Natural Rate of Interest, Demographics and Income Inequalities

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Abstract

Empirical evidence compels us to believe that the fall of interest rates, inflation and growth rates that characterizes the “new normal” of the economy since the Great Recession of 2008 is the result of numerous secular shifts that preceded the global financial crisis (slower population growth, increasing income disparity and lower gains from new technology to name a few). In this paper, we aim to show that r^*, the natural interest rate (the real short-term interest rate associated with null output gap and stable inflation) has been following a steady downward trajectory in the past six decades and is affected by low-frequency shocks. Inspired by Holston, Laubach and Williams (2017), we apply the Kalman filter procedure to jointly estimate the natural rate of interest, potential output and trend growth rate in the U.S., Canada and the U.K. over the period 1960 to 2017. Our estimates capture the downward trend of natural interest rates for all economies as well as a significant and persistent drop in r^* in 2008. In the U.S., we find that the natural rate of interest experienced a recent transitory surge that coincided with the interest rate normalization process initiated by U.S. monetary authorities beginning in 2016. In the hope of uncovering the factors underlying the secular decline in the natural rate of interest, we discuss and provide anecdotal evidence on a potential link between r^*, population aging and increasing income inequality.

Résumé

La récente littérature macroéconomique suggère que la chute des taux d'intérêt, de l'inflation et des taux de croissance de la production qui caractérisent la «nouvelle normale» des économies occidentales depuis la Grande Récession de 2008 est une conséquence de nombreux changements structuraux séculaires qui ont précédés cette dernière (la croissance de la population plus lente, les inégalités de revenu croissantes ainsi que les gains plus faibles issus des nouvelles technologies pour ne nommer qu'eux). Dans ce mémoire, nous cherchons à démontrer que r^*, le taux d'intérêt naturel (le taux d'intérêt réel de court terme associé à une inflation stable et un écart de production nul) a chuté de manière soutenue au courant des six dernières décénnies. Le taux d'intérêt naturel serait donc affecté par des déterminants de long terme. Nous appliquons la procédure de Kalman tirée de Holston, Laubach et Williams (2017) aux États-Unis, au Canada et au Royaume-Uni dans le but d'estimer conjointement le taux d'intérêt naturel, le niveau production potentielle ainsi que le taux de croissance potentiel entre 1960 et 2017. Les taux d'intérêt naturels de tous ces pays suivent une tendance baissière forte tout au long de la période d'estimation ainsi qu'une chute marquée en 2008. Aux États-Unis, le taux d'intérêt naturel subit une hausse transitoire qui coïncide avec le processus de normalisation des taux d'intérêt déclanché par les autorités monétaires américaines en 2016. Afin d'explorer les facteurs à l'origines de la chute de r^*, nous suggérons une potentielle relation entre le taux d'intérêt naturel, les changements démographiques et les inégalités de revenu croissantes.

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1 Introduction

At the onset of the Great Recession, few would have predicted that the subsequent recovery would be as sluggish as it was. Almost a decade later and despite years of near-zero interest rates, advanced economies are only beginning to see more optimistic inflation numbers with real interest rates remaining historically low. Moreover, real GDP growth in most countries is still lower than pre-crisis levels and forecasts do not point at any important surge in productivity.

Among the explanations suggested in order to make sense of the economic environment since 2008, the old theory of *secular stagnation* was reintroduced by Larry Summmers in 2013. The general premise of the secular stagnation hypothesis is that a variety of structural factors might be the root causes for the recent fall in real interest rates globally.[1] It is believed that slower population and technological growth rates, increasing inequalities, lower price of capital goods and increased demand for safe assets affect aggregate savings and investments in a way that tends to curtail the natural rate of interest i.e. the short-term real interest rate associated with a neutral stance of monetary policy.[2]

Inspired by secular stagnation literature, the purpose of this dissertation is two-fold. First, we aim to estimate the natural rate of interest (r^*) using the methodology developed by Holston et al. (2017), which we apply to an extended sample period that includes the decade that follows the beginning of the Great Recession. Being an unobservable measure of neutral monetary policy, the natural rate of interest must be estimated. To do so, we apply the Kalman filter approach to a sample of U.S., Canadian and British data that covers the last 60 years. Our estimates follow a steady downward trajectory throughout the sample in all countries, with sharp drops during the Great Recession. Consistent with contributions opting for different specifications and approaches, our estimates reach historically low levels in the last decade. However, our findings diverge from that of Holston et al. (2017) as we obtain increasing estimates of U.S. natural interest rates from 2013 onward.

Second, we explore potential explanations for the secular decline in the natural rates of interest.

[1]Secular stagnation remains a controversial topic where dissensions are vivid. As Eichengreen points out, finding a consensual definition for secular stagnation remains a challenge: "Secular stagnation, we have learned, is an economist's Rorschach Test. It means different things to different people." Whether secular stagnation is real or not is a challenge that is not direclty addressed in this paper.

[2]Hereafter, we use the terms "natural rate of interest", "neutral rate of interest" and "equilibrium real rate" interchangeably.

Indeed, we provide anecdotal evidence that r^* tends to move in a synchronized fashion with some of the structural factors suggested by the proponents of the secular stagnation hypothesis. We observe considerable negative comovement between r^*, income disparity and population aging. Particularly, the sharp drop in r^* amidst the Great Recession coincides with a greater than ever portion of the population entering retirement age around the year 2007.

The dissertation is organized as follows. Section 2 consists of a review of the literature on the natural rate of interest and its estimation. Section 3 presents the methodological approach and describes the state-space model. In Section 4, we discuss the data used in the estimation in addition to the data on demographic structure and economic disparities. The main results are reported in Section 5. In Section 6, we discuss the evolution of demographic and inequality trends in order to suggest the potential existence of a link between these structural factors and declining natural rates of interest. Section 7 concludes.

2 Literature Review

As explained previously, the natural rate of interest is an important concept in monetary economics because it can be interpreted as an anchor for monetary policy. Real rates below r^* would represent a situation of monetary expansion fostering inflation and output growth, and vice-versa (Woodford, 2003). The key difficulty however lies in the fact that we do not observe the natural rate of interest; it must be estimated. One could quite easily derive an estimate of r^* by taking the mean of real rates over a significantly long period if it was believe to be constant. Preferences and technology shocks however create time variation in r^*, hence the importance of more sophisticated estimation techniques. This has important monetary implications since policymakers have to forecast the level and path of r^* in order to implement effective inflation targeting measures. This is all the more true as the zero lower bound on nominal interest rates limits the effectiveness of traditional monetary policy tools.

There exist two main econometric approaches to estimate the natural rate of interest and they differ regarding the time horizon they focus on.[3] The first is usually carried out using dynamic stochastic

[3] There exists several more methods based on historical averages and Taylor rules, in addition to approaches based on a more financial markets perspective. These methods are not discussed in this paper. See Giammarioli and Valla (2004) for a comprehensive review.

general equilibrium (DSGE) models and focuses on the short-run aspect of the neutral interest rate. Albeit useful in forecasting the cyclical behavior of r^*, the vast majority of DSGE models abstract from permanent shocks and are less adequate in estimating trends in r^*. This is because these models typically define r^* as the period-by-period measure of the real rate that would prevail if all prices were perfectly flexible, thereby necessitating the use of detrended data. Nevertheless, one key advantage of deriving r^* from such structural models is the ability to identify the shocks that affect changes in the equilibrium real rate. For instance, Neiss and Nelson (2003) calibrate a small-scale DSGE model to the U.K. economy with the aim of assessing the response of the natural interest rate to technology and demand shocks. As a more recent example, Cúrdia et al. (2015) build their structural model to show that monetary feedback rules responding to what they call "the efficient real rate" fit the data better than traditional Taylor rules that respond to the output gap. Del Negro, Giannone, Giannoni and Tambalotti (2017) tackle the phenomenon of falling r_t^* using a medium-scale DSGE model that features nominal, real and financial frictions. Their work contributes in demonstrating that the natural interest rate has experienced a steady decline since 1980. The authors, however, attribute much of this decline to the increase of the premium for safe and liquid assets (also known as the *convenience yield*). They capture the safety and liquidity premia by comparing the trends in yields on securities that vary on their level of safety and liquidity. Owing to the short-term focus of the DSGE-based approach, our work is not inspired by this branch of literature.

The second general (and in our case, more adequate) approach focuses on the longer-term aspect of the equilibrium real rate. Pioneering this approach, Laubach and Williams (2003) were the first to document significant time variation in r^*. To put it differently, the natural rate of interest is assumed to be affected by low-frequency shocks with protracted effects. The authors jointly estimate the unobserved variables r^*, potential output (y^*) and its trend growth rate (g) through the Kalman filter. Identification is achieved using an output gap and an inflation equation that serve as IS and Phillips curves as well as observation equations in the state-space model. Based on economic theory, the Laubach-Williams (LW) model postulates that r^* is determined by g and by the component z that accounts for all other determinants of the natural interest rate. Despite inherently significant uncertainty in the estimates due in part to the number of unobserved variables that are jointly estimated, Laubach and Williams (2003) find that r^* experiences considerable variation and a downward trend over time.

As a consequence, a large body of work concentrated on applying variants of the LW methodology to different regions of the world, though mostly focusing on the United States. As illustrations, Manrique and Marqués (2004) apply the LW method to German data and Daníelsson et al. (2016) do the same to Iceland. Other researchers build on the initial framework by estimating less restrictive forms of the LW model. Mesonnier and Renne (2007) estimate r^* for the Euro area, assuming that the natural interest rate and potential output growth follow highly persistent but stationary processes whereas the original LW model assumes both variables to be non-stationary. A secular downward trajectory with recent record-low levels for r^* is a common finding to the majority of the papers applying this approach. This conclusion is reinforced in Laubach and Williams (2016) where the authors update their 2003 findings by feeding-in updated data to their original model as well as in Holston et al. (2017).

Before Laubach and Williams (2003) popularized the Kalman filter approach to estimate low-frequency movements in r^*, most research revolved around estimating the real potential GDP, neglecting the effects of interest rates. For instance, Watson (1986) models the output gap as following an AR process and assumes the trend growth rate of output g to be constant. Clark (1987) builds on Watson's model by dropping the time-invariant trend growth rate assumption. This is done by decomposing U.S. output data into a nonstationary trend and a stationary cycle component using the Kalman filter. Other papers deal with interest rates by assuming no relation between r^* and structural factors. By way of illustration, Enders and Siklos (2001) assume that real rates follow a GARCH process. Such models are not appropriate in our case since they lack any form of structural interpretation.

As described in Laubach and Williams (2016), it is possible to use univariate time-series techniques to isolate trend and short-term variations in real interest rates using, for instance, the Hodrick-Prescott (HP) or bandpass filter. However, these methods contain several flaws. First, in order to reflect changes in the r^*, the univariate approach requires that price and output dynamics remain stable. This is because this method does not control for inflation and output variations that can affect r^*. Second, this approach seems to mechanically assign extended periods of weak interest rates to the trend component (Hamilton et al., 2016).

Because the goal of this dissertation is to draw some parallels between the structural factors suggested by secular stagnation and declining natural rates of interest, we use the version of the LW model presented in Holston et al. (2017). We describe the model in greater detail in the next section.

3 Methodology

The empirical methodology we use for the estimation of the neutral rate of interest follows very closely that of Holston et al. (2017). Consequently, we adopt a similar definition of the natural rate of interest. Inspired by Wicksell (1936), the natural rate of interest r^* will hereby be defined as "the real short-term interest rate consistent with output equaling its natural rate and constant inflation". Moreover, we estimate the unobserved variables of potential output, trend growth rate of potential output and natural interest rate through the dynamics of the IS and Phillips curves. The first part of the following section presents the theoretical background of our approach, while the second part thoroughly describes the model that will be used in the estimation of the natural rate of interest for the United States, Canada and the United Kingdom.

3.1 Theoretical background

One can view the neoclassical growth model as an adequate starting point for our approach. Indeed, it provides us with our initial definition of r^*. The model suggests that, in the steady state, the natural rate of interest depends on household preferences and the growth rate of output per capita. Household intertemporal utility maximization yields:

$$r^* = \frac{1}{\sigma} g_c + \theta \tag{3.1}$$

where σ captures the intertemporal elasticity of substitution in consumption, g_c represents the steady-state growth rate of per capita consumption, and θ is the rate of time preference. Laubach and Williams argue that this equation is too simplistic and yields a restrictive definition of r^*. They instead assume that the natural rate of interest is a function of a time-varying growth rate of per capita output and some unobserved determinants that potentially include the rate of time preference.[4] Consequently, we posit that:

$$\begin{aligned} r_t^* &= \dot{g}_t + \bar{z}_t \\ &= g_t - \mu_t + \bar{z}_t \\ &= g_t + z_t \end{aligned} \tag{3.2}$$

where, $\bar{z}_t$ captures all determinants of r_t^* other than the trend growth rate of per capita output, $\dot{g}_t$. g_t is the trend growth rate of production, and μ_t is the trend growth rate of population. One can, therefore, interpret z_t as a linear combination of the trend population growth rate, the rate of time preference and all the other determinants of r_t^*. Moreover, we assume a one-for-one relationship between the trend growth rate of per capita output and the neutral rate of interest. This is analogous to assuming a coefficient of $\sigma = 1$ in Equation (3.1). Laubach and Williams (2003) estimate the relationship between the trend growth rate of output and natural rate of interest. They find a coefficient $\sigma \approx 1$. Thus, we consider the preceding assumption not to be overwhelmingly restrictive.

Laubach and Williams (2003) estimate r^* in a similar fashion with two different specifications for z_t. In one case, z_t follows an AR(2) process and in another case, it is I(1). Both specifications yield very similar results. Furthermore, the authors argue that the random-walk specification corresponds more closely to the low-frequency characterization of r^*. Consequently, we choose to set $z_t \sim$ I(1). We also assume that g_t follows a first-order random walk processes:

$$g_t = g_{t-1} + \epsilon_{g,t} \tag{3.3}$$

$$z_t = z_{t-1} + \epsilon_{z,t} \tag{3.4}$$

We model y_t^* as a random walk with stochastic drift $g \sim$ I(1).

$$y_t^* = y_{t-1}^* + g_{t-1} + \epsilon_{y^*,t} \tag{3.5}$$

$$= y_{t-1}^* + g_{t-2} + \epsilon_{g,t-1} + \epsilon_{y^*,t} \tag{3.6}$$

[4]This implicit underlying assumption is that the growth rate of y and c are highly correlated.

Log potential output is assumed to follow a second-order integrated process with $\epsilon_{y^*,t}$ having a permanent effect on the level of potential output, but only a contemporaneous effect on the rate of change of y_t^*. Shocks $\epsilon_{g,t}$ have a persistent effect on trend growth rate of potential output g_t.[5] Stock and Watson (1998) find evidence of a slow-moving nonstationary trend growth rate for the U.S. log real output over the post-WWII period.[6] We take an agnostic stance by assuming a second-order integrated process for log potential output. We do this for the purpose of estimation. Finally, we assume that $\epsilon_{y^*,t}$, $\epsilon_{g,t}$ and $\epsilon_{z,t}$ are all gaussian and independently distributed with standard deviation σ_{y^*}, σ_g and σ_z. The absence of serial correlation in these error terms is also assumed.

Because the data do not correspond to the long-run realization of economic variables, we need a specification that captures their cyclical variations. More specifically, we use reduced forms of IS and Phillips curves, taken from the standard New Keynesian framwork of (Galí, 2008), to model these short-term dynamics.[7] The following equations will serve as the basis for the observation equations of our state-space model:

$$\tilde{y}_t = E_t[\tilde{y}_{t+1}] - \sigma^{-1}(i_t - E_t[\pi_{t+1}] - r_t^*) \tag{3.7}$$

$$\pi_t = \beta E_t[\pi_{t+1}] + \kappa \tilde{y}_t \tag{3.8}$$

Equations (3.7) and (3.8) are the New Keynesian IS equation and Phillips curve, respectively. The output gap is denoted by $\tilde{y}_t$ and i_t is the short term risk-free nominal interest rate. Inflation is denoted by π_t. r_t^* represents the one-period natural interest rate. σ and κ are composite parameters that themselves depend on underlying structural parameters describing household preferences and technology.

The IS equation and Phillips curve used in our model are actually reduced and less restrictive forms of Equations (3.7) and (3.8). Using reduced-form IS and Phillips equations alleviates misspecification problems. We follow Laubach and Williams (2003) and let the output gap be determined by its first two lags. The authors demonstrate that, under such an assumption, the relationship between output gap and real rate gap is correctly specified. We thus estimate the following

[5] When substituting (3.3) in (3.5) we get (3.6). This clearly shows why the element (1,1) of covariance matrix **Q** is equal to $\sigma_g^2 + \sigma_{y^*}^2$.

[6] Using our sample, we reject the null hypothesis of $y_t \sim$ I(2). However, we find that log real output follows a first-order integrated process. See Table A.1 for all ADF-test results.

[7] An alternative interpretation of our approach is that we estimate θ by controlling for output and inflation dynamics. In this case, demographic and inequality shocks induce changes in preferences.

observation equations:

$$\tilde{y}_t = a_{y,1}\tilde{y}_{t-1} + a_{y,2}\tilde{y}_{t-2} + \frac{a_r}{2}\sum_{j=1}^{2}\left(r_{t-j} - r^*_{t-j}\right) + \epsilon_{\tilde{y},t} \tag{3.9}$$

$$\pi_t = b_\pi \pi_{t-1} + (1 - b_\pi)\pi_{t-2,4} + b_y \tilde{y}_{t-1} + \epsilon_{\pi,t} \tag{3.10}$$

Here, $\tilde{y}_t$ is the output gap which equals to $100 \times (y_t - y^*_t)$, where y_t and y^*_t are the logarithms of real output and potential output, respectively. The short-run real interest gap is captured by $(r_{t-j} - r^*_{t-j})$ with r_t being the short-term real interest rate and r^*_t being the real interest rate that is consistent with neutral inflation and potential real GDP growth i.e. the natural rate of interest. Consumer price inflation is denoted by π_t, and $\pi_{t-2,4}$ is the mean of the second to fourth lag of inflation.

Error terms $\epsilon_{\tilde{y},t}$ and $\epsilon_{\pi,t}$ allow us to capture low-frequency movements in r^*. These stochastic error terms capture shocks specific to the output gap and inflation and are useful in modeling persistent changes in the natural rate of interest (Williams, 2003). Equations (3.9)-(3.10) constitute the observation equations of our Kalman filter model.[8]

3.2 State-space model and estimation method

We now describe the state-space model used as input in the Kalman filter process and its initialization. The subsequent portion of the section presents the estimation method used in the paper.

3.2.1 State-space model

We use a notation similar to that of Hamilton (1994). Our dynamic system can be represented by the following system of linear equations:

$$\xi_t = \mathbf{F} \cdot \xi_{t-1} + \mathbf{v}_t \tag{3.11}$$

$$\mathbf{y}_t = \mathbf{A}' \cdot x_t + \mathbf{H}' \cdot \xi_t + \mathbf{w}_t \tag{3.12}$$

Equation (3.11) is the state equation and (3.12) is known as the observation equation. ξ_t is a $(r \times 1)$ vector of potentially unobserved states comprised of contemporaneous and lagged values of

[8]Section 3.2 describes the state-space model in detail.

potential output, the trend growth rate and z_t. $\mathbf{y}_t$ is a $(n \times 1)$ vector containing the endogenous variables (real output and inflation). x_t is a $(k \times 1)$ vector of exogenous variables. Matrices **F**, **Q**, **A**, **H** and **R** are time-invariant parameter matrices.

$$\xi_t = [y_t^*, y_{t-1}^*, y_{t-2}^*, g_{t-1}, g_{t-2}, z_{t-1}, z_{t-2}]' \tag{3.13}$$

$$\mathbf{y}_t = [y_t, \pi_t]' \tag{3.14}$$

$$x_t = [y_{t-1}, y_{t-2}, r_{t-1}, r_{t-2}, \pi_{t-1}, \pi_{t-2}, \pi_{t-2,4}]' \tag{3.15}$$

$$\mathbf{v}_t = \begin{bmatrix} \epsilon_{y^*,t} \\ 0 \\ 0 \\ \epsilon_{g,t} \\ 0 \\ \epsilon_{z,t} \\ 0 \end{bmatrix} \tag{3.16}$$

$$\mathbf{w}_t = \begin{bmatrix} \epsilon_{\tilde{y},t} \\ \epsilon_{\pi,t} \end{bmatrix} \tag{3.17}$$

$$\mathbf{H}' = \begin{bmatrix} 1 & -a_{y,1} & -a_{y,2} & \frac{-a_r}{2} & \frac{-a_r}{2} & \frac{-a_r}{2} & \frac{-a_r}{2} \\ 0 & -b_y & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.18}$$

$$\mathbf{A}' = \begin{bmatrix} a_{y,1} & a_{y,2} & \frac{a_r}{2} & \frac{a_r}{2} & 0 & 0 \\ b_y & 0 & 0 & 0 & b_\pi & 1-b_\pi \end{bmatrix} \tag{3.19}$$

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{3.20}$$

$$\mathbf{Q} = \begin{bmatrix} \sigma_g^2 + \sigma_{y^*}^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_g^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_z^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.21}$$

$$\mathbf{R} = \begin{bmatrix} \sigma_{\tilde{y}}^2 & 0 \\ 0 & \sigma_\pi^2 \end{bmatrix} \tag{3.22}$$

Furthermore, $\mathbf{v}_t$ and $\mathbf{w}_t$ are vectors of error terms satisfying:

$$E(\mathbf{v}_t\mathbf{v}_\tau') = \begin{cases} \underset{(r\times r)}{\mathbf{Q}} & \text{if } t = \tau \\ 0 & \text{if otherwise} \end{cases} \tag{3.23}$$

$$E(\mathbf{w}_t\mathbf{w}_\tau') = \begin{cases} \underset{(n\times n)}{\mathbf{R}} & \text{if } t = \tau \\ 0 & \text{if otherwise} \end{cases} \tag{3.24}$$

$$E(\mathbf{v}_t\mathbf{w}_\tau') = 0 \qquad \forall\ t \text{ and } \tau \tag{3.25}$$

We assume that $\mathbf{v}_t$ and $\mathbf{w}_t$ are mutually and serially uncorrelated and follow a Gaussian distribution centered at zero with covariance matrices $\mathbf{Q}$ and $\mathbf{R}$.

3.2.2 Estimation method

To start the Kalman iterative process, we use our best guess for the vector of unobserved variables and the matrix of forecasting errors:

$$\hat{\xi}_{1|0} = \mathbb{E}(\xi_1) \tag{3.26}$$

$$\mathbf{P}_{1|0} = \mathbb{E}\left\{[\xi_1 - \mathbb{E}(\xi_1)][\xi_1 - \mathbb{E}(\xi_1)]'\right\} \tag{3.27}$$

The initialization of matrix $\hat{\xi}_{1|0}$ is done by populating it with trend estimates of real GDP. To do so, we apply the HP filter, with $\lambda = 360000$, to y_t and obtain y_2^*, y_1^* and y_0^*. For the next two elements of $\hat{\xi}_{1|0}$, we take the first difference of y_t^* and use it for g_1 and g_0. We set the starting values of unobserved z_t in matrix $\hat{\xi}_{1|0}$ equal to zero. Using the initial values, we construct the mean squared error matrix $\mathbf{P}_{1|0}$.

The estimation of our model's parameters is done through maximum likelihood. Our vector of estimated parameters is:

$$\boldsymbol{\theta} = \left[a_{y,1}, a_{y,2}, a_r, b_\pi, b_y, \sigma_{\tilde{y}}, \sigma_\pi, \sigma_{y^*}, \sigma_g, \sigma_z\right]$$

Because $\mathbf{v}_t$ and $\mathbf{w}_t$ are Gaussian:

$$\mathbf{y}_t|\mathbf{x}_t, \mathbf{Y}_{t-1} \sim \mathrm{N}\left((\mathbf{A}'\mathbf{x}_t + \mathbf{H}'\hat{\xi}_{t|t-1}), (\mathbf{H}'\mathbf{P}_{t|t-1}\mathbf{H} + \mathbf{R})\right) \tag{3.28}$$

where, $\mathbf{Y}_{t-1} = (\mathbf{y}'_{t-1}, \mathbf{y}'_{t-2}, ..., \mathbf{y}'_1, \mathbf{x}'_{t-1}, \mathbf{x}'_{t-2}, ..., \mathbf{x}'_1)$.
Consequently, the likelihood function that we construct iteratively is[9]:

$$\begin{aligned} f(\mathbf{y}_t|\mathbf{x}_t, \mathbf{Y}_{t-1}) = (2\pi)^{-n/2}|\mathbf{H}'\mathbf{P}_{t|t-1}\mathbf{H} + \mathbf{R}|^{-1/2} \\ \times \exp\left[-\tfrac{1}{2}(\mathbf{y}_t - \mathbf{A}'\mathbf{x}_t - \mathbf{H}'\hat{\xi}_{t|t-1})'(\mathbf{H}'\mathbf{P}_{t|t-1}\mathbf{H} + \mathbf{R})^{-1}(\mathbf{y}_t - \mathbf{A}'\mathbf{x}_t - \mathbf{H}'\hat{\xi}_{t|t-1})\right] \end{aligned} \tag{3.29}$$

for $t = 1, 2, ..., T$ and n dimensions of matrix $\mathbf{y}_t$. We impose constraints on the slopes of the IS and Phillips curve. We set $a_r < -0.0025$ and $b_y > 0.025$ to simplify numerical convergence. We carry OLS estimations of the IS equation (3.9) and Phillips curve (3.10) individually in order to obtain provisional estimates of $a_{y,1}$, $a_{y,2}$, a_r, b_π, b_y, $\sigma_{\tilde{y}}$ and σ_π. We then populate $\boldsymbol{\theta}^{(0)}$ with these starting values for the maximum likelihood estimation of our state-space model. In order to compute standard errors for our estimates of state variables, we follow the Monte Carlo procedure presented in Hamilton (1986).

4 Data

In this section, we describe the raw data used to estimate the unobserved neutral rate of interest as well as the data on demographic structure and economic inequalities for each economy. We discuss the preparation and manipulation of the data used to fit our model. Detailed information on the correction of irregularities and errors in the raw data is also provided when needed.

4.1 Raw data used in the estimation of the natural rate

Our model is estimated on three industrialized economies i.e. the United States, Canada and the United Kingdom. Measures of output, the real interest rate and inflation are needed for the estimation of the unobserved variables through the Kalman filter. Our raw data consist of quarterly data on output, nominal short-term interest rates and consumer price indices. We define the *ex-*

[9] Numerical optimization is carried out with a variant of an L-BFGS algorithm which belongs in the quasi-Newton class. The estimation is done on `R` using the optimization package `nloptr`.

ante short-term real interest rate as the difference between the short-term nominal interest rate and *ex-ante* inflation expectation. The latter is constructed as the average of the current value to the third lag of inflation.[10] All inflation measures are constructed as the annualized quarterly growth rate of consumer price indices and interest rates are computed on a 365-day annualized basis. Ending dates of estimation vary across countries because of the availability of raw data. However, estimations always start on 1961:I for each country, four periods after the sample starting date. For the United States, our series cover the period 1961:I to 2017:IV, whereas for Canada and the United Kingdom, the sample spans the period 1961:I to 2017:I.

United States

We use the personal consumption expenditures (PCE) index excluding food and energy to measure the U.S. price level. We obtain data on real GDP and core PCE from the Bureau of Economic Analysis (BEA). Prior to 1965, we use the International Monetary Fund's (IMF) International Financial Statistics (IFS) database to get measures of the New York Federal Reserve Bank's discount rate to use as short-term nominal interest rate. After 1965, we use the federal funds rate that is available from the Board of Governors. We do so because the federal funds rate regularly fell under the discount rate before 1965. All data on real GDP and price levels are seasonally adjusted by the publishing bodies.

Canada

Canadian inflation is computed as the quarterly growth rate of the Bank of Canada's (BoC) core CPI. Data unavailability forces us to use the BoC's CPI containing all items before 1984. Measures of Canadian short-term nominal interest rates consist of the BoC's bank rate for the period before 2001, and of the overnight rate for the period after 2001. Real GDP data can be found in the IMF's IFS database. All other data are taken from Statistics Canada. Furthermore, all real GDP and price levels are seasonally adjusted by the publishing bodies.

[10] Inflation expectation can thus be expressed by the equation: $\pi_t^e = \frac{\pi_t + \pi_{t-1} + \pi_{t-2} + \pi_{t-3}}{4}$

United Kingdom

For the United Kingdom, price level data are constructed by splicing together the OECD's data on all-item CPI from 1960 to 1970 and core CPI from 1970 to 2017. CPI values from the OECD are not seasonally adjusted. We thus need to manually deseasonalize them.[11] We also use seasonally adjusted real GDP data that we retrieved from the Office of National Statistics' (ONS) website. Our measure of nominal short-term interest rate consists of the Bank of England's Official Bank Rate.

4.2 Demographic and income distribution data

Quarterly population data for the U.S. and Canada are taken from the BEA and Statistics Canada, respectively. Only annual population estimates are available for the United Kingdom. Consequently, we convert the annual data to a quarterly basis through quadratic interpolation following Forstythe, Malcolm and Moler (1977). Quarterly growth rates of population are then computed for all countries following:

$$\mu_t = 400 \times \frac{\log n_t}{\log n_{t-1}} \tag{4.1}$$

where n_t is population at time t.[12] We then apply an HP filter to μ_t in order to smooth the series. Various adjustments to the smoothened series were necessary to reduce the impact of eccentric values. These outliers are often the products of changes in the computing method of the raw data by the different official statistical institutions. For Canada, quarterly estimates of population are intercensal and unadjusted for census net undercoverage before 1971:III. From the third quarter of 1971 onward, all estimates are adjusted for census net undercoverage. The first of these adjustments is retroactive. Consequently, there is a significant increase in the population estimate for 1971:III that captures the net undercoverage of the census for all periods preceding that date.[13] This results in a one-period abnormally high value of quarterly population growth. To get around this problem, we interpolate quarterly population for 1971:III by replacing it with the average of population growth for 1971:II and 1971:IV.

[11] Seasonal adjustment is carried out by applying the Census Bureau's `X-13ARIMA-SEATS` procedure through the `R` package `seasonal`.

[12] Figures for raw population data are available in the Appendix.

[13] Canadian quarterly population estimates are taken form Statistics Canada's Table 051-0005 *Estimates of population, Canada, provinces and territories*.

All annual demographic data on life expectancy, fertility rate, age-group population and dependency ratios used hereafter are published by the World Bank. To illustrate the growing concentration of economic resources at the top of the distribution, we use the World Inequality Database's (WID) data on annual pre-tax income and net-wealth share distributions.[14] Pre-tax income comprises pre-tax labor, capital and pension income. Net wealth is the difference between assets (financial and non-financial) and debt.

5 Estimation Results

In the following section, we present the estimation results of our state-space model. We first report parameter estimates found by maximum likelihood. Filtered series of the unobserved variables (output gap, real rate gap and trend growth rate) are detailed in the second part of this section.[15] We conclude the section by discussing the estimates of the natural interest rate for all three countries. For the United States, the estimated period counts 228 quarters starting in 1961:I and ending in 2017:IV. For Canada and the United Kingdom, the estimation starts on the same date as the U.S. but ends three quarters earlier on 2017:I, totaling 225 quarters.

5.1 Parameter estimates

Table 1 reports parameter estimates obtained through the Kalman filter process. For the United States, estimation results of Σa_y and b_π seem to suggest that both output gap and inflation follow significantly persistent processes. Indeed, both parameters are greater in term of magnitude for the U.S. than for any other economies. Furthermore, the statistical significance as well as the fairly important size of slope coefficients a_r and b_y are evidence that both the IS and the Philips curve are reasonably well identified. However, owing to both the considerable filter uncertainty and parameter uncertainty in estimating our model through the Kalman filter, the statistical significance of coefficients a_r and b_y do not translate into accurate estimates of the natural rate of interest and potential output. As seen in the bottom part of Table 1, average standard errors for r^* and y^* are

[14] Linear interpolation was used to solve the problem of missing values for three non-consecutive periods for U.K. top 1 per cent income share. Our income disparity data end in 2010 for Canada and in 2014 for the United States and United Kingdom.

[15] The filtered (one-sided) estimate is the forecast of the state vector ξ_t conditional on previous observations; $\hat{\xi}_{t|t-1} = \mathbb{E}(\xi_t|\mathbf{y}_{t-1})$. On the other hand, the smoothed (two-sided) estimate is the forecast based on the full sample; $\hat{\xi}_{t|T} = \mathbb{E}(\xi_t|\mathbf{y}_T)$.

Table 1: Parameter Estimates

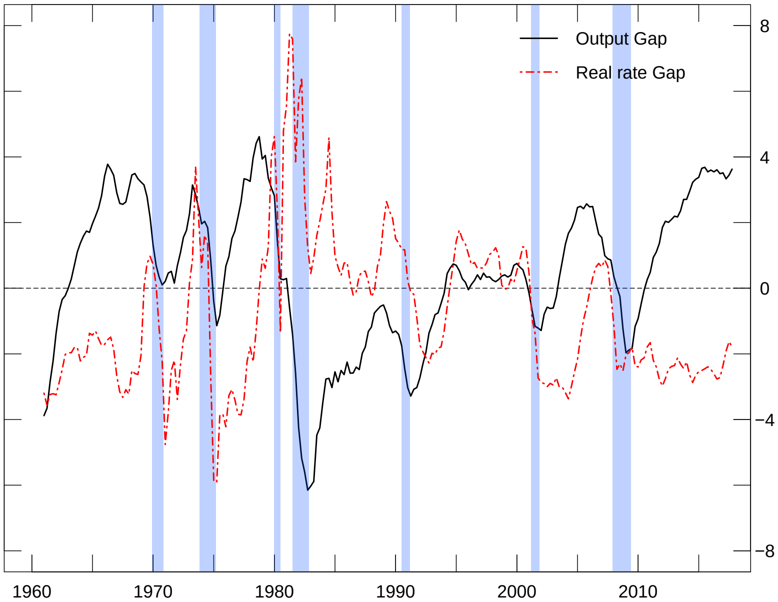
Parameter	United States	Canada	United Kingdom
Last quarter of estimation	2017:IV	2017:I	2017:I
Σa_y	0.942	0.939	0.908
$a_{y,1}$	1.446***	1.518***	1.863***
$a_{y,2}$	−0.504***	−0.579***	−0.956***
a_r	−0.082***	−0.067***	−0.003
b_π	0.664***	0.444***	0.567***
b_y	0.075***	0.025	0.630**
$\sigma_{\tilde{y}}$	0.415***	0.314***	0.084**
σ_π	0.793***	1.478***	2.720***
σ_{y*}	0.524***	0.638***	0.877***
σ_g	0.045**	0.035	0.016
σ_z	0.015	0.212	0.519
$\sigma_{r^*} = \sqrt{\sigma_g^2 + \sigma_z^2}$	0.047	0.215	0.519
Average Standard Error (%)			
r^*	2.067	4.779	17.705
g	0.536	0.643	0.470
y^*	1.948	3.884	1.394

Notes: σ_g is presented at an annual rate. *: significance at 90% confidence level. **: significance at 95% confidence level. ***: significance at 99% confidence level.

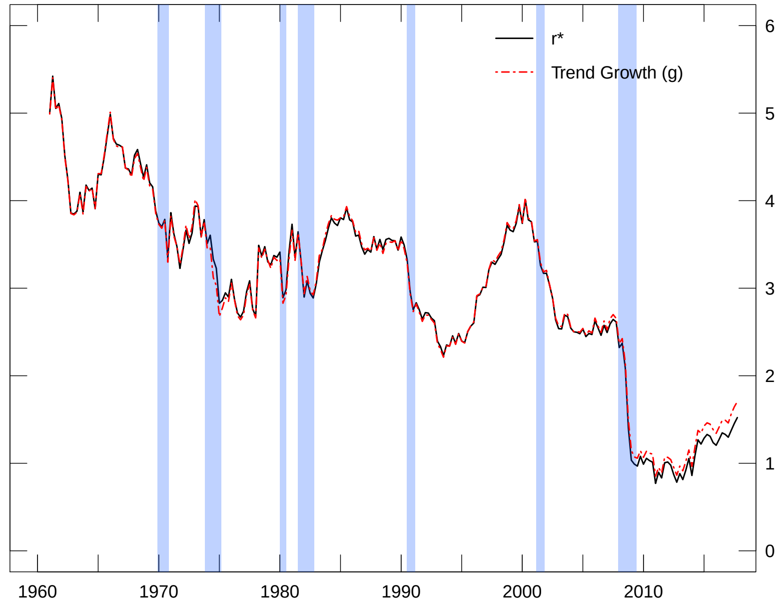
both close to 2 per cent.

In Canada, estimates of b_π suggest that inflation is less persistent than in any other economies. Furthermore, the estimate of Σa_y suggests that U.S. and Canadian output gaps follow almost equally persistent processes. The slope of the IS curve a_r is also similar in size to that of the United States and is precisely estimated. This is not the case for the slope of the Philips curve. Our estimated b_y is low and far from statistical significance at the 95% confidence level. Consequently, estimates of r^* and y^* are also imprecise with sample average standard deviations of 4.8 and 3.9 per cent, respectively.

In the case of the United Kingdom, output gap dynamics also display high persistence though not to the extent estimated for the U.S. and Canada. Inflation is found to be very strongly responsive to the output gap. Parameter b_y for the U.K. is almost 10 times as large as for the United States. Our estimation results seem to demonstrate a very weak and statistically insignificant relation between the output gap and the real rate gap, as shown by a_r. This imprecision in estimating a_r is



(a) Output Gap and Real Rate gap.



(b) Natural Rate of Interest and Trend Growth Rate.

Figure 1: Estimation Results for the United States.

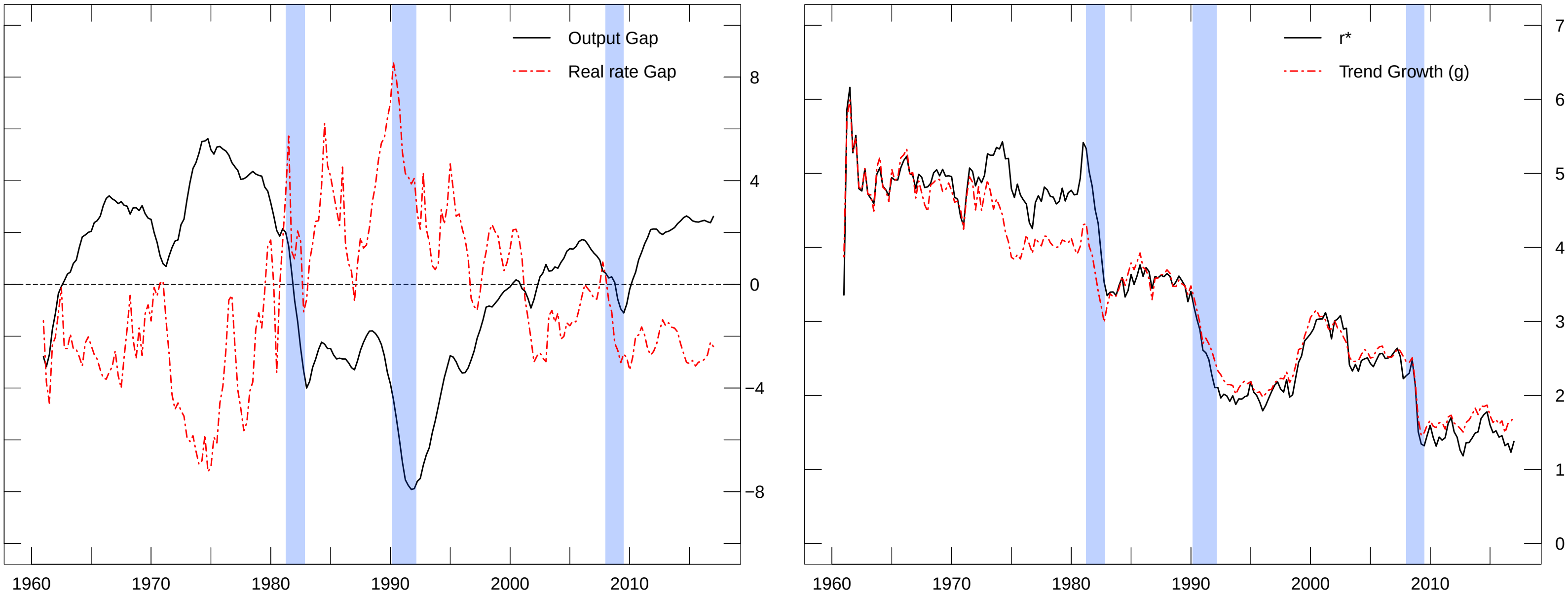
ultimately reflected in the extremely large sample average standard error of 17.7 per cent for r^*.

5.2 Estimation results for the output gap, real rate gap and trend growth rate

Figures 1a, 2a and 3a illustrate filtered estimates of output gap $\tilde{y}_t$, generated by the Kalman filter for all three countries. One-sided estimates represent imperfect approximations of a policymaker's real-time estimates. This is for two reasons. First, model parameters are estimated using not only current and past observations but the full sample. Second, state vectors are forecasted conditional on past and present observations. Filtered estimates are thus less subject to data revision than would be a "real" policymaker's estimation.[16] Blue-shaded regions span recessions from peak to trough. Beginning and end dates for recessions are taken from the NBER for the United States and from the Economic Cycle Research Institute for Canada and the United Kingdom.

The dynamics of output gap obtained through the estimation of the recursive process suggest that our model captures business cycle movements reasonably well. One can easily notice that, in most cases, large negative output gaps coincide with recessions. Notwithstanding the variations in the size of the gaps and in the subsequent recoveries, all countries experienced a steep drop in $\tilde{y}_t$ dur-

[16] Laubach and Williams (2003, 2016) estimate a state-space model akin to ours and present the filtered and smoothed estimates of r^*. The authors observe a similar downward trend in both cases.



(a) Output Gap and Real Rate Gap.

(b) Natural Rate of Interest and Trend Growth Rate.

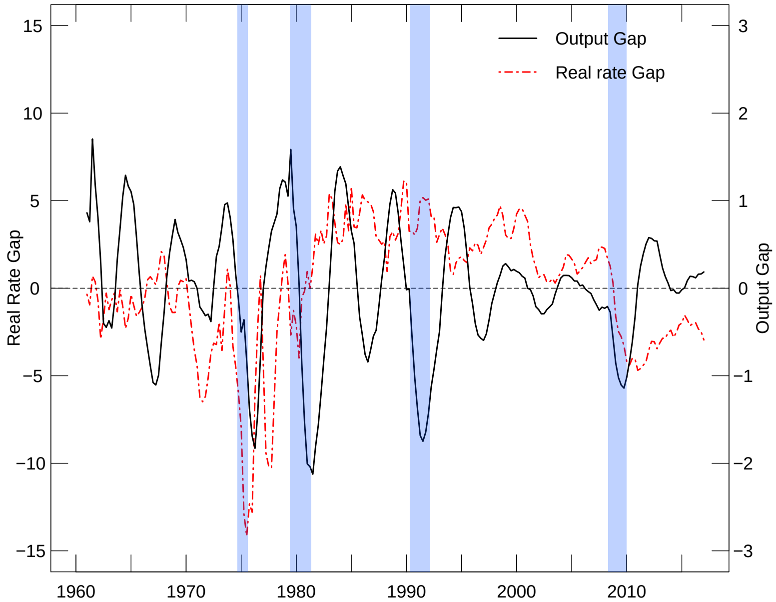
Figure 2: Estimation Results for Canada.

ing the Great Recession of 2008. Filtered estimations show that Canada experienced the shortest period of economic contraction with only five quarters of negative output gap. In contrast, U.S. and U.K. output gap took nine quarters to return to their levels of 2008:III. Furthermore, Canadian output gap did not go as far in negative territory as did its counterparts. It reached a post-2000 minimum of -1.10 per cent versus -1.97 and -1.14 per cent for the U.S. and U.K., respectively. Periods of negative output gaps also followed the first and second energy crises of the early 1970s and 1980s, as well as the dot-com bubble in the United States. The same is true for the Canadian recessions of the early 1980s and 1990s, in addition to the stagflation years in the United Kingdom during the 1980s.

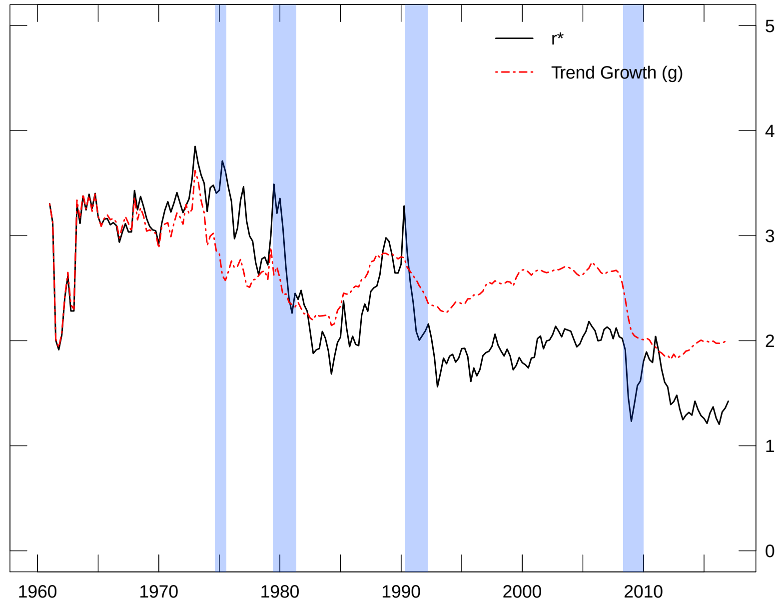
The same figures show the evolution of our estimates of the real rate gap throughout the same time frame. The real rate gap, $(r_t - r_t^*)$, is defined as the difference between the *ex ante* real interest rate and the estimate of natural interest rate.[17] As expected, periods of negative real rate gap seem to precede periods of economic boom for all countries. Conversely, periods of positive real rate gap coincide with restrictive monetary policy and are generally followed by economic slowdowns, as

[17]The *ex ante* real interest rate is itself defined as the difference between the nominal interest and the four-period average inflation.

$$r_t = i_t - \pi_t^e = i_t - \frac{\sum_{j=0}^{3} \pi_{t-j}}{4}$$



(a) Output Gap and Real Rate Gap.



(b) Natural Rate of Interest and Trend Growth Rate.

Figure 3: Estimation Results for the United Kingdom.

is the case for the period of high inflation in the United States in the 1980s. At the dawn of the 2007 recession, we can see monetary policy going from restrictive to expansionary as our estimates of the real rate gap move from positive to negative territory.[18] We also capture, in Figure B.12, the multiple hikes in U.S. nominal interest rate since 2016 towards the end of our estimation sample. From this figure, we can also see that recent increases in Canadian real rate gap are a result of weaker inflation as nominal rates of interest are constant and we see only tepid movements in r_t^*. Figures 1b, 2b and 3b show one-sided estimates of trend potential output growth g_t. Estimates for specific years are also reported on Table 2. These figures illustrate the persistent slowdown in productivity growth experienced by most advanced economies since the beginning of the post-war era. We draw the same conclusions as Holston et al. (2017) concerning estimates of g_t. First, since 1961, we witness a somewhat steady decline in the trend growth rate of potential GDP for all countries with a modest increase around the year 2000 reflecting the transitory impact of innovations in computer technology on total factor productivity. Gordon (2016) documents trend productivity slowdown in the United States and contends that the U.S. economy faces so-called headwinds such as the demographic transition, increasing income disparities, an underperforming educational system and a growing debt to GDP ratio. Gordon postulates that persistent low

[18] Figure B.12 in the Appendix shows movements in nominal, real and natural interest rates as well as the real rate gaps.

rates of growth are to be expected for the foreseeable future. Second, the Great Recession appears to have had a persistent and important effect on g_t. The period between 2007 and 2009 is marked with a sharp decline of about 1.0 percentage point in all three economies.

5.3 Estimates of the natural rate of interest

Filtered estimates of the natural rate of interest are shown in Figures 1b, 2b and 3b. Table 2 reports estimated values of the annual natural rate of interest for all countries for various years. We can see that periods of economic downturns have a permanent and negative impact on r^*. In between recessions, estimates remain relatively stable. A shared steady fall in natural interest rates is nonetheless clearly observable throughout the sample for all three economies. Table 2 shows that estimates of r^* were at there peak at the very beginning of our sample and hovered around roughly 3.0 per cent by 1990 in all countries. By 2007, all countries saw their natural rate of interest decrease to values ranging from about 2 to 2.5 per cent, with the U.S. and Canada experiencing the most important drops during the crisis. The right-hand side of the Table 2 shows that in the United States, the decline in trend growth rate of potential GDP accounts for virtually all of the decline in r^* between 1990 and 2007. In the U.K. however, the decline in g_t contributes to the fall of r^* to a much lesser extent. In Canada, the slump in estimated trend growth rate more than fully accounts for the fall in natural interest rate.

Table 2: Estimates of the Natural Rate of Interest and Trend Growth Rate of Potential Output

							Change	
	1965	1980	1990	2000	2007	2017	1990-2007	2007-2017
United States								
r_t^*	4.5	3.2	3.3	3.8	2.6	1.4	-0.8	-1.2
g_t	4.5	3.1	3.3	3.8	2.6	1.6	-0.7	-1.0
Canada								
r_t^*	5.0	4.8	3.1	2.9	2.5	1.4	-0.6	-1.1
g_t	5.0	4.0	3.2	3.1	2.6	1.7	-0.7	-0.9
United Kingdom								
r_t^*	3.1	2.9	2.9	1.8	2.1	1.4	-0.8	-0.7
g_t	3.2	2.5	2.7	2.7	2.7	2.0	-0.1	-0.7

Notes: Annual estimates are averages of quarterly estimates - Canadian and English values for 2017 consist of estimates for 2017:I.

From 2007 to 2017, natural rates of interest follow quite different paths in each economy. However, all economies initially experienced a sharp drop of about a percentage point in r^* as an immediate consequence of the Great Recession. In the U.S., a period of somewhat stagnating natural rates followed. By 2013, U.S. r^* started to increase again, stabilizing at around 1.5 per cent in 2017. On the other hand, estimates of the Canadian neutral interest rate hovered around the 1.4 per cent mark since 2008, experiencing only tepid variations. Right after the crisis, the U.K. saw its natural rate of interest rapidly reach back its pre-crisis level. Howbeit, this upward movement was short-lived as the following Euro crisis caused yet another sharp decline in the British natural rate, maintaining r^* around 1.4 per cent since then. In short, during the last decade all economies saw their natural interest rate fall by about 1 percentage point. Most if not all of the decline can be accounted for by the drop in the trend potential growth rate.

5.4 Comparing our estimates of r^*

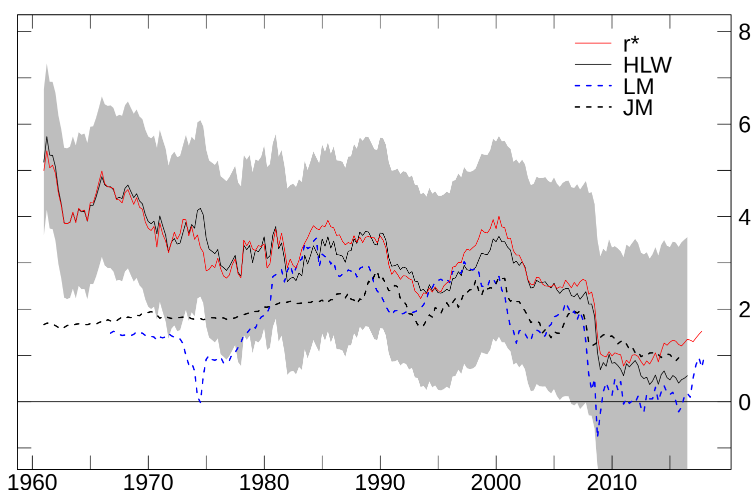


Figure 4: Comparison of estimates of U.S. r^*. *Notes:* The black line denoted by "HLW" is the estimate taken from Holston et al. (2017) with the grey-shaded area representing the 95% confidence interval. The blue (black) dashed line denoted by "LM" ("JM") is the estimate taken from Lubik and Matthes (2015) (Johannsen and Mertens (2016)).

Despite substantial imprecision in estimating r^*, a large number of papers using a broad range of approaches find strong evidence of a steady decline in natural rates of interest since 1980. Figure 4 compares our estimates of U.S. r^* to those of various authors. As shown in Figures 4 and B.1, we find estimates of r^* that are very close to that found by Holston et al. (2017). Per contra, the authors do not find upward movement in U.S. r^* after the Great Recession. By allowing for parameter time-variation in the LW method, Lubik and Matthes (2015) find natural interest rate estimates that are almost always lower than ours but move in similar fashion in the last 40 years.[19] They also find a more pronounced drop during the global crisis of 2008 and a steeper, more recent surge in r^*. Furthermore, Johannsen and Mertens (2016) estimate r^* by explicitly accounting for the effective lower bound on nominal interest and integrating yield curve data in their model. Notwithstanding weaker variation in their estimates, their model forecasts also capture the significant fall in long-term real rates. Lewis and Vazquez-Grande (2017) estimate several variations of the LW model with Bayesian methods and loose priors on z and g. They find a less pronounced secular decline in r^*.

In summary, despite the observed heterogeneity in the paths of r^* across the three countries under consideration, we detect a common downward trend that is also documented by earlier works. This persistent trend is also more pronounced in the last 25 years of our sample.

6 Demographics and Inequalities

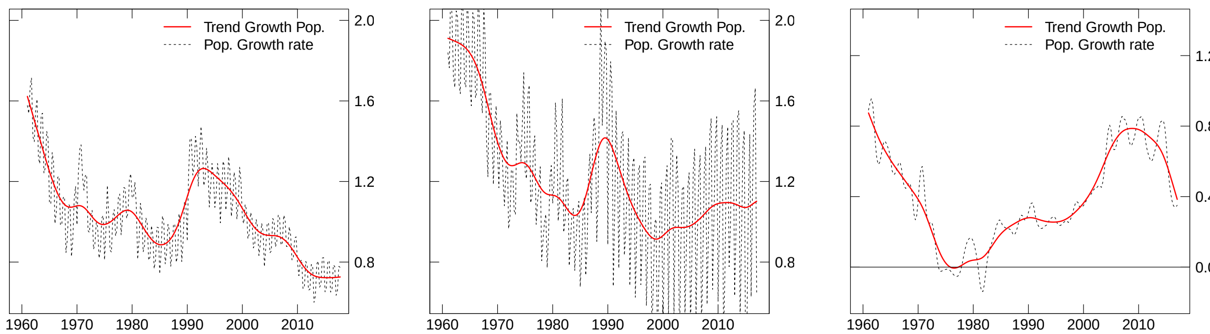
In the remainder of the dissertation, we discuss two of the factors that are, according to literature, potential sources of downward pressure on the natural rate of interest. In particular, we document changes in the demographic structure and distribution of economic resources in all economies, spanning the whole estimation period. We then present anecdotal evidence on a potential relationship between these structural changes and persistently low estimates of natural interest rates.

6.1 Measures on demographic trends and economic inequalities

Demographics

In virtually all countries, improvements in standards of living and advancements in healthcare technology have led to significantly longer lives and fewer children per women in the past 55 years. Figure B.4 in the Appendix shows life expectancy and fertility rate for various countries between 1960 and 2017. Since 1960, life expectancy has increased by more than 10 years and the number of births per women has fallen from about 3.2 to around 1.7 in the OECD. With rapid industrialization, we can also see emerging economies catch-up more advanced countries as the

[19]The estimates we present from Lubik and Matthes (2015) are actually updated estimates published on the Richmond Federal Reserve and may differ from the original paper.



(a) United States **(b)** Canada **(c)** United Kingdom

Figure 5: Quarterly Population Growth Rate

gaps that separate them shrink, despite persisting higher levels in both measures for less advanced countries.[20] While relatively long lifespans and low birth rates are nothing new for the United States, Canada and the United Kingdom in the past quarter of century, the consequences of these demographic changes are being felt to a greater degree today than before. Most industrialized economies are now facing the problem of populations that are both growing slower than they used to and aging much faster. We discuss how the effects of the demographic transition are now more deeply felt as the older generations leave the labor force, making place to a smaller younger generation of workers.

Figure 5 shows annualized quarterly population growth rates as well as the underlying trends for all three countries.[21] Table 3 reports trend population growth for the years 1965, 1980, 1990, 2000, 2007 and 2017 along with changes that occur between these periods. The United States and Canada display quite similar patterns of trend quarterly population growth from 1961 to 2017 whereas the United Kingdom experiences much less variations. Trend quarterly population growth rate ranges from zero to one per cent in the U.K. during that period.

For the first 20 years of the sample, population growth across all three countries shows a steep decline that is largely explained by falling fertility rates due to increased female participation into the labor force. The right panel of Figure B.4 depicts the fall in births per women during that period

[20] In Figure B.4, we present data on India and Brazil to show that the catch-up happens very quickly. Data on less industrialized regions such as Sub-Saharian countries could have also been used to show that this is a global trend.

[21] As described in Section 4, Canadian quarterly population growth was adjusted in 1971:III and U.K. quarterly population data was interpolated from annual data published by the ONS. The computation of population trend quarterly growth rates is also detailed in the same section.

Table 3: Trend Quarterly Growth Rate of Population

	1965	1980	1990	2000	2007	2017	Change 1990-2007	Change 2007-2017	Change 1965-2017
United States	1.22	1.04	1.16	1.05	0.92	0.73	-0.24	-0.19	-0.49
Canada	1.82	1.13	1.39	0.94	1.03	1.10	-0.36	0.07	-0.72
United Kingdom	0.59	0.04	0.28	0.38	0.78	0.38	0.50	-0.39	-0.20

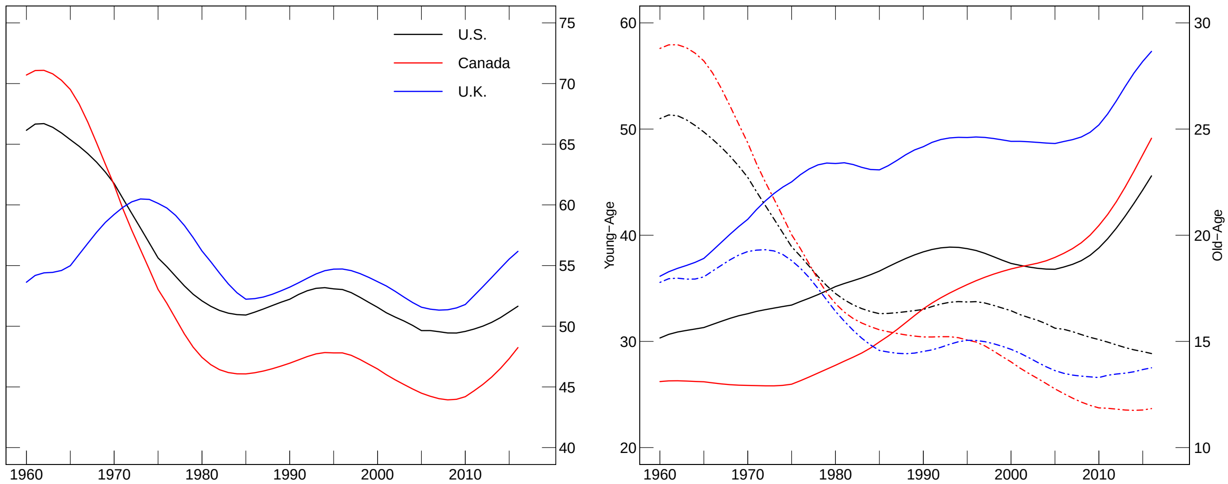
Notes: Annual rates of population growth are averages of quarterly trend population growth rates. Canadian and British rates for 2017 are quarterly growth rates for 2017:I.

and Figure B.3 displays increased female participation in the labor force for our three countries. From 1961 to 1981, the fall of population growth is more pronounced in the United Kingdom with a decrease of 0.80 per cent compared to 0.55 and 0.78 per cent for the United States and Canada, respectively, despite British fertility rate experiencing the smallest drop of all three economies during the same period.

From the mid-1980s to the early 1990s, all three economies share an observable acceleration in population growth rate. In the U.S. and Canada, these increased rates of population growth are consistent with the higher fertility rates of the 1990s. Indeed, Canada reaches its post-1970 peak population growth rate in 1989 with 1.41 per cent and the United States attains its peak three years later with 1.26 per cent. These higher rates of population increase are however short-lived for both countries. In fact, we witness population growth returning to its level of 1980 around the end of the millennium. A slowdown also characterizes U.K. around 1995 as the speed of population growth stabilizes around $\frac{1}{4}$ for some time.

Since the beginning of the 21st century, we can observe a surge in Canadian trend population growth, which reaches 1.1 per cent in 2017:I. Indeed, as of 2016, Canada had the fastest growing population of the G7 (Statistics Canada, 2017). Canada's faster growing population does not however immunize it against the impact of its aging population. Conversely, the U.S. has experienced a sharp decline of 0.32 percentage point between 2000 and 2017. In the United Kingdom, trend population growth has risen from 0.38 per cent in 2000 to 0.78 per cent in 2007 and has fallen back again to 0.38 per cent by 2017.

Of course, structural demographic changes encompasses more that mere population growth rate. To illustrate the importance of the changes in the age structure of the different countries, we use



(a) Total

(b) Old-age (full line) and young-age (dashed line)

Figure 6: Dependency Ratios in per cent

dependency ratios that we take from the World Bank. The dependency ratio is the quotient of the population that is usually not in the labor force (between 0 and 14 years old and over 65) over the proportion that usually is (between 15 an 64 years old). It can be interpreted as the number of *dependents* per 100 working-age population, or *non-depends*. A higher dependency ratio generates additional strain on workers to produce for those who require resources but are economically inactive. In our case, we present all three variants of the dependency ratio (total, old-age and young-age dependency ratios) in order to capture changes in all major age-groups.[22] Figure 6 shows the evolution of dependency ratios for our period of interest. The Appendix contains Figure B.5 that shows the evolution of the age-groups used in the computation of the dependency ratios. Owing to definition of the dependency ratio, Figures 6 and B.5 look very similar.

These measures allow us to distinguish between the different demographic phases through which all three countries went through. First, we can observe a period of high fertility rates, population growth and dependency ratio around the 1960s. This is due to the abundance of young people under the age of 15 and a lower proportion of elderly as shown in Figures 6b and B.5b. The drastic decline in fertility rates that follows a decade later is also timed with a steadily increasing proportion of elderly dependents, setting the stage for the upcoming demographic transition.

Secondly, we can observe a widening gap between the proportion of younger and older dependents around the 1980s in all countries. Indeed, in the United States and Canada, the proportion

[22]Old-age dependency ratio = $\frac{\text{65 y.o. and over}}{\text{15 to 64 y.o}}$; Young-age dependency ratio = $\frac{\text{0 to 14 y.o}}{\text{15 to 64 y.o}}$.

of 65+-year-olds becomes greater than the proportion of the population under 15 in the 10-year interval around the year 1980. In the U.K., even though the elderly population seems to outnumber the youth from the beginning of the sample, the gap between the two starts to grow substantially faster around that time.

Another key observation is that, from 1980 to 2007, the falling proportion of 0-to-14-year-olds seems to compensate, at least partly, for the growing number of individuals older than 65, maintaining the total dependency ratio within a somewhat stable range during 27 years. This dynamic of stable but low dependency ratios is however broken around the Great Recession. This is because baby boomers start to massively exceed 65 around that time, driving old-age dependency ratios to all-time highs, while young-age dependency stays low. This mass-aging is ultimately reflected in the total dependency ratio as it starts to accelerate significantly faster in 2007. Older individuals now enter retirement-age quicker than the youngest age-groups grow.

In other words, the U.S., U.K. and Canada have passed from a period of high dependency due to a rapidly growing young population to a period of accelerating (but still lower) dependency ratios. Forecasts of total dependency ratios predict record-high levels in the near future. This is because of an important mass of older individuals entering retirement age at the same time and a slower population growth.

There is compelling evidence that the end of the baby boom, experienced by the industrialized world after WWII, was accompanied by a sharp decline in population growth in the U.S., the U.K. and in Canada. This population growth slowdown finds its origin in falling fertility rates. For the United States, Canada and the United Kingdom respectively, trend population growth as lost 0.86, 0.80 and 0.46 percentage point from 1961 to 2017 i.e. from the beginning to the end of our estimation sample. This slowdown in population growth also created a situation that is today peculiar because the old-age component of the total dependency ratio is now driving it higher much faster than the young-age component is slowing it down. To put it simply, the populations of all three economies are growing older much faster than before. The reasons behind this phenomenon are two-fold. First, population growth is too low. Second, the older generations are representing an increasingly larger portion of the total population. This trend is expected to continue in the foreseeable future, maintaining growth rates of labor supply low (Aaronson et al., 2014).

Inequalities

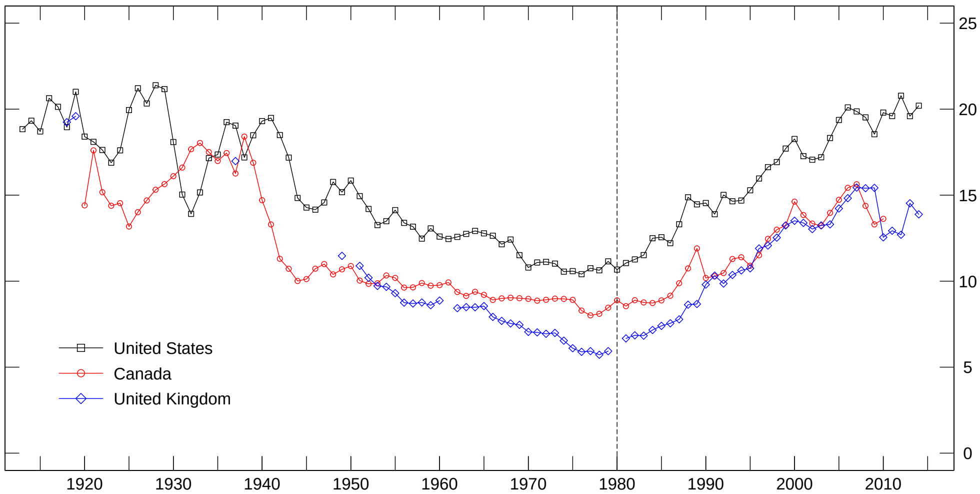


Figure 7: Income Share of the Top Percentile in per cent.

The rise in income disparity has been well-documented for industrialized economies (Autor, Katz and Kearney, 2008) as well as for emerging ones (Alvaredo et al., 2017) in the past few decades. It is also widely accepted that, in most advanced economies, wealth and income inequalities follow a U-shape with a high concentration at the top of the distribution until the 1950s. Economic disparities reach a trough around 1980, only to quickly increase until today (Atkinson and Leigh, 2010). This pattern is visible on Figure 7 which shows the evolution of the share of income captured by the top percentile of earners in each country. Figures B.6-B.8 report additional information on economic disparities within each country across time.

These figures illustrate income share and net wealth share distribution for various portion of earners. One can see that income inequality tends to move in a synchronized fashion across all three economies, with the U.S. almost always being the country where the top 1% of earners gets the highest portion of income. Furthermore, American data shows that the top percentile has increased its income and net wealth share of about 9 and 15 per cent respectively since 1980. On the other hand, the next 5 percentiles have seen their income share stay the same and their net wealth share drop of about 3 per cent. This, along with declining shares of income and net wealth for both the bottom 50% and middle 40% of earners during the same time-span, clearly shows that increased economic disparities are the results of a higher concentration of economic resources at the very top of the distribution. Indeed, Figure B.6 shows that all groups from the top 1% to the top 0.001% of earners have seen their income share grow since 1980. This observation is all the more clear when looking at net wealth share.

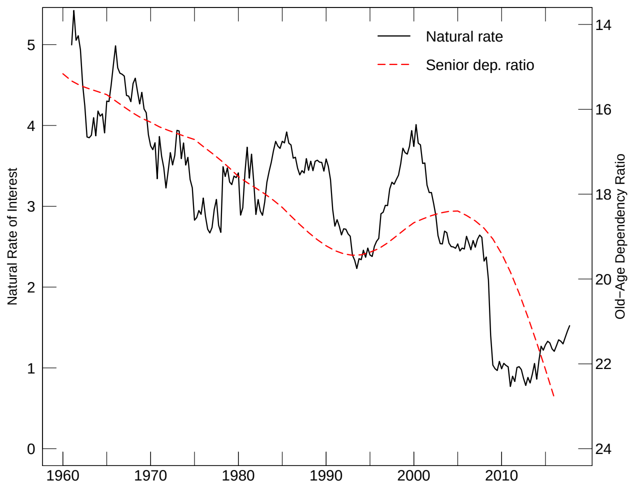
The WID does not have data on net wealth distribution for Canada and the data it has for the U.K. is rather incomplete. We can nonetheless conclude, from the available data, that income disparities have steadily increased in the U.K. and in Canada as well, albeit at a slower pace than in the United States.

Numerous explanations regarding the increase of income and wealth disparities as been proposed over the years. Fortin et al. (2012) argue that the factors driving growing inequalities encompass the increased demand for more educated workers, the demographic transition, the "off-shoring" of labor and institutional factors such as minimum wages and unionization. Labor-market polarization - the process of increasing demand for low-skilled and high-skilled jobs combined with decreasing demand for "middling" jobs - is also believed to be a potential contributor to the growing wage discrepancy between high-earning and low-earning workers (see also Goos, Manning and Salomons, 2009).

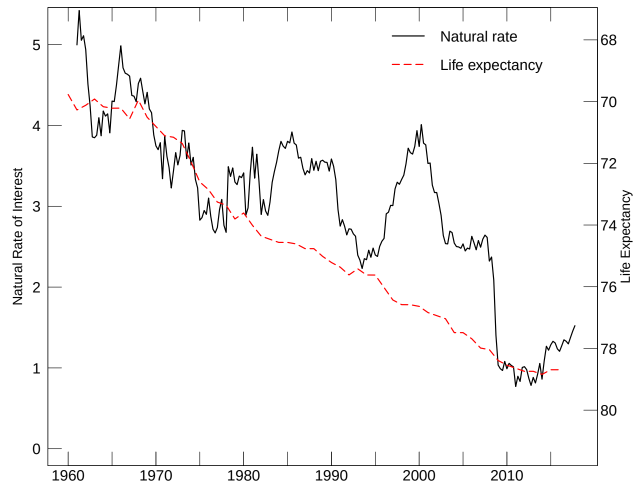
Lemieux, MacLeod and Parent (2009) show that the proportion of male workers on performance-pay schedule went from 30 per cent to more than 40 per cent between the late 1970s and the late 1990s in the United States. Jobs on performance-pay schedule tend to pay higher wages and be less equally distributed. The authors argue that the increased reliance on performance-pay has contributed to roughly 25 per cent of the increase in the variance of log wages from the late 1970s to the early 1990s, with most of the additional dispersion in earnings being observable above the 80th percentile.

The various papers by Piketty, Saez and their co-authors bring further contribution to the body of work regarding increased income concentration at the very top of the distribution by showing that many anglo-saxon countries (including the U.S., Canada and the U.K.) experience similar patterns of increasing income disparity in the last four decades whereas other advanced non-anglo-saxon economies like France, Germany and Japan do not (Alvaredo et al., 2013). Consequently, they suggest that the upward trend in income inequality cannot be explained solely by technological advancements and increased globalization. Institutional and policy differences are suggested to have important explanatory roles. For instance, lower top tax rates, greater bargaining incentives for high earners, increasing capital income and the stronger relation between earned and capital income are the four main factors put forth in order to explain increasing income inequality.

6.2 Discussion on the relation between demographics, income inequalities and r^*

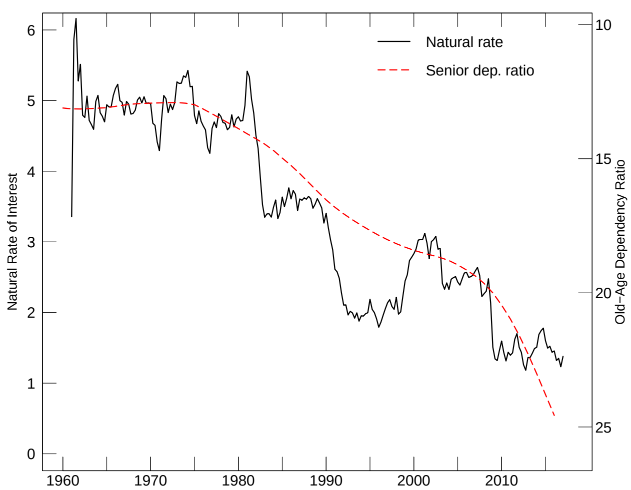


(a) Old-age dependency ratio (inverted axis)

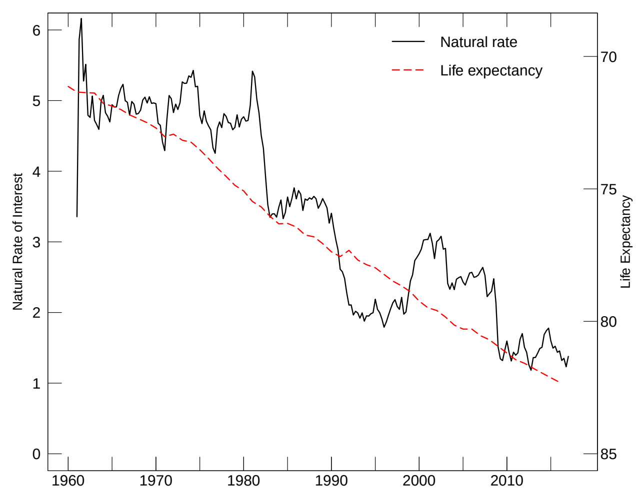


(b) Life expectation (inverted axis)

Figure 8: U.S. natural rate of interest and demographic measures.



(a) Old-age dependency ratio (inverted axis)

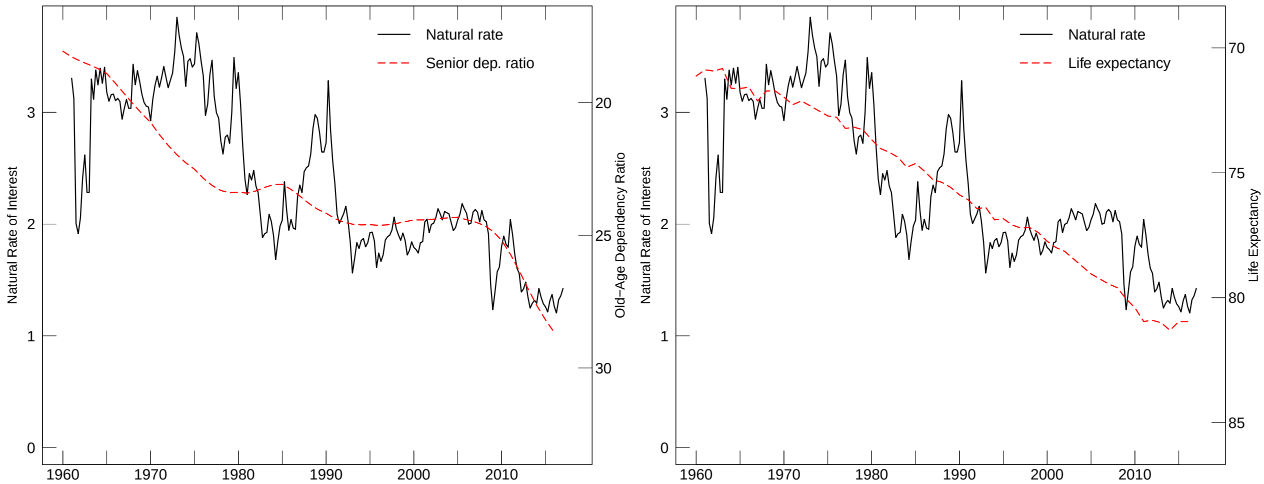


(b) Life expectation (inverted axis)

Figure 9: Canadian natural rate of interest and demographic measures.

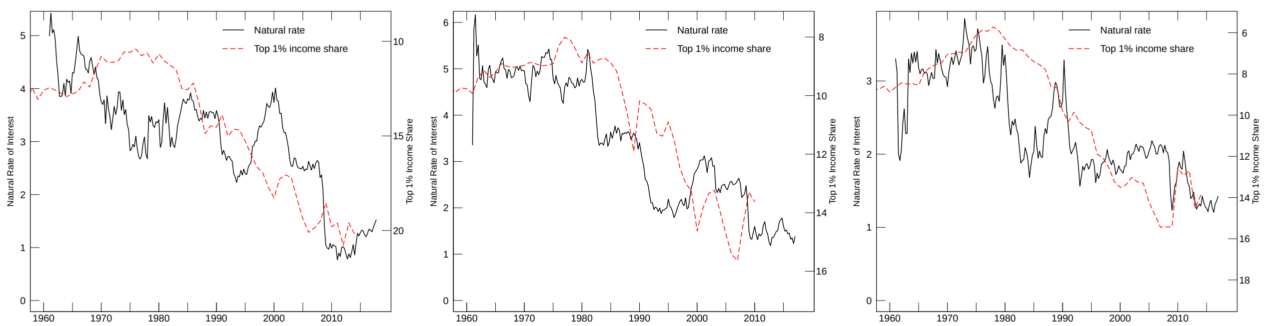
This section discusses the underlying channels through which demographic factors affect the neutral real rate interest.

Around the beginning of our sample, the baby-boom generation reached maturity and a large mass of individuals entered the workforce. Specifically, women labor force participation sky-rocketed (Figure B.3) and drove labor force growth for a considerable time. This led to increased aggregate



(a) Old-age dependency ratio (inverted axis) **(b)** Life expectation (inverted axis)

Figure 10: U.K. natural rate of interest and demographic measures.



(a) United States **(b)** Canada **(c)** United Kingdom

Figure 11: Natural rate of interest and income share (inverted axis).

labor supply and higher output growth rates. It also raised the marginal product of capital as the ratio of capital per worker stayed low because of rapid growth in the labor force. This generated supplementary incentives for investment, creating upward pressure on r^* and real GDP growth rate.

Today, as the baby boomers leave the labor market and the ratio of capital per worker surges, we witness a situation of excess capital relative to labor and lower marginal returns on capital. This leads to weaker levels of aggregate investment. This mechanism represents the "supply-side" effect of demographic changes on r^*. With record-low levels of fertility rates, resources that would otherwise be used for children consumption are now more likely to be allocated towards savings. Moreover, as innovations in healthcare increase lifespan, workers now ought to save for substantially more years of expected retirement than previous generations did. These two channels can be interpreted as having a joint "demand-side" effect on natural rates of interest since they increase aggregate savings. We believe that these phenomenons contribute in maintaining the dynamic of persistently low natural interest rates in the United States, Canada and the United Kingdom.

A large body of work aims to link the current environment of slow growth, lower-than-anticipated inflation and near-zero real interest rates to the effects of demographic factors through theoretical models. These contributions support our claims by quantifying the effects of demographics on r^*. Gagnon, Johannsen and Lopez-Salido (2016) propose a model of overlapping generations augmented with a complex demographic structure which abstracts from transitory shocks and focuses on long-term trends. They predict decades of low rates of interest and real GDP growth to come for the Unites States. The authors argue that, since 1980, demographics account for a 1.25-per-cent fall in the real output growth rate and the natural interest rate. Their model attributes pronounced declines in the past decade to demographic factors associated with the post-WWII baby boom and the end of the 2000s' technology boom.[23] Indeed, they find lower fertility rates and weaker employment growth to be the two main contributors to the decline of r^*, each shaving off $\frac{1}{2}$ percentage point from the equilibrium real rate since 1980.

Carvalho, Ferrero and Nechio (2016) develop a life-cycle model with uncertainty on idiosyncratic retirement and death risks. Their model brings support to the idea that increased expected longevity drags down r^* as agents anticipate longer retirements. They also argue that, on one hand, reduced population growth rate has a negative effect on equilibrium real rates through its "supply-side" effect on investments. On the other hand, as retirees dissave, upward pressure is created on r^* because of reduced aggregate savings. The model predicts that the overall effect of the demographic transition on equilibrium interest rates between 1990 and 2014 is a decline of 1.5 percentage points, with increased life expectancy accounting for $\frac{3}{4}$ of the drop. In sum, while Gagnon et al. (2016) argues that declining fertility rates drive r^* down, the Carvalho et al. (2016) concludes that it is rather increased longevity that is responsible for current interest rate dynamics.

While not being robust evidence of a causal relationship between natural rates of interest and demographics, the following might be considered a hint towards the existence of a potential relation. At first glance, Figures 8-10 seem to suggest that there is inverse comovement between natural

[23] Ignoring business cycles, their model yields an estimate of natural interest rate that starts in 1960 by increasing all the way to its peak in 1980, whereas we estimate declining rates all along the sample. More importantly, the authors' estimate and ours both capture an accelerated decline in r^* since the early 2000s.

interest rates and old-age dependency ratio, as well as between r^* and life expectancy. The interesting thing to keep in mind here is that, while both r^* and the inverse of old-age dependency follow a steady trend throughout the sample, there seems to be a common acceleration (upward for the dependency ratio and downward for r^*) that occurs at the onset of the Great Recession in all three economies. This observation is in line with the generally accepted idea that the effects of demographics are nowadays more deeply felt than three decades ago (Gagnon et al., 2016). Furthermore, the inverse of life expectancy seems to follow a steady trend very similar to that of the neutral rate of interest for all three countries.

In a similar fashion, other contributions propose models that serve as a theoretical basis for our assessment of a potential link between inequality and the natural rate of interest. For instance, Eggertsson, Mehrotra and Robbins (2017) formalize the secular stagnation hypothesis through an overlapping generation model with young, middle-aged and old cohorts exchanging resources between them. They aim to show that under certain conditions, the natural rate of interest can be indefinitely negative, resulting in lower than expected growth, inflation below target and binding zero lower bound on nominal interest rates. Their model is consistent with the idea that an income shift from the poorer to the wealthier households contribute in reducing r^* if higher earners have a greater propensity to save.

On the other hand, papers such as Busetti and Caivano (2017) investigate the empirical impact of the different demographic and inequality factors mentioned previously on real interest rates. Through a band spectrum regression approach and allowing for country fixed effects, the authors study the relationship between low frequency movements in the real interest rate and its determinants. They find that inequalities play a limited explanatory role in the short-to-medium run. However, they suggest the possibility that the effects of income disparity might be relevant in the very long run.

We show, in Figure 11, that there seems to be a secular trend that is somewhat similar throughout our estimates of r^* and the inverse of the top percentile's income share in each economy. Just as inequalities start to increase in all countries in the late 1970s, natural interest rates also begin their decline. Once again, the timing and size of the comovement between the respective measures of inequality and our estimates allow us believe that such a link is potentially plausible. Obviously, we do not consider this to be any form of evidence through which we could infer the impact of rising inequality on the natural rate of interest. Nonetheless, the rapid increase in income concentration to the highest earners in the last 30 years and historically low levels of r^* in most industrialized economies suggest that these two phenomena could be related. Future research that formally integrate structural changes in the state-space estimation of r^* would constitute a more valid basis for such a claim.

7 Conclusion

In this dissertation, we estimate the natural rate of interest for the United States, Canada and the United Kingdom. In all countries, document a steady downward trend throughout the sample. Despite considerable uncertainty, our results are in line with recent literature suggesting that several industrialized economies have experienced falling r^* in the past 25 years. In the exploratory portion of our dissertation, we hint at potential explanations for such declines. As population aging becomes more pronounced in western countries and life expectancy increases, aggregate savings rise and private investments fall. Moreover, a greater concentration of income in the most well-off groups of earners further decreases r^* as it encourages savings.

Our results have important implications for monetary and fiscal policy. First, policymakers may need to reconsider their inflation targets. Indeed, in the current economic environment the probability that monetary policy is constrained by the nominal zero bound is considerably greater than it used to be. Second, with increased strain on public health systems due to population aging, public debt ratios are projected to increase significantly and remain high. Broad fiscal reforms will be necessary in the foreseeable future to mitigate the effects of population aging.[24]

[24] However, some authors advocate that government spending could somewhat mitigate the fall of real rates by acting as a substitute to private demand (Summers, 2014).

References

Aaronson, S., Cajner, T., Fallick, B., Galbis-Reig, F., Smith, C. and Wascher, W. (2014), 'Labor force participation: Recent developments and future prospects', *Brookings Papers on Economic Activity* **45**(2 (Fall)), 197–275.

Alvaredo, F., Atkinson, A. B., Piketty, T. and Saez, E. (2013), 'The Top 1 Percent in International and Historical Perspective', *Journal of Economic Perspectives* **27**(3), 3–20.

Alvaredo, F., Chancel, L., Piketty, T., Saez, E. and Zucman, G. (2017), 'Global Inequality Dynamics: New Findings from WID.world', *American Economic Review* **107**(5), 404–409.

Atkinson, T. and Leigh, A. (2010), The Distribution of Top Incomes in Five Anglo-Saxon Countries over the Twentieth Century, Iza discussion papers, Institute for the Study of Labor (IZA).

Autor, D. H., Katz, L. F. and Kearney, M. S. (2008), 'Trends in U.S. Wage Inequality: Revising the Revisionists', *The Review of Economics and Statistics* **90**(2), 300–323.

Busetti, F. and Caivano, M. (2017), Low frequency drivers of the real interest rate: a band spectrum regression approach, Temi di discussione (economic working papers), Bank of Italy, Economic Research and International Relations Area.

Carvalho, C., Ferrero, A. and Nechio, F. (2016), Demographics and real interest rates: inspecting the mechanism, Working Paper Series 2016-5, Federal Reserve Bank of San Francisco.

Clark, P. K. (1987), 'The cyclical component of u.s. economic activity', *The Quarterly Journal of Economics* **102**(4), 797–814.

Cúrdia, V., Ferrero, A., Ng, G. C. and Tambalotti, A. (2015), 'Has U.S. monetary policy tracked the efficient interest rate?', *Journal of Monetary Economics* **70**(C), 72–83.

Daníelsson, A., Helgason, O. S. and Thórarinsson, S. (2016), Estimating the Natural Interest Rate for Iceland: An Exploratory Study, Economics, Department of Economics, Central bank of Iceland.

Del Negro, M., Giannone, D., Giannoni, M. and Tambalotti, A. (2017), Safety, liquidity, and the natural rate of interest, Staff Reports 812, Federal Reserve Bank of New York.

Dickey, D. and Fuller, W. A. (1981), 'Likelihood ratio statistics for autoregressive time series with a unit root', *Econometrica* **49**(4), 1057–72.

Eggertsson, G. B., Mehrotra, N. R. and Robbins, J. A. (2017), A Model of Secular Stagnation: Theory and Quantitative Evaluation, Nber working papers, National Bureau of Economic Research, Inc.

Enders, W. and Siklos, P. (2001), 'Cointegration and threshold adjustment', *Journal of Business and Economic Statistics* **19**(2), 166–76.

Forstythe, G. E., Malcolm, M. A. and Moler, C. B. (1977), *Computer Methods for Mathematical Computations*, Wiley.

Fortin, N., Green, D. A., Lemieux, T., Milligan, K. and Riddell, W. C. (2012), 'Canadian Inequality: Recent Developments and Policy Options', *Canadian Public Policy* **38**(2), 121–145.

Gagnon, E., Johannsen, B. K. and Lopez-Salido, J. D. (2016), Understanding the New Normal : The Role of Demographics, Finance and Economics Discussion Series 2016-080, Board of Governors of the Federal Reserve System (U.S.).

Galí, J. (2008), *Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework*, Princeton University Press.

Giammarioli, N. and Valla, N. (2004), 'The natural real interest rate and monetary policy: a review', *Journal of Policy Modeling* **26**(5), 641–660.

Goos, M., Manning, A. and Salomons, A. (2009), 'Job polarization in europe', *American Economic Review* **99**(2), 58–63.

Gordon, R. J. (2016), *The Rise and Fall of American Growth: The U.S. Standard of Living since the Civil War*, Princeton University Press.

Hamilton, J. D. (1986), 'A standard error for the estimated state vector of a state-space model', *Journal of Econometrics* **33**(3), 387 – 397.

Hamilton, J. D. (1994), *Time Series Analysis*, Princeton University Press.

Hamilton, J. D., Harris, E. S., Hatzius, J. and West, K. D. (2016), 'The Equilibrium Real Funds Rate: Past, Present, and Future', *IMF Economic Review* **64**(4), 660–707.

Holston, K., Laubach, T. and Williams, J. (2017), 'Measuring the natural rate of interest: International trends and determinants', *Journal of International Economics* **108**(S1), S59–S75.

Johannsen, B. K. and Mertens, E. (2016), A Time Series Model of Interest Rates With the Effective Lower Bound, Finance and Economics Discussion Series 2016-033, Board of Governors of the Federal Reserve System (U.S.).

Laubach, T. and Williams, J. C. (2003), 'Measuring the natural rate of interest', *The Review of Economics and Statistics* **85**(4), 1063–1070.

Laubach, T. and Williams, J. C. (2016), Measuring the Natural Rate of Interest Redux, Finance and Economics Discussion Series 2016-11, Board of Governors of the Federal Reserve System.

Lemieux, T., MacLeod, W. B. and Parent, D. (2009), 'Performance Pay and Wage Inequality', *The Quarterly Journal of Economics* **124**(1), 1–49.

Lewis, K. F. and Vazquez-Grande, F. (2017), Measuring the Natural Rate of Interest : Alternative Specifications, Finance and Economics Discussion Series 2017-059, Board of Governors of the Federal Reserve System (U.S.).

Lubik, T. A. and Matthes, C. (2015), 'Calculating the Natural Rate of Interest: A Comparison of Two Alternative Approaches', *Richmond Fed Economic Brief* (Oct), 1–6.

Manrique, M. and Marqués, J. M. (2004), An empirical approximation of the natural rate of interest and potential growth, Working Papers 0416, Banco de España;Working Papers Homepage.

Mesonnier, J.-S. and Renne, J.-P. (2007), 'A time-varying "natural" rate of interest for the euro area', *European Economic Review* **51**(7), 1768–1784.

Neiss, K. and Nelson, E. (2003), 'The real-interest-rate gap as an inflation indicator', *Macroeconomic Dynamics* **7**(02), 239–262.

Statistics Canada (2017), 'Population size and growth in canada: Key results from the 2016 census.'.

Stock, J. H. and Watson, M. W. (1998), 'Median unbiased estimation of coefficient variance in a time-varying parameter model', *Journal of the American Statistical Association* **93**(441), 349–358.

Summers, L. H. (2014), Reflections on the new secular stagnation hypothesis, *in* 'Secular Stagnation: Facts, Causes and Cures', VoxEU.org, pp. 27–38.

Watson, M. W. (1986), 'Univariate detrending methods with stochastic trends', *Journal of Monetary Economics* **18**(1), 49–75.

Wicksell, K. (1936), *Interest and Prices*, Macmillan.

Williams, J. C. (2003), 'The natural rate of interest', *Federal Reserve Bank of San Francisco Economic Letter* .

Woodford, M. (2003), *Interest and Prices: Foundations of a Theory of Monetary Policy*, Princeton University Press, Princeton.

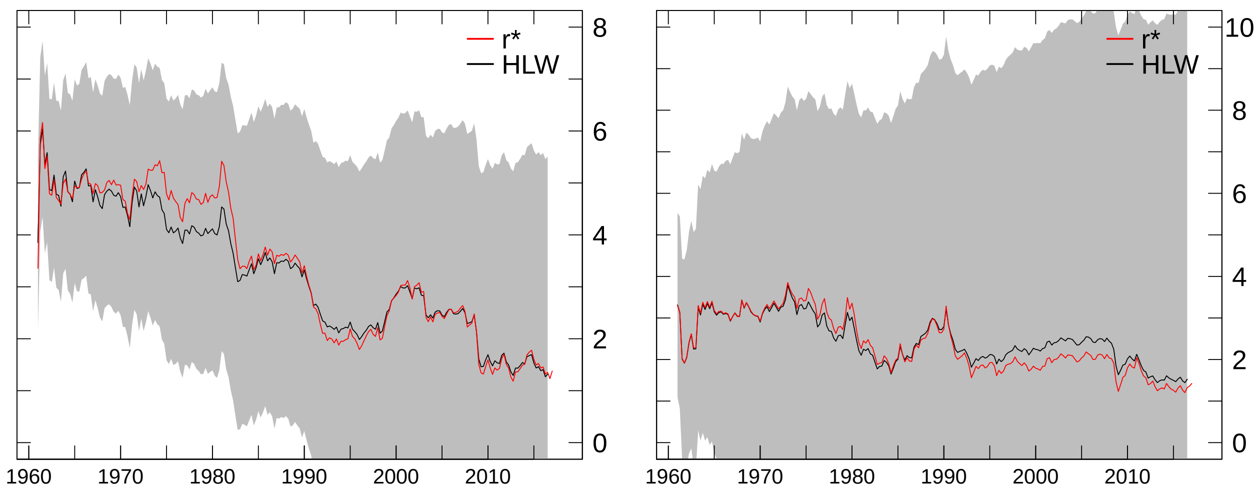
Appendix A Tables

Table A.1: ADF-Tests for y_t.

	United States	Canada	United Kingdom	*Crit.Val.*
Δy_t	-7.26	-8.01	-8.41	$\tau^{RWD} = -2.88$
y_t	-1.65	-2.53	-1.91	$\tau^{RWDT} = -3.43$

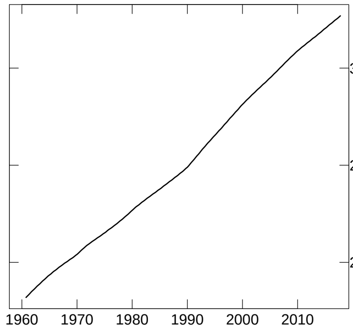
Notes: Critical values are taken from Dickey and Fuller (1981). All tests are based on a specification that includes one lag. The number of lags is selected using the BIC criteria.

Appendix B Figures

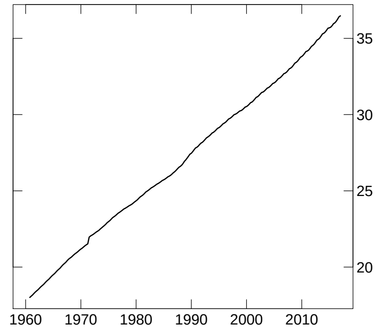


(a) Canada **(b)** United Kingdom

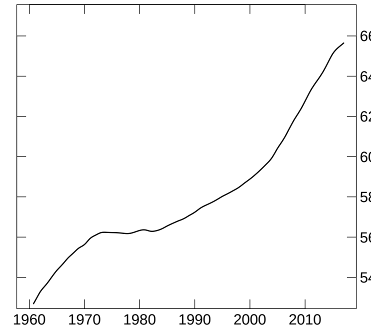
Figure B.1: Comparison of estimates of r^*. *Notes:* The black lines are the estimates taken from Holston et al. (2017) with grey-shaded areas representing the 95% confidence intervals.



(a) United States

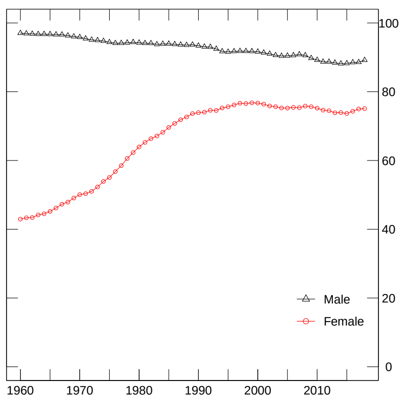


(b) Canada

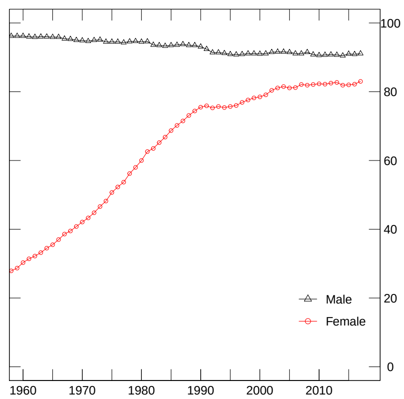


(c) United Kingdom

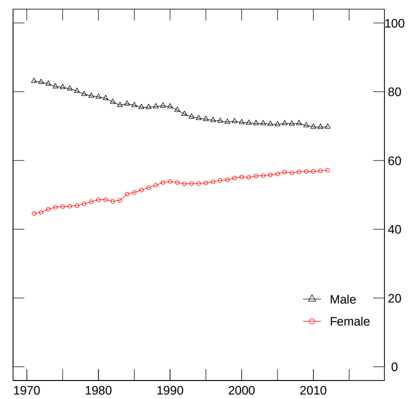
Figure B.2: Population in millions.



(a) United States

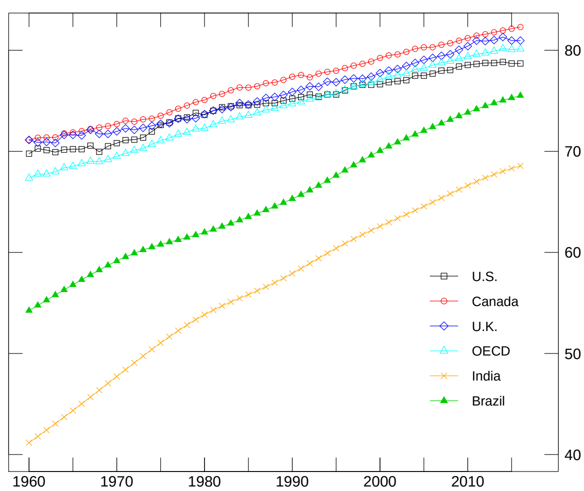


(b) Canada



(c) United Kingdom

Figure B.3: Labor Force Participation Rate in per cent. *Notes:* U.S. and Canadian data is for individuals between 25 and 54 years old. U.K. data is for individuals 16 and over.



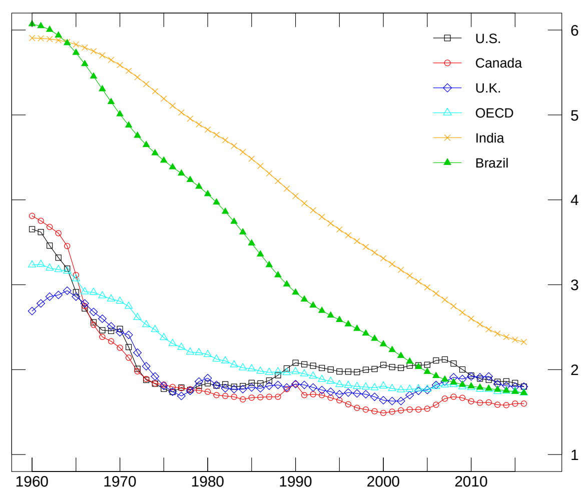
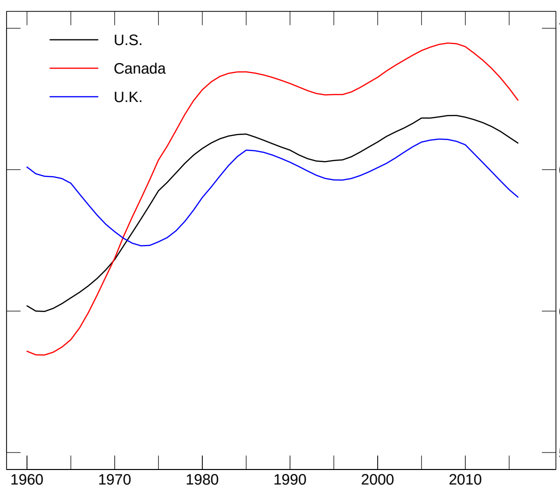
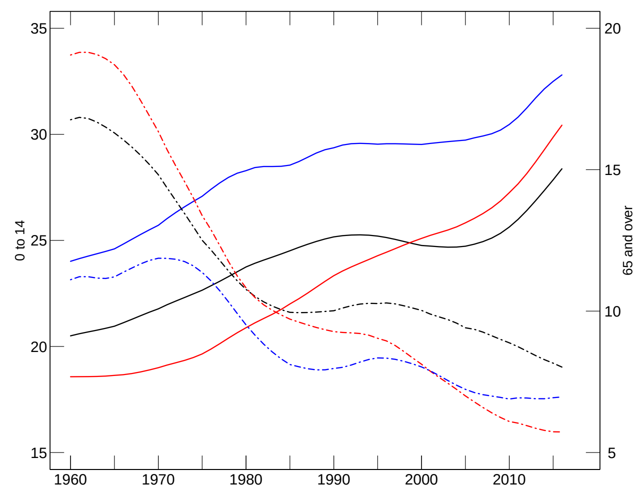


Figure B.4: Life Expectancy at Birth in years (left) and Fertility Rate in number of births per women (right).



(a) 15-64 y.o.



(b) 65+ y.o. (full line) and 0-14 y.o. (dashed line)

Figure B.5: Population Structure in per cent of total population.

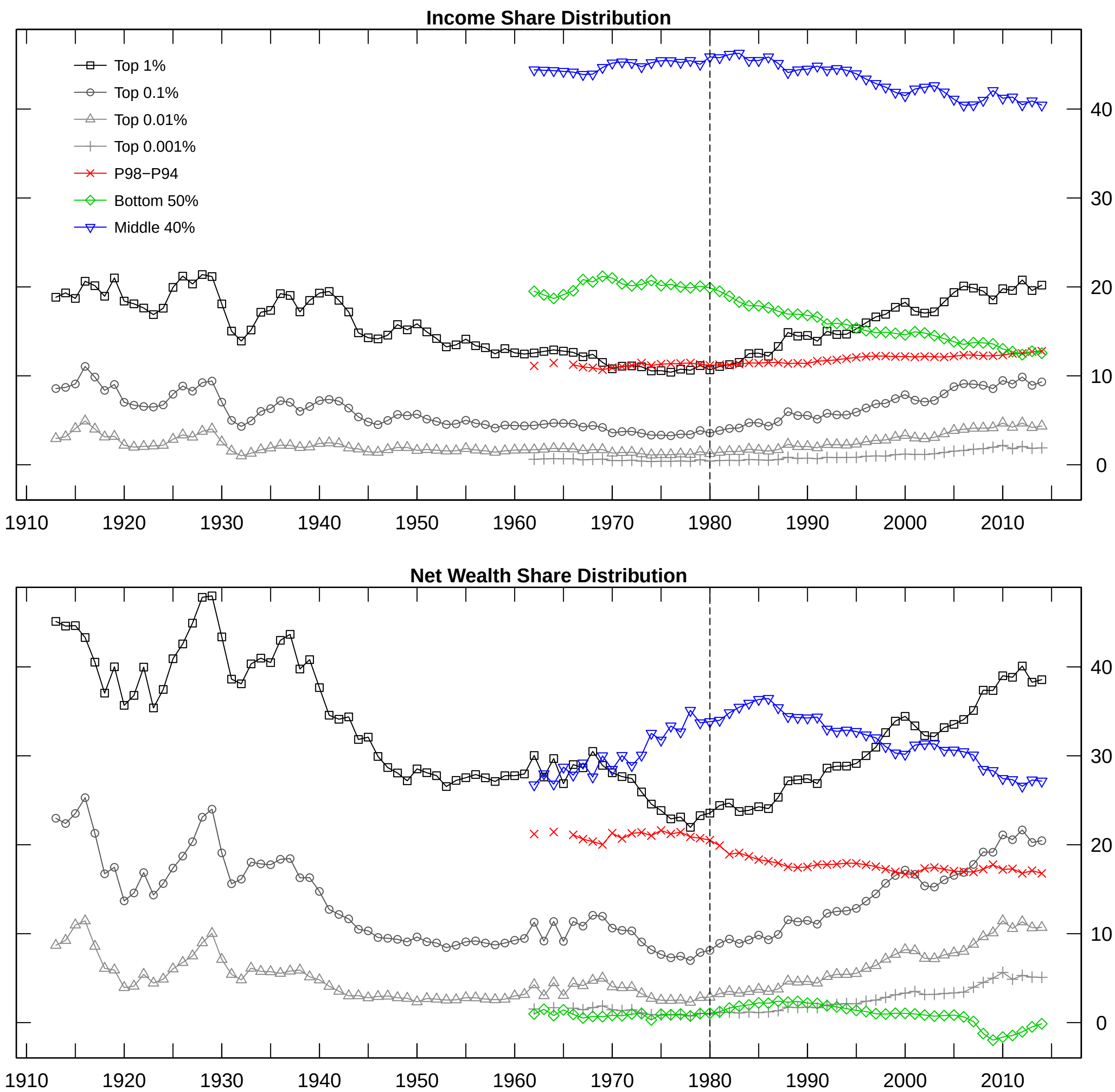


Figure B.6: Income and Net Wealth Share Distribution in the United States in per cent. *Notes:* Income consists of pre-tax labor, capital and pension income. Net wealth is the difference between assets (financial and non-financial) and debt.

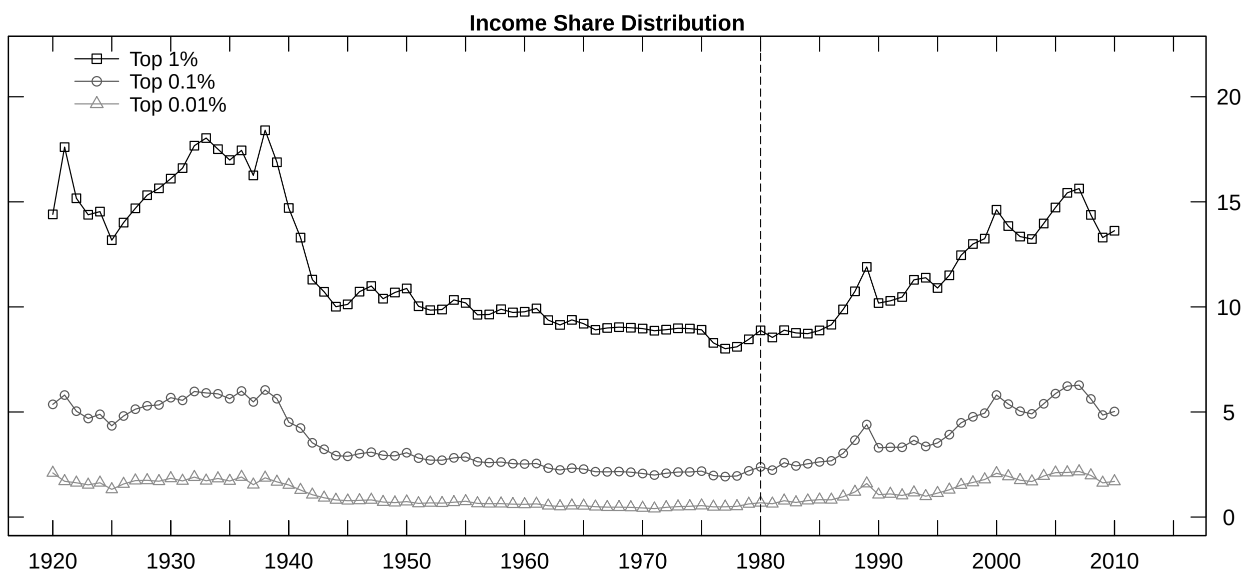
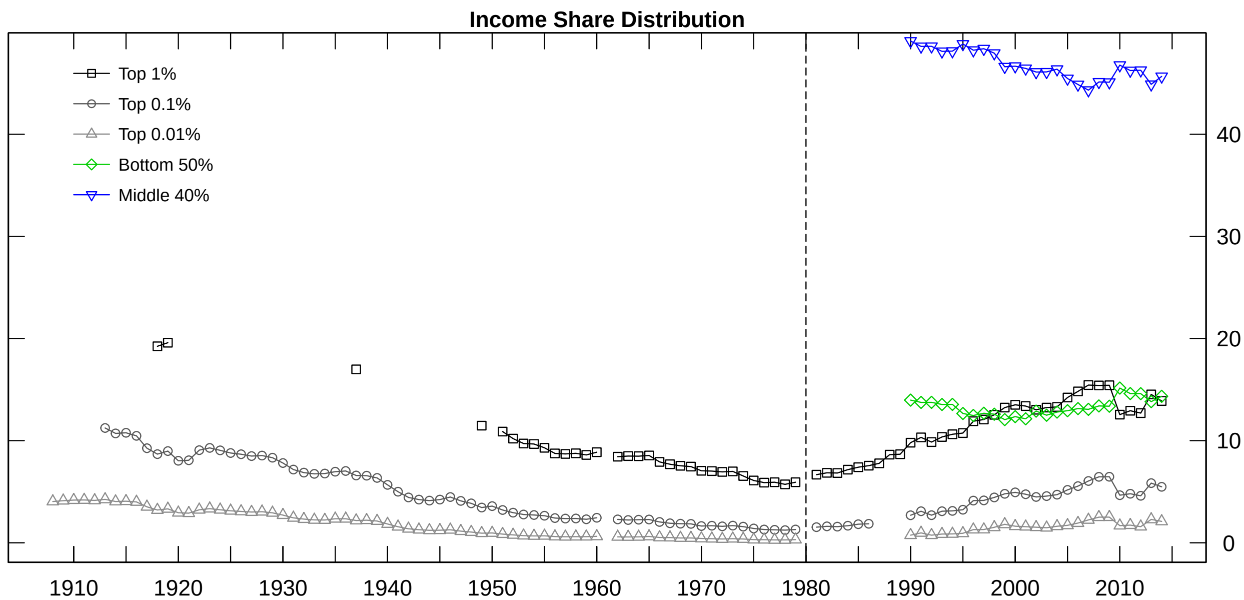


Figure B.7: Income Share Distribution in Canada in per cent. *Notes:* Income consists of pre-tax labor, capital and pension income.



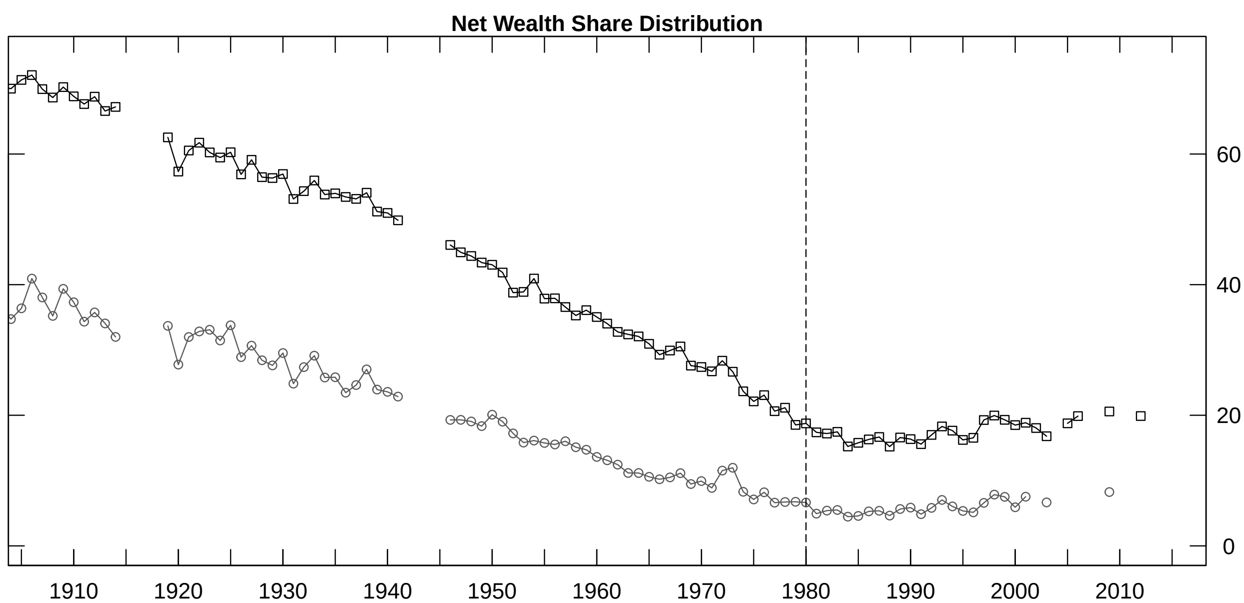
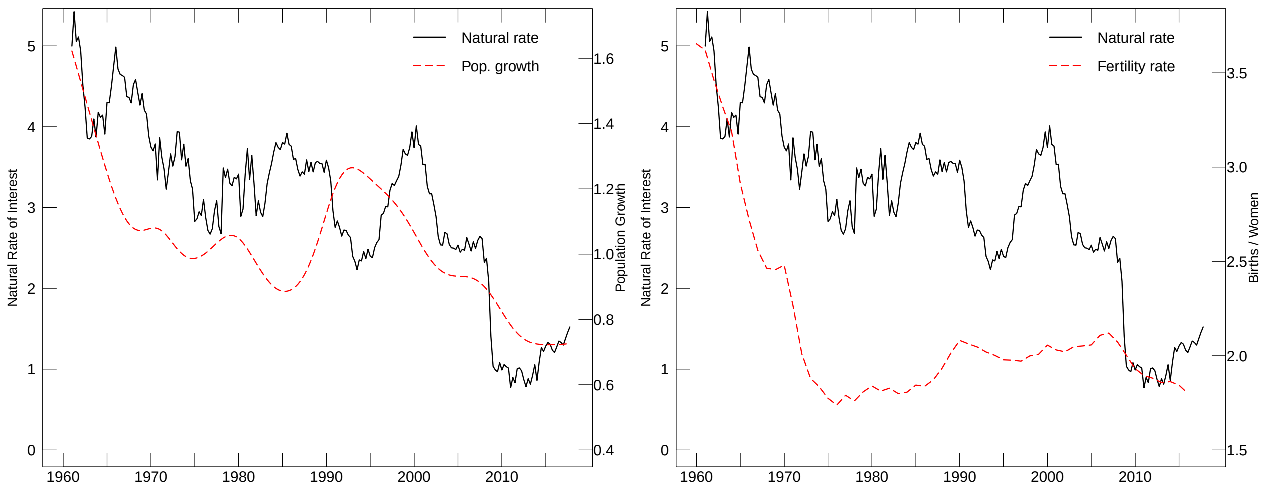
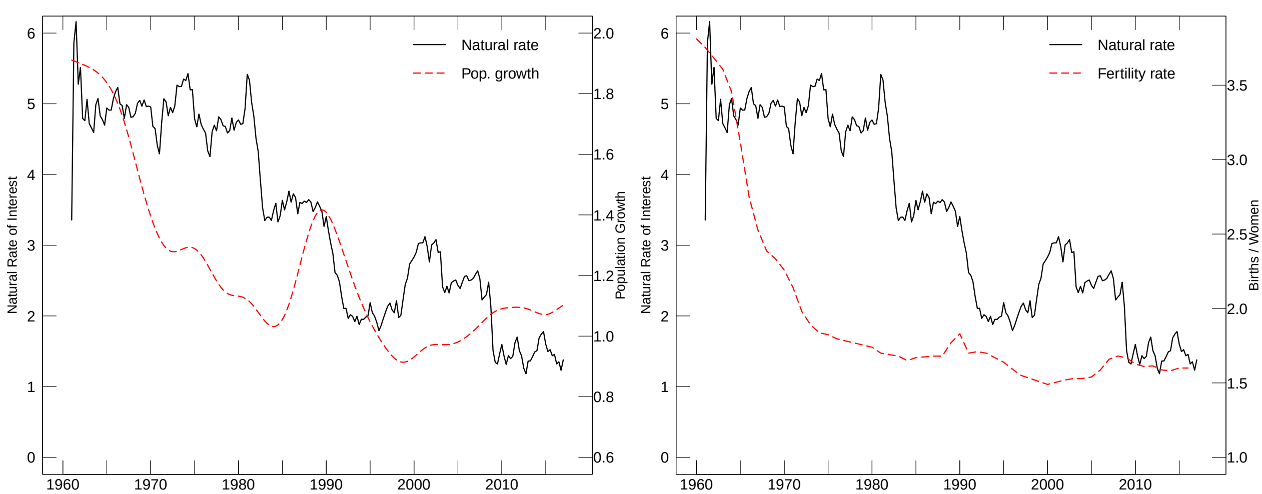


Figure B.8: Income and Net Wealth Share Distribution in the United Kingdom in per cent. *Notes:* Income consists of pre-tax labor, capital and pension income. Net wealth is the difference between assets (financial and non-financial) and debt.



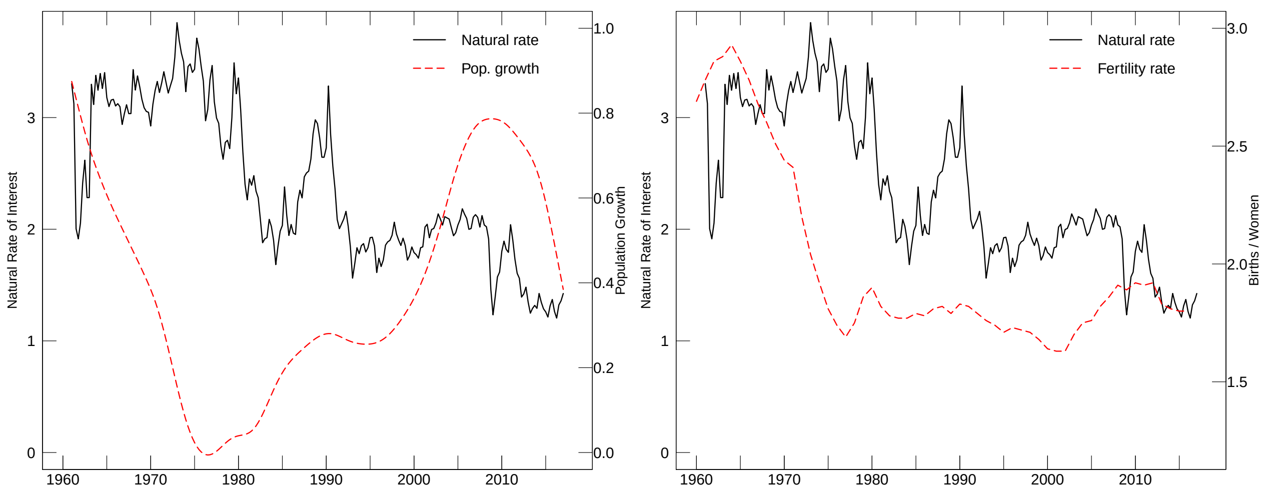
(a) Trend population growth **(b)** Fertility rate

Figure B.9: U.S. natural rate of interest and demographic measures.



(a) Trend population growth **(b)** Fertility rate

Figure B.10: Canadian natural rate of interest and demographic measures.



(a) Trend population growth

(b) Fertility rate

Figure B.11: U.K. natural rate of interest and demographic measures.

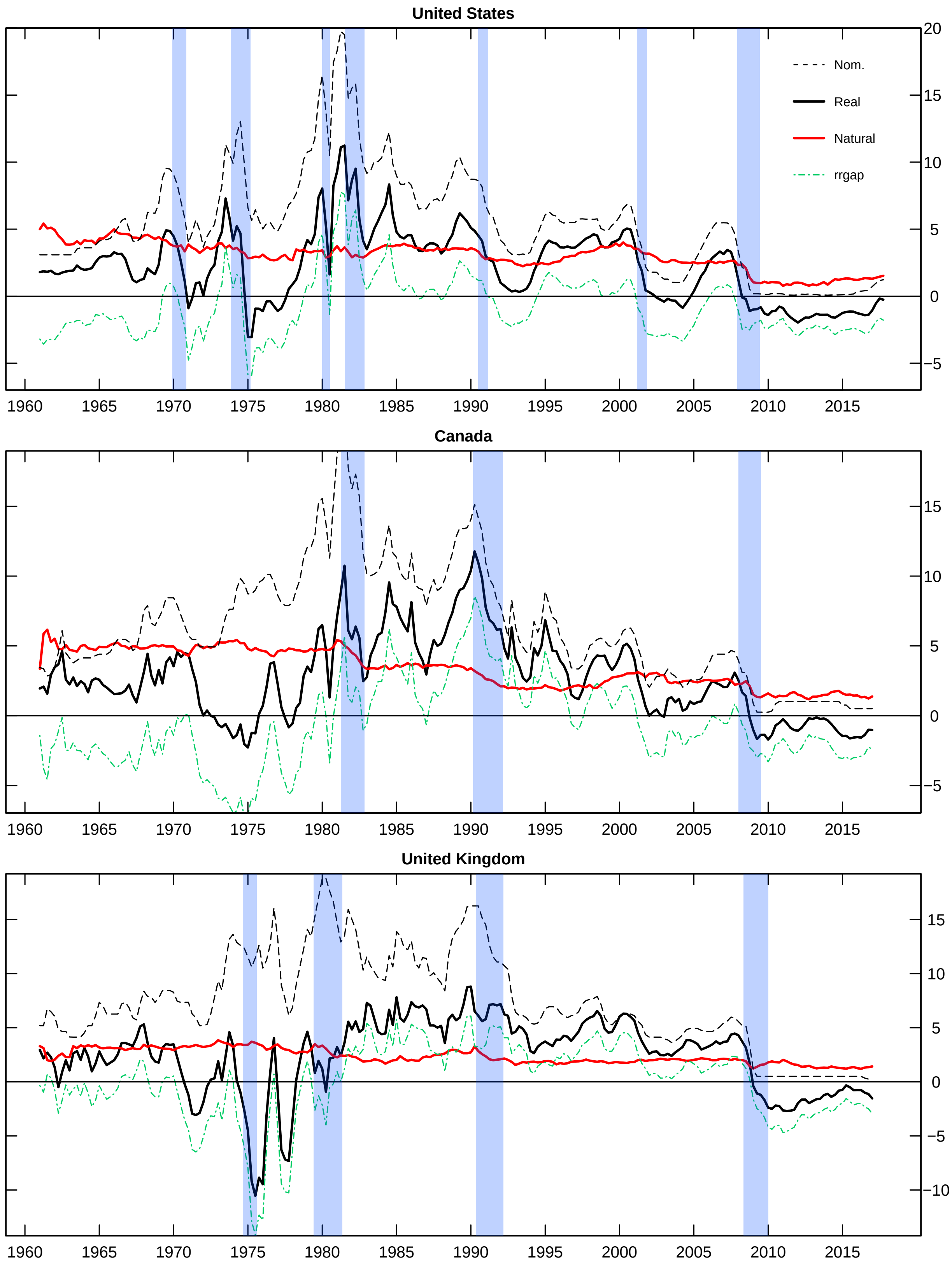


Figure B.12: Interest Rates and Real Rate Gap.