The log-likelihood estimates for housing log-return dynamics and the correlation between Z and $Z^{(H)}$ show relatively small estimation errors. The estimation errors for the Ornstein-Uhlenbeck process parameters are larger, consistent with Remark 5.2.3 in Rémillard [7], but remain within a reasonable range.

The resulting parameter estimates are shown in Table 1.[21]

4.2 Prepayment Parameter Estimation

To estimate the parameters in the prepayment intensity equation (29), i.e. a_π, b_π and c_π, we proceed as follows:

- We run the dynamic programming algorithm described in section 3.2 and, at each time step t_j, we store, for each pair of interest rate level and housing price with index mn, the conditional prepayment rate returned by the prepayment intensity function (29).

- Starting from the valuation date t_0, we compute, at each time step t_j, the cumulative probability that the interest rate level and the housing price level reach node mn. Let $\bar{p}_{t_j}^{(mn)}$ denote the probability for the housing price and the interest rate to be jointly in state m and n respectively at time t_j, given they are in known market-observed states at time t_0. And denote $\bar{\mathbf{p}}_{t_j}$ the vector of such probabilities at time t_j. For all $j \in \{1, J\}$, we have

$$\bar{\mathbf{p}}_{t_j} = \bar{\mathbf{p}}_{t_{j-1}}^{\intercal} \mathbf{p}$$

  where $\mathbf{p}$ is the matrix of the bivariate transition probabilities $p_{(mn,uv)}$, similar to the risk-neutral probabilities $q_{(mn,uv)}$ defined in section, but under a real-world probability measure $\mathbb{P}$ 3.1.3.

[21] The reasonableness of the log-likelihood estimates has been tested using Monte Carlo simulation. See Appendix A.

- At each time step, we compute our model monthly conditional prepayment rate, noted Ψ_{t_j}, as being the conditional prepayment rates returned by the prepayment intensity function defined by equation (29) on every node mn, weighted by the probabilities $\bar{p}_{t_j}^{(mn)}$ to be on such nodes respectively. In equation:

$$\Psi_{t_j} = \sum_{u=1}^{M} \sum_{v=1}^{N} \bar{\mathbf{p}}^{\intercal} \mathbf{p}_{(:,uv)} \exp\left(-\Lambda_{t_j}^{(uv)}\right),$$

  where $\mathbf{p}_{(:,uv)}$ is a vector column listing the transition probabilities of moving from every pair of states mn to the state uv.

- We calculate the sum of the squared differences between each element in the vector of our model conditional probabilities Ψ_{t_j} and the monthly conditional prepayment rate ("CPR") observed on the market in the corresponding month t_j.[22]

- We obtain a_π, b_π and c_π estimates by minimizing such a squared difference using the MATLAB function $fmincon$ and the interior-point algorithm.

To illustrate, consider the Fannie Mae pool with Bloomberg ticker FN 890404.[23] Such a pool was issued on May 1, 2012, with a 6.5% coupon. Trading above par, it is helpful in estimating prepayment parameters, as its higher-than-market undelying coupons are expected to trigger prepayments. Its weighted average coupon ("WAC"), or $\bar{C}$ as defined by equation (8), was 7.055% on issuance, and this is the coupon we use.

Seven years of observation are available, yielding a time series of 72 monthly conditional prepayment rates as at April 30, 2018. Over such a period, the default rate has been nil in the pool, based on the default rate reported by Bloomberg. This allows us to temporarily ignore the default parameters, and estimate the prepayment parameters separately.

Recall that our model stipulates, as a simplifying assumption, a constant risky spread. Observe figure 1 and note how such an assumption has held historically in the US since 1991 on relatively short periods of two years or so, but should be reconsidered for longer periods. This is a limitation of our model that we acknowledge. Such an assumption is however reasonable for the period May 2012-April 2018, when the 15-year mortgage spread over LIBOR 1-month, on a monthly basis, ranges from 174 basis points ("bps") to 338 bps, with a 265 bps average and a 43 bps standard deviation.

We first estimate the constant spread S, as defined in equation (22), by taking the difference between the market average mortgage rate and the 1-month US LIBOR rates prevailing on May 1, 2012, the beginning of our estimation period.[24] We estimate S to be equal to

[22] The market-observed CPR is obtained from Bloomberg. It is the ratio of the aggregate amount prepaid on a mortgage pool to $B_{t-1} - P$, where B and P are as defined in equations (6) and (7), while $t-1$ refers to the previous month.

[23] The pool CUSIP is 31410LGM8.

[24] The mortgage rate is obtained from the Federal Reserve Bank of St. Louis. The 1-Month London Interbank Offered Rate (LIBOR), based on U.S. Dollar [USD1MTD156N], from ICE Benchmark Administration Limited (IBA), and the US average 15-year mortgage rate were retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/USD1MTD156N, June 9, 2018.

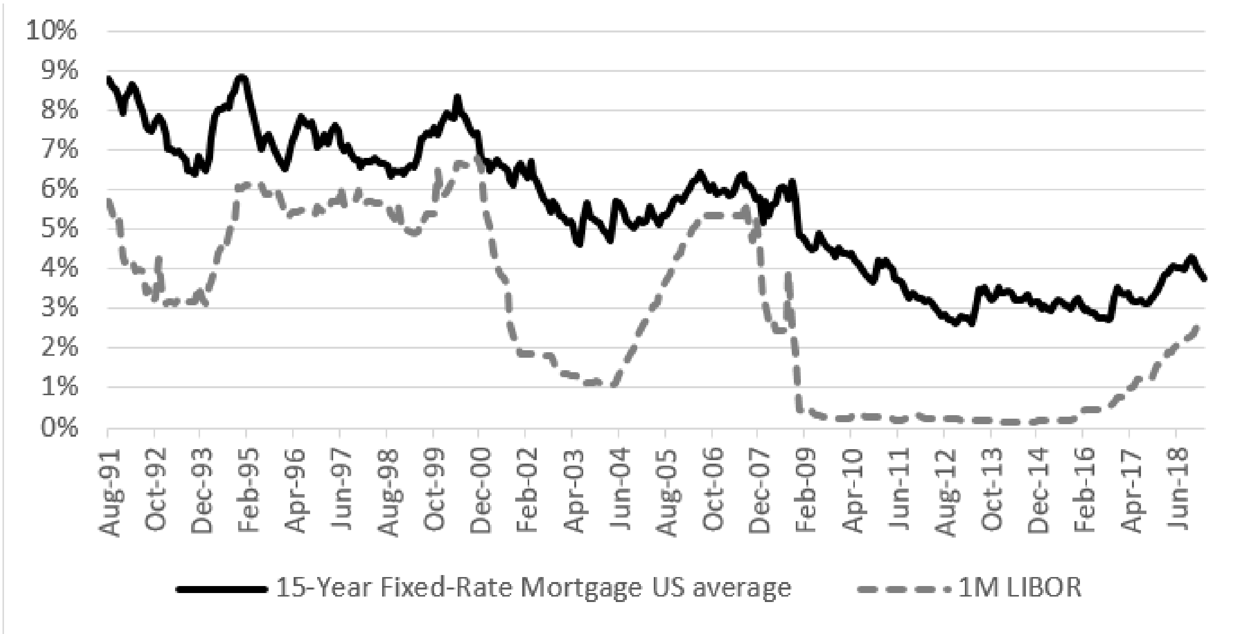

Figure 1: Historical spread between 15-year mortgage rate and LIBOR 1-Month (1991-2017)

2.7674%.

Then we follow the steps in the estimation procedure described further above. Minimizing the sum of squared differences between the Ψ_{t_j} values for all $j \in \{1, 72\}$ and the 72 monthly CPR observed on the market [25]for the pool being valued yields the following estimates for for a_π, b_π and c_π respectively: 0.000088732, 0.0011 and 1.2336.

Observe in figure 2 how the implied prepayment rates follow a much smoother trend than the actual CPR observed for the sample pool and don't appear to overfit. Note also how the model prepayment rates start high and decrease slowly from about 30% to 20% approximately, which is consistent with the historical trend, as well as the fact that much lower interest rates than the pool coupons prevailing at the time of issuance would trigger a lot of early payments.[26]

4.3 Default Parameters Estimation

Now that prepayment parameters have been estimated, it is possible to identify pools with non-null default rates and estimate default parameters a_η, b_η and c_η by minimizing the squared difference between the monthly default rate implied by such parameters in our model and the series of default rates observed historically.

[25]The CPR are obtained from Bloomberg.

[26]Recall we are dealing with 15-year Fannie Mae MBS, which are typically backed by refinancing mortgages. The pattern of prepayment rates starting low and increasing for two years and a half before plateauing, which is typical of many new mortages and reflected in the well-known Public Securities Association (PSA) model thus do not apply.

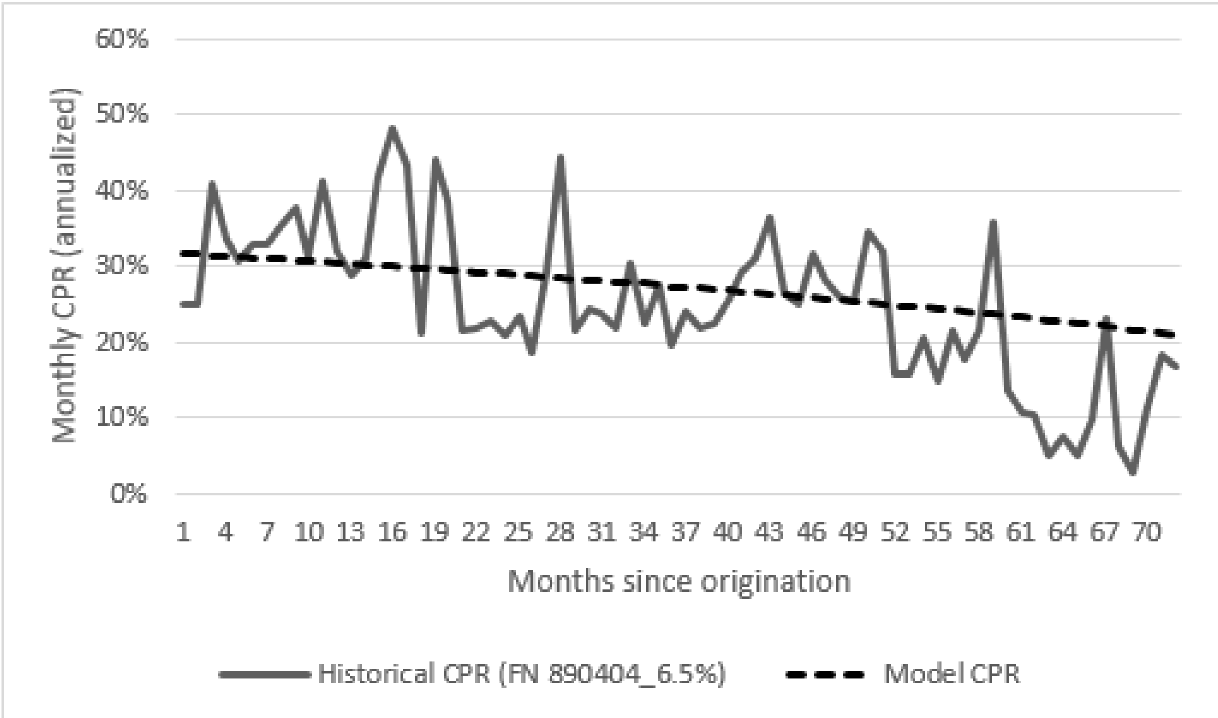

Figure 2: Model Prepayment Rate vs Historical CPR time series (FN 890404 Pool example)

5 Model Tests and Stress Tests

In a base case, mortgages currently issued carry a coupon that will make them trade at par. Alternatively, they can be priced at a premium (when carrying a higher coupon) or at a discount (when carrying a lower coupon).

To test our model, consider a mortgage carrying a 3.2% coupon. We use the parameter estimates in table 1 as baseline parameters. In the baseline scenario, we set the current one-month LIBOR to β, or 1.74%, resulting in a fair spread of 146 basis points. Setting prepayment and default intensities to 0 for this test purposes, our model yields $1,000.95, as shown in Figure 9 in appendix. A less than 0.1% difference is deemed reasonable.[27]

Now, narrow or expand the spread by 100 bps, to 46 bps and 246 bps respectively, which results in a parallel shift in the whole term structure. The resulting increase (decrease) in value should be approximately equal to the mortgage value times its duration, calculated to be 6.94 years,[28] yielding a 69.45 dollar-duration. Our model yields $1,073.81 and $934.82, or

[27] Using smaller interest rate physical increments in the grid results in values closer and closer to par, as can be seen from table 1. A less than 0.1% difference is however considered reasonable, and we stick to a 33 bps increment for test purposes.

[28] Recall that the prepayment and default intensities are set to 0 in the baseline setting.

0.4% and 0.5% from the expected approximate values, which again is reasonable.

We stress the current interest rate to a high (low, and even negative) level and verify that the value make sense, falling to \$711.23 (increasing to \$1177.30).

If the term structure is flat, a mortgage trading at par should not be sensitive to prepayment. We set the prepayment background intensity parameter a_π to $\frac{1}{\Delta_t}$ and verify than the model yields a \$1,000 value.

A mortgage trading at a premium (discount) should decrease (increase) in value as prepayments increase and early-returned capital gets to be reinvested at a lower (higher) rate. But, assuming there is no risk of default, a mortgage value cannot fall below (rise above) par. So we set the intensity of default to zero, and vary a_π from $\frac{1}{\Delta_t}$ to $\frac{1}{1000\Delta_t}$ for both a premium and a discount mortgage. Note how $a_\pi = \frac{1}{\Delta_t}$ is an extreme case, as it translates into a 63.21% monthly prepayment rate. Figure 3 shows how both mortgage values are bounded between par and the no-prepayment value, and exhibit a decreasing or increasing trend to par as the prepayment rate increases (from right to left on the graph).[29]

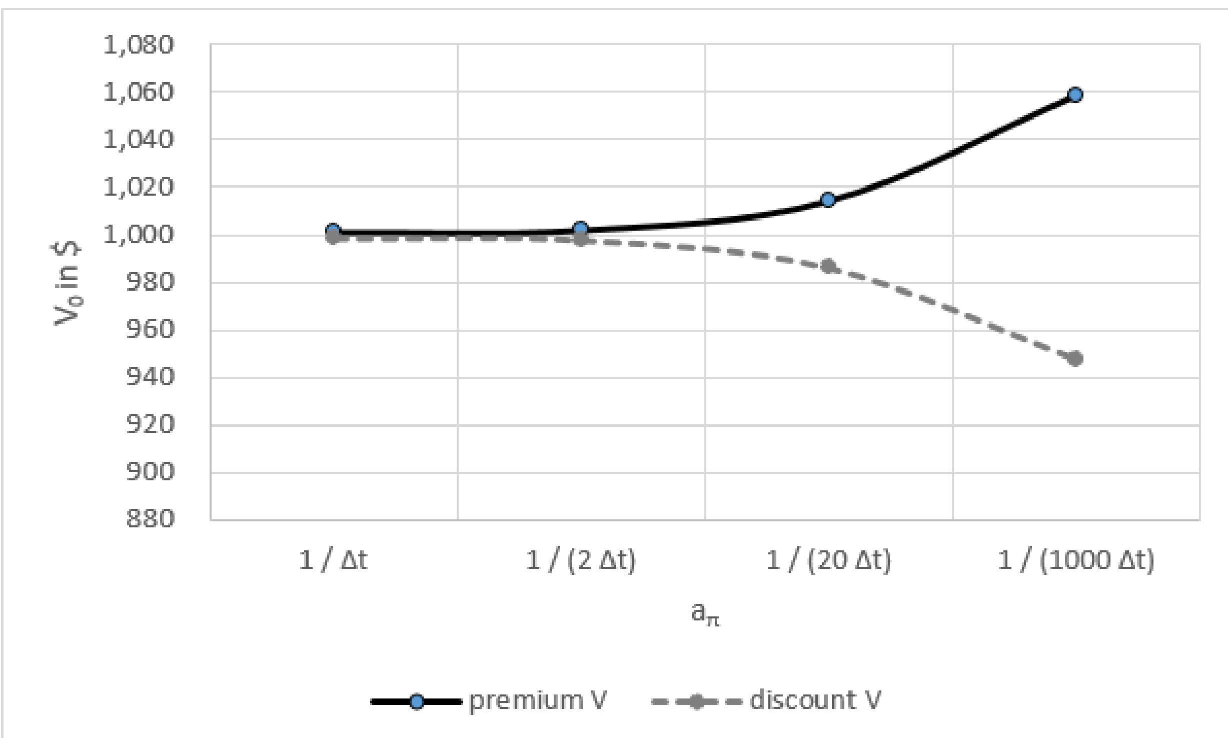

Figure 3: Impact of increasing a_π on discount and premium mortgage values.

We expect the multiplying factor in Λ^π, b_π as defined in equation (29) to have a similar

[29]For the exact values used to generate this graph, the various parameter precise settings, as well as further comments about the results reasonableness, please refer to the table in figure 10 in the appendix. The same or similar tables in the appendix back all figures in this stress-testing section.

effect as a_π. Focusing on the premium mortgage, and setting a_π and the exponent c_π to neutral values (0 and 1 respectively), we can verify in the table labeled Figure 10 that, again, the mortgage value decreases from the no-prepayment premium value to par as b_π increases (from bottom to top in the table).

More generally, the model returns values that are consistent with our reasonable expectations as can be see in figures 9, 10, 11,12 and 13 in Appendix B.

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Appendix A. Verifying the reasonableness of the housing and interest rate process parameter estimates

We verify whether the estimates presented in section 4.1 are reasonable and our log-likelihood formula implementation is accurate as follows:

- Using the parameters in table 1, we simulate 100 paths of monthly housing index values and interest rates over five years of housing prices and interest rates (resulting in 100 bivariate series of 60 time points);
- We estimate the sample parameters using the log-likelihood function and minimization approach described above;
- We compare the parameter sample means to the theoretical parameters and test the hypothesis that they are similar using a Student's t-test.
- We compare the sample variance for each parameter to the corresponding mean squared standard deviation, i.e.

$$\bar{S}^2_{\hat{\theta}} = \frac{1}{n}\sum_{i=1}^{n} S^2_{\hat{\theta}_i} \tag{38}$$

  where $S_{\hat{\theta}_i}$ is the standard deviation of any given parameter θ (i.e either μ_H,[30] σ_H, β, α, σ, or ρ) across the 100 simulations performed.

As can be seen in Figure 4, the estimates appear to be reasonable at a 95% confidence level, except for μ_H and α. With respect to μ_H however, the sample mean is arguably reasonably close to the theoretical value, being 3.45% compared to 3.68%. With respect to α, the test negative result is consistent with the large estimation error noted in Table 1. Further work could be devoted to explore how to better estimate Vasicek dynamic parameters. This is beyond the scope of this Master's thesis.

As shown in Figure 5, $\bar{S}^2$ looks reasonably close to the sample variance for every parameter, again except for α.

One can also verify visually in Figures 6, 7 and 8 that, except for alpha, for reasons already explained, the parameter estimates resulting from the 100 simulations performed reasonably bracket the theoretical value.

[30] For the purpose of this test, we used a constant housing drift parameter. Further work could be devoted to estimating the risk premium.

Parameter	μ_H	σ_H	β	α	σ	ρ
Theoretical value	0.0368	0.0226	0.0174	0.0745	0.0104	0.1999
Sample mean	0.03446691	0.0227145	-0.71849886	1.0260152	0.0104735	0.201835
Sample variance	0.00008428	0.00000428	20.81851377	1.49245113	0.00000099	0.02059304
Student statistic	2.54138085	-0.5533381	1.61284737	-7.7887122	-0.7400848	-0.1348407
Student critical value	1.984	1.984	1.984	1.984	1.984	1.984
Conclusion	Fail	Pass	Pass	Fail	Pass	Pass

Figure 4: Student's t-test to assess reasonableness of housing and interest-rate parameter estimates

Parameter	μ_H	σ_H	β	α	σ	ρ
$\bar{s}^2$	0.000079	0.000004	20.271496	0.577925	0.000001	0.020589
Sample variance	0.000084	0.000004	20.818514	1.492451	0.000001	0.020593

Figure 5: Simulated Housing and Interest Rate Parameter Mean Squared Standard Deviation vs. Sample Variance

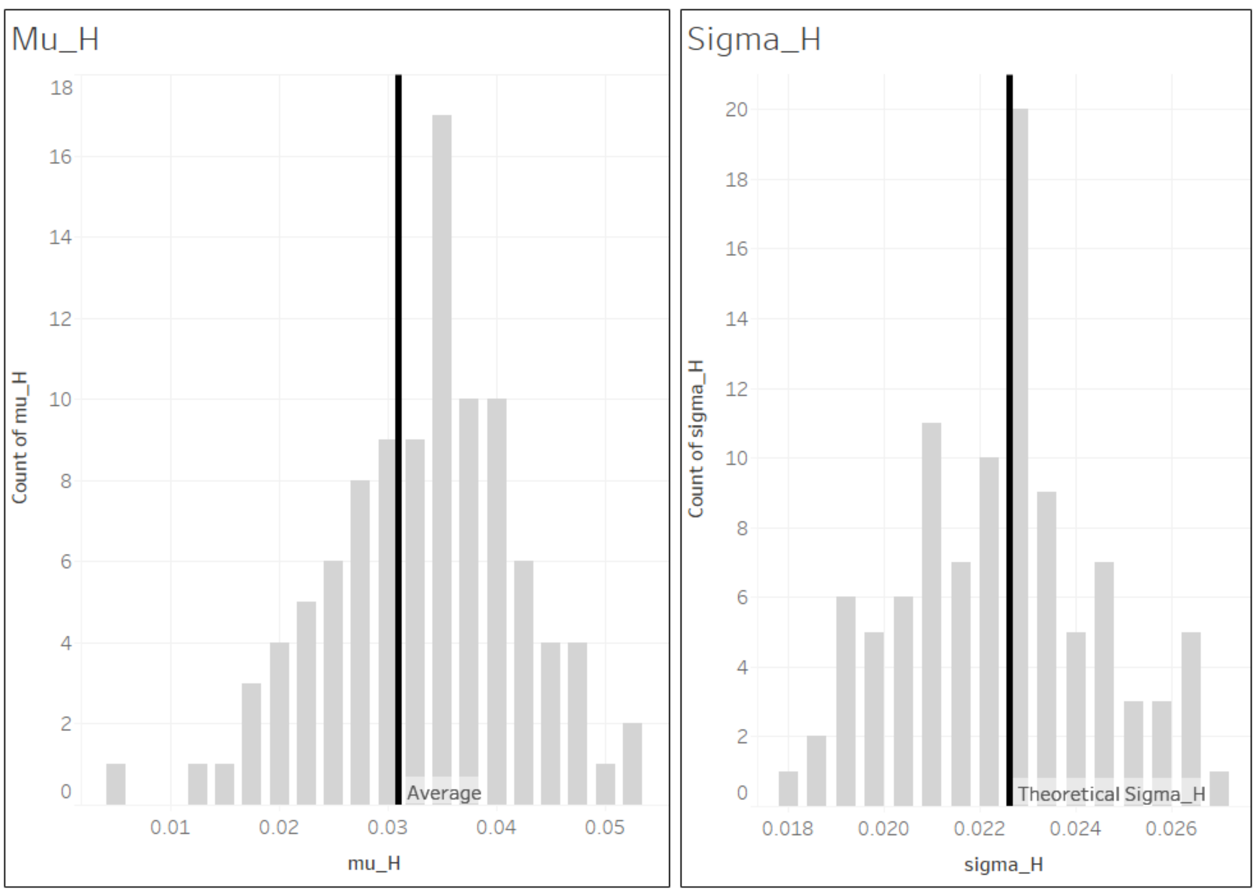

Figure 6: Simulated Housing Parameter Estimates

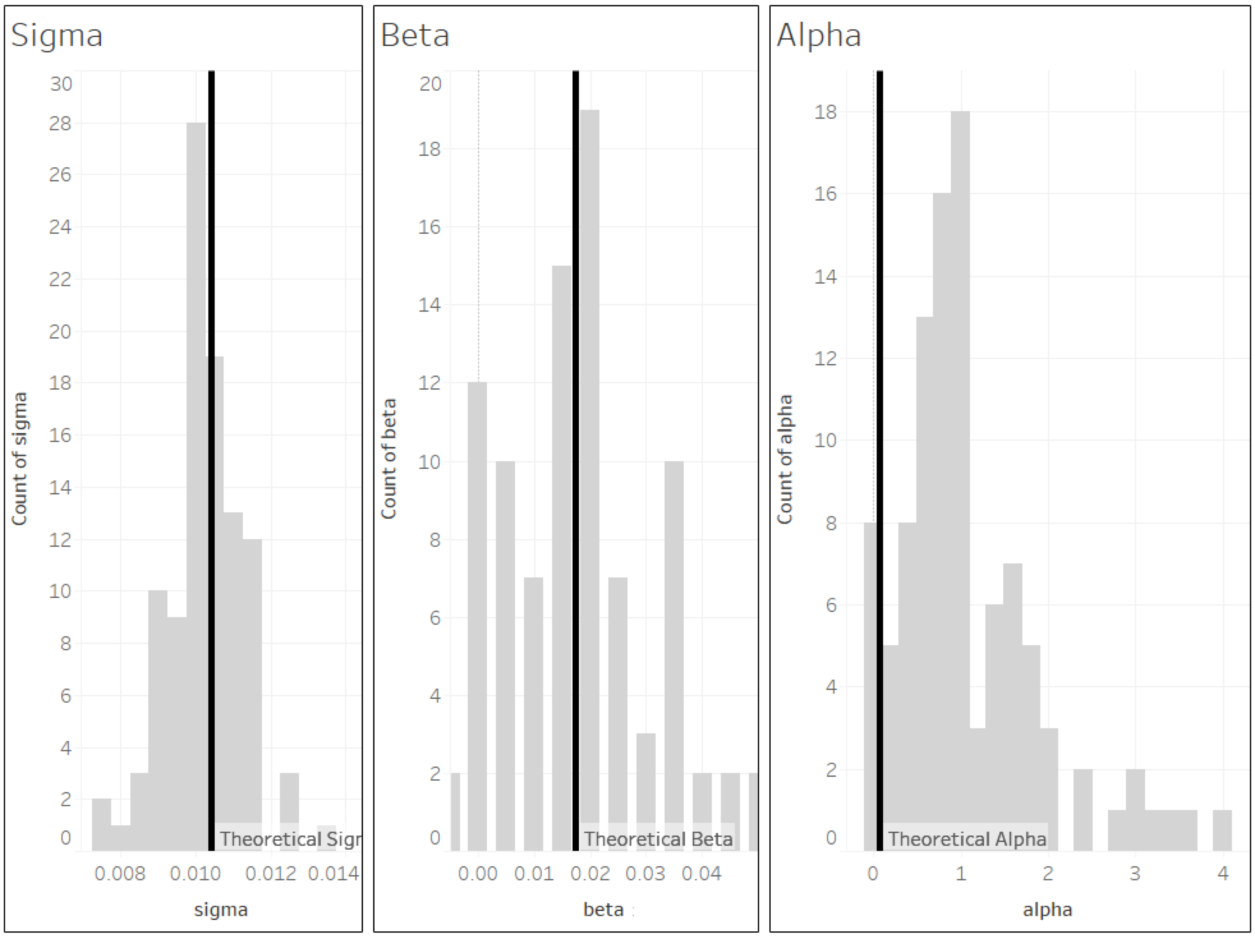

Figure 7: Simulated Interest Rate Parameter Estimates

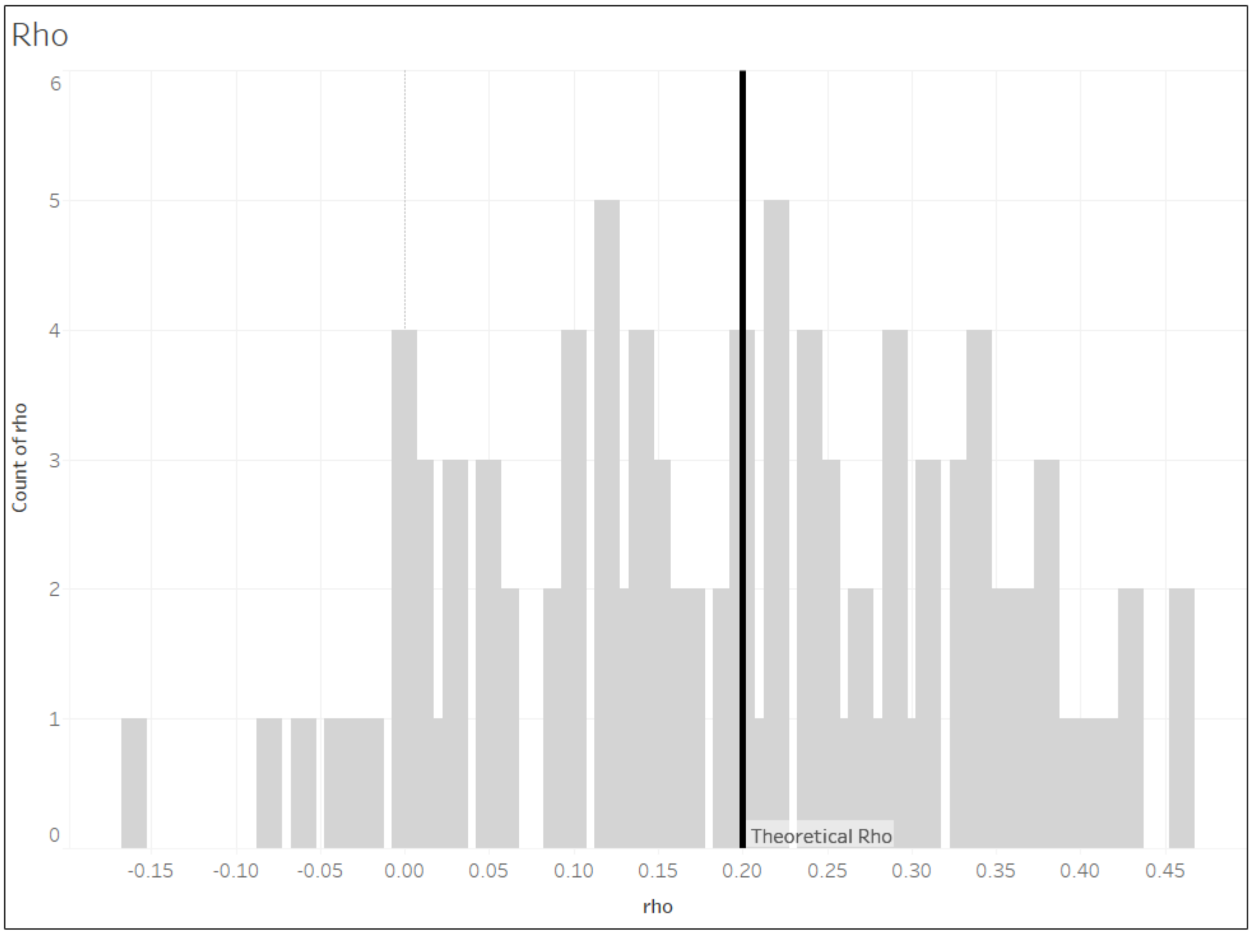

Figure 8: Simulated Correlation Parameter Estimates

Appendix B. Stress Tests

				Default parameters				Prepayment parameters				Interest rate parameters			Housing						
Variable	r_0 (in %)	Spread S	Δ_r (in bps)	a_η	b_η	c_η	LTV	α_π	b_π	c_π	κ	α	β (in %)	σ	μ_h	σ_h	V (model output)	Expected output	Rationale for expectation	Difference	Conclusion
interest rate physical increment Δ_r	1.74	1.46	33.0	0	0	1	0.8	0	0	1	1	0.0745	1.74	0.0104	0.0368	0.0214	1,000.95	1,000.00	at par	0.095%	reasonable
interest rate physical increment Δ_r	1.74	1.46	16.5	0	0	1	0.8	0	0	1	1	0.0745	1.74	0.0104	0.0368	0.0214	1,000.78	1,000.00	at par, and lower than above, from convergence	0.078%	reasonable
interest rate physical increment Δ_r	1.74	1.46	11.0	0	0	1	0.8	0	0	1	1	0.0745	1.74	0.0104	0.0368	0.0214	1,000.69	1,000.00	at par, and lower than above, from convergence	0.069%	reasonable
interest rate physical increment Δ_r	1.74	1.46	8.3	0	0	1	0.8	0	0	1	1	0.0745	1.74	0.0104	0.0368	0.0214	1,000.65	1,000.00	at par, and lower than above, from convergence	0.065%	reasonable
interest rate physical increment Δ_r	1.74	1.46	6.6	0	0	1	0.8	0	0	1	1	0.0745	1.74	0.0104	0.0368	0.0214	1,000.63	1,000.00	at par, and lower than above, from convergence	0.063%	reasonable
interest rate physical increment Δ_r	1.74	1.46	3.3	0	0	1	0.8	0	0	1	1	0.0745	1.74	0.0104	0.0368	0.0214	N/A	1,000.00	at par	N/A	out of memory error
spread	1.74	0.46	33.0	0	0	1	0.8	0	0	1	1	0.0745	1.74	0.0104	0.0368	0.0214	1,073.81	1,069.45	- dollar duration, calculated to be -$69.45	0.4%	reasonable
spread	1.74	2.46	33.0	0	0	1	0.8	0	0	1	1	0.0745	1.74	0.0104	0.0368	0.0214	934.82	930.55	+ dollar duration	0.5%	reasonable
interest rate	0.74	1.46	33.0	0	0	1	0.8	0	0	1	1	0.0745	1.74	0.0104	0.0368	0.0214	1,061.70	1,069.45	- dollar duration, calculated to be -$69.45	-0.7%	reasonable
interest rate	2.74	1.46	33.0	0	0	1	0.8	0	0	1	1	0.0745	1.74	0.0104	0.0368	0.0214	944.76	930.55	+ dollar duration	1.5%	reasonable
interest rate (stress testing)	8.0	1.46	33.0	0	0	1	0.8	0	0	1	1	0.0745	1.74	0.0104	0.0368	0.0214	711.23	< 930.55	smaller discounting factors than with r_0 = 3% as above	N/A	reasonable
interest rate (stress testing)	-1.0	1.46	33.0	0	0	1	0.8	0	0	1	1	0.0745	1.74	0.0104	0.0368	0.0214	1,177.30	< 1069.45	greater discounting factors than with r_0 = 1% as above	N/A	reasonable

Figure 9: Baseline parameters stress-testing

				Default parameters				Prepayment parameters				Interest rate parameters			Housing						
Variable	r_0 (in %)	Spread S	Δ_∞ (in bps)	a_η	b_η	c_η	LTV	α_π	b_π	c_π	κ	α	β (in %)	σ	μ_h	σ_h	output)	Expected output	Rationale for expectation	Difference	Conclusion
prepayment background intensity	1.74	1.46	33.0	0	0	1	0.8	$1/\Delta_1$	0	1	1	0.0745	1.74	0.0104	0.0368	0.0214	1,000.00	"≈ par"	at par, prepayment intensity should not significantly impact value	N/A	reasonable
prepayment background intensity	2.74	1.46	33.0	0	0	1	0.8	$1/\Delta_1$	0	1	1	0.0745	1.74	0.0104	0.0368	0.0214	998.70	944.76 < V <= par	at a discount => prepayment increases value; upper-bounded by par value	N/A	reasonable
prepayment background intensity	2.74	1.46	33.0	0	0	1	0.8	$1/(2\Delta_1)$	0	1	1	0.0745	1.74	0.0104	0.0368	0.0214	997.93	944.76 < V <= 998.7	b_π halved relative to above, but monthly prepayment rate is still 0.39 vs. 0.63	N/A	reasonable
prepayment background intensity	2.74	1.46	33.0	0	0	1	0.8	$1/(20\Delta_1)$	0	1	1	0.0745	1.74	0.0104	0.0368	0.0214	986.21	944.76 < V <= 997.93	b_π greatly reduced relative to above, but monthly prepayment rate is still 0.05	N/A	reasonable (see below for symmetrical effect)
prepayment background intensity	2.74	1.46	33.0	0	0	1	0.8	$1/(1000\Delta_1)$	0	1	1	0.0745	1.74	0.0104	0.0368	0.0214	947.43	very slightly > 944.76	at 0.0001, the monthly prepayment rate is only very slightly above nil.	N/A	reasonable (see below for symmetrical effect)
prepayment background intensity	0.74	1.46	33.0	0	0	1	0.8	$1/\Delta_1$	0	1	1	0.0745	1.74	0.0104	0.0368	0.0214	1,001.31	1000 <= V < 1061.7	at a premium => prepayment reduces value; but liability cannot fall below face value	N/A	reasonable
prepayment background intensity	0.74	1.46	33.0	0	0	1	0.8	$1/(2\Delta_1)$	0	1	1	0.0745	1.74	0.0104	0.0368	0.0214	1,002.08	1,001.31 <= V < 1061.7	b_π halved relative to above, but monthly prepayment rate is still 0.39 vs. 0.63	N/A	reasonable
prepayment background intensity	0.74	1.46	33.0	0	0	1	0.8	$1/(20\Delta_1)$	0	1	1	0.0745	1.74	0.0104	0.0368	0.0214	1,014.33	1,002.08 <= V < 1061.7	b_π greatly reduced relative to above, but monthly prepayment rate is still 0.05	N/A	reasonable, all the more given the ≈ +/- $14 symetrical effect of a 0.05 a_π on the value of mortgages with values symmetrical around par ($986.21 and $1,014.33).
prepayment background intensity	0.74	1.46	33.0	0	0	1	0.8	$1/(1000\Delta_1)$	0	1	1	0.0745	1.74	0.0104	0.0368	0.0214	1,058.52	very slightly < 1061.7	at 0.0001, the monthly prepayment rate is only very slightly above nil.	N/A	reasonable

Figure 10: Prepayment parameters stress-testing

				Default parameters				Prepayment parameters				Interest rate parameters			Housing						
Variable	r_0 (in %)	Spread S	Δ_∞ (in bps)	a_η	b_η	c_η	LTV	a_π	b_π	c_π	κ	α	β (in %)	σ	μ_L	σ_L	model outp	Expected output	Rationale for expectation	Difference	Conclusion
prepayment multiplying factor	0.74	1.46	33.0	0	0	1	0.8	0	1 / (2 Δ_t)	1	1	0.0745	1.74	0.0104	0.0368	0.0214	1,001.54	nodel output) <= V < 10	b_π halved relative to above, but monthly prepayment rate is still 0.39 vs. 0.63	N/A	reasonable
prepayment multiplying factor	0.74	1.46	33.0	0	0	1	0.8	0	1 / (20 Δ_t)	1	1	0.0745	1.74	0.0104	0.0368	0.0214	1,004.24	1,001.54 <= V < 1061.7	b_π greatly reduced relative to above, but monthly prepayment rate is still 0.05	N/A	reasonable
prepayment multiplying factor	0.74	1.46	33.0	0	0	1	0.8	0	1 / (1000 Δ_t)	1	1	0.0745	1.74	0.0104	0.0368	0.0214	1,025.19	1,004.24 <= V < 1061.7	even multiplied by 0.0001, the $ difference between V^h and V^e in prepayment intensity formula remains significant, and greatly detracts from premium value without prepayment ($1,072.29)	N/A	reasonable
prepayment multiplying factor	0.74	1.46	33.0	0	0	1	0.8	0	1 / (100000 Δ_t)	1	1	0.0745	1.74	0.0104	0.0368	0.0214	1,060.30	very slightly < 1061.7	at 0.00001, the factor reduce the $difference between continuing value and prepayment value to almost nothing	N/A	reasonable
prepayment accelerating factor	0.74	1.46	33.0	0	0	1	0.8	0	1 / (1000 Δt)	0.5	1	0.0745	1.74	0.0104	0.0368	0.0214	1,046.00	1,025.19 < V <= 1,061.70	for b_π fixed, slower prepayment than with c_π = 1, so higher and closer to premium value without prepayment	N/A	reasonable
prepayment accelerating factor	0.74	1.46	33.0	0	0	1	0.8	0	1 / (1000 Δt)	0.25	1	0.0745	1.74	0.0104	0.0368	0.0214	1,054.20	1,046.00 < V <= 1,061.70	for b_π fixed, slower prepayment than with c_π = 0.5, so higher and closer to premium value without prepayment	N/A	reasonable
prepayment accelerating factor	0.74	1.46	33.0	0	0	1	0.8	0	1 / (1000 Δt)	1.5	1	0.0745	1.74	0.0104	0.0368	0.0214	1,014.19	1,000 < V <= 1,025.19	for b_π fixed, and trading at a premium, faster prepayment than with c_π = 1, so lower value	N/A	reasonable
prepayment accelerating factor	0.74	1.46	33.0	0	0	1	0.8	0	1 / (1000 Δt)	2	1	0.0745	1.74	0.0104	0.0368	0.0214	1,009.35	1,000 < V <= 1,014.19	for b_π fixed, and trading at a premium, faster prepayment than with c_π = 1.5, so lower value	N/A	reasonable

Figure 11: Prepayment parameters stress-testing (continued)

				Default parameters				Prepayment				Interest rate parameters			Housing							
Variable	r_0 (in %)	Spread S	Δ_* (in bps)	a_η	b_η	c_η	LTV	α_π	b_π	c_π	κ	α	β (in %)	σ	μ_h	σ_h	model outp	Expected output	Rationale for expectation	ifferenc	Comment	Conclusion
background default intensity	0.74	1.46	33.0	$1/\Delta_t$	0	1	0.8	0	0	1	1	0.0745	1.74	0.0104	0.0368	0.0214	998.66	997.34 <= V < 1,061.70	combined with 100% recovery rate has similar effect as prepayment (reduces premium value); lower bound is lower than par, as one-period accrued and unpaid interest remains unpaid at default: with high default intensity, default is highly likely as early as t_1 and so lower bound is par times D_(t_0, t_1), or 997.34	N/A		reasonable
background default intensity	0.74	1.46	33.0	$1/(2\Delta_t)$	0	1	0.8	0	0	1	1	0.0745	1.74	0.0104	0.0368	0.0214	999.45	998.66 <= V < 1,061.70	while trading at a premium, slower default with no loss (so "prepayment", minus accrued and unpaid interest) than with a_eta = 1 on a monthly basis, so higher value	N/A		reasonable
background default intensity	0.74	1.46	33.0	$1/(10\Delta_t)$	0	1	0.8	0	0	1	1	0.0745	1.74	0.0104	0.0368	0.0214	1,005.51	999.45 <= V < 1,061.70	while trading at a premium, slower default with no loss (so "prepayment", minus accrued and unpaid interest) than with a_η = 0.5 on a monthly basis, so higher value	N/A		reasonable
background default intensity	0.74	1.46	33.0	$1/(100\Delta_t)$	0	1	0.8	0	0	1	1	0.0745	1.74	0.0104	0.0368	0.0214	1,037.45	1005.51 <= V < 1,061.70	while trading at a premium, slower default with no loss (so "prepayment", minus accrued and unpaid interest) than with a_η = 0.1 on a monthly basis, so higher value	N/A		reasonable
default intensity multiplying factor	0.74	1.46	33.0	$1/(100\Delta_t)$	$1/\Delta_t$	1	0.8	0	0	1	1	0.0745	1.74	0.0104	0.0368	0.0214	998.65	997.34 <= V < 1,037.45	while trading at a premium, for a_η set to 0.01 on a monthly basis, a non-null multiplying factor b_η decreases the mortgage value by accelerating default further, down to the lower bound set by par times D_(t_0, t_1)	N/A		reasonable
default intensity multiplying factor	0.74	1.46	33.0	$1/(100\Delta_t)$	$1/(10\Delta_t)$	1	0.8	0	0	1	1	0.0745	1.74	0.0104	0.0368	0.0214	1,004.86	998.65 <= V < 1,037.45	while trading at a premium, for a_η set to 0.01 on a monthly basis, a lower multiplying factor b_η than above increases the mortgage value from slowing down early return of principal, up to to the higher bound set by b_η = 0	N/A		reasonable
default intensity multiplying factor	0.74	1.46	33.0	$1/(100\Delta_t)$	$1/(100\Delta_t)$	1	0.8	0	0	1	1	0.0745	1.74	0.0104	0.0368	0.0214	1,025.60	1,004.86 <= V < 1,037.45	while trading at a premium, for a_η set to 0.01 on a monthly basis, a lower multiplying factor b_η than above increases the mortgage value from slowing down early return of principal, up to to the higher bound set by b_η = 0	N/A		reasonable
default intensity multiplying factor	0.74	1.46	33.0	$1/(100\Delta_t)$	$1/(1000\Delta_t)$	1	0.8	0	0	1	1	0.0745	1.74	0.0104	0.0368	0.0214	1,037.29	1,025.60 <= V < 1,037.45	while trading at a premium, for a_η set to 0.01 on a monthly basis, a lower multiplying factor b_η than above increases the mortgage value from slowing down early return of principal, up to to the higher bound set by b_η = 0	N/A		reasonable
default intensity accelerating factor	0.74	1.46	33.0	$1/(100\Delta_t)$	$1/(100\Delta_t)$	1.5	0.8	0	0	1	1	0.0745	1.74	0.0104	0.0368	0.0214	1,025.60	997.34 < V <= 1,025.60	while trading at a premium, for a_η and b_η both set to 0.01 on a monthly basis, a c_η higher than 1 is expected to accelerate the rate of return of principal when B_t > V_t and reduce the mortgage value increase, down to the lower bound set by par times D_(t_0, t_1)	N/A	Contrary to our expectation, a higher c_η results in a the same value. This is due to the ratio B_t/H_t being always below 1 in this test setting, even on the grid nodes with the lowest housing prices, based on the housing geometric Brownian motion estimated parameters and a loan-to-value ratio set to 0.8. As a result, in every state of the world, max(B_t/H_t, 1) returns 1, and 1^c_η = 1, for all c_η. Further testing is needed, in settings where B_t/H_t > 1. For example, the housing drift parameter could be estimated based on financial crisis periods such as 2008. A negative drift would then generate much lower values for H_t. Alternatively, a higher loan-to-value ratio than 0.8 could be used.	see testing with loan-to-value ratio set to 1 below
default intensity accelerating factor	0.74	1.46	33.0	$1/(100\Delta_t)$	$1/(100\Delta_t)$	1.5	1.2	0	0	1	1	0.0745	1.74	0.0104	0.0368	0.0214	1,025.28	997.34 < V <= 1,025.60	a lower loan-to-value ratio at origination decreases value	N/A		reasonable
default intensity accelerating factor	0.74	1.46	33.0	$1/(100\Delta_t)$	$1/(100\Delta_t)$	1.5	1.2	0	0	1	1	0.0745	1.74	0.0104	-0.2000	0.0214	1,019.64	997.34 < V <= 1,025.28	every other setting equal to the above settings, a negative μ_H depresses the loan-to-value ratio throughout the mortgage life and decreases value	N/A		reasonable
default intensity accelerating factor	0.74	1.46	33.0	$1/(100\Delta_t)$	$1/(100\Delta_t)$	2	1.2	0	0	1	1	0.0745	1.74	0.0104	-0.2000	0.0214	1,017.92	1,025.60 <= V < 1,019.64	every other setting equal to the above settings, accelerating the rate of prepayment as the outstanding balance-to_value ratio increases intensifies default and reduces value	N/A		reasonable
default intensity accelerating factor	0.74	1.46	33.0	$1/(100\Delta_t)$	$1/(100\Delta_t)$	3	1.2	0	0	1	1	0.0745	1.74	0.0104	-0.2000	0.0214	1,015.07	997.34 < V <= 1,017.92	every other setting equal to the above settings, accelerating the rate of prepayment as the outstanding balance-to_value ratio increases intensifies default and reduces value	N/A		reasonable

Figure 12: Default parameters stress-testing

Variable	r_0 (in %)	Spread S	Δ_π (in bps)	Default parameters a_η	b_η	c_η	LTV	Prepayment α_π	b_π	c_π	κ	Interest rate parameters α	β (in %)	σ	Housing μ_h	σ_h	V (model output)	Expected output	Rationale for expectation	Difference	Conclusion
recovery rate	0.74	1.46	33.0	0	0	1	0.8	0	0	1	0.8	0.0745	1.74	0.0104	0.0368	0.0214	1,061.70	1,061.70	same model value as when $\kappa = 1$, everything else equal, since default rate is nil	0.0%	reasonable
recovery rate	0.74	1.46	33.0	$1/\Delta_t$	0	1	0.8	0	0	1	0.8	0.0745	1.74	0.0104	0.0368	0.0214	933.00	< 998.66	lower value than when $\kappa = 1$, everything else equal, now that recovey rate less than 1	N/A	reasonable
recovery rate	0.74	1.46	33.0	$1/\Delta_t$	0.01	1	0.8	0	0	1	0	0.0745	1.74	0.0104	0.0368	0.0214	515.21	< 933.00	(significanlty) lower with $\kappa = 0$ than with $\kappa = 0.8$ as above	N/A	reasonable
long-term interest rate parameter	0.74	1.46	33.0	0	0	1	0.8	0	0	1	1	0.0745	0.74	0.0104	0.0368	0.0214	1,074.12	V > par	setting prepayment intensity constant to nil, lower long-term interest rate means higher mortgage value from higher discounting factors throughout the mortgage life	N/A	reasonable
long-term interest rate parameter	0.74	1.46	33.0	0	0	1	0.8	$1/(1000\Delta_t$	0	1	1	0.0745	0.74	0.0104	0.0368	0.0214	1,026.50	par <= V < 1,074.12	intensity rate reduces the mortgage value relative to the above	N/A	reasonable
long-term interest rate parameter	1.74	1.46	33.0	0	0	1	0.8	0	0	1	1	0.0745	2.74	0.0104	0.0368	0.0214	989.81	V < par	setting prepayment intensity constant to nil, higher long-term interest rate means lower mortgage value from lower discounting factors throughout the mortgage life	N/A	reasonable
long-term interest rate parameter	1.74	1.46	33.0	0	0	1	0.8	$1/(1000\Delta_t$	0	1	1	0.0745	2.74	0.0104	0.0368	0.0214	990.62	989.81 <= V < par	every else constant, a non-nil prepayment intensity rate increases the mortgage value relative to the above discount price	N/A	reasonable

Figure 13: Recovery, Ornstein-Uhlenbeck and Housing log-return process parameters stress-testing