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A model of corporate debt valuation under stochastic interest rates.

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Abstract

The objective of this thesis is to use the dynamic programming method, coupled with spectral interpolation, to evaluate corporate bonds with discrete coupons in the presence of credit risk and under a stochastic interest rate environment. Our evaluation approach considers the optionality of coupon payments, which [EHH04] confirms is a more accurate methodology than [LS95] because it does not overestimate the probability of default. Our approach is simple and robust, and our results show convergence of the model. Additionally, we tested the [LS95] linear regression on credit spreads using a new database derived from Trace with data from July 2002 to November 2014. We found correlation between the asset returns of the firm and interest rates; however, this correlation was not significant in predicting credit spreads. We find that the impact of this correlation on the value of the debt is dependent on the capital structure.

Résumé

L'objectif du projet est d'évaluer les obligations corporatives avec coupons discrets en présence de risque de crédit lorsque le taux d'intérêt est stochastique. L'approche utilisée est la programmation dynamique couplée à l'interpolation spectrale de la fonction valeur. Notre approche d'évaluation tient compte de l'optionalité des paiements de coupons, selon [EHH04] qui estime que cette méthodologie est plus précise parce que, par rapport à [LS95], elle ne surestime pas la probabilité de défaut. Notre approche est simple et robuste et nos résultats montrent la convergence du modèle. En plus, nous avons estimé, comme dans [LS95], une relation linéaire expliquant les écarts de crédit de la firme en utilisant la base de données Trace pour la période allant de juillet 2002 à novembre 2014. Nous avons trouvé une corrélation entre les rendements des actifs et le taux d'intérêt, mais cette corrélation n'était cependant pas significative pour prédire les écarts de crédit. Nous observons que l'impact de cette corrélation sur la valeur de la dette dépend de la structure du capital.

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Introduction

A bond is a financing tool used by companies to raise capital. It can be interpreted as a loan, where the firm commits to paying interests (more commonly known as coupons) and the face value (or principal) at a known future date, called the maturity. The firm that borrows the money is known as the issuer of the bond; the issuer signs an indenture, containing the terms of the contract, with the investor. The investor is then subject to credit risk, that is, the probability of the issuer defaulting on its loan and not being able to meet its contractual obligations. Investors then face the challenge of evaluating corporate bonds, which consists of accurately determining future cash flows, their probability, and finally their present value.

A branch of evaluation methods for corporate bonds is based on the structural model, which uses contingent-claim analysis (application of option pricing theory to the valuation of assets) to determine the proper credit spreads of corporate bonds. Since there are no analytical formulas for valuing American derivatives, most models are based on numerical methods [BBKL07]. Traditional approaches of evaluation include binomial trees, finite difference methods and Monte-Carlo simulation. The drawbacks of some approaches can be, among others, numerical instability [SIL16-a], bias [F14], or increased programming complexity. Dynamic Programming (DP) is a flexible and efficient method that solves recursively, via the no-arbitrage principle of asset pricing, a Markovian decision processes, that is, a stochastic dynamic programming (SDP) problem [BBKL07]. This approach is straightforward and simple to implement in problems with low dimensions.

Additionally, DP allows us to evaluate corporate bonds whilst considering stochastic interest rates, which many existing models assume constant. [LS95] explain that there is a correlation between the firm's asset value and the interest rates, which can help explain why there are differences in credit spreads for companies with the same credit rating. We reproduce the empirical analysis of [LS95] with current market data (from June 2002 to November 2014) to evaluate the impact of changes in interest rates on credit spreads. We find that correlation is important and that bond prices are sensitive to changes in interest rates.

Even though [LS95] propose a model that corrects for the assumption of constant interest rates, they ignore the pattern of coupon payments, which was shown [EHH04] to be an important determinant for average spread prediction.

In the spirit of contributing to the development of more effective methods for evaluating corporate debt, in this thesis we propose an evaluation approach based on dynamic programming (DP), coupled with spectral interpolation, to evaluate corporate bonds with discrete coupons. Our evaluation method allows us to: price at any point in time during the life of the bond; relax the assumption of constant interest rates; consider both default and interest rate risk; and, most importantly, it allows us to consider the importance of the pattern of cash flow payments (and their compounding effect). The compounding effect of coupons is reflected in the recursive computation of the forward holding value of the stock option. The stockholders at every coupon date decide whether it is profitable to pay the coupon and continue, or default. This dependence of coupons reduces the default probabilities with respect to those implied by the [LS95] model [EHH04].

This thesis is organized as follows: Chapter one presents a literature review of the different numerical methods proposed for evaluating default risk and interest rate risk, independently or jointly, most of which are derived from the Merton [M74] model. Chapter 2 briefly introduces our approach, and how it overcomes limitations of other models presented in the literature review. It also motivates our use of the Vasicek dynamics as our interest rate model. Chapter 3 contains the basic notions necessary for the evaluation of corporate debt under our model, including the mathematics behind the joint distribution between the value of the assets of the firm and the interest rates observed in the market (which are stochastic by nature), interpolation techniques and integration methods. Chapter 4 is a detailed exposition of our algorithm. Chapter 5 analyses the performance of our model, its convergence and the implications of the exogenous constant barrier of default of the [LS95] model. Chapter 6 analyzes the sources of the data used to construct the database that feeds our empirical analysis. Chapter 7 presents our empirical results and addresses the importance of considering stochastic interest rates for corporate debt evaluation. Finally, Chapter 8 briefly tests our model's predictions using historic market prices under the risk-neutral probability measure.

Notation

Z: Normally distributed $\mathcal{N}(0,1)$ random variable

R_t: Stochastic interest rate diffusion process

V_t: Firm asset process

Y_t: $\log(V_t)$

y: log asset value state variable

r: Interest rate state variable

W_t: Equity process

W: Equity value

W^h: Equity holding or continuation value

W^e: Equity exercise value

D_t: Debt process

D: Debt value

D^h: Debt holding or continuation value

D^e: Debt exercise value

$\mathbb{Q}$: Risk-neutral probability measure

$\mathbb{Q}_T$: T-Forward risk-neutral probability measure

T: Maturity of the debt

P: Principal of the debt

C: Coupon of the debt

: Number of coupons of the debt

eta: Recovery rate of the debt

$delta$: Coupon recurrence (e.g. delta=1 is a yearly coupon)

$delta0$: Lag between pricing date and next coupon date.

Chapter 1

Literature review

Over the past years, several scholars have attempted to resolve the complex problem of valuing corporate debt. Models in the literature that attempt to resolve this matter can be categorized into structural models, which use contingent-claim analysis (application of option pricing theory to the valuation of assets) to determine the proper credit spreads of corporate bonds, or reduced-form models, which use market credit spreads as well as an exogenous stochastic default process to value corporate bonds. In this thesis, we focus on structural models, which can be further classified into one-factor (for instance [M74] and [G77]) or multi-factor models (for instance [LS95] and [CDG01]).

In the Merton model [M74], a simple capital structure for the firm is assumed [BEN15]: A zero-coupon bond and a stock, whose values sum up to the asset value of the firm. [M74] considers a perfect market in which these two assets (stock and bond) are traded in continuous time during the period $[0; T]$.

Capital Structure:

<table>
<tr><td rowspan="2">Assets
$\{V\}$</td><td>Zero-coupon bond
{D}</td></tr>
<tr><td>Equity
$\{v\}$</td></tr>
</table>

The stock $\{v\}$ is interpreted as a European call option on the firm's assets $\{V\}$, where the strike of the option is the face value of the debt. The [BS73] model is then used to evaluate the stock under a closed-form.

Even though this was a breakthrough model, Merton had to impose restraining and unrealistic hypotheses to facilitate the mathematics behind the evaluation. One of the main critiques of this model is the assumption that equity is valued like a European option, and that default can only occur at the maturity of the debt. [JMR84] and [FT89] showed that this assumption results in smaller credit spreads than the ones observed in the market [LS95].

In order to allow default before maturity of the bond, Geske [G77] developed a methodology for evaluating the stock of a firm with coupon bonds as a compound option (giving its holder the possibility to acquire another option at each coupon date). For instance, at $\{t = 1\}$, the first coupon date, the stockholders have the option to pay the coupon, and as a reward receive another option that expires on the next coupon date, with strike price equalling the second coupon payment; or they can decide to default on the payment and liquidate all assets. At the final coupon date, the stockholders have the option to pay the coupon plus the principal and repurchase the claims of the firm from bondholders, or again, default on the debt.

[G77] operates under the assumptions that there are only coupon bonds and common stock outstanding, that the firm issues new equity to refinance the coupon payments, that default can only occur at the coupon payment dates, and that there are no dividends. The risk free rate is assumed constant, and the cash flows and the discount factor are jointly lognormal meaning we have known joint density function. Additionally, there is one single-price for all assets traded in the market (no arbitrage allowed) and non-satiation is present (a set of random variables $\{Z\}$ exists), meaning there must be positive discount rates in the market so that any security can be priced [G77]. The evaluation of corporate debt in [G77] consists of solving an -variate normal integral, being the number of coupon dates. This approach is very computer intensive and its programming is complex because of the several numerical integrations involved [DEC93].

For constant interest rates, the empirical evidence presented in [EHH04] shows that considering the optionality of coupons is certainly important, and an improvement over the simple portfolio of zeroes approach (where each coupon is evaluated as an individual [M74] model). This is because the compound option takes into account the conditional probabilities of default [EHH04]; indeed, in the simple portfolio of zeroes approach, the value of each coupon is computed irrespective of whether the previous coupon was paid or not. This overestimates the probability of default. Therefore, the higher the probability of default the higher the variability and therefore the larger the overestimation of credit spreads [EHH04].

To part from the constant interest rates assumption in [M74] and [G77], [LS95] propose a closed-form evaluation method for coupon paying bonds that considers stochastic interest rates and their correlation to the value of the firm. Their main conclusion is that credit spreads are not only driven by the asset factor and the current level of interest rates, but are also driven by the correlation between these two variables. This implies that besides default risk, interest rate risk (changes in interest rate) also explains the variation in credit spreads. They argue that this correlation can explain why bonds with similar credit rating can exhibit different credit spreads.

It is worthwhile noting that [LS95] do not consider some exogenous variables that have been proven in the literature to be relevant to the valuation of corporate bonds, for example the bargaining power of bondholders, the existence of an equity committee, and the strength of ties between managers and shareholders. Most importantly, [LS95] suffers from the same problem as [M74] in that it ignores the compound option nature of equity.

When determining the credit spread for a corporate bond, [CDG01] states that it is important to consider the current liability structure of the firm, and primarily the fact that this structure can be altered through time. [CDG01] argue that there is a mean-reversion in the leverage, because companies that have an increasing asset value tend to issue more debt. Their paper consists of comparing the effectiveness of considering a mean-reverting leverage ratio when calculating bond prices and bond spreads versus the constant boundary approach from [LS95]. They highlight that the [LS95] formula approximates the value of the debt based on the Fortet 1943 formula, which is only valid for one-dimensional Markov processes. [CDG01] main conclusion is that considering the firm's ability to modify its capital structure leads to higher credit spreads than the [LS95] model, by a substantial amount. The [CDG01] spreads are more consistent with empirical findings for both zero coupon and coupon paying bonds.

To the best of our knowledge, a compound option on equity model has not been used yet to evaluate corporate debt with discrete coupons under a stochastic interest rate environment. This approach would be interesting because it would account for both the impact of stochastic interest rates, which according to [LS95] is important because of its correlation to the firm's asset value, and the optionality of the coupon, which is shown in [EHH04] to entail major improvements in credit spreads prediction.

Chapter 2

Valuation Model

Given that most models in the literature either assume constant interest rates or ignore coupon payment patterns by using an exogenous barrier of default, in this thesis we propose an evaluation method that considers not only stochastic interest rates, but also the stream of coupon payments and potential default before maturity.

2.1 Model hypothesis

The key assumptions taken in our model are the following:

(1) the asset value of the firm V follows a standard Wiener process [LS95],

(2) the short-term riskless rate R follows a Vasicek dynamic [LS95],

(3) there is a perfect frictionless market, in which securities are traded in continuous time [M74],

(4) the value of the firm is independent of its capital structure, therefore coupon payments are financed by issuing new equity (this assumption is essential for the Modigliani-Miller theorem to hold) [G77],

(5) default can occur at coupon payment dates only [G77],

(6) the volatility of the underlying asset as well as of the interest rate are constant and known [BS73],

(7) there is a known and constant recovery rate ω [LS95],

(8) there are no problems with indivisibility of assets [BS73],

(9) the market is always liquid [BS73],

(10) the stock does not pay dividends [G77],

(11) there are no limits to borrowing at the risk free rate[BS73], and

(12) there are no restrictions to shortselling [BS73].

2.2 Model Construction

Let V_t denote the process of the value of the assets of the firm and R_t denote the process of the short-term interest rate. According to our assumptions, the dynamics of V and R are given by

$$dV_t = \left(R_t - \frac{1}{2}\sigma_s^2\right)dt + \sigma_s V_t dZ_1 \tag{2.0}$$

$$dR_t = \kappa(\theta - R_t)dt + \sigma_r dZ_2 \tag{2.1}$$

where σ_s, σ_r, κ and θ are constants, Z_1 and Z_2 are standard Wiener processes and where the instantaneous correlation between Z_1 and Z_2 is denoted by ρ. The debt of the firm consists of a bond maturing at a future date T.

Denote by

$W_t(y, r)$: the equity value at date t if the log of the asset value is y and the short interest rate is r,

$D_t(y, r)$: the debt value at date t if the log of the asset value is y and the short interest rate is r,

c: the coupon rate of the bond,

P: the principal of the bond,

δ: the time interval between coupon payments,

$P_t(r, T)$: the price at date t of a zero coupon bond maturing at T if the short interest rate is r,

ω: the recovery rate in case of default.

At maturity, the equity value is the payoff of a European call option.

$$W_T(y,r) = \max((exp(y) - (P + cP); 0) \tag{2.2}$$

At maturity, the value of the debt depends on the value of the assets. If $V_T \leq (P + cP)$, then the firm will not be able to make the final coupon and principal payment and therefore must default.

$$D_T(y,r) = \begin{cases} P + cP & \text{if } \ V_T > P + cP \\ mi \ \ (V_T; \omega P) & \text{otherwise.} \end{cases} \tag{2.3}$$

Assume that the value of the debt and equity at date t are known. Their holding or continuation value at $t - \delta$ is expressed as the expected discounted value of their date t value.

$$D^h{}_{t-\delta}(y,r) = E^Q_{t-\delta}\left[e^{-\int_{t-\delta}^{t} r_s ds} D_t(Y_t, R_t)\right] \tag{2.4}$$

$$W^h{}_{t-\delta}(y,r) = E^Q_{t-\delta}\left[e^{-\int_{t-\delta}^{t} r_s ds} W_t(Y_t, R_t)\right] \tag{2.5}$$

where $E_t{}^{\mathbb{Q}}[.]$ denotes the expected value under the risk-neutral measure $\mathbb{Q}$, conditional to the available information in the filtration $\mathcal{F}_t$ (the observable state variables).

As it is demonstrated further in Section (3.1.), under the T-forward risk-neutral measure $\mathbb{Q}T$ we can extract the zero coupon bond price from the expected value. Formula 2.6 shows the resulting expression for the debt value. The procedure is equivalent for the equity.

$$D^h{}_{t-\delta}(y,r) = P_{t-\delta}\,(r,t) E_{t-\delta}{}^{\mathbb{Q}T}[D_t(Y_t, R_t)] \tag{2.6}$$

At each coupon date, equity holders have the option to pay the coupon c and detain the holding value $W^h{}_t\,(y,r)$ or to default and receive nothing. Thus, the equity value at each coupon date t is the maximum between the holding value, minus the coupon paid, and zero.

$$W_{t-\delta}(y,r) = \max\left(W^{h}_{t-\delta}(y,r) - cP; 0\right) \tag{2.7}$$

After computing the forward holding value of the equity, we proceed to compute the value of the debt as

$$D_{t-\delta}(y,r) = \mathbb{I}_{\left(W^{h}_{t-\delta} - \mathrm{cP} \leq 0\right)} \min(\ W_{t-\delta}(y,r)\ ; \omega P) + \ \mathbb{I}_{\left(W^{h}_{t-\delta} - \mathrm{cP}\ \ 0\right)} (D^{h}{}_{t-\delta}(y,r) + cP) \tag{2.8}$$

Chapter 4 will develop on the algorithm and its solution. Before that, we explore key concepts needed for the evaluation.

2.3 Vasicek Interest Rate Model

As mentioned in the literature review, most [M74] derived models assume that the short-term interest rate is constant. One main essential argument in favour of the use of stochastic interest rates is given in [LS95] where the authors ascertain that the correlation between the firm's assets value and the current level of interest rates has a significant impact on the value of risky fixed income securities and, therefore, a high impact on credit spreads. There are numerous models for the stochastic dynamics of interest rates. [LSD10] analyze the pros and cons of several of them and highlight their main strengths and weaknesses.

One of the most common, and currently utilized, models is the Vasicek model, due to its mean reverting affine form, Gaussian solution and tractability [VER10]. The Gaussian solution of this model is a key feature because it allows for a closed-form in the joint density [SIL12], Chapter 5, Section 5.2.3.

The dynamics or stochastic differential equation of the Vasicek model is

$$dR_t = \kappa(\theta - R_t)dt + \sigma_r dZ_t \tag{2.9}$$

where $\kappa, \theta,$ and σ_r are know positive constants; κ represents the speed at which the process returns to the historical mean; θ is the historical mean; σ_r is the volatility.

The drift of this SDE captures the mean-reverting behaviour of interest rates. When rates are below the historical mean, $R_t < \theta$, their difference will be positive, pushing the process towards higher interest rate values. When $R_t > \theta$, their difference will be negative, bringing down the level of interest rates. This is most commonly referred to as an *Ornstein-Uhlenbeck* process.

A possible solution to the Vasicek model is found by applying Ito's lemma with the information until date , $\leq t$, and applying the change of variable $y_t = R_t e^{\kappa t}$. We get:

$$R_t \mathcal{F} \ = \theta + e^{-\kappa(t- \ )}(R(\ ) - \theta) + \sigma \int^t e^{-\kappa(t-m)} dZ_m. \tag{2.10}$$

From the previous equation, we can see how R_t follows a Gaussian law where the first two moments are:

$$E[R_t \ \mathcal{F} \ ] = R(\ )e^{-\kappa(t- \ )} + \theta\left(1 - e^{-\kappa(t- \ )}\right) \tag{2.11}$$

$$Var[R_t \ \mathcal{F} \ ] = \frac{\sigma}{2\kappa}(1 - e^{-2\kappa(t- \ )}). \tag{2.12}$$

Additionally, the Vasicek model allows for a closed-form solution for the evaluation of a zero coupon bond $P_t(r, T)$

$$P_t(r, T) = \quad (t, T)e^{- \ (t,T)r} \tag{2.13}$$

where

$$(t, T) = e^{\left(\ -\frac{\sigma}{2\kappa}\right)\left(\ (t,T)-(T-t)\right)-\frac{\sigma}{\kappa} \ (t,T)}$$

$$B(t, T) = \frac{1}{\kappa}(1 - e^{-\kappa(T-t)}).$$

The Vasicek model is the most analytically tractable model in the market, with simple estimation parameters, but the use of a Gaussian law in the solution of R_t allows for negative interest rates, which is generally not an acceptable assumption for investors [SIL12]. Additionally, the Vasicek model can generate only monotonically increasing or decreasing yield curves, or curves that have a small hump, when in reality yield curves that are more complex are observed in the market. However, as argued by [LS95], the occurrence of negative interest rates can be controlled through the effective calibration of parameters and, furthermore, the primary effect of interest rates on credit risk models is through its expected future value, which is mainly positive.

2.4 Dynamic Programming (DP)

There are many different solution approaches in the literature for the evaluation of derivative products, but some of them may not be reliable because they are numerically instable. For example, the binomial tree approach proposed in [CRR79] is a simple and flexible valuation method based on the martingale property and the risk-neutral probability measure. However, even though the binomial method is simple to implement and converges towards the BMS model when Δt 0, the convergence is slow, oscillatory and has generally a bias.

Other approaches like Explicit and Implicit Finite Difference approaches solve the partial differential equations satisfied by derivative products when there is no explicit solution. Respectively, the Explicit method although characterized by its simple programming, requires a lot of time for convergence and has a high risk of instability, whereas the Implicit method reduces this instability but increases the programming complexity.

Monte Carlo Simulation has also proven to be a popular method in derivatives evaluation because it can handle problems with high dimension. However, Monte Carlo simulation requires a relatively large computational effort and is subject to statistical error.

Dynamic programming (DP) is the methodology of choice for our evaluation approach. We have chosen to use DP because it allows us to price debt under two stochastic factors; it is analytically tractable; and can easily allow for the correlation between interest rates and asset value [SIL16-a]. DP is a current methodology used in the pricing of American options under a Markovian decision process, which captures the trade-off between low present costs and the undesirability of high future costs. This approach consists in breaking down an optimization problem into smaller, and more manageable, problems using the Bellman's optimality principle [BER95]. This approach always assumes optimal decision making for subsequent stages with two main conventions; (1) Decisions are made at discrete times, and (2) there is a cost function that is additive through time.

A Dynamic Programming approach to evaluate a derivative product where decisions are to be taken over the life of the product consists in expressing the value recursively, as the sum of an immediate reward and a future value, which often corresponds to a holding value, as in Equations (2.4)-(2.8). In the following paragraphs, we highlight the general approach using the simple example of a Bermudan option.

For debt evaluation while using dynamic programming, we rely on [G77] structural method as an illustrative example. Notice [G77] is applicable to the evaluation of American options, but since we are working with a discrete time dynamic system, evaluating an American option at each discretization of time Δt is equivalent to evaluating a Bermudian option. Additionally, as the discretization becomes smaller and smaller we can say that the price of a Bermudian option is converging into that of an American option.

Consider a vector $\mathrm{X}_t \in \mathbb{R}^n$ containing all the information at t of the value of the different variables affecting the option. We then define

$$v_t(x): \mathbb{R}^n \quad \mathbb{R}$$

as the value of the option at date t when the observed state vector is x. We suppose as well that $t = t_0 < \quad < t_m < \quad < t_M = T$, are the decision dates. At all dates (except at maturity), the holder of the option can decide between exercising the option or holding it for at least one supplementary period, i.e. from t_m to $t_{m \quad 1}$. Because the holder of any option wishes to maximize his gain, we can define the value of the option as:

$$v_{t}\ \ (x) = \max\left(v^{h}{}_{t}\ \ (x), v^{e}{}_{t}\ \ (x)\right) \tag{2.14}$$

where v^h and v^e are the holding and exercise value of the stock option, respectively.

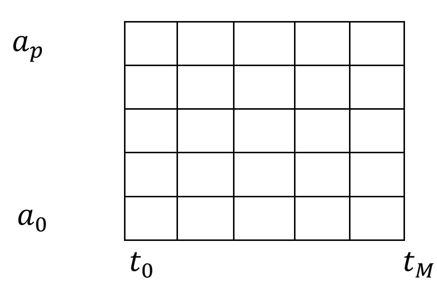
Assuming that the value function $v_{t \quad +1}$ is known on $\mathbb{R}^n$, the numerical implementation of a Dynamic Program consists of evaluating the holding value of the stock option on a finite grid of evaluation points $a_i, i = 0, \ldots, p$, at each decision date:

$$\tilde{v}^{h}{}_{t}\ \ (a_i) = E^{Q}[e^{-\int_{t}^{t \quad +1} r_s ds} v_{t \quad +1}(\mathrm{X}_{t \quad +1})\ a_i, t_m], \tag{2.15}$$

and of determining an interpolation function $\hat{v}$ using the grid as interpolation nodes to obtain an evaluation at any $x \in \mathbb{R}^n$.

Step-by-Step solution:

Step 1: Trace a grid where the $x - axis$ represents the discretization of time Δt and follows the index t_m, and the $y - axis$ represents the state variables (log asset value) indexed by a_i. (Note that we focus on a one dimension dynamic program to simplify the notation, where the only state variable is the underlying asset level.)



The boundaries of the grid have to be chosen such that the probability of the value of the assets being outside of the grid are small enough:

$$P(X < a_0) \approx 0 \quad \text{and} \quad P(X > a_p) \approx 0.$$

Step 2: Since DP uses analysis by recursion, we begin the analysis at the last date, or in other words the maturity t_M. The option value at t_M is given by:

$$v_{t_M}(a_i) = v^e{}_{t_M}(a_i) = \max(a_i - K, 0), i \in [0, p]$$

Step 3: Using the known values on the grid $v_{t\ \ }(a_i)$, compute the interpolation function $\hat{v}_{t\ \ }(x)$.

There are several interpolation methods, among which piecewise constant, piecewise linear, polynomial, piecewise polynomial.

Step 4: Go back in time to t_{m-1}. Find the holding value at each node.

$$\tilde{v}^h{}_{t\ \ -1}(a_i) = E_t^Q[e^{-r\Delta t}\hat{v}_{t\ \ }(X_{t\ \ })\ a_i].$$

Step 5: Calculate the exercise value at each node a_i.

$$v^e{}_{t\ \ -1}(a_i) = \max(a_i - K, 0).$$

Step 6: Find the value of the option at t_{m-1} for each node by comparing the exercise to the holding value.

$$\tilde{v}_{t\ \ -1}(a_i) = \max(\tilde{v}^h{}_{t\ \ -1}(a_i), v^e{}_{t\ \ -1}(a_i)).$$

Step 7: Repeat this recursive process until t_0 to find the value of the option at inception.

Chapter 3

Tool Box

In this chapter, we explain the importance of the T-Forward risk-neutral measure for evaluating corporate debt under a stochastic interest rate environment. We then derive the joint distribution of the asset value and the interest rate under the T-Forward risk-neutral probability measure. Finally, we explain the Chebyshev polynomials, their use in two dimensions and the Clenshaw-Curtis integration method used to approximate the density function.

3.1 Passage to the *Forward-Neutral* measure

In this paragraph, to simplify notation, we consider the evaluation of a derivative at t when the next decision date is T. Since we are evaluating a derivative product D_t written on the underlying assets of the company, where interest rates are stochastic, we can express the price of a zero-coupon bond as:

$$P_t(r,T) = E_t^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)ds\right) \; r\right] \tag{3.1}$$

The holding value of the debt is:

$$D^h{}_t(r,y) = E^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)ds\right) D_T \; \mathcal{F}_t\right] \tag{3.2}$$

Since the interest rate and the debt value are correlated, we cannot state that the expected value of the product is the product of the expected values. Therefore, we proceed into a change of probability measure where all forward prices are martingales, and this correlation disappears [F14]. In this new probability measure, we can evaluate any contingent claim as the product between the discount factor and the excepted value of the debt under the measure $\mathbb{Q}_T$, named forward risk-neutral measure.

$$D_t = P_t(r,T)E^{\mathbb{Q}_T}[D_T \ \mathcal{F}_t] \tag{3.3}$$

To pass to the forward risk-neutral measure we apply the following transformation to the geometric Brownian motion Z_t^1 [CLE13]:

$$dZ_t^1 = d\tilde{Z}_t^1 - \frac{\sigma_r}{\kappa}(1 - e^{-\kappa(T-t)}) \tag{3.4}$$

3.2 Construction of the Joint Distribution

As per [CLE13] we consider the following generalized model:

$$dR_t = \mu(t,R_t)dt + \eta(t,R_t)dZ_t^1 \tag{3.5}$$

$$dV_t = (R_t)V_t dt + \sigma_s V_t dZ_t^2 \tag{3.6}$$

where R_t is the interest rate, V_t is the asset value, Z_t^1 and Z_t^2 are two correlated Brownian motions with $\mathbb{E}(dZ_t^1 dZ_t^2) = \rho dt$; $\mu(t,R_t)$ and $\eta(t,R_t)$ are continuous on $\mathbb{R}$; σ_s the volatility.

Under the risk-neutral probability measure $\mathbb{Q}$, it is more practical to work with independent Brownian movements ($Z_t^1\ a\ \ d\ Z_t^2$). To obtain the SDE we use the Cholesky decomposition:

$$dR_t = \mu(t,R_t)dt + \eta(t,R_t)dZ_t^1 \tag{3.7}$$

$$dV_t = (R_t)V_t dt + \sigma_s V_t(\rho dZ_t^1 + \sqrt{1-\rho^2}dZ_t^2) \tag{3.8}$$

For modeling purposes, it is simpler to use $Y_t = \log(V_t)$ than V_t, because it allows us to find a Gaussian solution instead of a lognormal one. By applying Itô's lemma, we can find the stochastic differential equation of the underlying asset:

$$dY_t = dl\ \ g(V_t) = \left(R_t - \frac{1}{2}\sigma_S^2\right)dt + \sigma_s\left(\rho dZ_t^1 + \sqrt{1-\rho^2}dW_t^2\right) \qquad (3.9)$$

After using *T-Forward risk-neutral* passage, we are left with the dynamic for the interest rate and asset value as:

$$dR_t = \left(\theta - \frac{\sigma_R^2}{\kappa}\left(1 - e^{-\kappa(T-t)}\right) - \kappa R_t\right)dt + \sigma_r dW_t^1 \qquad (3.10)$$

$$dY_t = \left(R_t - \frac{\sigma_S^2}{2} - \frac{\rho\sigma_r\sigma_s}{\kappa}\left(1 - e^{-\kappa(T-t)}\right)\right)dt + \sigma_s\left(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2\right) \qquad (3.11)$$

3.2.1 Solution of R_t

The solution of R_t, where $\ \ < t < T$, under the *T-Forward Risk-neutral* measure is

$$R_t = R\ \ e^{-\kappa(t-\ \ )} + \frac{\theta}{\kappa}\left(1 - e^{-\kappa(t-\ \ )}\right) - \frac{\sigma_r^2}{\kappa^2}\left(1 - e^{-\kappa(t-\ \ )} - \frac{e^{-\kappa(T-t)} - e^{-\kappa(T\ \ t-2\ \ )}}{2}\right) + \sigma_r \int^t e^{-\kappa(t-s)}\, dW_s^T$$

(3.12)

Distribution

$$E^{\mathbb{Q}^T}[R_t|\mathcal{F}\ \ ] = R\ \ e^{-\kappa(t-\ \ )} - \frac{\theta}{\kappa}\left(1 - e^{-\kappa(t-\ \ )}\right) - \frac{\sigma_r^2}{\kappa^2}\left(1 - e^{-\kappa(t-\ \ )} - \frac{e^{-\kappa(T-t)} - e^{-\kappa(T\ \ t-2\ \ )}}{2}\right)$$

(3.13)

$$Var^{\mathbb{Q}^T}[R_t \ \mathcal{F} \ ] = \sigma_r{}^2 \int^t e^{-2\kappa(t-s)}\, ds$$

(3.14)

(See [CLE13] for mathematical development)

3.2.2 Solution of Y_t

The solution of Y_t under the *T-Forward Risk-neutral* measure T is

$$Y_t = Y \ \ + \eta(\ \ ,t) + \frac{\sigma_r}{\kappa}\int^t \left(1 - e^{-\kappa(t-m)}\right) dW_m^1 - \left(\frac{\sigma_X^2}{2} + \frac{\rho\sigma_r\sigma_X}{\kappa}\right)(t - \ \ ) + \frac{\rho\sigma_r\sigma_X}{\kappa}\left(e^{-\kappa(T-t)} - e^{-\kappa(T- \ \ )}\right) + \sigma_X\, \rho \int^t d\, W^1 + \sigma_X\sqrt{1-\rho^2}\int^t d\, W^2$$

(3.15)

where

$$\eta(\ \ ,t) = R \ \ \frac{(1 - e^{-\kappa(\ \ - \ \ )})}{\kappa} + \left(\frac{\theta}{\kappa} - \frac{\sigma_r^2}{\kappa^2}\right)\left((t - \ \ ) - \frac{1 - e^{-\kappa(t- \ \ )}}{\kappa}\right) + \frac{\sigma_r^2}{2\kappa^3}\left(e^{-\kappa(T-t)} - 2e^{-\kappa(T- \ \ )} + e^{-\kappa(T \ \ t-2 \ \ )}\right)$$

(3.16)

Distribution

$$E^T[Y_t|\mathcal{F} \ ] = Y \ \ + \ \eta(\ \ ,t) - \left(\frac{\sigma_X^2}{2} + \frac{\rho\sigma_r\sigma_X}{\kappa}\right)(t - \ \ ) + \frac{\rho\sigma_r\sigma_X}{\kappa}\left(e^{-\kappa(T-t)} - e^{-\kappa(T- \ \ )}\right) \quad (3.17)$$

$$Var^T[Y_t|\mathcal{F} \ ] = \left(\frac{\sigma_r^2}{\kappa^2} + \frac{2\rho\sigma_r\sigma_X}{\kappa} + \sigma_X^2\right)(t - \ \ ) - \frac{\sigma_r^2}{2\kappa^3}\left(3 - 4e^{-\kappa(T- \ \ )} + e^{-2\kappa(t- \ \ )}\right) - \frac{2\rho\sigma_r\sigma_X}{\kappa^2}\left(1 - e^{-\kappa(t- \ \ )}\right)$$

(3.18)

(See [CLE13] for mathematical development)

3.2.3 Joint Density

Given that both our variables $Y_t \; \mathcal{F}$ and $R_t \; \mathcal{F}$ have a Gaussian distribution, to construct their join density we have to find the variance/covariance matrix. For this, it is sufficient to find the quadratic variation between both dynamics (this comes back to multiplying the stochastic terms of our solutions (3.12) an (3.15)).

To do so we use [CLE13] notation.

Pose a vector ψ_t of dimension 2×1 that contains the values of Y_t and R_t respectively.

$$\psi_t = \begin{bmatrix} Y_t \\ R_t \end{bmatrix}$$

Given our previous solutions, and the information at date   and t, where $\; < t$, we can state that our joint distribution is bivariate normal with expected value of

$$\mu_\psi(\;, t) = \begin{bmatrix} \sigma_Y(\;, t) \\ \sigma_R(\;, t) \end{bmatrix}$$

And a variance-covariance matrix of

$$\Sigma_\psi(\;, t) = \begin{bmatrix} \sigma_Y^2(\;, t) & \sigma_{Y,R}(\;, t) \\ \sigma_{Y,R}(\;, t) & \sigma_R^2(\;, t) \end{bmatrix} \tag{3.19}$$

where $\sigma_Y(\;, t)$ and $\sigma_R(\;, t)$ are equal to (3.13) and (3.17) and $\sigma_Y^2(\;, t)$ and $\sigma_R^2(\;, t)$ are defined by (3.14) and (3.18) respectively.

The quadratic variation $\sigma_{Y,R}(\;, t)$ is given by

$$\langle Y, R \rangle_t = \langle \int^t \left(\sigma_Y \rho + \left(\frac{\sigma_r}{\kappa} \left(1 - e^{-\kappa(t-m)} \right) \right) \right) dW_m^1 ; \sigma_r \int^t e^{-\kappa(t-s)} \, dW_m^1 \rangle$$

$$+ \langle \sigma_Y \sqrt{1-\rho^2} \int^t d\, W_m^1 ; \sigma_r \int^t e^{-\kappa(t-s)} \, dW_m^2 \rangle$$

$$= \langle \int^{t} \left(\sigma_Y \rho + \left(\frac{\sigma_r}{\kappa} \left(1 - e^{-\kappa(t-m)}\right) \right) \right) dW_m^1 ; \sigma_r \int^{t} e^{-\kappa(t-s)} \, dW_m^1 \rangle$$

$$= \int^{t} \left(\frac{\sigma_r}{\kappa} \left(1 - e^{-\kappa(t-m)}\right) + \sigma_Y \rho \right) \left(\sigma_r e^{-\kappa(t-s)} \right) dm$$

$$= \frac{\sigma_r^2}{\kappa^2} \left(1 - e^{-\kappa(t- \quad)}\right) - \frac{\sigma_r^2}{2\kappa^2} \left(1 - e^{-2\kappa(t- \quad)}\right) + \frac{2\rho \sigma_r \sigma_Y}{\kappa} \left(1 - e^{-\kappa(t- \quad)}\right) \tag{3.20}$$

3.3 Chebyshev Polynomials

The chosen evaluation methodology for this thesis paper is dynamic programming. This method, due to the discretization of the state variables, computes the value function at the given grid nodes only. To be able to calculate the value function for asset and interest rate values that are in between the grid nodes, we must perform an interpolation, and we have chosen the spectral interpolation.

While doing polynomial interpolation, some approaches, like the Lagrange interpolation, can result in the Runge phenomenon (oscillation in the edges of the interval). Chebyshev polynomials are widely used in approximation theory because they can be coupled with the Gauss-Lobatto points (to avoid the Runge phenomenon and increase the convergence of the error) and the Fast Fourier Transform (to increase the speed of the algorithm).

Remark 1 (First Tchebychev Polynomials) *The first Tchebychev polynomials are*

$$\begin{aligned} T_0(X) &= 1 \\ T_1(X) &= X \\ T_2(X) &= 2X^2 - 1 \\ T_3(X) &= 4X^3 - 3X \\ T_4(X) &= 8X^4 - 8X^2 + 1 \\ T_5(X) &= 16X^5 - 20X^3 + 5X. \end{aligned}$$

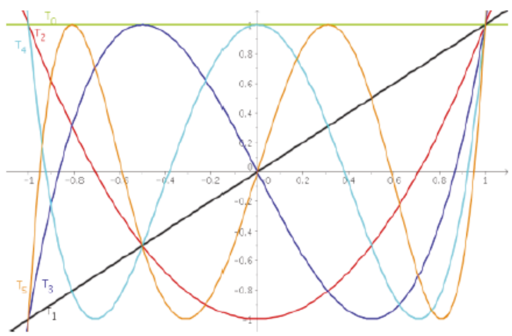


Figure 5.1: First Tchebytchev Polynomials graphical representation

Remark 2 (Particular values)

$$\begin{aligned} \forall k &\in \mathbb{N}\ T_k(1) = 1. \qquad (5.7) \\ \forall k &\in \mathbb{N}\ T_k(-1) = (-1)^k. \end{aligned}$$

Graph taken from [SIL12].

3.3.1 One dimension polynomial

When handling a one variable interpolation problem, the Chebyshev polynomial must be defined over the interval $[-1,1]$, and it can be evaluated by either recursion or trigonometry. Using [CLE13] notation, let $T_k(x)$ be a polynomial of degree k, where $x \in [-1,1]$.

When evaluation by recursion:

$T_0(x) = 1.$

$T_1(x) = x.$

$T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x).$

When evaluating by trigonometry:

$T_k(x) = \cos(karc\ \ s(x)).$

3.3.2 One dimension Interpolation

Consider a function $g(x)$ which we know only at $m+1$ points, $x_i, i = 0, \dots, m$. Chebyshev Interpolation of the function $g(x)$ consists of approximating the function by a polynomial $q_m(x)$ of degree m such that

$$g(x) \simeq q_m(x) = \sum_{k=0}^{m} c_k T_k\,(x) \qquad \forall x \in [-1,1] \tag{3.21}$$

This equation represents the Chebyshev decomposition where T_k represents the different polynomials and c_k the spectral coefficients. To be able to use equation (3.21) we must find the value of all the spectral coefficients, and we do this by resolving a system of linear equations

$$y = \boldsymbol{T}c$$

where

$$y = \begin{bmatrix} v(x_0) \\ v(x_1) \\ \vdots \\ v(x_m) \end{bmatrix}$$

$$c = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_m \end{bmatrix}$$

$$\boldsymbol{T} = \begin{bmatrix} T_0(x_o) & T_1(x_o) & & T_m(x_o) \\ \vdots & \vdots & \ddots & \vdots \\ T_0(x_N) & T_2(x_N) & & T_m(x_m) \end{bmatrix} \tag{3.22}$$

The solution of this linear system can be done by the inversion of matrix $\boldsymbol{T}$ but the cost of inversing a matrix is elevated in terms of computing time or by LU decomposition. Choosing Chebyshev or Gauss-Lobatto nodes allows the inversion of T by *Fast Fourier*

Transform (*FFT*), which is the most efficient (it reduces the number of operations from $O(m^2)$ to $O(ml\ \ g(m))$).

3.3.3 Gauss-Lobatto

As mentioned previously the Runge's phenomenon can be avoided with the use of the Gauss Lobbatto points, which has as a main feature that the points in the optimal grid are not equidistant. They are closer together at the edges of the interval and more spaced in the center. Additionally, the Gauss-Lobatto points, unlike the Chebyshev nodes, do include the extreme values of the grid (-1 and 1). The Gauss-Lobatto points are defined as $x_i \in [x_{min}, x_{ma}\ \ ]$ and $i \in \{0,1, \dots, m\}$:

$$x_i = \frac{1}{2}\left(x_{ma}\ \ + x_{min} + (x_{ma}\ \ - x_{min})\cos\left(\frac{\pi_i}{m}\right)\right). \tag{3.23}$$

3.3.4 Two dimensions Interpolation

Since our model includes two different variables (stochastic log asset value and stochastic interest rates), we need a polynomial with two variables for the interpolation.

Like before, we consider that there is a function $g(y,r)$ (this time with two variables in its argument), which we interpolate over $(m_1+1)\times(m_2+1)$ points. To simplify notation, we assume that $m_1 = m_2 = m$ and that m is an even number, but clearly this need not be the case. When doing interpolation we use a polynomial of degree $= m^2$, $q_m(y,r)$, using the points $\{y_i : i = 0,1, \dots, m\}$ and $\{r_j : j = 0,1, \dots, m\}$ where $(y,r) \in [-1,1]\times[-1,1]$.

$$g(y,r) \simeq q_m(y,r) = \sum_{k=0}^{m}\sum_{l=0}^{m} c_{k,l} T_k(h_1 + d_1 y) T_l(h_2 + d_2 r) \tag{3.24}$$

where we applied a linear transformation to have our boundaries be $[-1,1]$ and the result is:

$$h_1 = -\frac{y_{ma}\ \ + y_{min}}{y_{ma}\ \ - y_{min}}$$

$$d_1 = \frac{2}{y_{ma} \quad - y_{min}}$$

$$h_2 = -\frac{r_{ma} \quad + r_{min}}{r_{ma} \quad - r_{min}}$$

$$d_2 = \frac{2}{r_{ma} \quad - r_{min}}$$

To apply the FFT in our spectral interpolation we must proceed in two stages. The first one consists of rearranging the terms and calculating the internal sum and then external sums as follows.

$$g(y_i, r_j) = \sum_{k=0}^{m} \left(\sum_{l=0}^{m} c_{k,l} T_l(h_2 + d_2 y_i) \right) T_k(h_1 + d_1 r_j) \tag{3.25}$$

We can define as per [CLE13]

$$F_k(y_i) = \sum_{l=0}^{m} c_{k,l} T_l(h_2 + d_2 y_i) \tag{3.26}$$

After calculating $F_k(y_i)$ we can proceed to calculate

$$g(y_i, r_j) = \sum_{k=0}^{m} F_k(y_i)\, T_k(h_1 + d_1 r_j). \tag{3.27}$$

The general idea is to perform a Chebyshev one-dimensional interpolation on dimension r_j for some known points $y_o, y_1, \dots, y_m$ to find $F_k(y_j)$, which then represents the spectral coefficients in a new one-dimensional interpolation on dimension y_i for some known points $r_o, r_1, \dots, r_m$.

3.4 Clenshaw-Curtis Integration

The Clenshaw-Curtis quadrature is a numerical method for calculating the integral of the function $f(x)$ via a change of variable $x = \cos(\theta)$. However, the cosine series that we use to implement this approximation is given by the Chebyshev polynomials. The quadrature is calculated at the Chebyshev nodes. This method weights the value of the function at each node, and uses the fast Fourier transform algorithm to reduce computational time to $\mathcal{O}(m\log m)$.

Define the set of even indices $L = \{0,2,4,\dots,m\}$ (recall that m is an even number). The transition density function from (x_t, t) to (U_T, T) is denoted by $f_{\ ,t}(U,T)$. The holding value of a derivative product D_t is defined as its discounted expected value, which in turn is expressed as the integral of the value of the derivative at the next evaluation date weighted by transition density function.

$$E^{\mathbb{Q}T}_{t-\delta,\ }[D_t(U_t)] \simeq \int_{-\infty}^{\infty} D_t(\ )f_{\ ,t-\delta}(\ ,t)d$$

$$\simeq \sum\nolimits_{l\in L} \frac{2c_l}{1-l^2} \times \frac{x_{ma}\ \ - x_{min}}{2} \tag{3.28}$$

Chapter 4

Model Solution

In Chapter 2, we did a brief introduction of our model. However, since the expected value of equation (2.6) is unknown we proceed to use the numerical approximation method proposed in Chapter 5. The procedure used for this approximation is based on the construction of an observation grid that contains the log of asset price and risk-free interest rate. The first step consists on choosing the boundaries of the grid such that the probability of the log asset value or the interest rate being outside of the grid $[y_{min}, y_{ma}\ \ ]\times$ $[r_{min}, r_{ma}\ \ ]$ is zero or close to zero.

We can now write the expected holding value of the debt as the double integral over the y and x domain, of the future value multiplied by the transition density function.

$$D^{h}{}_{t-\delta}(y_i, r_j) = P_{t-\delta}(r_j, t)\left(\int_{y\quad in}^{y\quad ax}\int_{r\quad in}^{r\quad ax} D_t(y,r) f\ (y,r) dy dr\right) \tag{4.1}$$

Since we are working with gaussian variables, we know that their joint density will be a bi-variate normal distribution with the moments defined in Section 3.2.3. Since we are working with the Vasicek model for interest rates we have a closed form for $P_{t-\delta}(r_j, t)$ and we only need to work on the double integrals. Following the notation and methodology of [SIL16-a] the double integrals are recursively calculated using the Clenshaw-Curtis integration (Section 3.4), approximating the integrand by its truncated Chebyshev series of order m^2, where m is even.

We then use the Chebyshev coefficients to evaluate the integral in a closed-form. The steps needed to approximate the debt holding value are shown below. The methodology for approximating the holding value of the equity is equivalent.

1: **Initialization.**

a) Define the boundaries of the grid for all $i, i = 0, \dots, m$ for $\{y\}$ variable that will follow for the log value of the assets and $j, j = 1, \dots, m$ for $\{r\}$ variable that will follow for the interest rate. Define the set L of even indices.

b) Create the grid using the Gauss-Lobatto nodes, explained in Chapter 3, equation (3.23)

$$y_i = \frac{1}{2}\left(y_{ma} \quad + y_{min} + (y_{ma} \quad - y_{min})\cos\left(\frac{\pi i}{m}\right)\right), \qquad i = 0,1, \dots, m$$

$$r_j = \frac{1}{2}\left(r_{ma} \quad + r_{min} + (r_{ma} \quad - r_{min})\cos\left(\frac{\pi j}{m}\right)\right), \qquad j = 0,1, \dots, m$$

c) Denote Y_i as the set of Gauss-Lobatto points for factor $i, i = 1,2$. The set of interpolation points is then defined as $Y = Y_1 \times Y_2$.

d) For every maturity node in Y, calculate the value of the debt $D_T(y_i, r_j)$ using formula (2.3)

2: **Approximation.** Once the value of the debt is known on Y at t, we step back in time so that $t = t - \delta$, and proceed to approximate the holding value.

a) The holding value of the debt is equal to its discounted expected future value. The expected value can be calculated using its transition density function $f_{i,j,t-\delta}(y, r)$

b) For each coordinate (y_i, r_j) of Y, calculate the function

$$g_{i,j,t-\delta}(y_k, r_l) = f_{i,j,t-\delta}(y_k, r_l) \times D_t(y_k, r_l) \tag{4.2}$$

c) Using the Fast Fourier Transform (FFT), find the multidimensional truncated Chebyshev series of order m interpolating the values $g_{i,j,t-\delta}(y_k, r_l)$ on Y.

d) For every combination of nodes in the set Y evaluate the holding value (Formula 4.1), using the Clenshaw-Curtis integration for the forward expectation.

$$D^h{}_{t-\delta}(y_i, r_j) = P_{t-\delta}(r_j, t)\left(\int_{y\quad in}^{y\quad ax}\int_{r\quad in}^{r\quad ax} D_t(y, r) f_{i,j.t-\delta}\,(y, r)dydr\right)$$

$$\simeq \int_{y\quad in}^{y\quad ax}\int_{r\quad in}^{r\quad ax} g_{i,j,t-\delta}(y, r)dydr$$

$$= \frac{(y_{ma}\quad - y_{min})}{2}\ \frac{(r_{ma}\quad - r_{min})}{2}\sum_{k\in L}\sum_{l\in L}\frac{4c_{k,l}}{(1-k^2)(1-l^2)} \qquad (4.3)$$

3: **Final step.** Once we have calculated the expected holding value of the debt and the equity, in the T-forward risk-neutral measure, at $t = t - \delta$, we calculate the value of the debt using formula (2.8).

$$D_{t-\delta}(y_i, x_j) = \mathbb{I}_{(W^h{}_{t-\delta}(y_i,\ _j)-\mathrm{cP}\le 0)}\min\left(\ W_{t-\delta}(y_i, x_j)\ ; \omega P\right)$$
$$+\ \ \mathbb{I}_{(W^h{}_{t-\delta}(y_i,\ _j)-\mathrm{cP}\quad 0)}\left(D^h{}_{t-\delta}(y_i, x_j) + cP\right).$$

4: **Recursion.** Repeat 2 and 3, until $t = 0$.

Chapter 5

Theoretical analysis of the model

In this section we present the tests we performed to check the accuracy of our model and solution algorithm. First, we compare our results to [M74], second, we replicate the plain vanilla formula for coupon paying bonds and discuss the impact of the capital structure on correlation, next we perform an analysis of each parameter on our model and finally we visualize our endogenous barrier of default through the exogenous boundary of [LS95] and analyse its behaviour.

5.1 Merton Model

We adjust some parameters (no coupons and very small interest rate volatility) to compare the results obtained using our algorithm to the closed-form for debt evaluation of the Merton model, in this case there is only one step in our DP approach and the difference in value for $m = 100$ is attributed to interpolation error. Tables 1 and 2 present the convergence of our Dynamic Program to the [M74] closed-form solution for various asset values and interest rate levels. These tables show that the DP algorithm closely replicates the closed-form price.

Table 1: Convergence of our solution method to the closed-form of the Merton model as interpolation points m increase, for various asset values.

V	m=20	m=30	m=40	m=50	m=60	m=70	m=80	m=90	m=100	m=200	Closed Form
1.0	0.6819	0.6782	0.6770	0.6764	0.6761	0.6760	0.6759	0.6758	0.6757	0.6756	0.6758
1.5	0.7621	0.7593	0.7583	0.7579	0.7577	0.7575	0.7574	0.7574	0.7573	0.7572	0.7573
2.0	0.7912	0.7894	0.7888	0.7885	0.7884	0.7883	0.7882	0.7882	0.7882	0.7881	0.7881
2.5	0.8026	0.8015	0.8011	0.8010	0.8009	0.8008	0.8008	0.8008	0.8008	0.8007	0.8007
3.0	0.8073	0.8067	0.8065	0.8064	0.8063	0.8063	0.8063	0.8063	0.8063	0.8062	0.8063
3.5	0.8094	0.8090	0.8089	0.8088	0.8088	0.8088	0.8088	0.8088	0.8088	0.8088	0.8089
4.0	0.8103	0.8101	0.8100	0.8099	0.8099	0.8099	0.8099	0.8099	0.8099	0.8099	0.8102
4.5	0.8105	0.8104	0.8103	0.8103	0.8103	0.8103	0.8103	0.8103	0.8103	0.8102	0.8108
5.0	0.8103	0.8102	0.8102	0.8102	0.8101	0.8101	0.8101	0.8101	0.8101	0.8101	0.8112

Parameters : rbar =0.05, a=0.4, sigma_r=0.0001, sigma_V=0.3, rho=0, T=4.172, n=1, delta=1, delta0=T, C=0, P=1, eta=1

Table 2: Convergence of our solution method to the closed-form of the Merton model as interpolation points m increase, for various interest rate levels.

r	m=20	m=30	m=40	m=50	m=60	m=70	m=80	m=90	m=100	m=200	Closed Form
0.01	0.9126	0.9117	0.9115	0.9114	0.9113	0.9113	0.9113	0.9113	0.9113	0.9114	0.9129
0.02	0.8807	0.8797	0.8793	0.8792	0.8792	0.8791	0.8791	0.8791	0.8791	0.8792	0.8807
0.03	0.8478	0.8474	0.8474	0.8475	0.8476	0.8476	0.8477	0.8477	0.8477	0.8476	0.8491
0.04	0.8162	0.8164	0.8167	0.8170	0.8169	0.8168	0.8168	0.8168	0.8169	0.8168	0.8182
0.05	0.7860	0.7867	0.7870	0.7867	0.7867	0.7868	0.7868	0.7867	0.7867	0.7867	0.7881
0.06	0.7569	0.7579	0.7573	0.7574	0.7575	0.7574	0.7574	0.7575	0.7574	0.7575	0.7588
0.07	0.7290	0.7289	0.7288	0.7290	0.7289	0.7290	0.7289	0.7289	0.7289	0.7289	0.7302
0.08	0.7022	0.7010	0.7015	0.7012	0.7013	0.7012	0.7013	0.7012	0.7013	0.7012	0.7025
0.09	0.6746	0.6743	0.6743	0.6745	0.6743	0.6744	0.6744	0.6744	0.6744	0.6744	0.6756
0.1	0.6482	0.6486	0.6483	0.6483	0.6484	0.6483	0.6483	0.6483	0.6483	0.6483	0.6495

Parameters : a=0.4, sigma_r=0.0001, sigma_V=0.3, rho=0, V=2, T=4.172, n=1, delta=1, delta0=T, C=0, P=10, eta=1

Table 3 below shows the impact of various parameters on the results of the dynamic program and the solution of the [M74] model. Since both results are obtained under the assumption of constant interest rates, we observe constant prices whilst modifying interest rate parameters. (C.F = Closed-form, O.A = Our Approach).

Table 3: Impact of parametrization of stochastic interest rates in the Merton model.

Parameter	Method	r=0.01	r=0.02	r=0.03	r=0.04	r=0.05	r=0.06	r=0.07	r=0.08	r=0.09	r=0.1
Default	C.F	0.9113	0.8791	0.8477	0.8169	0.7867	0.7574	0.7289	0.7013	0.6744	0.6483
Default	O.A	0.8680	0.8410	0.8142	0.7877	0.7614	0.7356	0.7102	0.6853	0.6608	0.6369
a ↓ 0.1	C.F	0.8680	0.8410	0.8142	0.7877	0.7614	0.7356	0.7102	0.6853	0.6608	0.6369
a ↓ 0.1	O.A	0.9113	0.8791	0.8477	0.8169	0.7867	0.7574	0.7289	0.7013	0.6744	0.6483
eta 0.5	C.F	0.8680	0.8410	0.8142	0.7877	0.7614	0.7356	0.7102	0.6853	0.6608	0.6369
eta 0.5	O.A	0.8653	0.8409	0.8161	0.7830	0.7588	0.7345	0.7102	0.6860	0.6580	0.6350
rho 0.5	C.F	0.8680	0.8410	0.8142	0.7877	0.7614	0.7356	0.7102	0.6853	0.6608	0.6369
rho 0.5	O.A	0.9113	0.8791	0.8476	0.8168	0.7867	0.7574	0.7289	0.7013	0.6744	0.6483

Default : rbar =0.05, a=0.4, sigma_r=0.0001, sigma_V=0.3, rho=0, V=2, T=4.172, n=1, delta=1, delta0=T, C=0, P=1, eta=1, m=100

5.2 Plain Vanilla Formulation

In addition to comparing our model performance to the [M74], we compared it to the plain vanilla bond price, which is known in closed-form, when default risk is negligible. To eliminate default risk, it suffices to make the value of the firm very large (i.e. V=200) relative to the principal of the debt, so that the probability of the value of the assets being below the exogenous barrier of default can be assumed to be close to zero. Results are presented in Table 4. We observe convergence of our model to the plain vainilla formula when default risk is negligible. The analysis of parameterization is consistent with the analysis done below in Section 5.3.

Table 4: Impact of parametrization of stochastic interest rates for a scenario without default risk.

Parameter	Method	r=0.01	r=0.02	r=0.03	r=0.04	r=0.05	r=0.06	r=0.07	r=0.08	r=0.09	r=0.1
Default	C.F	1.1497	1.1282	1.1071	1.0864	1.0661	1.0462	1.0267	1.0076	0.9888	0.9704
Default	O.A	1.1498	1.1282	1.1071	1.0864	1.0661	1.0462	1.0267	1.0076	0.9888	0.9705
rbar ↓ 0.01	C.F	1.1720	1.1501	1.1285	1.1074	1.0867	1.0664	1.0465	1.0269	1.0078	0.9890
rbar ↓ 0.01	O.A	1.1720	1.1500	1.1285	1.1074	1.0866	1.0663	1.0464	1.0269	1.0078	0.9890
a ↓ 0.1	C.F	1.1837	1.1579	1.1326	1.1079	1.0837	1.0601	1.0371	1.0145	0.9925	0.9710
a ↓ 0.1	O.A	1.1839	1.1581	1.1328	1.1080	1.0839	1.0603	1.0372	1.0147	0.9927	0.9712
σ_r ↑ 0.1	C.F	1.3124	1.2876	1.2633	1.2395	1.2162	1.1933	1.1708	1.1488	1.1273	1.1061
σ_r ↑ 0.1	O.A	1.3287	1.3040	1.2796	1.2557	1.2323	1.2093	1.1867	1.1645	1.1427	1.1213
rho -0.5	C.F	1.1497	1.1282	1.1071	1.0864	1.0661	1.0462	1.0267	1.0076	0.9888	0.9704
rho -0.5	O.A	1.1498	1.1283	1.1072	1.0865	1.0662	1.0463	1.0268	1.0076	0.9889	0.9705
σ_V ↑ 0.1	C.F	1.1497	1.1282	1.1071	1.0864	1.0661	1.0462	1.0267	1.0076	0.9888	0.9704
σ_V ↑ 0.1	O.A	1.1497	1.1282	1.1071	1.0864	1.0661	1.0462	1.0267	1.0076	0.9888	0.9704

Default : rbar =0.05, a=0.4, sigma_r=0.13, sigma_V=0.3, rho=0.5, V=200, T=4.172, n=5, delta=1, delta0=0.172, C=0.0425, P=1, eta=1, m=100

5.3 Convergence and Parametrization tests

After ensuring our approach could replicate existing models in the literature, we proceeded to evaluate the convergence of the dynamic program and the impact of parameterization. As shown in Tables 5 and 6, for a principal of 1, and asset value of 2 and 4 respectively, convergence is achieved for $m = 100$. In Table 6, the relative increase in accuracy achieved by increasing m from 100 to 200 is 0.02%.

Now we proceed to analyzing the effects of the key parameters in our model. Each parameter will be analyzed through its effect on short and long maturity bonds.

Parameters:

-The effect of maturity $\{T\}$: The value of the bond is positively correlated to maturity in general terms: As maturity increases there is more time for the variability of the assets to dissipate, more coupon payments to discount and more time for the interests rates to come back to their long-run mean. Since expected value of the company increases exponentially through time, the value of the bond increases with maturity. However, the effect of maturity depends on the other parameters. We explore its impact coupled with the others parameters below.

-The effect of asset value $\{V\}$: As the asset value of the company increases from 2 to 4, in Tables 5 and 6, we can see an increase in the value of the debt. The reason behind this effect is straightforward, the higher the value of the assets, the further away the company is from a potential default, and the debt becomes more secure and therefore more valuable.

-The effect of interest rates $\{\bar{r}\}$: When we increase the long run interest rate average, the value of the debt decreases. According to [LS95] there are two effects that drive the price of bonds, the discounting effect (as interest rates increase, the price of the zero coupon bond decreases) and the decrease in the risk-neutral probability (as interest rates increase there is less volatility and the debt value increases). For the parameters in Figure 1 the discounting effect is dominant.

Table 5: Debt prices for various values of m (number of interpolation points) and Interest Rate with an Asset Value of 2.

r	m=30	delta	m=40	delta	m=50	delta	m=60	delta	m=70	delta	m=80	delta	m=90	delta	m=100	delta	m=200	CPU time
0.01	4.51	1.11%	4.46	1.23%	4.41	1.36%	4.35	1.49%	4.28	1.62%	4.21	1.75%	4.14	1.88%	4.06	2.00%	3.98	13.630
0.02	1.84	1.38%	1.81	1.41%	1.78	1.45%	1.76	1.49%	1.73	1.53%	1.71	1.57%	1.68	1.60%	1.65	1.64%	1.62	12.320
0.03	1.35	1.39%	1.33	1.42%	1.31	1.44%	1.29	1.46%	1.27	1.49%	1.26	1.51%	1.24	1.54%	1.22	1.56%	1.20	12.630
0.04	1.13	1.40%	1.11	1.42%	1.10	1.44%	1.08	1.46%	1.07	1.48%	1.05	1.50%	1.03	1.52%	1.02	1.54%	1.00	13.350
0.05	1.04	1.42%	1.02	1.43%	1.01	1.45%	0.99	1.46%	0.98	1.48%	0.97	1.49%	0.95	1.50%	0.94	1.52%	0.92	13.670
0.06	1.01	1.43%	0.99	1.44%	0.98	1.45%	0.97	1.46%	0.95	1.47%	0.94	1.49%	0.92	1.50%	0.91	1.51%	0.90	12.780
0.07	1.00	1.43%	0.99	1.44%	0.97	1.45%	0.96	1.46%	0.95	1.47%	0.93	1.48%	0.92	1.49%	0.90	1.50%	0.89	13.170
0.08	1.00	1.43%	0.99	1.44%	0.97	1.45%	0.96	1.46%	0.94	1.47%	0.93	1.48%	0.92	1.49%	0.90	1.50%	0.89	13.450
0.09	1.00	1.43%	0.98	1.44%	0.97	1.45%	0.96	1.46%	0.94	1.47%	0.93	1.48%	0.91	1.49%	0.90	1.50%	0.89	13.770
0.1	1.00	1.43%	0.98	1.44%	0.97	1.45%	0.96	1.46%	0.94	1.47%	0.93	1.48%	0.91	1.49%	0.90	1.50%	0.89	12.720

Default : rbar =0.05, a=0.4, sigma_r=0.13, sigma_V=0.3, rho=0.5, T=4.172, n=5, delta=1, delta0=0.172, C=0.0425, P=1, eta=1

Table 6: Debt prices for various values of m (number of interpolation points) and Interest Rate with an Asset Value of 4.

r	m=30	delta	m=40	delta	m=50	delta	m=60	delta	m=70	delta	m=80	delta	m=90	delta	m=100	delta	m=200	CPU time
0.01	2.02	25.78%	1.50	16.14%	1.26	8.04%	1.16	2.82%	1.13	0.73%	1.12	0.13%	1.12	0.00%	1.12	0.02%	1.12	13.730
0.02	1.99	25.81%	1.48	16.16%	1.24	8.07%	1.14	2.84%	1.11	0.73%	1.10	0.13%	1.10	0.00%	1.10	0.02%	1.10	13.310
0.03	1.96	25.83%	1.45	16.19%	1.22	8.09%	1.12	2.85%	1.09	0.74%	1.08	0.13%	1.08	0.00%	1.08	0.02%	1.08	12.360
0.04	1.93	25.84%	1.43	16.21%	1.20	8.10%	1.10	2.86%	1.07	0.75%	1.06	0.13%	1.06	0.00%	1.06	0.02%	1.06	12.930
0.05	1.89	25.84%	1.40	16.22%	1.18	8.12%	1.08	2.87%	1.05	0.75%	1.04	0.14%	1.04	0.00%	1.04	0.02%	1.04	12.670
0.06	1.86	25.83%	1.38	16.23%	1.16	8.12%	1.06	2.88%	1.03	0.75%	1.02	0.14%	1.02	0.00%	1.02	0.02%	1.02	13.810
0.07	1.83	25.81%	1.36	16.24%	1.14	8.12%	1.04	2.88%	1.01	0.75%	1.01	0.14%	1.00	0.00%	1.00	0.02%	1.00	12.920
0.08	1.80	25.78%	1.33	16.24%	1.12	8.12%	1.03	2.88%	1.00	0.75%	0.99	0.14%	0.99	0.01%	0.99	0.02%	0.99	12.470
0.09	1.76	25.74%	1.31	16.23%	1.10	8.11%	1.01	2.87%	0.98	0.75%	0.97	0.14%	0.97	0.01%	0.97	0.02%	0.97	12.900
0.1	1.73	25.69%	1.29	16.22%	1.08	8.10%	0.99	2.86%	0.96	0.75%	0.95	0.14%	0.95	0.01%	0.95	0.02%	0.95	13.080

Default : rbar =0.05, a=0.4, sigma_r=0.13, sigma_V=0.3, rho=0.5, T=4.172, n=5, delta=1, delta0=0.172, C=0.0425, P=1, eta=1

-The effect of speed of mean-reversion$\{\alpha\}$: As the speed of mean-reversion decreases, the interests rates in the Vasicek dynamic are more volatile; they can stay away from the long-run mean for longer periods of time (Figure 2). For shorter maturities, this additional volatility decreases the debt value. For longer maturities, the interest rate has more time to eventually return to the long-run average and therefore value increases, as prices are positively related to maturity. The current level of interest rate also plays an important role. When $r_0 > \bar{r}$, for shorter maturities the tendency is for r_0 to decrease, creating a downward shift in the risk-neutral probability of default, making the investment a riskier one, and hence with less value. When $r_0 < \bar{r}$, for shorter maturities the tendency is for r_0 to increase, creating an upward shift in the risk-neutral probability of default, making the investment less risky, and hence more valuable.

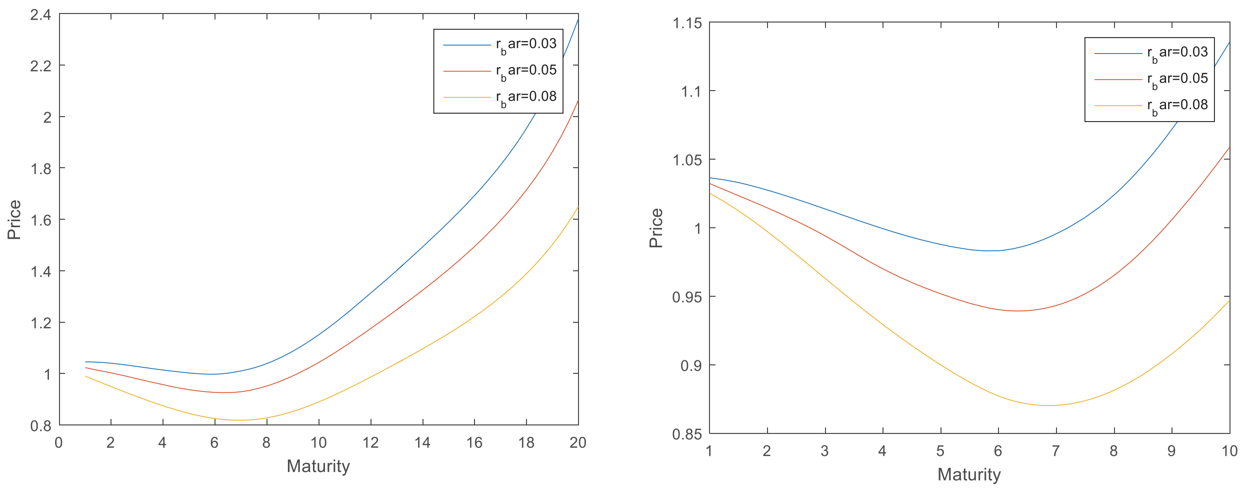
-The effect of interest rate volatility $\{\sigma_r\}$: For the parameters in Figure 3 as the volatility of interest rates decreases, the value of the corporate bond decreases. This can be because the for these specific parameters the discounting effect is dominant.

-The effect of asset value volatility $\{\sigma_V\}$: As the volatility of asset value increases, the price of the coporate debt decreases. This is explained through the additional risk the investor undertakes. This effect is amplified with maturity because the value of the firm (with high volatility) has more time to fall underneath the default threshold (Figure 4).

-The effect of correlation $\{\rho\}$: We can observe that as correlation increases from 0 to 0.5, the debt value decreases because there is an added covariance term to the total variance of the coporate bond [LS95]. The reverse effect is present when correlation decreases from 0 to -0.5, therefore reducing volatility and increasing the value of the bond (Figure 5). However, it is important to highlight here that this effect can be attenuated through the level of leverage of the firm. As we can observe in Figure 6, we have increased the asset value to 40, and the variability in the price for various correlation values is substantially reduced, specially for shorter maturities. This leaves us with one key conclusion: There are capital structure effects to consider when evaluating corporate bonds.

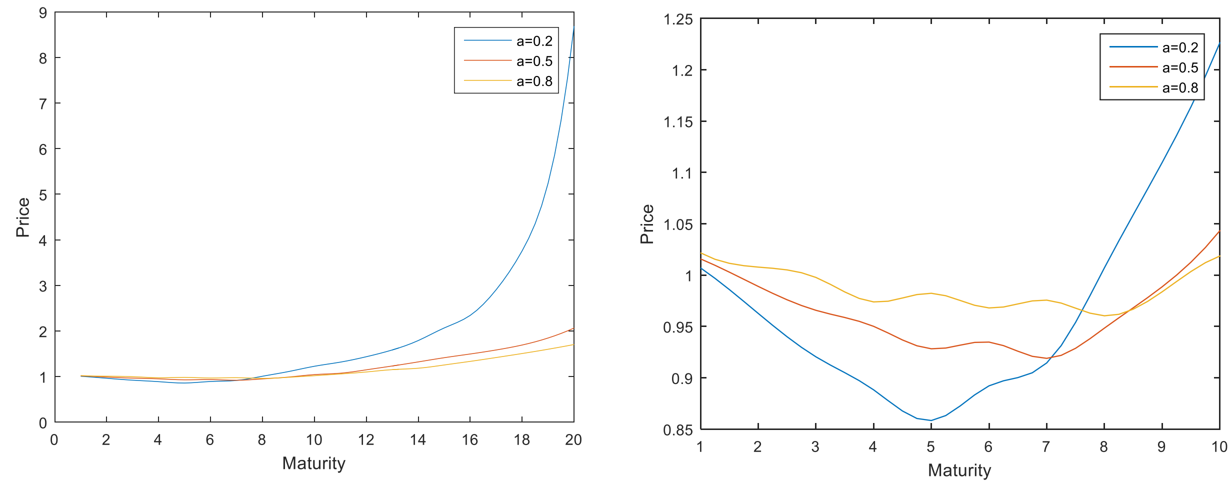
-The effect of the recovery rate $\{\omega\}$: as the recovery rate decreases, the value of the debt decreases, because a lower recovery rate implies a riskier investment (Figure 7). However, when maturity increases, leaving everything else constant, the value of the debt increases because the risk can be attenuated through time and the probability of default decreases.

Figure 1. Value of the corporate debt for various values of $\bar{r}$, the long run interest rate average, for a debt principal of 1 and an asset value of the company of 2.



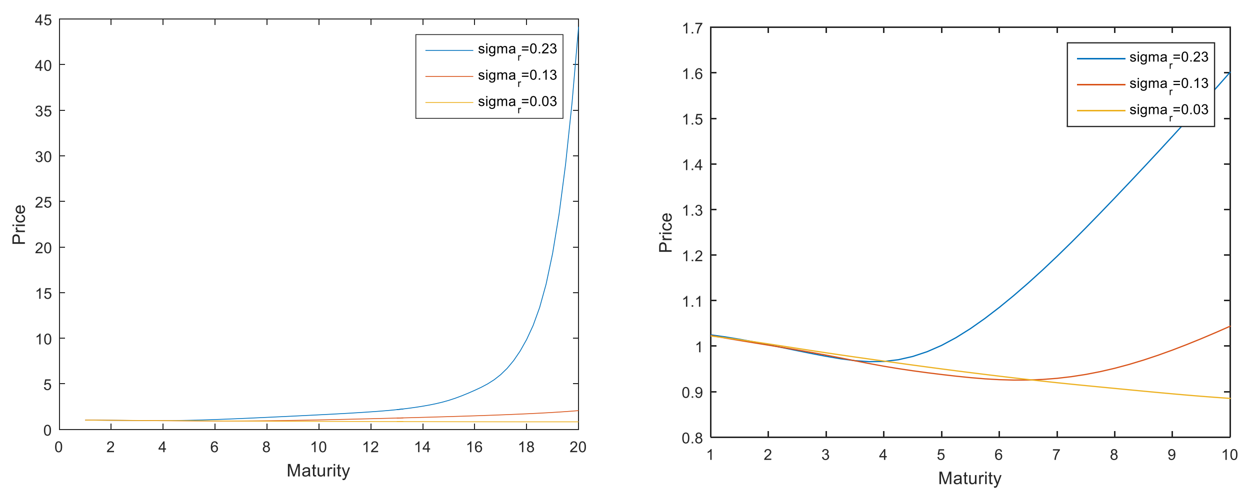
Parameters: a=0.4, sigma_r=0.13; r0 =0.03; sigma_V=0.3; rho=0.5; V0_list=2; C=0.0425; P=1; ω =0.5; m=100;

Figure 2. Value of the corporate debt for various values of α, the mean-speed reversion, for a debt principal of 1 and an asset value of the company of 2.



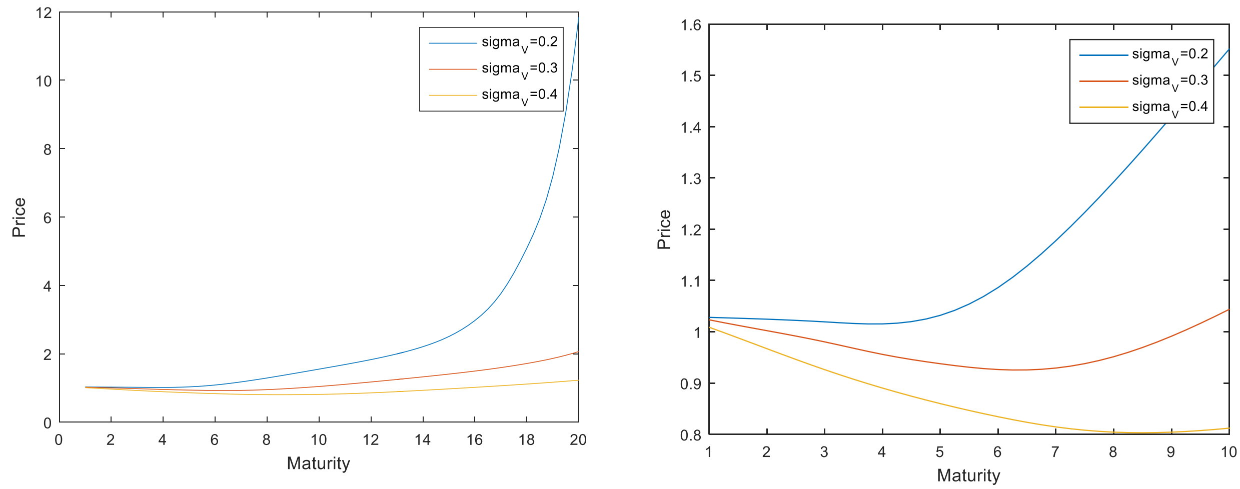
Parameters: rbar=0.05, sigma_r=0.13; r0 =rbar; sigma_V=0.3; rho=0.5; V0_list=2; C=0.0425; P=1; ω =0.5; m=100;

Figure 3. Value of the corporate debt for various values of σ_r, volatility of interest rates, for a debt principal of 1 and an asset value of the company of 2.



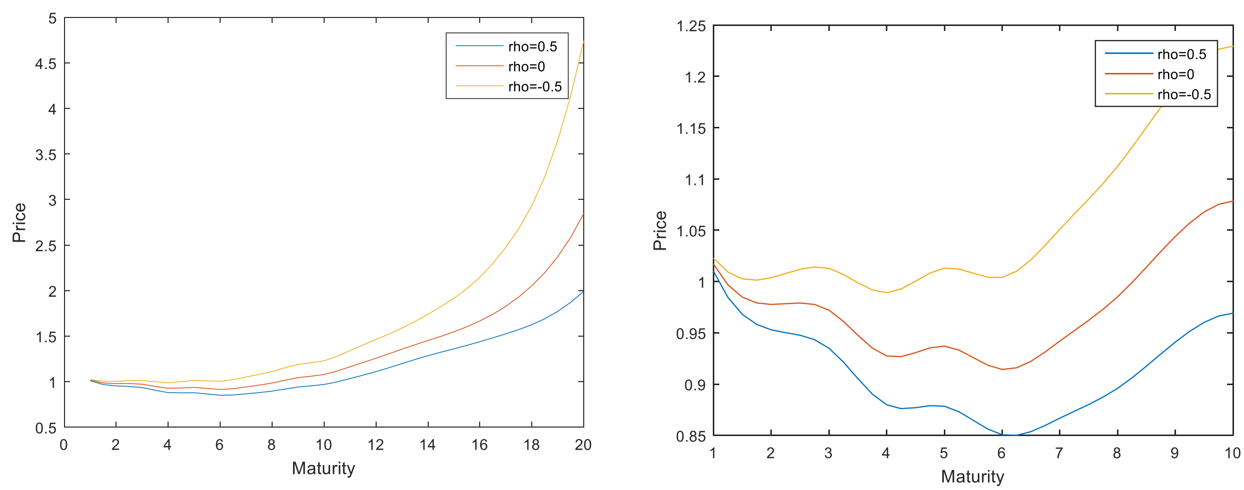
Parameters: rbar=0.05, a=0.4, r0 =rbar; sigma_V=0.3; rho=0.5; V0_list=2; C=0.0425; P=1; ω =0.5; m=100;

Figure 4. Value of the corporate debt for various values of σ_V, volatility of assets, for a debt principal of 1 and an asset value of the company of 2.



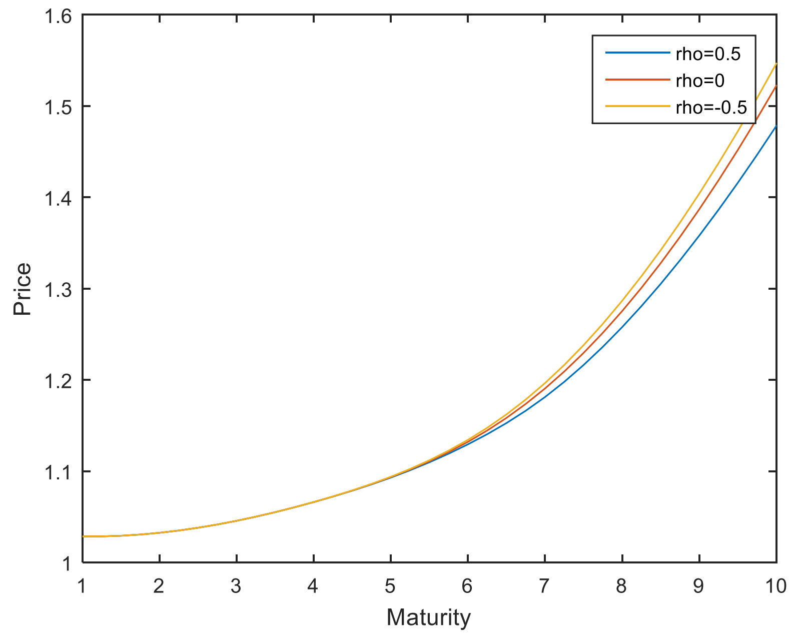
Parameters: rbar=0.05, a=0.4, sigma_r=0.13; r0 =rbar; rho=0.5; V0_list=2; C=0.0425; P=1; ω =0.5; m=100;

Figure 5. Value of the corporate debt for various values of ρ, correlation, for a debt principal of 1 and an asset value of the company of 2.



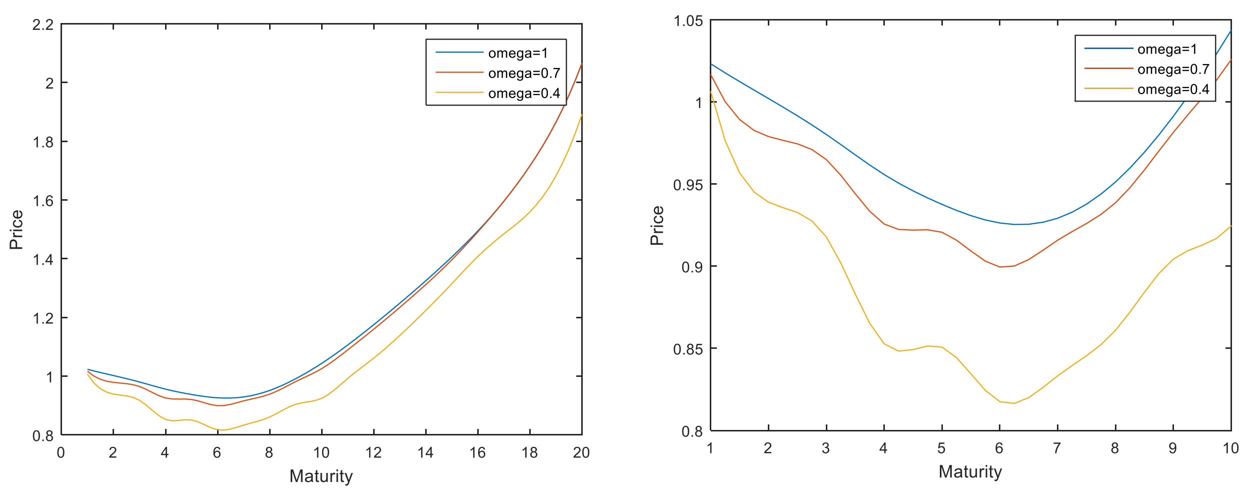
Parameters: rbar=0.05, a=0.4, sigma_r=0.13; r0 =rbar; sigma_V=0.3; V0_list=2; C=0.0425; P=1; ω =0.5; m=100;

Figure 6. Value of the corporate debt for various values of ρ, correlation, for a debt principal of 1 and an asset value of the company of 40.



Parameters: rbar=0.05, a=0.4, sigma_r=0.13; r0 =rbar; sigma_V=0.3; V0_list=40; C=0.0425; P=1; ω=0.5; m=100;

Figure 7. Value of the corporate debt for various values of ω, recovery rate, for a debt principal of 1 and an asset value of the company of 2.



Parameters: rbar=0.05, a=0.4, sigma_r=0.13; r0 =rbar; sigma_V=0.3; rho=0.5; V0_list=2; C=0.0425; P=1; m=100;

5.4 The LS95 Exogenous Barrier Replication

Our DP approach uses an endogenous barrier of default; before maturity the barrier is the coupon payment and at maturity the coupon payment plus the principal of the debt. On the other hand, [LS95] uses an exogenous constant barrier of default which is not disclosed in the paper. To understand the behaviour of the exogenous barrier we replicate the price of a risky discount bond of [LS95] and then varied certain parameters and analyzed their impact on the barrier. Table 7 shows our price replication results. These are representative of the price of the discount bond in [LS95] page 799.

Table 7: Price of a risky discount bond of [LS95] with the same parameters as page 799.

	Maturity			
rate	1 year	2 year	3 year	4 year
0.04	0.28	0.23	0.20	0.18
0.07	0.30	0.24	0.21	0.19
0.10	0.32	0.26	0.22	0.20

After confirming the accuracy of our replication code, we proceed to calculate the value of the debt under our DP approach with the default parameters presented in Table 8. We then proceed to solve for the exogenous barrier of default that matches the [LS95] debt value to our calculated debt value (using *fsolve)*. This allows us to visualize what would have been the exogenous barrier of default of the [LS95] model and analyse it. We proceed to do this analysis in two separate categories. In the first section we analyse the barrier for Zero Coupon bonds and in the second section for Coupon paying bonds (which according to [LS95] can be considered as a portfolio of zero coupon bonds, where at each coupon date we have a zero coupon bond with the face value equaling the coupon payment, and at maturity the face value equal to the principal plus the coupon).

Zero Coupon Bonds:

For Zero Coupon bonds, we can observe that the default barrier is, for all maturities, smaller than the face value of the debt and it is decreasing in maturity. This occurs because in our model default can only occur at coupon dates and assumes that coupon payments are dependent of one another. These two effects combined reduce the probability of default in our model. When probability of default is smaller, we see that the [LS95] exogenous barrier decreases in time, to compensate for the reduction of risk. On the other hand, as interest rates increase the price of the debt increases because of the upward drift in the risk-neutral probability of default. This causes an increase in the barrier of default because even though the barrier does not necessarily have to equal the face value of the debt, they are both related [CDG01].

Table 8: The constant barrier of default for different maturities and spot rates for a zero coupon bond.

r\T	1yr	2yr	3yr	4yr	5yr	6yr	7yr	8yr	9yr	10yr
0.03	0.9388	0.8973	0.8771	0.8370	0.8170	0.7697	0.7733	0.7390	0.7084	0.7325
0.04	0.9389	0.8976	0.8778	0.8379	0.8182	0.7710	0.7748	0.7405	0.7098	0.7342
0.07	0.9396	0.8987	0.8800	0.8409	0.8218	0.7747	0.7792	0.7449	0.7141	0.7393
0.08	0.9400	0.8990	0.8807	0.8418	0.8230	0.7760	0.7806	0.7463	0.7156	0.7410
0.1	0.9411	0.8997	0.8821	0.8438	0.8254	0.7784	0.7835	0.7492	0.7184	0.7443

Default : rbar =0.06, a=1, sigma_r=0.0316, sigma_V=0.2, rho=-0.25, V=1.5, n=1, delta=1, delta0=T, C=0, P=1, eta=0.9, m=100, w=0.5

Coupon Bonds:

The first difference to the zero coupon bond evaluation is that here the value of the boundary is no longer smaller than the principal, in this case our boundary seems to be below the asset value of the firm; $V = 1.5$. Another difference is that the constant default boundary of [LS95] is increasing in maturity. This is because as maturity increases the number of coupons payments the compnay must honor increases. Therefore there is an accumulation of all the contractual obligations the firm has to make, increasing default boundary. As interest rates increase the default barrier increases and the reasoning is consistant with the explanation provided for zero coupon bonds.

Table 9: The constant barrier of default for different maturities and spot rates for an eight percent (8%) coupon bond.

r\T	1yr	2yr	3yr	4yr	5yr	6yr	7yr	8yr	9yr	10yr
0.03	1.0262	1.1209	1.1664	1.2163	1.2668	1.3127	1.3567	1.3968	1.4383	1.4753
0.04	1.0262	1.1252	1.1715	1.2216	1.2719	1.3175	1.3608	1.4001	1.4402	1.4759
0.07	1.0262	1.1390	1.1873	1.2379	1.2876	1.3317	1.3730	1.4097	1.4462	1.4781
0.08	1.0262	1.1440	1.1928	1.2434	1.2928	1.3365	1.3770	1.4128	1.4481	1.4790
0.1	1.0264	1.1542	1.2040	1.2546	1.3032	1.3458	1.3849	1.4189	1.4520	1.4807

Default : rbar =0.06, a=1, sigma_r=0.0316, sigma_V=0.2, rho=-0.25, V=1.5, n=1, delta=1, delta0=T, C=0.08, P=1, eta=0.5, m=100, w=0.5

Chapter 6

Data Description

Like [LS95] we perform an empirical test of the relevance of stochastic interest rates and their impact through correlation. To build the dataset we used the Trace, Compustat, Crsp and Fisd databases from the Wharton Research Data Services (WRDS) research platform. These databases are built with daily trading data, which can contain reporting errors and therefore may misleadingly represent a more liquid market. This is why we have used the Jens Dick-Nielsen algorithm to clean our data. To compile the information, we have chosen to use SAS software due to its efficiency at sequential data access, processing power and database access through SQL. For data analysis, data manipulation and summary graphics and tables we have chosen to use *Matlab* software because its basic data element is the matrix which facilitates data manipulation, its user-friendly graphical interactive tools, its detailed documentation, and the numerous built in algorithms.

6.1 Jens Dick- Nielsen Algorithm

According to Jens Dick-Nielsen [DIC09] in his paper "Liquidity Biases in Trace", there is an overestimation of the liquidity of the corporate bond markets because close to 7.7% of the records in the Trade Reporting and Compliance Engine (Trace) are reporting errors. The reported errors are mainly: a record that has been input twice, identified by the same message sequence number; a correction of a previous record done in the same day, it can be either a cancellation or a correction (if we have a cancellation both records should be deleted, if we have a correction only the original should be deleted); or a full reversal of the record done at a later date, both the reversal and the original should be deleted. If we fail to account for these errors, this will lead to double counting and hence higher liquidity.

Even though this algorithm attempts to reduce reporting errors, some limitations endure. For instance, it is not always possible to match the same-day corrections or the reversals to the original report. In addition, it is possible that a broker breaks down a large trade into smaller trades, all for the same client, where the negotiated price was done on the larger package rather than on the small trades. Finally, manual and human error can occur, where traders can, by mistake, input the wrong date, rate, time record, etc. However, the performance of this algorithm remains significant; the algorithm is capable of eliminating 7.5% of the total 7.7% error rate. Note that the algorithm is available to the public.[1]

6.2 The SAS Library

Using SAS, we have managed to merge all the WRDS data sources mentioned above. We began with all corporate bonds transactions since July 2002 recorded in Trace. Our database contained over 99 million observations to date (November 2014) excluding agency (144a) trades. The exclusion of agency transactions is a measure taken to avoid an overstatement of the liquidity in the bond market [DIC09]. When an agent facilitates the transaction between a customer and another broker, he usually charges a commission, which is not always visible in Trace, and therefore the system ends up with two consecutive records with the same price. After implementing the Dick-Nielsen cleaning algorithm to our database, we have implemented other filters to retain only the data relevant to our analysis. Table 10 describes the evolution of our data as we implemented each filter.

[1] How to Clean Trace data, Department of Finance Copenhagen Business School, URL: http://sf.cbs.dk/jdnielsen/how_to_clean_trace_data

Table 10. Data observations at each filter step.

Step	Category or Filter	No. Obs
0	Trace initial data (since July 2002)	99,465,669
1	Eliminate true duplicates	99,461,324
2	Eliminate transactions that where reversed	98,095,852
3	Eliminate same day corrections	94,072,285
4	Leave only closing price & average daily price	15,092,763
5	Fixed coupon, non-callable, non-redeemable & positive debt principal	658,464
6	Bonds with available quarterly accounting data & S&P credit rating	229,302
7	Bonds with available asset value and volatility (parameters obtained from [SIL16-b])	88,099
8	Bonds with available credit spread change and interest rate change. Deleted all first entries where we could calculate the change from one period to the next	88,738

Our final daily sample consists of 88,738 filings from July 2002 to November 2014, which is our sample period. Our final monthly sample consists of 11,157 filings. These entries are the averages of our dataset per Cusip9, per month. From these filings, we proceed to compute the monthly average of prices, asset returns, asset volatility, etc., per industry and credit rating to be able to replicate the regression from [LS95], which is done with monthly credit spread changes, monthly interest rate changes and monthly stock return. The size of the final sample is 2,446. Table 11 presents the summary statistics of our final sample data on a daily basis, and Table 12 presents the summary statistics of our final sample data on an average monthly basis.

We can observe in these tables that there is a larger variability in the data for daily filings. For example, we observe for the Transportation and Communications industry, credit rating CC, a 45.72% increase in standard deviation of credit spreads for daily data with respect to monthly data. For the Manufacturing industry, daily data, credit rating BBB, we observe skewness in the credit spreads of 106.08 implying that the data is asymmetric and a kurtosis in the spreads of 11,642.13, implying that the data is heavily tailed relative to a normal distribution. The average monthly data on the other hand, weighted according to the traded monthly volume of each security, reflects lower variation in spreads relative to daily data (with the exception of Transportation and Communications, credit rating D).

There are some anomalies in our dataset for credit ratings CC and D. Our credit spreads and their standard deviation are unrealistically high. This is explained because the records we have for these two credit ratings go only until 2005. However, in our regressions, both of these credit ratings were not statistically significant, and therefore were not considered in our analysis.

Table 11. Summary statistics for the daily filings from July 2002 to November 2014.

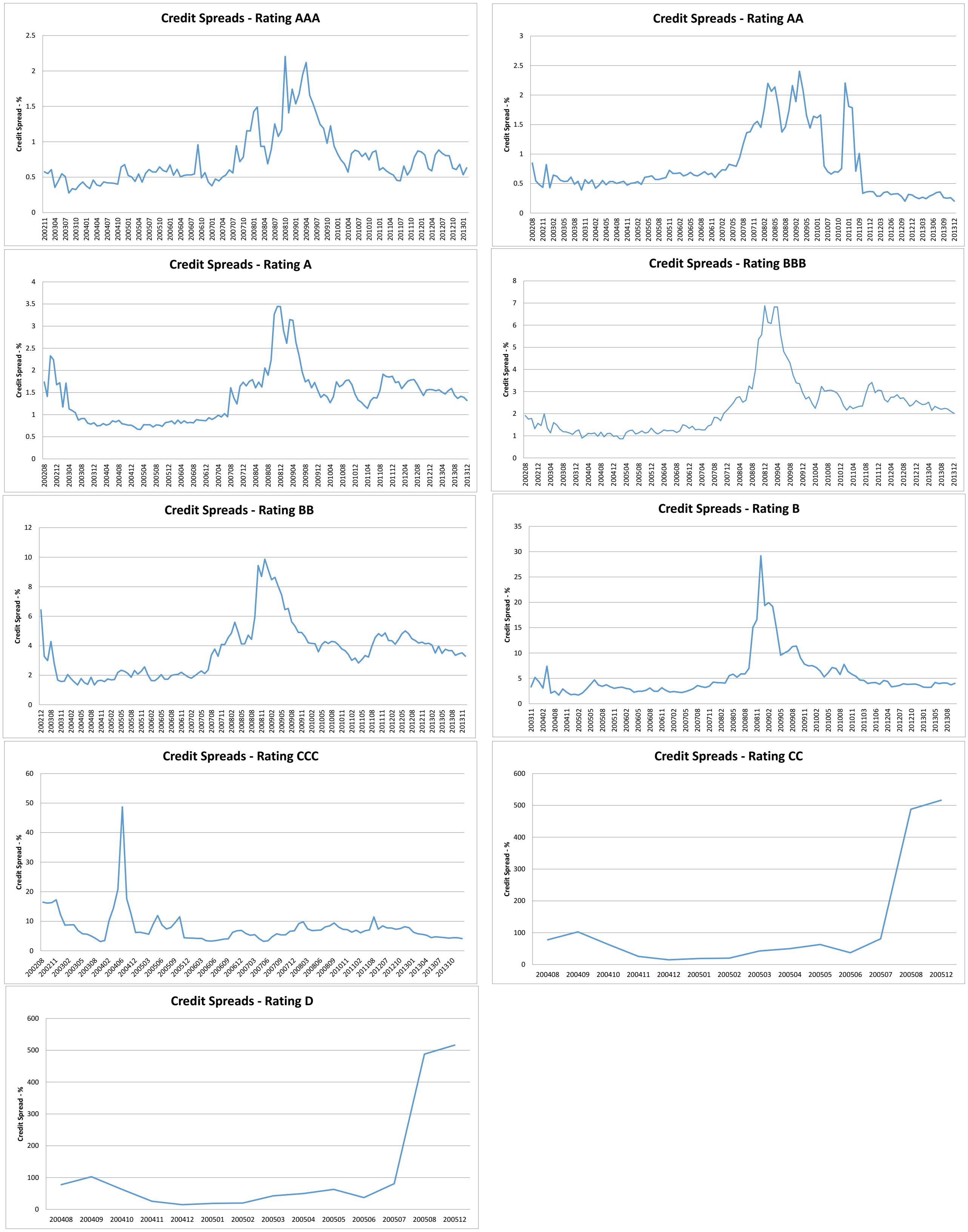
Rating	Indsutry	N	Avg Price	Std Price	Avg Yield	Std Yield	Avg Spr	Std Spr	Skewness Spr	Kurtosis Spr
BBB	Construction	149	107.15	4.35	4.77	1.72	0.01	1.67	-0.15	1.50
BB	Construction	212	105.68	11.33	6.05	2.52	2.81	2.23	0.45	2.13
A	Financials	134	128.00	4.14	4.47	0.39	1.53	0.37	-0.71	4.98
BBB	Financials	2805	110.37	11.64	6.32	1.51	2.29	1.64	1.03	5.16
AAA	Manufacturing	3302	118.89	11.62	4.55	1.02	0.28	0.91	-0.79	3.68
AA	Manufacturing	6194	108.80	7.00	3.58	1.66	-0.83	1.29	0.00	2.16
A	Manufacturing	23323	117.24	13.10	4.88	1.41	0.56	1.38	3.33	143.79
BBB	Manufacturing	12463	108.33	10.99	5.56	9.29	1.54	9.27	106.08	11642.13
BB	Manufacturing	5701	102.19	9.75	6.97	1.88	3.28	1.99	0.75	4.23
B	Manufacturing	1158	98.66	11.17	8.00	2.33	3.86	2.31	1.49	6.41
CCC	Manufacturing	1073	98.18	8.62	7.98	2.48	3.25	2.49	1.29	4.12
CC	Manufacturing	4	93.68	2.52	10.30	0.57	6.25	0.57	-0.30	1.70
A	Mining	1889	121.11	11.90	4.99	1.14	0.80	0.95	-0.26	5.28
BBB	Mining	1858	112.67	12.66	6.02	1.45	2.10	1.48	-0.05	3.54
BB	Mining	6	86.16	1.97	8.55	0.27	4.83	0.24	-1.37	3.55
AA	Retail Trade	983	112.13	10.03	5.19	0.77	0.64	0.77	0.69	4.62
A	Retail Trade	6686	110.04	8.19	4.72	1.58	0.30	1.69	-0.24	2.93
BBB	Retail Trade	3298	107.82	8.09	5.04	1.81	0.52	1.87	1.07	7.24
BB	Retail Trade	1920	99.76	8.65	6.26	2.29	1.93	2.22	0.86	5.77
B	Retail Trade	881	97.95	11.41	8.74	4.25	4.39	4.45	2.93	13.66
CCC	Retail Trade	1428	86.74	13.23	9.71	3.18	5.52	3.17	1.92	10.51
CC	Retail Trade	694	60.52	16.60	16.06	9.66	12.52	9.61	3.45	15.71
AA	Services	1270	125.17	7.53	4.56	1.27	0.10	0.82	-1.84	6.18
A	Services	2181	119.06	8.70	4.58	1.33	0.28	1.15	-0.73	4.32
BBB	Services	43	101.97	1.17	4.75	0.69	0.13	0.64	1.04	4.14
BB	Services	317	97.23	6.40	6.32	1.40	1.87	1.43	0.55	2.69
B	Services	781	95.21	7.73	6.38	1.40	1.59	1.33	-0.15	1.90
AAA	Transportation & Comm.	169	106.85	7.33	5.34	0.77	0.56	0.76	-1.14	9.56
AA	Transportation & Comm.	20	101.60	2.66	5.46	1.53	0.87	1.51	-0.39	1.73
A	Transportation & Comm.	1475	105.42	4.69	4.54	1.35	-0.17	1.40	0.27	5.17
BBB	Transportation & Comm.	2050	104.73	4.90	4.35	1.23	-0.25	1.25	0.53	5.41
BB	Transportation & Comm.	125	102.21	1.68	5.96	0.42	0.99	0.40	2.19	14.11
B	Transportation & Comm.	1	99.00	-	8.05	-	3.07	-	-	-
CCC	Transportation & Comm.	864	77.91	15.65	20.13	8.96	15.51	9.00	0.59	3.34
CC	Transportation & Comm.	53	50.41	15.72	97.41	122.89	92.67	122.89	3.24	12.75
D	Transportation & Comm.	220	78.32	14.85	58.84	88.13	54.25	88.12	5.68	39.59
A	Wholesale Trade	698	106.67	3.61	2.33	0.61	-2.45	0.59	1.49	10.30
BBB	Wholesale Trade	1080	106.59	10.04	6.33	1.79	2.55	1.98	1.09	18.79
BB	Wholesale Trade	427	106.31	3.14	5.42	1.18	0.68	1.27	0.07	1.61
B	Wholesale Trade	25	99.87	1.13	6.85	0.71	1.96	0.74	1.05	3.45

Table 12. Summary statistics for the average monthly filings from July 2002 to November 2014.

Rating	Indsutry	N	Avg Price	Std Price	Avg Yield	Std Yield	Avg Spr	Std Spr	Skewness Spr	Kurtosis Spr
BBB	Construction	44	107.86	2.82	4.97	1.71	1.47	0.53	0.02	2.25
BB	Construction	46	102.65	12.13	6.74	2.65	4.53	1.86	1.19	4.33
A	Financials	7	126.82	6.96	4.45	0.26	1.99	0.21	-0.24	1.78
BBB	Financials	112	111.81	11.78	6.42	1.48	2.75	1.48	1.59	5.71
AAA	Manufacturing	122	120.33	10.11	4.63	0.87	0.71	0.39	1.73	6.04
AA	Manufacturing	107	108.37	5.49	3.99	1.80	0.70	0.46	1.37	3.72
A	Manufacturing	137	120.26	9.17	5.05	0.92	1.42	0.58	1.10	4.49
BBB	Manufacturing	137	108.89	8.01	5.39	1.49	2.20	1.25	1.71	6.33
BB	Manufacturing	119	101.89	9.11	7.01	1.47	3.75	1.74	1.24	5.09
B	Manufacturing	82	97.00	12.78	8.44	2.42	4.95	2.19	1.73	5.89
CCC	Manufacturing	45	95.37	7.47	9.10	2.05	4.93	2.16	0.65	3.18
CC	Manufacturing	1	93.89	0.00	10.37	0.00	6.90	0.00	-	-
A	Mining	129	121.82	10.60	4.92	1.08	1.40	0.61	1.51	5.44
BBB	Mining	128	108.45	11.76	6.18	1.24	2.42	1.28	1.24	4.56
BB	Mining	1	91.48	0.00	7.87	0.00	3.92	0.00	-	-
AA	Retail Trade	30	114.03	5.99	5.23	0.44	0.93	0.54	1.64	4.21
A	Retail Trade	133	113.13	9.32	4.87	0.82	1.45	0.78	1.46	6.73
BBB	Retail Trade	105	109.07	7.30	5.04	1.53	2.02	1.55	2.03	7.23
BB	Retail Trade	113	94.80	14.12	7.17	2.73	3.90	2.36	1.73	6.26
B	Retail Trade	76	97.79	14.38	8.89	4.97	5.75	5.43	2.40	8.53
CCC	Retail Trade	81	84.14	13.80	10.15	3.17	6.42	3.22	1.76	5.72
CC	Retail Trade	43	55.84	19.59	18.70	12.09	15.21	11.98	1.88	4.97
AA	Services	41	125.25	3.65	4.56	1.13	0.76	0.17	2.71	13.25
A	Services	69	117.40	7.18	4.44	1.13	1.11	0.67	1.75	5.67
BBB	Services	9	101.93	0.92	5.10	0.85	1.18	0.61	0.62	2.33
BB	Services	34	99.93	3.61	7.29	1.48	4.32	2.15	0.28	1.75
B	Services	24	96.96	4.97	6.55	0.99	2.44	1.14	3.68	16.71
AAA	Transportation & Comm.	38	106.42	6.14	5.46	0.50	0.97	0.52	1.42	4.56
AA	Transportation & Comm.	5	101.33	0.89	5.13	1.26	2.68	0.58	1.07	2.68
A	Transportation & Comm.	84	104.33	6.78	5.15	1.17	1.32	0.78	1.12	3.56
BBB	Transportation & Comm.	114	107.75	6.93	4.73	1.27	1.66	1.02	0.41	1.92
BB	Transportation & Comm.	9	101.50	1.58	5.91	0.20	1.07	0.27	0.36	2.16
B	Transportation & Comm.	1	99.00	0.00	8.05	0.00	5.20	0.00	-	-
CCC	Transportation & Comm.	27	87.15	11.79	15.48	9.59	11.71	10.17	1.74	7.42
CC	Transportation & Comm.	2	37.39	6.49	114.81	77.09	110.47	77.17	0.00	1.00
D	Transportation & Comm.	14	69.15	20.51	117.27	166.69	114.36	166.35	1.94	4.97
A	Wholesale Trade	18	106.83	2.66	2.30	0.48	0.47	0.27	2.01	5.81
BBB	Wholesale Trade	109	104.64	11.25	6.62	1.65	3.00	1.49	1.35	5.93
BB	Wholesale Trade	36	106.27	2.03	5.15	0.92	1.68	0.73	2.10	6.87
B	Wholesale Trade	14	100.06	1.17	6.83	0.77	2.34	0.68	0.69	3.39

To be able to further validate the quality of our monthly dataset, we have plotted the time series of credit spreads for each S&P rating category and we observe consistent data. For example, the credit spreads for the S&P credit rating BBB increased drastically during the 2008 crisis and was in fact close to 7% during that period. Figures 8-16 highlight the direct effects of changes in credit quality on the prices of corporate bonds, and can signal changes in the market prices of OTC derivative positions [DS03].

Figures 8-16. Time series of monthly Credit Spreads per Standard and Poor's credit rating.



Chapter 7

Empirical Results and Analysis

We attempt to replicate, using the data described in Chapter 6, the results from [LS95] (page 808), which state that there is an important correlation to consider, between asset returns and interest rates, when evaluating corporate debt. According to [LS95], this correlation can explain why companies in different industries may have different spreads whilst having the same credit rating. The correlation was calculated between the average asset return per industry per month and the average interest rate change, per industry per month. The results are presented in Table 13.

We then proceed to perform the [LS95] regression (7.1) on the changes of credit spreads with respect to changes in interest rates and stock returns with a 95% confidence interval.

$$\Delta S = a + b\Delta Y + cI + \varepsilon \tag{7.1}$$

where ΔS is the change in credit spreads (taken from the Trace database built in chapter 6), ΔY is the change in interest rates (in our case the 20-year Treasury Bond) and I is the return on the stock (taken from the Trace database built in chapter 6). This regression provides us with coefficients b and c. Coefficient b measures the sensitivity of the change of credit spreads to changes in interest rates. Results are presented in Table 14. We can observe that all coefficients are negative; this means that as interest rates increase, credit spreads decrease, as predicted by our model and that of [LS95]. Additionally, c coefficients are also negative; as the return on the stock increases, the credit spreads decrease.

Another implication of our model (Chapter 4) and that of [LS95] is that, as correlation increases, the sensitivity of the credit spreads towards changes in interest rates must also increase. The explanation behind this relationship is that as when correlation is negative, changes in the interest rate should be reversed by changes in the asset value, therefore a change in r, has less of an effect on the credit spreads than when correlation is zero or positive [LS95].

We have mirrored their methodology beginning with the following hypothesis:

$$H_0: b = 0 \text{ vs. } H_1: b \neq 0 \text{ at } \alpha = 0.05$$

Considering only those regressions that were statistically significant we did not observed the same pattern (as ρ increases sensitivity, b, increases) in our dataset. Performing the regression with daily data did not coincide with [LS95] either, most likely because of the additional noise in daily trades. Other tests where implemented, on both daily and monthly data. For instance, we calculated both the correlation and the [LS95] regression for bonds that had been traded sequentially on a daily basis, allowing for a maximum gap of 4 days in the data (to avoid deleting entries because of long weekends where no trade occurs). A similar reasoning was done on a monthly basis: We performed regression considering only those bonds that had been traded every month during their lifetime. In all scenarios, we did not observe the results presented by [LS95]. One possible explanation for this discrepancy could be the drastic movements in the markets during the 2008 financial crisis. As an attempt to rationalize this difference, regressions dividing the dataset into 'before' and 'after' the crisis were performed. However, this substantially reduced the number of data points in the regression and our results were not statistically significant.

We can observe that the confidence interval is very large for all entries on Table 14. This allows us to conclude that even though we considered only those industries and credit ratings that had statistically significant results, their difference is not.

It is still important to highlight that we do observe a relation between credit spreads and the correlation between interest rates and asset value. The analysis of the how sensitive each industry is to correlation is a future research project.

Table 13: Results of the calculation of correlation between the return on the assets, and the monthly changes in the 20-year Treasury Bond Yield

Rating	Industry	N	Rho	Mean of Credit Spread	Std. Dev. of Credit Spread
A	Services	69	0.09100	1.11340	0.67416
A	Mining	129	0.14717	1.40187	0.61176
A	Retail Trade	133	0.19449	1.44619	0.78149
B	Retail Trade	76	0.03451	5.75019	5.43219
B	Manufacturing	82	0.29514	4.94620	2.19236
BB	Construction	46	0.07224	4.52795	1.85785
BB	Services	34	0.77598	4.32032	2.14694
BB	Retail Trade	113	0.18296	3.90131	2.36054
BB	Manufacturing	119	0.30017	3.74715	1.73581
BBB	Retail Trade	105	0.35186	2.02382	1.54555
BBB	Finance, Insurance, And Real Estate	112	0.49865	2.74820	1.48311

Table 14: Results from regressing monthly changes in Credit Spreads on monthly changes in the 20-year Treasury Bond Yield and the return on the equity.

The term ΔS is the change in the credit spread, the term ΔY is the change in the 20-year Treasury bond yield, and the term I is the return on the equity.

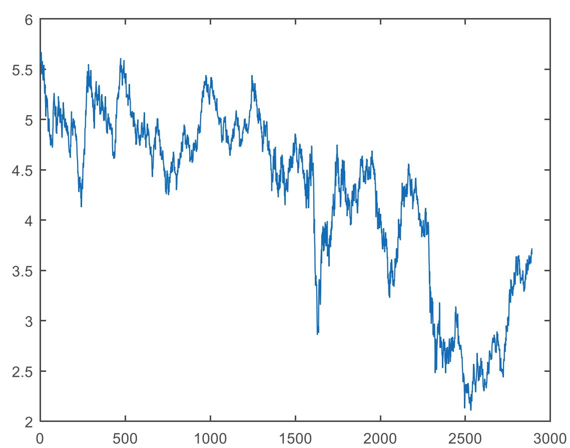
Rating	Industry	Rho	a	b	c	t_a	t_b	t_c	R^2	N	lower 95% bound	upper 95% bound
A	Services	0.09100	-0.0152	-0.4471	-0.0105	-0.4794	-3.1472	-2.0582	0.1828	68	-0.7308	-0.1634
A	Mining	0.14717	0.0201	-0.1639	-0.0089	0.8858	-2.0512	-3.5011	0.1243	128	-0.3221	-0.0058
A	Retail Trade	0.19449	0.0005	-0.5164	-0.0076	0.0183	-3.6372	-1.9419	0.1275	132	-0.7973	-0.2355
B	Retail Trade	0.03451	-0.0510	-2.9304	-0.0160	-0.1650	-2.1359	-0.8236	0.0709	75	-0.3546	-0.1991
B	Manufacturing	0.29514	0.0378	-1.3797	-0.0438	0.4636	-3.7918	-7.1840	0.5400	81	-0.2560	0.1207
BB	Construction	0.07224	-0.2223	-1.8840	-0.0178	-1.5118	-4.4889	-2.5297	0.3791	45	-2.7309	-1.0370
BB	Services	0.77598	-0.0833	-1.1607	0.0027	-0.5591	-2.6397	0.1875	0.3077	33	-2.0587	-0.2627
BB	Retail Trade	0.18296	0.0252	-1.1549	-0.0081	0.2552	-2.1820	-1.2479	0.0675	112	-2.2039	-0.1059
BB	Manufacturing	0.30017	-0.0155	-0.8764	-0.0078	-0.2625	-2.6216	-1.4744	0.0919	118	-1.5386	-0.2142
BBB	Retail Trade	0.35186	-0.0039	-0.7207	-0.0156	-0.0695	-2.4061	-2.2798	0.1312	104	-1.3149	-0.1265
BBB	Finance	0.49865	0.0188	-0.4595	-0.0174	0.4930	-2.0355	-5.1500	0.3077	111	-0.9070	-0.0120

Chapter 8

Replicating Historical Prices

To be able to compare our approach to real life scenarios we parameterized our model using maximum Likelihood estimation for the 20-year Treasury bond under the Vasicek dynamic. The details of the Vasicek dynamic are presented in Chapter 2, Section 2.3. Figure 17 depicts the evolution of this security from 2002 until 2014 (period under which we focus our analysis).

Figure 17. 20-Year Treasury bond historical evolution



The parameterization of the Vasicek model is done using the historical data collected (Chapter 6) and MLE (maximum likelihood estimation). Here below find the parameters of formula (2.1), and the results of both daily and monthly parameterizations are presented in Table 15.

$$\hat{\alpha} = \frac{\sum_{i=1}^{n} r_i r_{i-1} - \sum_{i=1}^{n} r_i \sum_{i=1}^{n} r_{i-1}}{\sum_{i=1}^{n} r_{i-1}^2 - \left(\sum_{i=1}^{n} r_{i-1}\right)^2}$$

$$\hat{\hat{r}} = \frac{\sum_{i=1}^{n}(r_i - \hat{\alpha} r_{i-1})}{(1-\hat{\alpha})}$$

$$\widehat{V^2} = \frac{1}{-}\sum_{i=1}^{n}\left(r_i - \hat{\alpha} r_{i-1} - \hat{\hat{r}}(1-\hat{\alpha})\right)^2$$

$$\hat{k} = -\frac{l\ \ g(\hat{\alpha})}{\Delta t}$$

$$\widehat{\sigma^2} = \frac{2\hat{k}\widehat{V^2}}{\left(1 - e^{-2\hat{k}\Delta t}\right)}$$

Table 15: Results from the parametrization of the Vasicek model using the dataset built (description on Chapter 6)

	20-year yield (Monthly Data)	**20-year yield (Daily Data)**
κ	0.3532	0.6788
σ	0.007463	0.009419
$\bar{r}$	0.038214	0.040059

Using the calculated parameters for the Vasicek model and parametrization for the asset value and asset volatility provided by [SIL12], we price nine different bonds at various dates. We observe relative errors of the order of 16%. Part of this error is certainly attributable to the fact that we do not apply any risk adjustment to the physical interest rate dynamics. This can be considered for future research. Additionally, we could not find on WRDS or Bloomberg the time series of the amount of the debt outstanding for each bond, we were only able to obtain the original size of the issue. This obstructed us from proceeding to a more in-depth analysis. Table 16 presents our results.

Table 16: Pricing with our DP model compared to actual bond prices and the plain vanilla formula.

Cusip_ID	Company Name	Bond Price	Our approach	Vanilla Formula
110122AB4	BRISTOL-MYERS SQUIBB CO	1,084.16	908.99	921.96
134429AG4	CAMPBELL SOUP CO	1,398.95	1,142.92	1,185.99
382388AK2	GOODRICH CORP	1,017.77	959.04	994.37
418056AH0	HASBRO INC	1,074.79	884.73	904.16
478160AJ3	JOHNSON & JOHNSON	1,454.55	962.94	983.23
500255AC8	KOHL'S CORP	1,017.02	985.78	1,054.43
655664AH3	NORDSTROM INC	775.39	994.20	958.45
902494AD5	TYSON FOODS INC -CL A	967.30	866.96	954.31
963320AH9	WHIRLPOOL CORP	1,155.08	1,003.32	1,073.50

Conclusion

In this thesis, we propose a robust methodology for evaluating corporate bonds under a stochastic interest rate environment and with discrete fixed rate coupons. This dynamic programming approach, coupled with spectral interpolation, has the benefit of being a more tractable and manageable approach than the N-dimensional integral proposed by [G77] and comprises the compounding effect of coupons through time, unlike the [LS95] model. [EHH04] have demonstrated that the optionality feature of the [G77] model results in a more accurate prediction of credit spreads and that an endogenous barrier of default like the one observed in the [G77] model is more effective in increasing the precision of pricing of corporate debt over an exogenous barrier of default like that of [LS95].

We have shown that our approach not only replicates others already existing models in the literature (under appropriate parameterization), but it also converges with only one hundred interpolation points on the grid. This model can account for the price of the debt at any point in time, and considers the lag between the pricing date and the first coupon date, if any.

We have been able to observe in our dataset that there is a relationship between current interest rates and the value of the company. From our regression of interest rate changes on credit spreads we found a negative relation between these two variables represented by $b < 0$. This implies that an increase in interest rate will decrease the credit spread of the corporate bond. However, we did not observe the [LS95] implication of a positive relation between correlation and interest rate sensitivity. We attribute this to the spikes in prices and credit spreads during the 2008 financial crisis, and a more in-depth analysis can be considered for further research.

There are some important concepts that were not considered in this thesis. To begin with, we have decided to use the one factor Vasicek model, as it is the simplest model and the most tractable one. We could have used the two factor model, now that the inclusion of the second factor explains an additional 10.3% of the variability in interest rates (from 85% to 95.3%) [SIL12]. Additionally, our methodology can be adapted to other interest rates models for which the joint density is known, it does not necessarily have to be one with a Gaussian solution [CLÉ13]. This implies that jump diffusion model or CIR could be used to model corporate debt under our valuation approach. Finally, we did not analyze the impact of parameterization on our endogenous barrier of default. This can be done through the [LS95] exogenous barrier, using *fsolve* in Matlab to match the bond prices. This would be interesting to better visualize the analysis of the impact of the parameters, done on Section 5.3, on the exogenous barrier of default.

Appendix

Table A: Industry classification according to the United Stated department of Labour.[2]

Division	Industry Name	SIC 2 Codes
A	Agriculture, Forestry, And Fishing	01, 02, 07, 08, 09
B	Mining	10, 12, 13, 14
C	Construction	15, 16, 17
D	Manufacturing	20-39
E	Transportation, Communications, Electric, Gas, And Sanitary Services	40-49
F	Wholesale Trade	50, 51
G	Retail Trade	52-59
H	Finance, Insurance, And Real Estate	60-65,67
I	Services	70, 72-73, 75-76, 78-84, 86-89
J	Public Administration	91-97, 99

Table B: Database elements and description

	Database Code	Name	Description
1	rating_dt_end	date_of_rating	Day the rating of the bond was made
2	issue_id	issue_id	A Mergent-generated number unique to each issue.
3	rating	StdPoors_rating	S&P bond issue rating as per trading date
4	cusip_id	cusip_id	The 9 digit cusip identifier of the issue. Issuer CUSIP + Issue CUSIP
5	bond_sym_id	TRACE_bond_symbol	Unique NASD identifier assigned to each bond issue. This identifier was was constructed by appending a unique issue code to the company's common stock ticker symbol. For example, "IBM.GT" is the TRACE Bond Symbol for a particular

[2] SIC Division Structure, US Department of Labour, URL: https://www.osha.gov/pls/imis/sic_manual.html

			IBM debenture.
6	trd_exctn_dt	trade_execution_date	Date the bond was traded
7	trd_exctn_tm	trade_execution_time	Time the bond was traded
8	msg_seq_nb	msg_seq_nb	Message sequence number
9	cmsn_trd	commission_indicator	Indicates if the reported price is inclusive of dealer commission (Y=yes, N=no)
10	ascii_rptd_vol_tx	reported_volume	Par value volume of the reported trade
11	rptd_pr	daily_closing_price	Originally reported price. For this database we chose closing day price
12	Avg_RPTD_PR	daily_average_price	Average daily price
13	yld_pt	security_yield	This field indicates the effective rate of return earned on a security, expressed as a percentage. The field will be blank if no yield is available.
14	diss_rptg_side_cd	reporting_side	Identifies the trade as either a customer buy or sell trade, or an inter-dealer (sell) trade. (B=bought, S=sold, D=Intra-dealer from sellers perspective)
15	chng_cd	price_change	Change Indicator High/Low/Last. This field describes the price change(s) that the transaction 16caused for the issue traded.
16	rptd_high_pr	daily_high_price	Th17is field represents the high price reported for th2e specific bond for the day. If the High Price is not available for a bond, this field will contain all zeroes (0000.000000)
17	rptd_low_pr	daily_low_price	This field represents the low price reported for the specific bond for the day. If the Low Price is not available for a bond, this field will contain all zeroes (0000.000000).
18	rptd_last_pr	last_selling_price	This field represents the last sale price reported for the specific bond for the day. If the Last Sale Price is not available for a bond, this field will contain all zeroes (0000.000000).
19	total_vol	total_vol	It is expressed in units of one share, for daily data, and on hundred shares for monthly data. Our data source for NYSE/AMEX reports the number rounded to the nearest hundred. For example, 12,345 shares traded will be reported on the Nasdaq Stock Exchange as 12,345 and on the NYSE or AMEX exchanges as 12,300. Volume is set to -99 if the value is missing. A volume of zero

			usually indicates that there were no trades during the time period and is usually paired with bid/ask quotes in price fields.
20	issuer_id	issuer_id	For FISD data, ID of the company that does the issue of both bonds and stocks
21	principal_amt	principal_amount	Principal of the corporate bond in question
22	dated_date	issue_date	Date the corporate bond was issued. Not the same as the transaction day.
23	first_interest_date	first_coupon_date	First coupon since the issuance of the bond
24	next_interest_date	next_coupon_date	Next coupon date with respect to the current trading day
25	last_interest_date	last_coupon_date	Last coupon until maturity of the bond
26	interest_frequency	coupon_frequency	How many times per year is the coupon payed
27	coupon (%)	coupon_rate_percentage	What is the coupon rate (%)
28	day_count_basis	day_count_basis	All are “30/360”
29	TTM	Maturity	Maturity of the bond in days
30	CUSIP_ID_6	CUSIP_ID_6	First 6 digits of the CUSIP pertain to the company which issued the obligation. The last 3 digits are particular to the obligation
31	ISSUNO	Nasdaq_issue_number	ISSUNO is a unique integer assigned by the National Association of Securities Dealers (NASD) to each listed security on the Nasdaq Stock Market(SM). It is this issue-specific identifier which differentiates securities issued by the same company. If the issue number is unknown, ISSUNO is set to zero. If an NYSE/AMEX security was ever traded on Nasdaq, this number is set to the latest issue number assigned when it was trading on Nasdaq. The ISSUNO in the CRSP file may change if Nasdaq assigns a new number to an issue CRSP considers to be a continuation of an existing issue.
32	PRC	stock_price	Prc is the closing price or the negative bid/ask average for a trading day. If the closing price is not available on any given trading day, the number in the price field has a negative sign to indicate that it is a bid/ask average and not an actual closing price. Please note that in this field the negative sign is a symbol and that the value of the bid/ask average is not negative.

33	RET	holding_period_return	A return is the change in the total value of an investment in a common stock over some period of time per dollar of initial investment. RET(I) is the return for a sale on day I. It is based on a purchase on the most recent time previous to I when the security had a valid price.
34	BID	closing_bid	The bid price from the last representative quote before the markets close for each trading date.
35	ASK	closing_ask	The ask price from the last representative quote before the markets close for each trading date.
36	SHROUT	shares_outstanding	SHROUT is the number of publicly held shares, recorded in thousands
37	RETX	return_without_dividends	RETX contains returns without dividends. Ordinary dividends and certain other regularly taxable dividends are excluded from the returns calculation. The formula is the same as for RET except d(t) is usually 0.
38	NewFiscalQTR	lagged_QTR	Since Financial statements are done quarterly we have corrected for the lag in financial reporting to match the trading date to the date where the revenues and expenses where actually accounted for.
39	NewFiscalYr	lagged_YR	Since Financial statements are done quarterly we have corrected for the lag in financial reporting to match the trading date to the date where the revenues and expenses where actually accounted for.
40	AccountingDate	accounting_date	Since Financial statements are done quarterly we have corrected for the lag in financial reporting to match the trading date to the date where the revenues and expenses where actually accounted for.
41	LOC	country_code	LOC -- Current ISO Country Code – Headquarters. This item contains the code that identifies the country where the company headquarters is located. The country codes are established by the International Standards Organization (ISO).
42	STATE	STATE	STATE – US State/Province
43	NAICS	North American Industry Class System	North American Industry Classification System Code (NAICS) is an 6-character code used to group companies with similar products or services. It was adopted in 1997 and implemented in 1999, by the Office of Management and

			Budget (OMB), to replace the U.S. Standard Industrial Classification(SIC) system.
44	SIC	SIC -- Standard Industry Classification Code	Standard Industrial Classification Code
45	CONM	company_name	Company Name
46	ACTQ	total_current_assets	ACTQ -- Current Assets – Total. (Quarterly)
47	ATQ	total_assets	ATQ -- Assets – Total. (Quarterly)
48	CEQQ	total_common_equity	CEQQ -- Common/Ordinary Equity – Total. (Quarterly)
49	CSHOQ	common_shares_outstanding	CSHOQ -- Common Shares Outstanding. Quarterly
50	DLTTQ	total_LT_debt	DLTTQ -- Long-Term Debt – Total. Quarterly
51	DVPSPQ	dividends_per_share	DVPSPQ -- Dividends per Share - Pay Date – Quarter. Quarterly
52	INTACCQ	interest_accrued	INTACCQ -- Interest Accrued. This is for utility companies only. This item represents the amount of interest accrued but not matured on all liabilities of the Utility. However, this does not include interest, which is added to the principal of the debt on which incurred.
53	LCTQ	total_current_liabilities	LCTQ -- Current Liabilities – Total. Quarterly
54	LLTQ	total_LT_liabilities	LLTQ -- Long-Term Liabilities (Total). Quarterly
55	LTQ	total_liabilities	LTQ -- Liabilities – Total. Quarterly
56	OPTDRQ	dividend_rate_percentage	OPTDRQ -- Dividend Rate - Assumption (%). Quarterly
57	id_cnum	id_cnum	Used to merge Axels database with the asset value and asset volatility of each company
58	obs	obs	Used to merge Axels database with the asset value and asset volatility of each company
59	asset_value	asset_value	Asset value following a standard Brownian motion, usually higher than the fundamental/accounting asset value
60	asset_vol	asset_vol	Asset volatility needed to calculate the standard Brownian motion
61	FF_O	fed_fund_rate	Interest rate needed to calculate correlation
62	TCMNOM_M3	tbill_3month	Treasury constant maturities 3 months
63	TCMNOM_Y20	tbill_20yr	Treasury constant maturities 20 years
64	TCMII_Y20	tbill_20yr_inflation_indexed	Treasury constant maturities 20 years, inflation indexed

65	sic2	Sic2	First two digits of the SIC. It is used to narrow down the industry categories
66	Industry_Name	Industry_Name	Industry_Name
67	Division	Industry_Division	Industry_Division
68	Days_TM	days_to_maturity	days_to_maturity
69	Calendar_Days_TM	calendar_days_to_maturity	calendar_days_to_maturity

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