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The impact of risk sensitivity and a large bank on interbank transactions and systemic risk

par

Yizhuo Liu

Dena Firoozi
HEC Montréal
Directrice de recherche

Sciences de la gestion
(Spécialisation M.Sc. Financial Engineering)

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Résumé

La théorie du jeu du champ moyen (MFG) étudie un système composé de nombreux individus, où l'action de chaque individu a un impact négligeable sur le système dans son ensemble. En outre, ces agents interagissent les uns avec les autres par le biais du comportement agrégé de l'ensemble de la population (par exemple, l'état moyen). Nous appliquons cette méthodologie au système de transactions interbancaires, où nous incorporons une grande banque dont l'action affecte l'état du système et chaque petite banque. Plus précisément, nous modélisons les activités d'emprunt et de prêt interbancaires entre une grande banque et de nombreuses petites banques. Les banques sont sensibles aux risques et contrôlent leur taux d'emprunt/de prêt auprès de la banque centrale afin de maintenir un certain niveau de réserves monétaires logarithmiques. La sensibilité au risque est représentée par la fonction de coût exponentiel quadratique. Nous considérons également différents types de petites banques, et l'état du marché est obtenu par une combinaison linéaire des réserves log-monétaires de la grande banque et de la moyenne empirique des réserves de toutes les petites banques. Les banques interagissent à travers l'état du marché car elles souhaitent suivre une fraction de l'état du marché. Les stratégies de transaction de meilleure réponse des grandes et des petites banques sont dérivées en utilisant les techniques d'analyse convexe et d'analyse variationnelle. Nous obtenons ensuite un équilibre de Nash pour le système avec les stratégies de meilleure réponse lorsque le nombre de petites banques tend vers l'infini. Nous étudions également l'impact d'une grande banque, la sensibilité au risque, la défaillance d'un sous-groupe de petites banques et plusieurs autres facteurs sur la probabilité de défaillance de la banque individuelle et le risque systémique, qui est le risque que l'état du marché tombe en dessous d'un seuil spécifique. Nous menons des expériences numériques pour illustrer les comportements des banques et du système dans différents scénarios. Nous observons que la présence d'une grande banque averse au risque améliore sa propre stabilité, celle des petites banques et celle du marché interbancaire. Cependant, les avantages d'une grande banque averse au risque sont compromis par l'augmentation de sa taille. En outre, la santé financière du système est améliorée lorsque la grande banque a un taux de croissance rapide, tandis qu'une grande banque alignant étroitement ses réserves monétaires sur la moyenne du marché augmente les risques de défaillance dans l'ensemble du système. En outre, un sous-groupe de banques mineures ayant une aversion au risque augmente généralement les probabilités de défaut sur le marché interbancaire en réduisant la liquidité. Des politiques et des réglementations peuvent être mises en œuvre en conséquence pour prévenir les défaillances graves.

Keywords

Jeux à champ moyen majeur-mineur; Système de transactions interbancaires; Sensibilité au risque; Risque systémique

Abstract

The mean-field game (MFG) theory studies a system that consists of many individuals, where each individual's action has a negligible impact on the system as a whole. Moreover, these agents interact with each other through the aggregate behaviour of the whole population (e.g. average state). We apply this methodology to the interbank transactions system, where we incorporate a large bank whose action affects the system's state and each small bank. Specifically, we model the interbank borrowing and lending activities between a large bank and many small banks. The banks are sensitive to risks and control their borrowing/lending rate with the central bank to maintain a certain level of log-monetary reserves. The risk sensitivity is featured through the exponential quadratic cost functional. We also consider different types of small banks, and the market state is obtained from a linear combination of the log-monetary reserves of the large bank and the empirical average of the reserves of all the small banks. The banks interact through the market state as they wish to follow a fraction of the market state. The best-response transaction strategies of the large and small banks are derived using the convex analysis and the variational analysis techniques. We then obtain a Nash equilibrium for the system with the best-response strategies when the number of small banks tends to infinity. We also investigate the impact of a large bank, risk sensitivity, the default of a subgroup of small banks, and several other factors on the individual bank's default probability and systemic risk, which is the risk that the market state falls below a specific threshold. We conduct numerical experiments to illustrate the behaviours of the banks and the system under different scenarios. We observe that the presence of a risk-averse large bank improves the stability of itself, small banks, and the interbank market. However, the benefits of a risk-averse large bank are compromised by the increase in its size. Furthermore, the financial health of the system is improved when the large bank has a fast growth rate, while a large bank closely aligning its monetary reserves with the market average raises default risks throughout the system. Moreover, a subgroup of minor banks being risk-averse generally increases the default probabilities in the interbank market by reducing liquidity. Policies and regulations can be implemented accordingly to prevent severe default events.

Keywords

Major-minor mean-field games; Interbank transactions system; Risk sensitivity; Systemic risk

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Introduction

The presence of banks dates back to thousands of years ago when the first currencies were invented. In its early days, the primary function of banks was to serve as a safe place for individuals to store their coins and valuable assets, while some banks were also documented to lend money. As the banking system evolves to be complete, more services are offered, such as foreign currency exchange, wealth management, and various investment instruments. Nevertheless, the fundamental role of banks has remained constant: accepting deposits and extending loans to retail customers. With the emergence of online banking, the services banks provide are becoming more automated. Therefore, banks must ensure that the withdrawal demands of their customers are fulfilled at any time. When a bank fails to meet these demands, it can borrow cash from another bank with excess funds. This is considered an interbank transaction, where banks trade directly with each other at the interbank rate. By engaging in the interbank market, banks can reduce their liquidity risk, which is the risk of being unable to cover their short-term debt. Therefore, interbank transactions typically take place on an overnight basis.

Banks also have the option to borrow from central banks in cases of liquidity shortages. In the United States, a commercial bank can borrow from the Federal Reserve (the Fed) through the discount window when it does not have enough cash to meet the daily reserves requirement. Banks also conduct transactions with the central bank through open market operations. When the Fed implements a contractionary monetary policy to cut the money supply in the system, it sells government securities to individual banks. If an expansionary monetary policy is required for the current economic state, the Fed will purchase securities from banks. In this sense, banks engage in lending and borrowing operations with the central bank, which sets the stage for a delicate financial ecosystem. Banks purchase these financial instruments in Canada at the Canadian overnight repo rate average (CORRA). The interest rate charged by the central bank when lending money to commercial banks is normally higher than the rate set by the banks when trading among themselves. However, banks still trade with the central bank since it is always available.

The Global Financial Crisis of 2007 - 2008 was a stark reminder of banks' vulnerability. Banks worldwide were distressed and struggling to raise funds, leading to a freeze in the interbank market. Therefore, banks had to resort to central banks to secure funds and maintain operations. In times of crisis, central banks and governments must intervene to ensure that the so-called "too big to fail" institutions do not collapse during a financial crisis. These banks are also labeled as systemically important financial institutions (SIFI) by the U.S. federal regulators since their failure would be catastrophic for the entire economy. The goal, therefore, is to establish an equilibrium in the interbank market to maintain system stability, prevent defaults, and mitigate systemic risk.

Each bank maintains a certain level of monetary reserves for its daily activities in the banking industry. Banks also want to utilize their cash and resources sufficiently to capture investment opportunities in the market. Therefore, banks need to keep the appropriate amount of cash to engage in trades and investments that benefit their long-term operation while not facing liquidity issues. To study this matter, we set up an interbank transactions system, drawing motivations from (Carmona et al., 2015) and (Chang et al., 2023). We incorporate one major bank, which can be seen as a systemically important financial institution in the U.S banking system, and many small banks. The major bank's action affects the system's state and each minor bank. We model the dynamics of monetary reserves and cost functionals of banks in the mean-field games (MFG) and risk-sensitive linear-quadratic Gaussian (LQG) setting. MFG is the study of the connection between the behaviours of a large finite population and the limiting case with an infinite population. It explores the strategies taken by each player in a game setting and the presence of approximate Nash equilibria formed by the strategies (Lasry and Lions, 2006, 2007; Caines et al., 2006, 2007; Carmona and Delarue, 2018). In this setting, we assume that matching the market average level indicates effective utilization of a bank's monetary reserves, as motivated by Carmona et al. (2015). Therefore, banks borrow when their monetary reserves are lower than the market average level and lend when their monetary reserve levels are higher. Each bank also engages in transactions with the central bank at additional costs at a borrowing/lending rate controlled by the bank itself, which is the frequency at which an individual bank trades with the central bank based on its needs. Therefore, banks want to find the optimal trading rate to maintain adequate funds while the costs are minimized. We demonstrate in the later sections that by operating at this optimal borrowing/lending rate, the interbank system reaches a Nash equilibrium in the limiting case where there is an infinite number of minor banks and no bank benefits from deviating from its optimal trading strategy. The set of optimal trading rates forms a ε-Nash equilibrium when a finite number of minor banks present in the interbank market.

Furthermore, banks are sensitive to liquidity risk and credit risk. Liquidity risk, as mentioned above, is the probability that a bank cannot cover its short-term debt due to the lack of cash, and credit risk is the risk of a borrower failing to repay its loans. There is no significant association between credit risk and liquidity risk in terms of contemporaneous and time-lagged relationships. Nonetheless, these two risks affect bank stability and might contribute to the system's failure individually and through their interaction (Ghenimi et al., 2017). Therefore, the banks in our model behave in a risk-averse fashion. The risk aversion is featured through an exponential quadratic cost functional, which also allows us to include the higher moments of the integral cost function like skewness and kurtosis. By modelling the higher moments, we can obtain a more robust framework for sensitivity analysis and decision-making, particularly for a system with many stochastic factor such as in this work. We also consider different types of minor banks, such as retail banks, commercial banks, and investment banks. These different types of banks target different groups of clients, specialize in various services, and are exposed to different sources of risks. The average state of the system could be obtained from the empirical distribution of the states of all the minor players. We mathematically determine the optimal rates of borrowing and lending of the major and minor banks, which serve as their best-response strategies, using the convex analysis and variational analysis approaches in Firoozi et al. (2020) and Liu et al. (2023). The MFG theory states that when the number of players in a finite population system increases, the system's behaviour mirrors that of an infinite system. Therefore, we solve our problem under an infinite system since it is often mathematically simpler while providing valuable insights into complex large-scale finite systems. The best-response strategies derived form a Nash equilibrium under the infinite-population system, and an approximated Nash (ε-Nash) equilibrium for the finite-population system.

Additionally, we define the default probability of the major and minor banks, which is the likelihood of a bank's monetary reserves falling below a certain default threshold. We also study systemic risk, the probability that the market state, a combination of the monetary reserves of the major bank and the average reserves of all minor banks, falls below the defined default threshold. We conduct Monte Carlo simulations to analyze the impact of risk sensitivity degrees of the major bank and a subgroup of minor banks, the size of the major bank, the default of the major bank or a subgroup of minor banks, how closely the major bank tracks the market state, and the major bank's growth rate on banks' default probabilities and systemic risk in the finite population. Our findings imply that the presence of a risk-averse major bank improves the stability of itself, minor banks, and the interbank market when the major bank's size takes up less than or equal to half the market size. However, a larger major bank is associated with increased default probabilities across the system, especially when the major bank fails. The increase in the size of a risk-sensitive major bank counteracts the benefits of its risk aversion. Furthermore, our simulations show that a major bank closely aligning its monetary reserves with the market average raises default risks throughout the system. In contrast, a major bank with a faster growth rate improves the financial health of other banks and the system. This growth rate can be seen as government support to prevent SIFIs from going bankrupt. Moreover, a subgroup of minor banks being risk-averse generally increases the default probabilities of the major bank, minor banks in other subgroups, and the system since their more cautious behaviours can lead to a less liquid and robust interbank market. The collapse of a subgroup of small banks also impacts the stability of banks across the market. The larger the subgroup, the more substantial the influence its default has. These evaluations emphasize the significance of regulating the size of major financial institutions and optimizing other features of banks to maintain the interbank market's stability, owing to its interconnected nature.

This thesis is greatly motivated by Chang et al. (2023) and Liu et al. (2023). Under a similar framework, Chang et al. (2023) propose a risk-neutral interbank system with a major bank and many minor banks. The results obtained using Monte Carlo simulations indicate that the presence of a major bank in the market increases the total default probability of a representative minor bank and the systemic risk. A larger major bank and a higher mean reversion rate also contribute to greater systemic risk. Therefore, having a large bank is not necessarily good for the economy, and policymakers should impose regulations on large banks to guarantee that they do not default. Liu et al. (2023) employ the variational method to address general LQG risk-sensitive optimal control problems and MFGs with one major agent and many minor agents of different types.

The contributions of our work are outlined as follows

- Building on the work of Firoozi et al. (2020) and Liu et al. (2023) in a risk-sensitive setting, we use the convex analysis and variational technique to derive the optimal trading strategies of banks, which yield a Nash equilibrium in the infinite-population system and a ε-Nash equilibrium in the finite-population system. We then verify if the obtained optimal strategies for the major bank and a representative minor bank simplify and admit a specific mean-reversion form for the risk-sensitive case.

- To the best of our knowledge, this is the first time that the effect of risk sensitivity on an interbank transactions system with one major bank and different types of minor banks is dynamically simulated.

- We perform Monte Carlo simulations for different scenarios to analyze how default probabilities of the major bank, minor banks, and systemic risk are influenced by
  - the degree of risk sensitivity of a major bank or a subgroup of minor banks,
  - the percentage of the market state the major bank tracks,
  - the growth/income rate of the major bank, and
  - the collapse of the major bank or a subgroup of minor banks.

The subsequent sections of the thesis are structured as follows: We first review the previous work on the relevant topics. Chapter 1 introduces the models used for the interbank transactions system, the market clearing condition, and Nash and ε-Nash equilibria. We then formulate the models under the MFG setting in Chapter 2 and present the analytical solutions and the derivation of banks' optimal borrowing and lending rates. We also derive a simplified version of banks' optimal transaction rates in the form of mean reversion rates in the risk-sensitive setting. Chapter 3 first presents the impact of banks' degree of risk sensitivity and volatility level on their optimal trading strategies. We then define the default probabilities we want to study and the market's systemic risk. Numerical experiments on different scenarios are then conducted to analyze further the impact of risk sensitivity degrees, the major bank's size, the percentage of the market state the major bank follows, the major bank's growth rate, the default of the major bank or a subgroup of minor banks on the major bank, minor banks, and the interbank market. Finally, we conclude our work and propose recommendations accordingly.

Literature review

Interbank lending and borrowing constitute a cornerstone of operations for most financial institutions and are pivotal to the overall stability of the financial system. The interbank market serves as a platform facilitating banks in managing and redistributing their money reserves, which enhances the efficiency of financial intermediation (d'Addio et al., 2020). Each bank can set its own interest rates, often contingent on its financial capacity and the prevailing market structure. Hence, there has been evidence of banks engaging in collusion and monopolistic pricing to increase lending rates, decrease deposit rates, and enhance their overall performance (Lartey et al., 2023). Despite the global consistency of interbank markets' bilateral nature, the system's structure might differ based on the economic setting. In'T Veld and Van Lelyveld (2014) identify two primary models of the interbank structure: the nested split graph (NSG) and the core-periphery (CP) model. The former model is built based on the reliability of each bank, and the latter is a framework in which core banks act as intermediaries between peripheral banks that rely on them. A hierarchical structure within the interbank market is proposed by Craig and Von Peter (2014), where lower-tier banks trade with each other through intermediaries acted by large money center banks. This particular structure connects the financial network closely, implying that the actions of a single bank can affect several entities beyond its immediate counterparty. Furthermore, the interbank market serves as a critical component in the transmission of monetary policy, mainly when asymmetric information prevails in the system, positioning it at the forefront of the sequence of policy effects (Freixas and Jorge, 2008).

The interbank transactions system has also been regarded as a platform where banks seek protection from idiosyncratic liquidity shocks that are specific to their operations (In'T Veld and Van Lelyveld, 2014). However, a network as connected and complicated as the interbank market may aggravate liquidity crises and financial contagion, as the distress within one institution can spread across the balance sheets of other banks and cause more defaults (Gai and Kapadia, 2010; Gai et al., 2011). The 2007 - 2008 Global Financial Crisis (GFC) highlighted this vulnerability. Following BNP Paribas' decision to temporarily halt withdrawals from several hedge funds invested in sub-prime mortgage-backed securities on August 9, 2007, major settlement banks increased their liquidity holdings by an average of 30%. The inefficient use of liquidity caused a rise in the interbank rates, ultimately leading to an interest-rate contagion in the interbank market (Acharya and Merrouche, 2013). Moreover, when there is a liquidity shock, banks tend to withdraw deposits from other banks rather than liquidate their long-term assets, which is more costly. However, given the linkage pattern of interbank lending, the liquidity shortfall may spread across the financial system, potentially leading to a freeze in the interbank market, much like during the GFC (Upper and Worms, 2004).

The collapse of Lehman Brothers during the GFC also raises concerns about the size of banks. The late 1990s saw a noticeable increase in the size of large banks, with more engagement in market-based operations. Compared to their smaller counterparts, large banks often operate with a lower capital level, less reliable funding, and a more complex structure, which contributes to a higher level of systemic risk. Laeven et al. (2014) discuss the challenges of regulating large banks, suggesting measures such as capital surcharges, reducing their participation in market-based activities, and simplifying their organizational structures. The authors also note that there is no one-size-fits-all approach to bank size since large banks offer economies of scale and scope that smaller banks cannot match. We will study the effect of the presence of a large bank in our model in the later sections.

There has been a substantial amount of research on the systemic risk of interbank markets since the failure of the market can be disastrous to the entire financial system. In addition to factors such as market power, asymmetric information, and the incompleteness of markets, Lucchetta (2010) also studies the impact of higher concentration of risks, often resulting from a concentrated market structure, on the breakdowns of the interbank market. Furthermore, the risk sensitivity of banks has a strong correlation with the presence of robust financial markets. Pausch (2012) reveals that in the absence of an interbank market and liquidity regulation, risk-neutral banks display risk-averse behaviour. Therefore, the presence of a robust interbank market has the effect of eliminating endogenous risk aversion, enabling banks to independently manage credit risk and liquidity risk. Policymakers and regulators have implemented various measures to mitigate systemic risk, including implicit insurance for interbank claims, discounted loans, purchase-and-assumption arrangements, and nationalizations. Another approach to reducing systemic risk is the centralization of liquidity management inside banks. In this case, the central bank takes on the role of a counterparty for each transaction, prevents the spread of credit risks through the market, even in the event of a bank's default (Rochet and Tirole, 1996).

To analyze systemic risk mathematically, Carmona et al. (2015) introduce a framework where the dynamics of the logarithmic monetary reserves of banks are simulated by diffusion processes coupled through the control in the drift terms. The system stability is contingent upon this control, which is the rate at which one bank conducts transactions with other banks. A game feature is integrated in which individual banks decide their borrowing and lending rates with the central bank to optimize its reserves level while minimizing costs. The central bank in this model is also shown to function as a clearing house. The study suggests that banks engaging in interbank borrowing and lending, as well as trading with the central bank, contribute to the overall stability of the financial system. Moreover, the likelihood of systemic risk, defined as the event of a significant number of banks surpassing a certain default threshold, is quantified using the large deviation theory. Sun (2017) presents a similar model that includes a growth rate/income rate and does not have a common noise component in the Brownian motions of the monetary reserves dynamics, unlike in Carmona et al. (2015). The liquidity created by the Markov-Nash equilibrium for a finite number of banks results in a flocking effect, leading to either market stability or systemic risk, contingent on the growth rate. Furthermore, the deposit rate incorporated in the model can hinder the growth of the overall monetary reserve, thus increasing the likelihood of bank defaults. Huang and Jaimungal (2017) restructure the model in Carmona et al. (2015) as a robust stochastic game and accounts for ambiguity aversion, meaning that each bank evaluates alternative models and has its own version of the log-reserves dynamics. The authors demonstrate that ambiguity-averse players might have lower default probabilities than ambiguity-neutral players. The existence of a ε-Nash equilibrium and the validity of a verification theorem for convex-concave cost functions are also proven. Interestingly, Zhou (1997) presents a method to analytically value securities that are exposed to risks and their default probabilities. The proposed technique involves modeling the dynamics of firm value as a jump-diffusion process, which combines the empirical flexibility of the reduced-form method with the ability to offer theoretical insights into the market mechanisms, as provided by the Merton-Black-Cox-Longstaff-Schwartz's structural approach.

To obtain decentralized control strategies in a noncooperative game setting, Huang (2010) presents a model involving a major agent and a large population of minor agents under the LQG framework. Mean-field game (MFG) is introduced as the influence of each individual player in the model is negligible on the whole system, and all players interact through the average state of the system. Since each player typically only has complete information on its own situation in a large population, the Nash certainty equivalence (NCE) approach is employed. The optimal strategies of agents obtained from this method form an asymptotic Nash equilibrium, where the strategy of each agent is determined only by its own state and some deterministic variables. Many other literatures on MFG account for the presence of a major agent since there is often a relatively dominant participant in the game in many practical scenarios (Nourian and Caines, 2013; Carmona and Zhu, 2016; Bensoussan et al., 2016; Carmona and Wang, 2016, 2017; Lasry and Lions, 2018; Firoozi et al., 2020). Carmona and Wang (2016) and Lasry and Lions (2018) study the MFG theory with one major agent and many small agents in a finite game setting, while Nourian and Caines (2013) consider a system with a nonlinear stochastic dynamic formulation. The alternative method of searching for fixed points for the optimal response functions proposed by Carmona and Wang (2017) underlines the importance of McKean-Vlasov (MV) dynamics. Nguyen and Huang (2012) focus on the continuum case where the mean field is calculated using a random process solely dependent on the major agent's initial state and Brownian motion. Another interesting case under the major-minor (MM) setting is the Stackelberg game setup, where individual follower's evolution is subject to delay effects from the leader and their own state and control variables (Bensoussan et al., 2017). The result establishes the existence and uniqueness of solutions for different forward-backward stochastic differential equations (FBSDE).

Furthermore, Firoozi et al. (2020) propose a convex analysis method to obtain the best-response strategies for all agents without any assumptions on the evolution of the mean-field for the MM LQG MFG system. Building on this method, Liu et al. (2023) use a variational technique and a change of measure to address the risk-sensitive optimal control problems with exponential cost functionals. We will approach the optimal control problem in our setting using these methods. An interesting case of the MM LQG MFG systems is when minor agents have partial observation of the state of the major agent, while the major agent has full knowledge of its own state (Şen and Caines, 2014, 2016; Caines and Kizilkale, 2017). Firoozi and Caines (2021) also examine the case where all agents have partial observations, meaning the major agent possesses partial information of its own state, and each minor agent partially observes both its own state and the major agent's state. Therefore, each minor agent recursively estimates the major agent's assessment of its own state. The presence of ε-Nash equilibria and the controls leading to the equilibria are established by the separation principle in the context of partially observed (PO) MM LQG MFG systems. Finally, while there are several methodologies devised to address the MM MFG problems, literature has shown that the obtained solutions are consistent using different approaches (Firoozi et al., 2020; Huang, 2021; Firoozi, 2022).

The concept of MFG is also used in many other topics. Firoozi and Caines (2017) adapt the PO MM LQG MFG system to the optimal execution problems in the finance field. When a major institutional investor liquidates a predetermined number of shares, the minor high-frequency traders in the market seek to either sell or purchase a specific quantity of shares within the period. Each trader only possesses partial information about its own state and the major investor's state. Hence, each agent aims to find the optimal trading rate to maximize its portfolio and performance function. Furthermore, market participants have different perspectives on the models and dynamics followed by financial instruments despite being presented with the same data. Based on this fact, Casgrain and Jaimungal (2020) turn the trading world into a stochastic game where each player aims to optimize their trading strategies under different probability measures. The Nash equilibrium under the MFG limit can be obtained by solving a nonstandard vector-valued FBSDE. The solar renewable energy certificate (SREC) market can also be conceptualized as a stochastic game in which companies strategically maximize their rates of SREC generation and trade to achieve optimal outcomes. All the players in the system interact through the SREC price, which is determined through the market clearing condition. The optimal controls derived are the solution to a MV FBSDE (Shrivats et al., 2022). Other applications of the MFG theory include optimal consumption and investment (Lacker and Zariphopoulou, 2019; Lacker and Soret, 2020; Min and Hu, 2021), financial engineering such as blockchain and cryptocurrency mining (Djehiche et al., 2019; Bertucci et al., 2020; Carmona, 2020; Li et al., 2022), trade crowding (Cardaliaguet and Lehalle, 2018; Carmona and Laurière, 2021), and moral hazard in an economic setting (Carmona, 2020; Elie et al., 2021; Fabrice, 2023).

Chapter 1

Interbank Transactions Model

In this chapter, we introduce the model used for the monetary transactions in an interbank system containing one major bank and N minor banks. This model is greatly inspired by the works of Carmona et al. (2015) and Chang et al. (2023). Each bank lends to or borrows from one another as well as the central bank when its log-monetary reserves surpass or fall below the average level of the market. The individual bank also controls its rate of transactions with the central bank to minimize the corresponding cost functional. We also discuss the market clearing condition that needs to be satisfied for the system to be a closed environment. Last but not least, we introduce the best-response strategies of banks and the Nash and ε-Nash equilibria that the strategies form in the interbank market.

1.1 Major Bank Model

A major bank is an institution whose decisions and actions impact the entire banking system due to its relatively large size. In Canada, we can identify institutes like Toronto-Dominion Bank and Royal Bank of Canada as major banks since their decisions can affect the Canadian economy.

For our model, we denote the major bank's log-monetary reserves at time t with x_t^0. The following stochastic differential equation (SDE) illustrates the dynamic of the log-monetary reserves

$$dx_t^0 = a_0\left[\theta_0\big(\lambda x_t^0 + (1-\lambda)x_t^{(N)}\big) - x_t^0\right]dt + u_t^0 dt + \gamma^0(t)dt + \sigma_0\, dw_t^0. \tag{1.1}$$

In this setting, $x_t^{(N)}$ indicates the average log-monetary reserves level of all minor banks, where $x_t^{(N)} = \frac{1}{N}\sum_{i=1}^{N} x_t^i$, and x_t^i is the log-monetary reserves a representative minor bank i possesses at time t. The parameter λ denotes the weight the major bank takes in the market, and $(1-\lambda)$ represents the weight of the average reserves of all minor banks in the system. The market state, $\lambda x_t^0 + (1-\lambda)x_t^{(N)}$, is defined as a linear combination of the log-monetary reserves of the large bank and the average reserves level of all small banks with the corresponding weights. In the interbank transactions market, banks are incentivized to maintain their monetary reserves close to the market state to minimize liquidity risk and efficiently utilize their funds. Therefore, the major bank borrows funds if its reserves are below the market state and lends money if the reserves are above it. Moreover, we use the parameter $\theta_0 \leq 1$ to indicate the proportion of the market state the major bank follows. If $\theta_0 = 1$, the major bank tries to stay close to the market state, whereas if $\theta_0 < 1$, the bank would match part of the market state. A bank might not want to follow the market trend entirely due to the fear that if the system's state decreases significantly, the bank's monetary reserves will be drained quickly as well.

We then denote the mean reversion rate of the major bank by a_0, which explains how fast the major bank's monetary reserves revert to the market state. A higher mean reversion rate indicates that the major bank lends to or borrows from other banks more frequently to stay close to the market state. Furthermore, the variable u_t^0 stands for the frequency of monetary transactions incurred outside the major-minor bank network at time t, i.e., the lending and borrowing activities between the major agent and the central bank. Through open market operations, a commercial bank lends to the central bank by buying government bonds when a contractionary monetary policy is implemented to reduce the money supply in the economy. For our model, this is the control input we want to optimize to minimize the cost functional mentioned in the later section. We also denote $\gamma^0(t)$ as the growth rate of the major bank, where $\gamma^0(t)$ is a deterministic process. This can be the bank's income from its daily financial activities other than interbank transactions. Moreover, σ_0 defines the volatility of the bank's log-monetary reserves, which is affected by the daily activities of its retail customers. The level of volatility is driven by a Brownian motion, which is a continuous random process where the increments of the process are independent. The Brownian motion w_t^0 for the major player is defined on the filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t^0\}, \mathbb{P}^0)$, where $\mathscr{F}^0 := (\mathscr{F}_t^0)_{t\in[0,T]}$ is the filtration generated by the major agent's state.

As indicated earlier, banks also conduct monetary transactions with the central bank apart from each other. However, it does cost more than borrowing from another commercial bank. In the United States, the discount rate charged by the Fed during the discount window is normally 50 basis points higher than the federal funds rate, which is the interest rate that banks charge one another. Therefore, the major bank wants to control the number of transactions with the central bank to minimize the cost functional expressed as follows

$$J^0(u) = \frac{1}{\delta_0}\mathbb{E}\Bigg[\exp\Bigg(\frac{\delta_0}{2}\Big(c_0\big[\theta_0(\lambda x_T^0 + (1-\lambda)x_T^{(N)}) - x_T^0\big]^2 + \int_0^T \Big\{\varepsilon_0\big[\theta_0(\lambda x_t^0 + (1-\lambda)x_t^{(N)}) - x_t^0\big]^2 - 2q_0 u_t^0\big[\theta_0(\lambda x_t^0 + (1-\lambda)x_t^{(N)}) - x_t^0\big] + (u_t^0)^2\Big\}dt\Big)\Bigg) - 1\Bigg]. \quad (1.2)$$

The major bank is averse to risk factors such as liquidity and customer credit. Using exponential functions of quadratic costs to solve the optimal control problem under a risk-sensitive setting has been studied extensively (Jacobson, 1973; Pan and Basar, 1994; Moon and Basar, 2017; Liu et al., 2023). The exponentiated quadratic cost allows us to capture an agent's risk-averse or risk-seeking behaviours and the higher-order moments of the integral function, such as skewness and kurtosis. These higher-order moments of the cost functional provide more insights on a stochastic linear system like our model. To our knowledge, the exponential cost functional has not been considered in the interbank transactions market context in the literature. The degree of risk sensitivity of the major bank is represented by the constant δ_0. The bank behaves in a risk-neutral fashion if δ_0

tends to zero. The cost functional consists of both the running cost and the terminal cost. The positive constants c_0 and ε_0 impose the terminal and running penalty, respectively. If a bank keeps too much extra cash, it is not using its income efficiently for financial activities, and hence costs are incurred. It is also risky for a bank to keep the level of its reserves too low. Therefore, the bank wants to maintain its log-monetary reserves close to a fraction of the market state, i.e., the target level. This target level depicts the overall interbank market's current condition and financial health. It also signals to banks if funds in the interbank system are growing or draining so banks can adjust their operating strategies accordingly. The parameters c_0 and ε_0 increase trading costs if the major bank fails to keep its reserves close to the target level. Moreover, we denote $q_0 > 0$ as the incentive for borrowing and lending. If the major bank's monetary reserves are lower than the target level, it intends to borrow from the central bank, leading to u_t^0 being positive, and vice versa, which reduces the costs. Hence, the bank is encouraged to trade more with the central bank to maintain liquidity. We also assume $q_0^2 \leq \varepsilon_0$ to guarantee that the cost functional is convex to be able to find the minimum value.

1.2 Minor Bank Model

We now introduce the model used to describe the dynamic of a representative minor agent's log-monetary reserves. In contrast with the Canadian banking system, where only a few big banks dominate the industry, there are many small banks in the United States for retail customers. As of 2021, there are 4,238 commercial banks insured by the Federal Deposit Insurance Corporation (FDIC). With this large amount of banks in the system, only a few banks have the size and ability to be influential in the entire American financial industry. Our minor bank model can be applied to a banking system like this or the Chinese interbank market, where there are 4,561 banking institutions as of 2023. Each minor bank i has a negligible impact on the system as N increases, where $i \in \{1, ..., N\}$ and N represents the number of minor banks in the system. We also categorize banks into different sectors, and define $K < N$ as distinct types of minor banks. The index set of each subpopulation is denoted as

$$I_k = \{i : \rho_i = \rho^{(k)}, i \in \mathfrak{N}\}, k \in \mathfrak{K} := \{1, ..., K\} \tag{1.3}$$

where $\rho^{(k)}$ is the set of parameters of subpopulation k that we will define later in the section. The number of banks of each type is $N_k = |I_k|$, which is the cardinality of I_k. Furthermore, we denote the empirical distribution of different types of agents as $\pi^N = (\pi_1^N, ..., \pi_K^N)$, where $\pi_k^N = \frac{N_k}{N}$. Therefore, the average level of the log-monetary reserves of all minor banks in the system may be formulated as the weighted average of the average reserves under each subpopulation. The average monetary reserve of type k banks is given by: $x_t^{(N),k} = \frac{1}{N_k} \sum_{i \in I_k} x_t^i$.

We now define the dynamic of the log-monetary reserves of minor bank i as the following

$$dx_t^i = a_k \Big[\theta_k \big(\lambda x_t^0 + (1 - \lambda) x_t^{(N)} \big) - x_t^i \Big] dt + u_t^i dt + \gamma^k(t) dt + \sigma_k dw_t^i. \tag{1.4}$$

Banks in the interbank market aim to track the market state to keep an appropriate amount of funds and to have a clearer picture of the system's overall condition. Hence, a minor bank will borrow if its monetary reserves are less than the market state and lend if they are above that level. Minor banks of each type share the same mean reversion rate a_k, growth rate $\gamma^k(t)$, and volatility level σ_k. Each minor bank also trades with the central bank and controls its rate of borrowing and lending, u_t^i. We denote the information set generated by a minor bank's state as: $\mathscr{F}^i := (\mathscr{F}_t^i)_{t \in [0,T]}$. The Brownian motion w_t^i driving the volatility level of the bank is defined on the filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t^i\}, \mathbb{P}^i)$.

The cost functional of a minor agent follows a similar structure as that of a major agent, which is expressed as

$$\begin{aligned} J^i(u) = \frac{1}{\delta_k} \mathbb{E} \Bigg[\exp \Bigg(\frac{\delta_k}{2} \Big(& c_k \big[\theta_k (\lambda x_T^0 + (1 - \lambda) x_T^{(N)}) - x_T^i \big]^2 \\ & + \int_0^T \Big\{ \varepsilon_k \big[\theta_k (\lambda x_t^0 + (1 - \lambda) x_t^{(N)}) - x_t^i \big]^2 \\ & - 2 q_k u_t^i \big[\theta_k (\lambda x_t^0 + (1 - \lambda) x_t^{(N)}) - x_t^i \big] + (u_t^i)^2 \Big\} dt \Big) \Bigg) - 1 \Bigg]. \end{aligned} \tag{1.5}$$

The minor bank gets penalized if it deviates too much from the fraction of the market state that it targets and is incentivized to trade more with the central bank. Banks under the same type are assigned the same parameters denoting the running penalty ε_k, terminal penalty c_k, and incentive for trading q_k. δ_k is the risk sensitivity degree of the kth-type minor banks. Each bank controls its borrowing/lending rate with the central bank to minimize the cost functional. A larger q_k implies a lower cost for transactions with the central bank and vice versa. We again assume $q_k^2 \leq \varepsilon_k$ to ensure that the cost functional is convex.

1.3 Market Clearing Condition

In an economic setting, market clearing is defined as matching the quantity of supply to the exact amount of demand so that there is no excess demand or supply in the market. For the interbank transactions system, the banks only trade with each other or with the central bank. In other words, this is a closed network where if one bank is lending some money, another bank must be receiving this same amount of money. Therefore, the supply equals the demand, and the net activities of all interbank transactions should be zero. Mathematically, the sum of all borrowing and lending of the log-monetary reserves done by the major and minor agents at time t should equal zero, which is expressed as

$$\frac{1}{N}\sum_{i=1} a_k\Big[\theta_k\big(\lambda x_t^0 + (1-\lambda)x_t^{(N)}\big) - x_t^i\Big] + a_0\Big[\theta_0\big(\lambda x_t^0 + (1-\lambda)x_t^{(N)}\big) - x_t^0\Big] = 0. \quad (1.6)$$

As mentioned in the last section, the average log-monetary reserves of all minor banks is calculated by the weighted average reserves of each subpopulation, which can be shown mathematically by

$$x_t^{(N)} = \frac{1}{N}\sum_{i=1}^{N} x_t^i = \sum_{k\in K}\frac{1}{N_k}\frac{N_k}{N}\sum_{i\in I_k} x_t^i = \sum_{k\in K}\frac{N_k}{N}\frac{1}{N_k}\sum_{i\in I_k} x_t^i = \sum_{k\in K}\pi_k^N x_t^{(N),k}. \quad (1.7)$$

By using this property and the empirical distribution of minor banks defined before, we can rewrite equation (1.6) as the following

$$\big(\lambda \sum_{k\in K} \pi_k^N a_k \theta_k + a_0\theta_0\lambda - a_0\big)x_t^0 + \sum_{k\in K} \pi_k^N \big[(1-\lambda) \sum_{k\in K} \pi_k^N a_k \theta_k - a_k + a_0\theta_0(1-\lambda)\big] x_t^{(N),k} = 0. \quad (1.8)$$

In order for this equation to be valid for every value of the processes x_t^0 and $x_t^{(N),k}$, we need

$$a_0 = \frac{\lambda}{1-\theta_0\lambda} \sum_{k\in K} \pi_k^N a_k \theta_k, \quad (1.9a)$$

$$a_k = (1-\lambda) \sum_{k\in K} \pi_k^N a_k \theta_k + a_0\theta_0(1-\lambda). \quad (1.9b)$$

Substituting (1.9a) in (1.9b) we have

$$a_k = (1-\lambda)\frac{a_0(1-\theta_0\lambda)}{\lambda} + a_0\theta_0(1-\lambda) = \frac{(1-\lambda)}{\lambda}(a_0 - a_0\theta_0\lambda + a_0\theta_0\lambda). \quad (1.10)$$

Finally, we obtain an expression for a_k in terms of a_0, given by

$$a_k = \frac{(1-\lambda)}{\lambda} a_0, \quad (1.11)$$

which is a constant. This indicates that regardless of which subpopulation the bank belongs to, all minor banks share the same mean-reversion rate. They trade at the same frequency to prevent themselves from deviating too far from the market state. We also notice that when the relative size of the major bank is larger than that of all the minor banks combined, its mean-reversion rate is greater than a generic minor banks' rate, and vice versa. If the major bank's size takes up a large portion of the interbank system, it will likely have a higher level of required reserves set by the governors to sustain its operations or have more excess money to lend. Therefore, the large bank needs to conduct more transactions to keep its monetary reserves above the required reserves level and close to the market state since each small bank can only fulfill part of the required funds. From now on, we denote the mean-reversion rate of a minor bank as a for all a_k, $k \in \mathfrak{K}$.

It is also worth discussing the role of the central bank in our model. Numerous transactions occur daily among different parties in the interbank system, which imposes great operational and settlement risks. Therefore, it is beneficial to have a third party between the two banks to facilitate the transactions, which is the central bank in this case. This is known as the clearing house function of a central bank (Carmona et al., 2015). The central bank acts as an intermediary and keeps records of all the transactions for each bank throughout the day. At the end of a trading day, the central bank organizes all the payments in one account, and each bank only needs to pay or receive the net amount, i.e., to clear the accounts. This process greatly reduces the mistakes made during financial interactions, saves costs with fewer payments, and guarantees a transparent interbank transactions system.

1.4 Interbank Market Equilibrium and Best-Response Transaction Rates

After introducing the models that represent the dynamics of log-monetary reserves and the cost functionals of major and minor banks in the interbank market, we aim to find the optimal rate of borrowing and lending with the central bank for each bank, such that their cost functionals are minimized. This optimal trading rate is known as the best-response transaction rate of the bank. If all major and minor banks follow their respective best-response functions simultaneously, the cost functional of each bank is minimized, and no bank will benefit from changing its strategy while all other players remain the same. In this case, the interbank system reaches a Nash equilibrium in the setting with an infinite population. We now introduce the concept of Nash equilibrium. A Nash equilibrium is achieved when no player can gain any extra benefit by changing its current strategy unilaterally in a non-cooperative game. Therefore, given that other agents follow the equilibrium strategies in the system, the player has no incentive to deviate from its strategy. The strategy taken by each player in a Nash equilibrium is considered a best-response strategy in interactions with other agents. Mathematically, we define a Nash equilibrium as follows

Definition 1 (Nash Equilibrium) *For a set of N-tuple strategies* $(u^1,\dots,u^N)\in\mathscr{U}^N$*, it is said to be a Nash equilibrium if for every* $i\in\{1,\dots,N\}$ *and* $u\in\mathscr{U}$*,* $\mathscr{U}$ *and* $\mathscr{U}^N$ *are the admissible sets of strategies of agent i,*

$$J^i(u^1,\dots,u^i,\dots,u^N)\leq J^i(u^1,\dots,u^{i-1},u,u^{i+1},\dots,u^N), \tag{1.12}$$

or

$$J^i(u^i,u^{-i})\leq J^i(u,u^{-i}), \tag{1.13}$$

where

$$J^i(u,u^{-i})=J^i(u^1,\dots,u^{i-1},u,u^{i+1},\dots,u^N). \tag{1.14}$$

We then define the best-response strategy of agent i as:

$$u^i=\arg\min_{u\in\mathscr{U}^i}J^i(u,u^{-i}). \tag{1.15}$$

We also introduce the notion of an ε-Nash equilibrium for when there is a finite population in the interbank system. In a Nash equilibrium, players do not have the motivation to change their strategies, whereas, in an ε-Nash equilibrium, an agent may have some amount of incentive, ε, to change its behaviour, where ε is a small positive value. Furthermore, the ε-Nash approximation property states that when agents in a finite-population system use the infinite-population equilibrium strategies, they achieve an approximation of the infinite population equilibrium. An ε-Nash equilibrium is defined as

Definition 2 (ε-Nash Equilibrium) *A set of strategies* $(u^1,\dots,u^N)\in\mathscr{U}^N$ *is said to be an* ε*-Nash equilibrium for an N-player game if there exists* $\varepsilon>0$*, such that for every* $i\in\{1,\dots,N\}$ *and* $u\in\mathscr{U}$*,*

$$J^i(u^i,u^{-i})-\varepsilon\leq J^i(u,u^{-i})=\inf_{u\in\mathscr{U}^i}J^i(u,u^{-i}). \tag{1.16}$$

However, obtaining an exact ε-Nash equilibrium is very challenging, given that we are working with a model that consists of many agents. Therefore, we want to simplify the problem by considering a system with infinitely many banks for which $N \to \infty$, hoping to obtain some asymptotic properties in this limiting case (Carmona and Delarue, 2018), which could lead to an approximate Nash equilibrium. Hence, we employ the theory of mean-field games (MFG), which is the study of the asymptotic behaviour of games with a large number of players. We will introduce the theory in details in the next chapter.

Chapter 2

Optimal Control and Best-Response Strategies

In this chapter, we aim to obtain the optimal transaction strategy of each bank, which is the optimal rate of borrowing/lending between an individual bank and the central bank such that the cost functional of the bank is minimized. We first introduce the mean-field games (MFGs) concept and transit our model to the MFG setting. We then approach the problem with an infinite number of minor banks, using the dynamics and cost functionals under the mean-field setting derived in Section 2.1. This leads to a Nash equilibrium for the infinite-population system, where each bank is subject to its optimal strategy and interacts through the overall effect of the system. In this limiting case, we notice from equations (2.5) and (2.7) that the state of the major bank's log-monetary reserves is impacted by the state mean-field, and the minor bank's reserves are impacted by the state mean-field as well as the major bank's state. Therefore, we extend the major bank's dynamic to include the mean-field dynamics and extend each minor bank's dynamics to include the monetary reserves dynamics of both the major bank and the mean-field of the system. The stochastic game we initially have to solve for the system becomes a stochastic control problem for each bank. We will explain each step in detail in the following sections.

2.1 Mean-Field Game Formulation

In our model, each minor bank controls its lending/borrowing rate with the central bank and only has information on its own log-monetary reserves dynamic and cost functional. Therefore, it does not make decisions based on other banks' activities. However, to obtain an equilibrium or the optimal state for the whole system, banks need to consider their interactions with one another. Hence, we define the state mean-field and control mean-field of the system respectively as

$$\bar{x}_t = \mathbb{E}[x_t^{\cdot}|\mathscr{F}_t^0], \tag{2.1}$$

$$\bar{u}_t = \mathbb{E}[u_t^{\cdot}|\mathscr{F}_t^0]. \tag{2.2}$$

where $\bar{x}_t^\top = [(\bar{x}_t^1)^\top, ..., (\bar{x}_t^K)^\top]$ and $\bar{u}_t^\top = [(\bar{u}_t^1)^\top, ..., (\bar{u}_t^K)^\top]$ are vectors of mean-fields of the monetary reserves and control of each subpopulation $k \in \mathfrak{K}$. The mean-field of the log-monetary reserves of each type of minor banks is equivalently expressed as

$$\bar{x}_t^k = \lim_{N_k \to \infty} x_t^{(N),k} = \lim_{N_k \to \infty} \frac{1}{N_k} \sum_{i \in I_k} x_t^i, \tag{2.3}$$

which equals the market average state of the subgroup when the number of banks in this group goes to infinity if the limit exists. Similarly, the mean field of the borrowing/lending rate with the central bank of each subpopulation is equivalently written as

$$\bar{u}_t^k = \lim_{N_k \to \infty} u_t^{(N),k} = \lim_{N_k \to \infty} \frac{1}{N_k} \sum_{i \in I_k} u_t^i. \tag{2.4}$$

Consequently, the major bank and minor banks interact with the system through the mean-field. We can then rewrite the dynamic of monetary reserves and cost functional of a major bank defined before in the limiting case

$$dx_t^0 = \Big[a_0(\theta_0\lambda - 1)x_t^0 + (a_0\theta_0(1-\lambda) \otimes \pi)\bar{x}_t + u_t^0 + \gamma^0(t)\Big]dt + \sigma_0\, dw_t^0, \tag{2.5}$$

$$\begin{aligned} J^0(u) = \frac{1}{\delta_0}\mathbb{E}\Bigg[\exp\Bigg(\frac{\delta_0}{2}\Big(c_0\big[\theta_0(\lambda x_T^0 + ((1-\lambda)\otimes\pi)\bar{x}_T) - x_T^0\big]^2 \\ + \int_0^T \Big\{\varepsilon_0\big[\theta_0(\lambda x_t^0 + ((1-\lambda)\otimes\pi)\bar{x}_t) - x_t^0\big]^2 \\ - 2q_0\, u_t^0\big[\theta_0(\lambda x_t^0 + ((1-\lambda)\otimes\pi)\bar{x}_t) - x_t^0\big] + (u_t^0)^2\Big\}dt\Big)\Bigg) - 1\Bigg], \end{aligned} \tag{2.6}$$

where $(1-\lambda)\otimes\pi := [\pi_1(1-\lambda),...,\pi_K(1-\lambda)]$ is the Kronecker product, and we assume $\pi_k = \lim_{N\to\infty}\pi_k^N$. The mean-field of the interbank system $\bar{x}_t$ is then the weighted average of the mean-field monetary reserves of all subgroups of the minor banks.

The expressions for the log-monetary reserves and cost functional of a small bank when there is an infinite number of minor players in the system are

$$dx_t^i = \Big[-ax_t^i + (a\theta_k(1-\lambda)\otimes\pi)\bar{x}_t + a\theta_k\lambda x_t^0 + u_t^i + \gamma^k(t)\Big]dt + \sigma_k dw_t^i, \tag{2.7}$$

$$\begin{aligned} J^i(u) = \frac{1}{\delta_k}\mathbb{E}\Big[\exp\Big(\frac{\delta_k}{2}\Big(c_k\big[\theta_k(\lambda x_T^0 + (1-\lambda)\otimes\pi)\bar{x_T}) - x_T^i\big]^2 \\ + \int_0^T \Big\{\varepsilon_k\big[\theta_k(\lambda x_t^0 + (1-\lambda)\otimes\pi)\bar{x}_t) - x_t^i\big]^2 \\ - 2q_k u_t^i\big[\theta_k(\lambda x_t^0 + (1-\lambda)\otimes\pi)\bar{x}_t) - x_t^i\big] + (u_t^i)^2\Big\}dt\Big)\Big) - 1\Big]. \end{aligned} \tag{2.8}$$

We can then derive the dynamic of the mean-field. The dynamic (2.7) subject to the control u_t^i has a solution $x_t^{i,k}$ for type-k agent i as the following

$$\begin{aligned} x_t^{i,k} = e^{-at}x_0^i + \int_0^t e^{-a(t-\tau)}\Big[a\theta_k\big(\lambda x_\tau^0 + ((1-\lambda)\otimes\pi)\bar{x}_\tau\big)\Big]d\tau \\ + \int_0^t e^{-a(t-\tau)}(u_\tau^{i,k} + \gamma^k(\tau))d\tau + \int_0^t e^{-a(t-\tau)}\sigma^k dw_\tau^i. \end{aligned} \tag{2.9}$$

To calculate the mean-field of subpopulation k, we take the conditional expectation of $x_t^{i,k}$ with respect to the filtration $\mathscr{F}_t^0$, which yields

$$\begin{aligned} \bar{x}_t^k = \bar{x}_0 + \int_0^t e^{-a(t-\tau)}\Big[a\theta_k\big(\lambda x_\tau^0 + ((1-\lambda)\otimes\pi)\bar{x}_\tau\big)\Big]d\tau \\ + \int_0^t e^{-a(t-\tau)}(\bar{u}_\tau^k + \gamma^k(\tau))d\tau. \end{aligned} \tag{2.10}$$

The diffusion term vanishes since the expectation of Brownian motions is zero due to its independence property. We then calculate the infinitesimal variation of $\bar{x}_t^k$ according to (2.10) and finally obtain the mean-field of the log-monetary reserves of the interbank system as

$$d\bar{x}_t = [\breve{A}\bar{x}_t + \breve{G}x_t^0 + \bar{u}_t + \breve{m}(t)]dt, \tag{2.11}$$

where

$$\breve{A} = \begin{bmatrix} -a\mathbf{e}_1 + a\theta_1(1-\lambda)\otimes\pi \\ \vdots \\ -a\mathbf{e}_K + a\theta_K(1-\lambda)\otimes\pi \end{bmatrix}, \quad \breve{G} = \begin{bmatrix} a\theta_1\lambda \\ \vdots \\ a\theta_K\lambda \end{bmatrix}, \quad \breve{m}(t) = \begin{bmatrix} \gamma^1(t) \\ \vdots \\ \gamma^K(t) \end{bmatrix}, \quad (2.12)$$

and $\mathbf{e}_k = [0_{nxn}, ..., \mathbb{I}_n, ..., 0_{nxn}]$, with $\mathbb{I}_n$ being an identity matrix of size n at the k-th block.

2.2 Nash and ε-Nash Equilibria for the Interbank Market

We first present the optimal transaction strategies of the major bank and a representative minor bank in the interbank market that form a Nash equilibrium in the infinite-population system. The obtain optimal controls also admit an ε-Nash property in the finite-population system as shown in Huang (2010) and Carmona and Zhu (2016). We will show the detailed methodology in Section 2.3.

Theorem 1 (Best-Response Strategies) *For the interbank transactions system defined by equations* (2.5) - (2.8)*, the optimal borrowing and lending rates with the central bank for the major bank and a representative minor bank i are, respectively, given by*

$$u_t^{0,*} = (q_0 - \phi_t^0)\left[(\theta_0(1-\lambda)\otimes\pi)\bar{x}_t + (\theta_0\lambda - 1)x_t^0\right] - \mathbb{B}_0^\top s_0(t), \quad (2.13)$$

$$u_t^{i,*} = -\Big[\mathbb{N}_k^\top X_t^i + \mathbb{B}_k^\top\big(\Pi_k(t)X_t^i + s_k(t)\big)\Big]. \quad (2.14)$$

The deterministic processes $\phi_0(t)$, $s_0(t)$, $\Pi_k(t)$ and $s_k(t)$ in the optimal strategies satisfy the following sets of ordinary differential equations (ODEs) given by

$$\begin{cases} -\dot{\phi}_t^0 = 2\phi_t^0\big(a_0(\theta_0\lambda - 1) + H_0^\pi\bar{G}\big) - (\phi_t^0 - q_0)^2(\theta_0\lambda - 1) + \varepsilon_0(\theta_0\lambda - 1) \\ \qquad + \delta_0(\phi_t^0)^2(\sigma_0)^2(\theta_0\lambda - 1), \qquad \phi_T^0 = c_0(\theta_0\lambda - 1); \\ -\dot{s_0}(t) = \big((\mathbb{A}_0 - \mathbb{B}_0\mathbb{N}_0^\top)^\top - \Pi_0(t)\mathbb{B}_0\mathbb{B}_0^\top\big)s_0(t) + \Pi_0(t)\mathbb{M}_0 \\ \qquad + \delta_0\Pi_0(t)\Sigma_0\Sigma_0^\top s_0(t), \qquad s_0(T) = 0. \end{cases} \quad (2.15)$$

$$
\begin{cases}
-\dot{\Pi}_k(t) = \Pi_k(t)\mathbb{A}_k + \mathbb{A}_k^\top \Pi_k(t) - (\Pi_k(t)\mathbb{B}_k + \mathbb{N}_k)(\mathbb{B}_k^\top \Pi_k(t) + \mathbb{N}_k^\top) + \mathbb{Q}_k \\
\qquad\qquad + \delta_k \Pi_k(t)\Sigma_k\Sigma_k^\top \Pi_k(t), \qquad \Pi_k(T) = \mathbb{G}_k; \\
-\dot{s_k}(t) = \big((\mathbb{A}_k - \mathbb{B}_k\mathbb{N}_k^\top)^\top - \Pi_k(t)\mathbb{B}_k\mathbb{B}_k^\top\big)s_k(t) + \Pi_k(t)\mathbb{M}_k \\
\qquad\qquad + \delta_k \Pi_k(t)\Sigma_k\Sigma_k^\top s_k(t), \qquad s_k(T) = 0,
\end{cases}
\tag{2.16}
$$

where

$$
\begin{aligned}
&\mathbb{N}_0 = -[1, H_0^\pi]^\top q_0(\theta_0\lambda - 1), \quad \mathbb{N}_k = [1, -H_k, -\hat{H}_k^\pi]^\top q_k, \\
&\mathbb{Q}_k = [1, -H_k, -\hat{H}_k^\pi]^\top \varepsilon_k [1, -H_k, -\hat{H}_k^\pi], \\
&\mathbb{G}_k = [1, -H_k, -\hat{H}_k^\pi]^\top c_k [1, -H_k, -\hat{H}_k^\pi], \\
&H_0 = \frac{\theta_0(1-\lambda)}{\theta_0\lambda - 1}, \quad H_0^\pi = [\pi_1 H_0, ..., \pi_K H_0], \\
&H_k = \theta_k\lambda, \quad \hat{H}_k = \theta_k(1-\lambda), \quad \hat{H}_k^\pi = [\pi_1\hat{H}_k, ..., \pi_K\hat{H}_k]
\end{aligned}
\tag{2.17}
$$

$$
\mathbb{A}_0 = \begin{bmatrix} a_0(\theta_0\lambda - 1) & a_0\theta_0(1-\lambda)\otimes\pi \\ \bar{G} & \bar{A} \end{bmatrix}, \quad \mathbb{M}_0 = \begin{bmatrix} \gamma^0(t) \\ \bar{m} \end{bmatrix}, \tag{2.18}
$$

$$
\mathbb{A}_k = \begin{bmatrix} -a & [a\theta_k\lambda, a\theta_k(1-\lambda)\otimes\pi] \\ 0 & \mathbb{A}_0 - \mathbb{B}_0\big(\mathbb{N}_0^\top + [\phi_t^0(\theta_0\lambda - 1), \phi_t^0\theta_0(1-\lambda)\otimes\pi]\big) \end{bmatrix}, \tag{2.19}
$$

$$
\mathbb{M}_k = \begin{bmatrix} \gamma^k(t) \\ \mathbb{M}_0 - \mathbb{B}_0\mathbb{B}_0^\top s_0(t) \end{bmatrix}, \mathbb{B}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \Sigma_0 = \begin{bmatrix} \sigma_0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.20}
$$

$$
\mathbb{B}_k = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \Sigma_k = \begin{bmatrix} \sigma_k & 0 \\ 0 & \Sigma_0 \end{bmatrix}. \tag{2.21}
$$

Furthermore, the corresponding mean-field equation is given by

$$
d\bar{x}_t = (\bar{A}\bar{x}_t + \bar{G}x_t^0 + \bar{m})dt, \tag{2.22}
$$

where

$$
\begin{aligned}
&\bar{A}_k = (-a - q_k - \Pi_{11}^k)\boldsymbol{e}_k + a\theta_k(1-\lambda)\otimes\pi + q_k\theta_k(1-\lambda)\otimes\pi - \Pi_{13}^k, \\
&\bar{G}_k = a\theta_k\lambda + q_k\theta_k\lambda - \Pi_{12}^k, \\
&\bar{m}_k = \gamma^k - \mathbb{B}_k^\top s_k,
\end{aligned}
\tag{2.23}
$$

$$\bar{A} = \begin{bmatrix} \bar{A}_1 \\ \vdots \\ \bar{A}_K \end{bmatrix}, \quad \bar{G} = \begin{bmatrix} \bar{G}_1 \\ \vdots \\ \bar{G}_K \end{bmatrix}, \quad \bar{m} = \begin{bmatrix} \bar{m}_1 \\ \vdots \\ \bar{m}_K \end{bmatrix}. \tag{2.24}$$

We represent $\Pi_k(t)$, *and* $s_k(t)$ *as*

$$\Pi_k(t) = \begin{bmatrix} \Pi_{11}^k & \Pi_{12}^k & \Pi_{13}^k \\ \Pi_{21}^k & \Pi_{22}^k & \Pi_{23}^k \\ \Pi_{31}^k & \Pi_{32}^k & \Pi_{33}^k \end{bmatrix}, \quad s_k(t) = \begin{bmatrix} s_1^k \\ s_2^k \\ s_3^k \end{bmatrix}, \tag{2.25}$$

with $\Pi_{11}^k, \Pi_{22}^k \in \mathbb{R}$, $\Pi_{33}^k \in \mathbb{R}^{K\times K}$, $s_1^k(t), s_2^k(t) \in \mathbb{R}$, *and* $s_3^k(t) \in \mathbb{R}^K$.

Proof. See the methodology section.

Corollary 1 *The closed-loop dynamics of log-monetary reserves of the major bank and a representative minor bank i are respectively, given by*

$$dx_t^0 = \Big\{(a_0 + q_0 - \phi_t^0)\big[\theta_0\big(\lambda x_t^0 + ((1-\lambda)\otimes\pi)\bar{x}_t\big) - x_t^0\big] - \mathbb{B}_0^\top s_0(t) + \gamma^0(t)\Big\}dt \tag{2.26}$$
$$+ \sigma_0\, dw_t^0,$$

$$dx_t^i = \Big\{(a + q_k)\big[\theta_k(\lambda x_t^0 + ((1-\lambda)\otimes\pi)\bar{x}_t - x_t^i\big] - (\Pi_{11}^k x_t^i + \Pi_{12}^k x_t^0 + \Pi_{13}^k \bar{x}_t) \tag{2.27}$$
$$- s_1^k + \gamma^k(t)\Big\}dt + \sigma_k\, dw_t^i.$$

Proof. The closed-loop dynamics are obtained by direct substitution of the optimal strategies (2.13) and (2.14) in the associated dynamics.

2.3 Methodology

This section is devoted to deriving the optimal rates of transactions of the major bank and a representative minor bank under a subgroup k in the risk-sensitive setting. We follow the convex analysis approach developed by Firoozi et al. (2020) and the variatinal approach in Liu et al. (2023) to solve this problem.

We first perturb the control action of the major bank by a small value ξ_0 in the direction $\omega^0 \in \mathscr{U}^0$, where $\mathscr{U}^0$ is the admissible set of control actions of the major bank, generated by the filtration $\mathscr{F}^0$. Then we observe the effect this shock has on the major agent itself, other minor banks in the system, and the interbank system as a whole. The dynamics of the major bank's monetary reserves in the infinite population setting with the perturbed control, $u_t^0 + \xi_0 \omega_t^0$, is given by

$$dx_t^{0,\xi_0} = \Big[a_0(\theta_0\lambda - 1)x_t^{0,\xi_0} + (a_0\theta_0(1-\lambda)\otimes\pi)\bar{x}_t^{\xi_0} + (u_t^0 + \xi_0\omega_t^0) + \gamma^0(t)\Big]dt + \sigma_0\, dw_t^0. \tag{2.28}$$

We can see from the equation that a shock in the major bank's control action will affect the bank's log-monetary reserves level since there is a change in its trading rate with the central bank. In an economic setting, this can be seen as the central bank adjusting the interest rate following the current monetary policies, leading to banks borrowing less or more from the federal reserves. The mean-field state $\bar{x}_t$ is also affected since the decisions made by the major bank influence the entire system due to its large size. Therefore, the major agent's state and mean-field in a minor bank's dynamics are adjusted accordingly. This results in the following minor bank's monetary reserves dynamics subject to the perturbed control of the major bank

$$dx_t^{i,\xi_0} = \Big[-ax_t^{i,\xi_0} + (a\theta_k(1-\lambda)\otimes\pi)\bar{x}_t^{\xi_0} + a\theta_k\lambda x_t^{0,\xi_0} + u_t^{i,\xi_0} + \gamma^k(t)\Big]dt + \sigma_k\, dw_t^i. \tag{2.29}$$

The perturbed state mean-field is then expressed as

$$d\bar{x}_t^{\xi_0} = [\breve{A}\bar{x}_t^{\xi_0} + \breve{G}x_t^{0,\xi_0} + \bar{u}_t + \breve{m}(t)]dt, \tag{2.30}$$

where the control mean-field stays the same since the perturbed factor is only directed at the major bank's control action.

After defining the new dynamics of the agents and the mean-field subject to the perturbation of the major bank's control, we extend the major bank's monetary reserves state by including mean-field as in

$$X_t^{0,\xi_0} = \begin{bmatrix} x_t^{0,\xi_0} \\ \bar{x}_t^{0,\xi_0} \end{bmatrix}. \tag{2.31}$$

Subsequently, the perturbed joint dynamics of the major bank and mean-field's monetary reserves satisfy the following stochastic differential equation (SDE)

$$dX_t^{0,\xi_0} = (\tilde{A}_0 X_t^{0,\xi_0} + \mathbb{B}_0 u_t^0 + \tilde{B}_0 \bar{u}_t + \xi_0 \mathbb{B}_0 \omega_t^0 + \tilde{M}_0(t))dt + \Sigma_0 dW_t^0, \tag{2.32}$$

where

$$\tilde{A}_0 = \begin{bmatrix} a_0(\theta_0\lambda - 1) & a_0\theta_0(1-\lambda)\otimes\pi \\ \breve{G} & \breve{A} \end{bmatrix}, \quad \mathbb{B}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \tilde{B}_0 = \begin{bmatrix} 0 \\ \mathbb{I}_k \end{bmatrix}, \tag{2.33a}$$

$$\tilde{M}_0 = \begin{bmatrix} \gamma^0(t) \\ \breve{m}(t) \end{bmatrix}, \quad \Sigma_0 = \begin{bmatrix} \sigma_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad W_t^0 = \begin{bmatrix} w_t^0 \\ 0 \end{bmatrix}. \tag{2.33b}$$

This provides us a better understanding of how the major bank's reserves level interacts with the mean-field level. The cost functional of the major bank subject to the perturbed control action in terms of the extended state (2.31) is given by

$$J^0(u^0 + \xi_0\omega^0) = \frac{1}{\delta_0}\mathbb{E}\Bigg[\exp\Big(\frac{\delta_0}{2}\Big((X_T^{0,\xi_0})^\top \mathbb{G}_0 X_T^{0,\xi_0}$$

$$+ \int_0^T \big\{(X_s^{0,\xi_0})^\top \mathbb{Q}_0 X_s^{0,\xi_0} + 2(X_s^{0,\xi_0})^\top \mathbb{N}_0(u_s^0 + \xi_0\omega_s^0) + (u_s^0 + \xi_0\omega_s^0)^2\big\}ds\Big)\Big) - 1\Bigg], \tag{2.34}$$

where

$$\mathbb{G}_0 = [1, H_0^\pi]^\top c_0(\theta_0\lambda - 1)^2[1, H_0^\pi], \tag{2.35a}$$

$$\mathbb{Q}_0 = [1, H_0^\pi]^\top \varepsilon_0(\theta_0\lambda - 1)^2[1, H_0^\pi], \tag{2.35b}$$

$$\mathbb{N}_0 = -[1, H_0^\pi]^\top q_0(\theta_0\lambda - 1), \tag{2.35c}$$

$$H_0 = \frac{\theta_0(1-\lambda)}{\theta_0\lambda - 1}. \tag{2.35d}$$

Following Theorem 1 in Liu et al. (2023), we can write the Gâteaux derivative of the major bank's perturbed cost functional (2.34) in the direction ω^0 as

$$\langle DJ_0^\infty(u), \omega^0\rangle = \delta_0\mathbb{E}\Bigg[\int_0^T (\omega_t^0)^\top \Big\{\mathbb{B}_0^\top e^{-\tilde{A}_0^\top t} M_{2,t}^0$$

$$+ M_{1,t}^0\big(u_t^0 + \mathbb{N}_0^\top X_t^0 + \mathbb{B}_0^\top \int_0^t e^{\tilde{A}_0^\top(s-t)}(\mathbb{Q}_0 X_s^0 + \mathbb{N}_0 u_s^0)ds\big)\Big\}dt\Bigg], \tag{2.36}$$

where $M^0_{1,t}$ and $M^0_{2,t}$ are martingales on the filtration $\mathscr{F}^0_t$ given by

$$M^0_{1,t} = \mathbb{E}[e^{G(u^0)}|\mathscr{F}^0_t], \tag{2.37a}$$

$$M^0_{2,t} = \mathbb{E}\Big[e^{G(u^0)}\big(e^{\tilde{A}_0^\top T}\mathbb{G}_0 X^0_T + \int_0^T e^{\tilde{A}_0^\top s}(\mathbb{Q}_0 X^0_s + \mathbb{N}_0 u^0_s)ds\big)|\mathscr{F}^0_t\Big]. \tag{2.37b}$$

In equations (2.37a) and (2.37b), $G(u^0)$ is defined as

$$G(u^0) = \frac{\delta_0}{2}\Big((X^0_T)^\top \mathbb{G}_0 X^0_T + \int_0^T \{(X^0_s)^\top \mathbb{Q}_0 X^0_s + 2(X^0_s)^\top \mathbb{N}_0(u^0_s) + (u^0_s)^2\}ds\Big). \tag{2.38}$$

The major agent's unperturbed cost functional is expressed as

$$J^0(u^0) = \frac{1}{\delta_0}\mathbb{E}[\exp(G(u^0)) - 1], \tag{2.39}$$

and the respective unperturbed extended dynamics of the major bank is given as

$$dX^0_t = (\tilde{A}_0 X^0_t + \mathbb{B}_0 u^0_t + \tilde{B}_0 \bar{u}_t + \tilde{M}_0(t))dt + \Sigma_0 dW^0_t, \tag{2.40}$$

where we set the shock ξ_0 in equation (2.32) to zero.

By applying Theorem 3 in Firoozi et al. (2020) and Theorem 1 in Liu et al. (2023), which give the control action under a risk-sensitive LQG setting with a convex cost functional, we can derive the control action u^0_t of the major bank for the LQG system (2.39) - (2.40)

$$u^0_t = -\Big[\mathbb{N}_0^\top X^0_t + \mathbb{B}_0^\top\big(e^{-\tilde{A}_0^\top t}\frac{M^0_{2,t}}{M^0_{1,t}} - \int_0^t e^{\tilde{A}_0^\top (s-t)}(\mathbb{Q}_0 X^0_s + \mathbb{N}_0 u^0_s)ds\big)\Big]. \tag{2.41}$$

However, this is not an explicit representation of the major bank's control action. We aim to obtain the explicit optimal solution in the remainder of this section.

Furthermore, following Lemma 2 in Liu et al. (2023), we adopt the ansatz

$$p^0_t = \Pi_0(t)X^0_t + s_0(t), \tag{2.42}$$

where

$$\Pi_0(t) = \begin{bmatrix} \Pi^0_{11} & \Pi^0_{12} \\ \Pi^0_{21} & \Pi^0_{22} \end{bmatrix}, \quad s_0(t) = \begin{bmatrix} s^0_1 \\ s^0_2 \end{bmatrix}, \tag{2.43}$$

with $\Pi_{11}^0 \in \mathbb{R}$, $\Pi_{22}^0 \in \mathbb{R}^{K\times K}$, $s_1^0(t) \in \mathbb{R}$, and $s_2^0(t) \in \mathbb{R}^K$. Therefore, we can express the major agent's control action as

$$u_t^0 = -\Big[\mathbb{N}_0^\top X_t^0 + \mathbb{B}_0^\top \big(\Pi_0(t)X_t^0 + s_0(t)\big)\Big]. \tag{2.44}$$

However, we need to obtain $\bar{u}_t$ to calculate $\Pi_0(t)$ and $s_0(t)$. To address this problem, we proceed to define the control action for a generic minor agent.

For a representative minor bank i under the k-th subgroup in the infinite population, we apply a similar approach where we perturb the bank's control action u_t^i with a small value ξ_i in the direction $\omega^i \in \mathscr{U}^i$, where $\mathscr{U}^i$ is the admissible set of control actions of the minor bank. This leads to the perturbed state of the minor bank's log-monetary reserves satisfying

$$dx_t^{i,\xi_i} = \Big[-ax_t^{i,\xi_i} + (a\theta_k(1-\lambda)\otimes\pi)\bar{x} + a\theta_k\lambda x_t^0 + (u_t^i + \xi_i\omega_t^i) + \gamma^k(t)\Big]dt + \sigma_k\, dw_t^i. \tag{2.45}$$

We notice that since only the minor bank i's dynamic is perturbed, there are no changes in the dynamics of the monetary reserves of the major bank or the mean-field. This is because a single minor agent's size is so small in the large population that the impact of its actions is negligible, as defined previously for our setting. Therefore, a minor bank changing its rate of borrowing/lending with the central bank does not affect the states of the major bank or the whole system. We can then write the mean-field state when a minor bank's control is perturbed as

$$d\bar{x}_t^{\xi_i} = [\breve{A}\bar{x}_t + \breve{G}x_t^0 + \bar{u}_t + \breve{m}(t)]dt, \tag{2.46}$$

which is the same as the mean-field defined in Section 2.1.

Since a minor bank's state interacts with the system through both the state of the major bank and the mean-field, we extend its dynamics to include both processes as

$$X_t^{i,\xi_i} = \begin{bmatrix} x_t^{i,\xi_i} \\ x_t^0 \\ \bar{x}_t \end{bmatrix}, \tag{2.47}$$

which satisfies the following SDE

$$dX_t^{i,\xi_i} = (\tilde{A}_k X_t^{i,\xi_i} + \mathbb{B}_k u_t^i + \tilde{B}_k \bar{u}_t + \xi_i \mathbb{B}_k \omega_t^i + \tilde{M}_k(t))dt + \Sigma_k dW_t^i, \tag{2.48}$$

where

$$\tilde{A}_k = \begin{bmatrix} -a & [a\theta_k\lambda, a\theta_k(1-\lambda)\otimes\pi] \\ 0 & \tilde{A}_0 - \mathbb{B}_0\mathbb{N}_0 - \mathbb{B}_0\mathbb{B}_0^\top\Pi_0(t) \end{bmatrix}, \quad \mathbb{B}_k = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{B}_k = \begin{bmatrix} 0 \\ \tilde{B}_0 \end{bmatrix}, \tag{2.49a}$$

$$\tilde{M}_k(t) = \begin{bmatrix} \gamma^k(t) \\ \tilde{M}_0(t) - \mathbb{B}_0\mathbb{B}_0^\top s_0(t) \end{bmatrix}, \quad \Sigma_k = \begin{bmatrix} \sigma_k & 0 \\ 0 & \Sigma_0 \end{bmatrix}, \quad W_t^i = \begin{bmatrix} w_t^i \\ W_t^0 \end{bmatrix}, \tag{2.49b}$$

and $\Pi_0(t)$ and $s_0(t)$ are defined as in the ansatz of the major bank. The cost functional of the minor bank subject to the perturbed control action, in terms of the extended state (2.47), is expressed as

$$J^i(u^i + \xi_i\omega^i) = \frac{1}{\delta_k}\mathbb{E}\Bigg[\exp\Big(\frac{\delta_k}{2}\Big((X_T^{i,\xi_i})^\top \mathbb{G}_k X_T^{i,\xi_i} + \int_0^T \big\{(X_s^{i,\xi_i})^\top \mathbb{Q}_k X_s^{i,\xi_i} + 2(X_s^{i,\xi_i})^\top \mathbb{N}_k(u_s^i + \xi_i\omega_s^i) + (u_s^i + \xi_i\omega_s^i)^2\big\}ds\Big)\Big) - 1\Bigg], \tag{2.50}$$

where

$$\mathbb{G}_k = [1, -H_k, -\hat{H}_k^\pi]^\top c_k [1, -H_k, -\hat{H}_k^\pi], \tag{2.51a}$$

$$\mathbb{Q}_k = [1, -H_k, -\hat{H}_k^\pi]^\top \varepsilon_k [1, -H_k, -\hat{H}_k^\pi], \tag{2.51b}$$

$$\mathbb{N}_k = [1, -H_k, -\hat{H}_k^\pi]^\top q_k, \tag{2.51c}$$

$$H_k = \theta_k\lambda, \quad \hat{H}_k = \theta_k(1-\lambda). \tag{2.51d}$$

After defining the dynamics and cost functional of a minor agent subject to the perturbed control action in the direction ω^i, we can derive the Gâteaux derivative of the minor bank's cost functional following Theorem 1 in Liu et al. (2023)

$$(DJ_i^\infty(u), \omega^i) = \delta_k\mathbb{E}\Bigg[\int_0^T (\omega_t^i)^\top \Big\{\mathbb{B}_k^\top e^{-\tilde{A}_k^\top t} M_{2,t}^i + M_{1,t}^i \big(u_t^i + \mathbb{N}_k^\top X_t^i + \mathbb{B}_k^\top \int_0^t e^{\tilde{A}_k^\top (s-t)}(\mathbb{Q}_k X_s^i + \mathbb{N}_k u_s^i)ds\big)\Big\}dt\Bigg], \tag{2.52}$$

where $M^i_{1,t}$ and $M^i_{2,t}$ are martingales on the filtration $\mathscr{F}^i_t$ given by

$$M^i_{1,t} = \mathbb{E}[e^{G(u^i)}|\mathscr{F}^i_t], \tag{2.53a}$$

$$M^i_{2,t} = \mathbb{E}\Big[e^{G(u^i)}\big(e^{\tilde{A}_k^\top T}\mathbb{G}_k X^i_T + \int_0^T e^{\tilde{A}_k^\top s}(\mathbb{Q}_k X^i_s + \mathbb{N}_k u^k_s)ds\big)|\mathscr{F}^i_t\Big], \tag{2.53b}$$

Furthermore, $G(u^i)$ in above equations is defined as

$$G(u^i) = \frac{\delta_k}{2}\Big((X^i_T)^\top \mathbb{G}_k X^i_T + \int_0^T \big\{(X^i_s)^\top \mathbb{Q}_k X^i_s + 2(X^i_s)^\top \mathbb{N}_k(u^i_s) + (u^i_s)^2\big\}ds\Big). \tag{2.54}$$

The minor agent i's unperturbed cost functional is then expressed as

$$J^i(u^i) = \frac{1}{\delta_k}\mathbb{E}[\exp G(u^i) - 1], \tag{2.55}$$

and the corresponding unperturbed extended dynamic of the minor bank is given as

$$dX^i_t = (\tilde{A}_k X^i_t + \mathbb{B}_k u^i_t + \tilde{B}_k \bar{u}_t + \tilde{M}_k(t))dt + \Sigma_k dW^i_t. \tag{2.56}$$

Similar to the derivation of the major bank's control, we can obtain the control action of the minor bank u^i_t in the infinite-population limit using Theorem 3 in Firoozi et al. (2020) and Theorem 1 in Liu et al. (2023) which allow us to obtain control actions in the risk-sensitive LQG system (2.55) - (2.56)

$$u^i_t = -\Big[\mathbb{N}_k^\top X^i_t + \mathbb{B}_k^\top\big(e^{-\tilde{A}_k^\top t}\frac{M^i_{2,t}}{M^i_{1,t}} - \int_0^t e^{\tilde{A}_k^\top(s-t)}(\mathbb{Q}_k X^i_s + \mathbb{N}_k u^i_s)ds\big)\Big]. \tag{2.57}$$

We then adopt the ansatz following Lemma 2 in Liu et al. (2023)

$$p^i_t = \Pi_k(t)X^i_t + s_k(t), \tag{2.58}$$

where

$$\Pi_k(t) = \begin{bmatrix} \Pi^k_{11} & \Pi^k_{12} & \Pi^k_{13} \\ \Pi^k_{21} & \Pi^k_{22} & \Pi^k_{23} \\ \Pi^k_{31} & \Pi^k_{32} & \Pi^k_{33} \end{bmatrix}, \quad s_k(t) = \begin{bmatrix} s^k_1 \\ s^k_2 \\ s^k_3 \end{bmatrix}, \tag{2.59}$$

with $\Pi^k_{11}, \Pi^k_{22} \in \mathbb{R}$, $\Pi^k_{33} \in \mathbb{R}^{K\times K}$, $s^k_1(t), s^k_2(t) \in \mathbb{R}$, and $s^k_3(t) \in \mathbb{R}^K$. The control action of the minor agent can then be written as

$$u^i_t = -\Big[\mathbb{N}_k^\top X^i_t + \mathbb{B}_k^\top\big(\Pi_k(t)X^i_t + s_k(t)\big)\Big]. \tag{2.60}$$

By taking the average of (2.60) over all $i \in I_k$, we obtain

$$u_t^{(N_k)} = -\Big[\big(\mathbb{N}_k^\top + \begin{bmatrix} \Pi_{11}^k & \Pi_{12}^k & \Pi_{13}^k \end{bmatrix}\big) \begin{bmatrix} x_t^{(N_k)} \\ x_t^0 \\ \bar{x}_t \end{bmatrix} + \mathbb{B}_k^\top s_k(t)\Big]. \tag{2.61}$$

Since we have an infinite number of minor banks in each subpopulation, as $N_k \to \infty$, the control mean-field of subgroup k is given by

$$\bar{u}_t^k = -\Big[\big(\mathbb{N}_k^\top + \begin{bmatrix} \Pi_{11}^k & \Pi_{12}^k & \Pi_{13}^k \end{bmatrix}\big) \begin{bmatrix} \bar{x}_t^k \\ x_t^0 \\ \bar{x}_t \end{bmatrix} + \mathbb{B}_k^\top s_k(t)\Big]. \tag{2.62}$$

We then substitute the control mean-field into the dynamic of the mean-field of log-monetary reserves (2.11) and obtain

$$d\bar{x}_t = (\bar{A}\bar{x}_t + \bar{G}x_t^0 + \bar{m})dt \tag{2.63}$$

where

$$\bar{A} = \begin{bmatrix} \bar{A}_1 \\ \vdots \\ \bar{A}_K \end{bmatrix}, \quad \bar{G} = \begin{bmatrix} \bar{G}_1 \\ \vdots \\ \bar{G}_K \end{bmatrix}, \quad \bar{m} = \begin{bmatrix} \bar{m}_1 \\ \vdots \\ \bar{m}_K \end{bmatrix}, \tag{2.64}$$

and

$$\bar{A}_k = (-a - q_k - \Pi_{11}^k)\mathbf{e}_k + a\theta_k(1-\lambda) \otimes \pi + q_k\theta_k(1-\lambda) \otimes \pi - \Pi_{13}^k, \tag{2.65a}$$

$$\bar{G}_k = a\theta_k\lambda + q_k\theta_k\lambda - \Pi_{12}^k, \tag{2.65b}$$

$$\bar{m}_k = \gamma^k - \mathbb{B}_k^\top s_k. \tag{2.65c}$$

We notice that the control mean-field term $\bar{u}_t$ is gone, and the dynamic of the mean-field is denoted by a combination of the state mean-field, the major bank's monetary reserves, and an offset term $\bar{m}$.

We then replace the control mean-field in the major bank's unperturbed extended dynamics (2.40), which leads to the dynamics

$$dX_t^0 = (\mathbb{A}_0 X_t^0 + \mathbb{B}_0 u_t^0 + \mathbb{M}_0)dt + \Sigma_0 dW_t^0, \tag{2.66}$$

where

$$\mathbb{A}_0 = \begin{bmatrix} a_0(\theta_0\lambda - 1) & a_0\theta_0(1-\lambda)\otimes\pi \\ \bar{G} & \bar{A} \end{bmatrix}, \quad \mathbb{M}_0 = \begin{bmatrix} \gamma^0(t) \\ \bar{m} \end{bmatrix}, \tag{2.67}$$

and matrices $\mathbb{B}_0$ and Σ_0 are as defined in (2.33). Lastly, we perform the same procedures to the minor bank's extended monetary reserves dynamics (2.56) and obtain

$$dX_t^i = (\mathbb{A}_k X_t^i + \mathbb{B}_k u_t^i + \mathbb{M}_k)dt + \Sigma_k dW_t^i, \tag{2.68}$$

where

$$\mathbb{A}_k = \begin{bmatrix} -a & [a\theta_k\lambda, a\theta_k(1-\lambda)\otimes\pi] \\ 0 & \mathbb{A}_0 - \mathbb{B}_0(\mathbb{N}_0^\top + \mathbb{B}_0^\top\Pi_0(t)) \end{bmatrix}, \quad \mathbb{M}_k = \begin{bmatrix} \gamma^k(t) \\ \mathbb{M}_0 - \mathbb{B}_0\mathbb{B}_0^\top s_0(t) \end{bmatrix}, \tag{2.69}$$

and matrices $\mathbb{B}_k$ and Σ_k are as defined in (2.49).

Now in order to completely characterize the control actions of the major agent and a representative minor agent, we proceed to obtain the ODEs that $\Pi_0(t), s_0(t), \Pi_k(t)$ and $s_k(t)$ satisfy. We first start with the major agent. In equation (2.41) of the major bank's control, we have the term $\frac{M_{2,t}^0}{M_{1,t}^0}$, which is the quotient of two martingales under the measure $\mathbb{P}^0$. For this term to also behave like a martingale under a new measure, we follow Lemma 2 in Liu et al. (2023) and apply the change of measure technique, where we show that a random variable $\exp(G(u^0) - C_0^*(T))$ can define a Radon-Nikodym derivative and $G(u^0)$ is represented by following function

$$G(u^0) = -\frac{\delta_0^2}{2}\int_0^T (\nu_t^0)^2 dt + \delta_0 \int_0^T \nu_t^0 dW_t^0 + C_0^*(T) + \Phi_T^0, \tag{2.70}$$

where ν_t^0, $C_0^*(T)$, and Φ_T^0 are functions that we will define later on.

To approach this problem, following the methodology in Liu et al. (2023), we first define the process

$$V_t = \frac{\delta_0}{2}\left[(X_t^0)^\top \Pi_0(t) X_t^0 + 2(s_0(t))^\top X_t^0\right]. \tag{2.71}$$

We then apply Itô's lemma to the process above, and substitute dX_t^0 with equation (2.66), which leads to

$$\begin{aligned} dV_t &= \frac{\delta_0}{2}\Big(2(X_t^0)^\top \Pi_0(t) dX_t^0 + \Pi_0(t) d\left\langle X^0, X^0 \right\rangle_t + (X_t^0)^\top d\Pi_0(t) X_t^0 \\ &\quad + 2(s_0(t))^\top dX_t^0 + 2(ds_0(t))^\top X_t^0\Big) \\ &= \frac{\delta_0}{2}\Big(2(X_t^0)^\top \Pi_0(t)\left[(\mathbb{A}_0 X_t^0 + \mathbb{B}_0 u_t^0 + \mathbb{M}_0)dt + \Sigma_0 dW_t^0\right] \\ &\quad + tr\left\{[\sigma_0 dw_t^0, 0]\Pi_0(t)\begin{bmatrix} \sigma_0 dw_t^0 \\ 0 \end{bmatrix}\right\} + (X_t^0)^\top \dot{\Pi}_0(t) X_t^0 dt \\ &\quad + 2(s_0(t))^\top\left[(\mathbb{A}_0 X_t^0 + \mathbb{B}_0 u_t^0 + \mathbb{M}_0)dt + \Sigma_0 dW_t^0\right] + 2(\dot{s_0}(t))^\top X_t^0 dt\Big), \end{aligned} \tag{2.72}$$

where $tr\left\{[\sigma_0 dw_t^0, 0]\Pi_0(t)\begin{bmatrix} \sigma_0 dw_t^0 \\ 0 \end{bmatrix}\right\}$ defines the trace of the square matrix, which calculates the sum of entries on the main diagonal of the matrix. From the fundamental theorem of calculus, we integrate both sides from 0 to T

$$\begin{aligned} \int_0^T dV_t &= V_T - V_0 \\ &= \frac{\delta_0}{2}\int_0^T \left[(X_t^0)^\top \dot{\Pi}_0(t) X_t^0 + 2(X_t^0)^\top \Pi_0(t)\mathbb{A}_0 X_t^0 + 2(X_t^0)^\top \Pi_0(t)\mathbb{B}_0 u_t^0\right. \\ &\quad + 2(X_t^0)^\top \Pi_0(t)\mathbb{M}_0 + 2(\dot{s_0}(t))^\top X_t^0 \\ &\quad \left.+ 2(s_0(t))^\top(\mathbb{A}_0 X_t^0 + \mathbb{B}_0 u_t^0 + \mathbb{M}_0) + \Pi_{11}^0 \sigma_0^2)\right] dt \\ &\quad + \delta_0 \int_0^T \left[(X_t^0)^\top \Pi_0(t) + (s_0(t))^\top\right]\Sigma_0 dW_t^0, \end{aligned} \tag{2.73}$$

where Π_{11}^0 is an entry of the matrix $\Pi_0(t)$, defined in (2.43).

We then add $G(u^0)$ to both side of (2.73). With some algebraic manipulations we obtain $G(u^0) = G(u^0) - V_T + V_0 + \int_0^T dV_t$. We substitute $G(u^0)$ with (2.38) and u_t^0 with the control action in (2.44), leading to

$$
\begin{aligned}
G(u^0) = {} & \frac{\delta_0}{2}(X_T^0)^\top \mathbb{G}_0 X_T^0 + \frac{\delta_0}{2}\int_0^T \Big\{ (X_t^0)^\top \mathbb{Q}_0 X_t^0 \\
& - 2(X_t^0)^\top \mathbb{N}_0 \big(\mathbb{N}_0^\top X_t^0 + \mathbb{B}_0^\top \Pi_0(t) X_t^0 + \mathbb{B}_0^\top s_0(t) \big) + \big(\mathbb{N}_0^\top X_t^0 + \mathbb{B}_0^\top \Pi_0(t) X_t^0 \\
& + \mathbb{B}_0^\top s_0(t) \big)^2 \Big\} dt - \frac{\delta_0}{2}(X_T^0)^\top \Pi_0(T) X_T^0 - \delta_0 (s_0(T))^\top X_T^0 + \frac{\delta_0}{2}(X_0^0)^\top \Pi_0(0) X_0^0 \\
& + \delta_0 (s_0(0))^\top X_0^0 + \frac{\delta_0}{2}\int_0^T \Big\{ (X_t^0)^\top \dot{\Pi}_0(t) X_t^0 + 2(X_t^0)^\top \Pi_0(t) \mathbb{A}_0 X_t^0 \\
& - 2(X_t^0)^\top \Pi_0(t) \mathbb{B}_0 \big(\mathbb{N}_0^\top X_t^0 + \mathbb{B}_0^\top \Pi_0(t) X_t^0 + \mathbb{B}_0^\top s_0(t) \big) + 2(X_t^0)^\top \Pi_0(t) \mathbb{M}_0 \\
& + 2(\dot{s_0}(t))^\top X_t^0 + 2(s_0(t))^\top \mathbb{A}_0 X_t^0 - 2(s_0(t))^\top \mathbb{B}_0 \big(\mathbb{N}_0^\top X_t^0 + \mathbb{B}_0^\top \Pi_0(t) X_t^0 \\
& + \mathbb{B}_0^\top s_0(t) \big) + 2(s_0(t))^\top \mathbb{M}_0 + \Pi_{11}^0 \sigma_0^2 \Big\} dt + \delta_0 \int_0^T \big[(X_t^0)^\top \Pi_0(t) \\
& + (s_0(t))^\top \big] \Sigma_0 dW_t^0.
\end{aligned} \tag{2.74}
$$

After some simplifications we obtain

$$
\begin{aligned}
G(u^0) = {} & \frac{\delta_0}{2}(X_0^0)^\top \Pi_0(0) X_0^0 + \delta_0 (s_0(0))^\top X_0^0 + \frac{\delta_0}{2}(X_T^0)^\top \mathbb{G}_0 X_T^0 \\
& - \frac{\delta_0}{2}(X_T^0)^\top \Pi_0(T) X_T^0 - \delta_0 (s_0(T))^\top X_T^0 \\
& + \frac{\delta_0}{2}\int_0^T \Big\{ (X_t^0)^\top \mathbb{Q}_0 X_t^0 - (X_t^0)^\top \mathbb{N}_0 \mathbb{N}_0^\top X_t^0 - (X_t^0)^\top \Pi_0(t) \mathbb{B}_0 \mathbb{B}_0^\top \Pi_0(t) X_t^0 \\
& - (s_0(t))^\top \mathbb{B}_0 \mathbb{B}_0^\top s_0(t) + (X_t^0)^\top \dot{\Pi}_0(t) X_t^0 + 2(X_t^0)^\top \Pi_0(t) \mathbb{A}_0 X_t^0 \\
& - 2(X_t^0)^\top \Pi_0(t) \mathbb{B}_0 \mathbb{N}_0^\top X_t^0 + 2(X_t^0)^\top \Pi_0(t) \mathbb{M}_0 + 2(\dot{s_0}(t))^\top X_t^0 + 2(s_0(t))^\top \mathbb{A}_0 X_t^0 \\
& - 2(s_0(t))^\top \mathbb{B}_0 \mathbb{N}_0^\top X_t^0 - 2(s_0(t))^\top \mathbb{B}_0 \mathbb{B}_0^\top \Pi_0(t) X_t^0 + 2(s_0(t))^\top \mathbb{M}_0 \\
& + \Pi_{11}^0 \sigma_0^2 \Big\} dt + \delta_0 \int_0^T \big[(X_t^0)^\top \Pi_0(t) + (s_0(t))^\top \big] \Sigma_0 dW_t^0 \\
& - \frac{\delta_0^2}{2}\int_0^T \Big(\big[(X_t^0)^\top \Pi_0(t) + (s_0(t))^\top \big] \Sigma_0 \Big)^2 dt \\
& + \frac{\delta_0^2}{2}\int_0^T \Big(\big[(X_t^0)^\top \Pi_0(t) + (s_0(t))^\top \big] \Sigma_0 \Big)^2 dt.
\end{aligned} \tag{2.75}
$$

For $G(u^0)$ to admit the form of (2.70), we need

$$
v_t^0 = \big[(X_t^0)^\top \Pi_0(t) + (s_0(t))^\top \big] \Sigma_0, \tag{2.76}
$$

and

$$
\begin{aligned}
C_0^*(T) = \frac{\delta_0}{2}(X_0^0)^\top \Pi_0(0) X_0^0 + \delta_0 (s_0(t))^\top X_0^0 + \frac{\delta_0}{2}\int_0^T \Big\{ &-(s_0(t))^\top \mathbb{B}_0 \mathbb{B}_0^\top s_0(t) \\
+ 2(s_0(t))^\top \mathbb{M}_0 + \Pi_{11}^0 \sigma_0^2 + \delta_0 \Sigma_0^\top s_0(t)(s_0(t))^\top \Sigma_0 \Big\} dt, &
\end{aligned} \tag{2.77}
$$

which is a function that contains all the terms that do not depend on the state X_t^0. We also have Φ_T^0, which is a random variable given by

$$
\begin{aligned}
\Phi_T^0 = &\frac{\delta_0}{2}(X_T^0)^\top \mathbb{G}_0 X_T^0 - \frac{\delta_0}{2}(X_T^0)^\top \Pi_0(T) X_T^0 - \delta_0 (s_0(T))^\top X_T^0 \\
&+ \frac{\delta_0}{2}\int_0^T \Big\{ (X_t^0)^\top \mathbb{Q}_0 X_t^0 - (X_t^0)^\top \mathbb{N}_0 \mathbb{N}_0^\top X_t^0 - (X_t^0)^\top \Pi_0(t) \mathbb{B}_0 \mathbb{B}_0^\top \Pi_0(t) X_t^0 \\
&+ (X_t^0)^\top \dot{\Pi}_0(t) X_t^0 + 2(X_t^0)^\top \Pi_0(t) \mathbb{A}_0 X_t^0 - 2(X_t^0)^\top \Pi_0(t) \mathbb{B}_0 \mathbb{N}_0^\top X_t^0 \\
&+ 2(X_t^0)^\top \Pi_0(t) \mathbb{M}_0 + 2(\dot{s_0}(t))^\top X_t^0 + 2(s_0(t))^\top \mathbb{A}_0 X_t^0 - 2(s_0(t))^\top \mathbb{B}_0 \mathbb{N}_0^\top X_t^0 \\
&- 2(s_0(t))^\top \mathbb{B}_0 \mathbb{B}_0^\top \Pi_0(t) X_t^0 + \delta_0 \Pi_0(t) \Sigma_0 \Sigma_0^\top \Pi_0(t) X_t^0 \\
&+ 2\delta_0 \Pi_0(t) \Sigma_0 \Sigma_0^\top s_0(t) X_t^0 \Big\} dt.
\end{aligned} \tag{2.78}
$$

In order to define $\exp(G(u^0) - C_0^*(T))$ as a Radon-Nikodym derivative, we need $\Phi_T^0 = 0$ according to Lemma 2 in Liu et al. (2023). This leads to

$$
\begin{aligned}
\dot{\Pi}_0(t) + \Pi_0(t)\mathbb{A}_0 + \mathbb{A}_0^\top \Pi_0(t) - (\Pi_0(t)\mathbb{B}_0 + \mathbb{N}_0)(\mathbb{B}_0^\top \Pi_0(t) + \mathbb{N}_0^\top) + \mathbb{Q}_0 \\
+ \delta_0 \Pi_0(t) \Sigma_0 \Sigma_0^\top \Pi_0(t) = 0,
\end{aligned} \tag{2.79}
$$

$$
\dot{s_0}(t) + \big((\mathbb{A}_0 - \mathbb{B}_0 \mathbb{N}_0^\top)^\top - \Pi_0(t)\mathbb{B}_0\mathbb{B}_0^\top\big) s_0(t) + \Pi_0(t)\mathbb{M}_0 + \delta_0 \Pi_0(t)\Sigma_0\Sigma_0^\top s_0(t) = 0, \tag{2.80}
$$

as well as the terminal conditions

$$
\Pi_0(T) = \mathbb{G}_0, \tag{2.81a}
$$

$$
s_0(T) = 0. \tag{2.81b}
$$

Therefore, $G(u^0)$ satisfies the condition (2.70). We can then define the random variable $\exp(G(u^0) - C_0^*(T))$ as a Radon-Nikodym derivative that defines a probability measure $\hat{\mathbb{P}}^0$ equivalent to $\mathbb{P}^0$ based on Lemma 2 in Liu et al. (2023), where

$$
\begin{aligned}
\frac{d\hat{\mathbb{P}}^0}{d\mathbb{P}^0} &= \exp(G(u^0) - C_0^*(T)) \\
&= \exp\Big(-\frac{\delta_0^2}{2}\int_0^T (\nu_t^0)^2 dt + \delta_0 \int_0^T \nu_t^0 dW_t \Big).
\end{aligned} \tag{2.82}
$$

We can then define $\hat{M}_t^0$ as

$$\begin{aligned}\hat{M}_t^0 &= \frac{M_{2,t}^0}{M_{1,t}^0} \\ &= \frac{\mathbb{E}[\mathscr{L}^0(T)\exp(-\frac{\delta_0^2}{2}\int_0^T(\nu_t^0)^2dt + \delta_0\int_0^T \nu_t^0 dW_t)|\mathscr{F}_t^0]}{\mathbb{E}[\exp(-\frac{\delta_0^2}{2}\int_0^T(\nu_t^0)^2dt + \delta_0\int_0^T \nu_t^0 dW_t)|\mathscr{F}_t^0]},\end{aligned} \tag{2.83}$$

which is a martingale under the new measure $\hat{\mathbb{P}}^0$ where

$$\mathscr{L}^0(T) = e^{\mathbb{A}_0^\top T}\mathbb{G}_0 X_T^0 + \int_0^T e^{\mathbb{A}_0^\top s}(\mathbb{Q}_0 X_s^0 + \mathbb{N}_0 u_s^0)ds, \tag{2.84}$$

and

$$\hat{M}_t^0 = \hat{\mathbb{E}}[\mathscr{L}(T)|\mathscr{F}_t^0]. \tag{2.85}$$

Furthermore, based on the martingale representation theorem, we can express $\hat{M}_t^0$ as

$$\hat{M}_t^0 = \hat{M}_0^0 + \int_0^t Z_s^0 d\hat{W}_s^0, \qquad d\hat{M}_t^0 = Z_t^0 d\hat{W}_t^0, \tag{2.86}$$

where $d\hat{W}_t^0 = -\delta_0\big([(X_t^0)^\top\Pi_0(t) + (s_0(t))^\top]\Sigma_0\big)dt + dW_t^0$ under the new measure $\hat{\mathbb{P}}^0$. We can then rewrite the major agent's control action in (2.41) as

$$u_t^0 = -\Big[\mathbb{N}_0^\top X_t^0 + \mathbb{B}_0^\top\big(e^{-\mathbb{A}_0^\top t}\hat{M}_t^0 - \int_0^t e^{\mathbb{A}_0^\top(s-t)}(\mathbb{Q}_0 X_s^0 + \mathbb{N}_0 u_s^0)ds\big)\Big]. \tag{2.87}$$

We define the adjoint process under measure $\hat{\mathbb{P}}^0$ as

$$p_t^0 = e^{-\mathbb{A}_0^\top t}\hat{M}_t^0 - \int_0^t e^{\mathbb{A}_0^\top(s-t)}(\mathbb{Q}_0 X_s^0 + \mathbb{N}_0 u_s^0)ds. \tag{2.88}$$

Applying Itô's lemma on this process and we get

$$\begin{aligned}dp_t^0 &= \big[-\mathbb{A}_0^\top p_t^0 - (\mathbb{Q}_0 X_t^0 + \mathbb{N}_0 u_t^0)\big]dt + e^{-\mathbb{A}_0^\top t}Z_t^0 d\hat{W}_t^0 \\ &= \Big[-\mathbb{A}_0^\top(\Pi_0(t)X_t^0 + s_0(t)) - \big(\mathbb{Q}_0 X_t^0 - \mathbb{N}_0(\mathbb{N}_0^\top X_t^0 + \mathbb{B}_0^\top\Pi_0(t)X_t^0 + \mathbb{B}_0^\top s_0(t))\big)\Big]dt \\ &\quad + e^{-\mathbb{A}_0^\top t}Z_t^0 d\hat{W}_t^0 \\ &= \big[(-\mathbb{Q}_0 - \mathbb{A}_0^\top\Pi_0(t) + \mathbb{N}_0\mathbb{N}_0^\top + \mathbb{N}_0\mathbb{B}_0^\top\Pi_0(t))X_t^0 - \mathbb{A}_0^\top s_0(t) + \mathbb{N}_0\mathbb{B}_0^\top s_0(t)\big]dt \\ &\quad + e^{-\mathbb{A}_0^\top t}Z_t^0 d\hat{W}_t^0.\end{aligned} \tag{2.89}$$

We also apply Ito's lemma to the ansatz (2.42) which leads to

$$\begin{aligned}
dp_t^0 &= d(\Pi_0(t)X_t^0 + s_0(t)) = (\dot{\Pi}_0(t)X_t^0 + \Pi_0(t)dX_t^0 + \dot{s}_0(t))dt \\
&= \Big(\dot{\Pi}_0(t)X_t^0 + \Pi_0(t)\mathbb{A}_0X_t^0 - \Pi_0(t)\mathbb{B}_0\big(\mathbb{N}_0^\top X_t^0 + \mathbb{B}_0^\top\Pi_0(t)X_t^0 + \mathbb{B}_0^\top s_0(t)\big) \\
&\quad + \Pi_0(t)\mathbb{M}_0 + \delta_0\Pi_0(t)\Sigma_0(X_t^0)^\top\Pi_0(t)\Sigma_0 + \delta_0\Pi_0(t)\Sigma_0(s_0(t))^\top\Sigma_0 + \dot{s}_0(t)\Big)dt \\
&\quad + \Pi_0(t)\Sigma_0 d\hat{W}_t^0.
\end{aligned} \tag{2.90}$$

Since we have previously obtained the conditions (2.79) - (2.80), the drift and diffusion coefficients of both SDEs (2.89) and (2.90) are equal at each time t. Therefore, according to Theorem 3 in Liu et al. (2023), we have

$$\Pi_0(t)X_t^0 + s_0(t) = e^{-\mathbb{A}_0^\top t}\hat{M}_t^0 - \int_0^t e^{\mathbb{A}_0^\top(s-t)}(\mathbb{Q}_0X_s^0 + \mathbb{N}_0u_s^0)ds, \tag{2.91}$$

under measure $\hat{\mathbb{P}}^0$.

Finally, following Theorem 3 in Firoozi et al. (2020) and Corollary 4 in Liu et al. (2023), we obtain that $u_t^{0,*}$ given by the form of (2.44) where $\Pi_0(t)$ and $s_0(t)$ satisfying the conditions (2.79) - (2.80) is the optimal trading strategy under the original measure $\mathbb{P}^0$ for the major bank in a risk-sensitive interbank transactions system defined by (1.1) and (1.2).

We now apply a similar method to obtain the optimal trading strategy for a representative minor bank and the ODEs that $\Pi_k(t)$ and $s_k(t)$ satisfy. Following Lemma 2 in Liu et al. (2023), for the random variable $\exp(G(u^i) - C_i^*(T))$ to define a Radon-Nikodym derivative, we aim to obtain the representation

$$G(u^i) = -\frac{\delta_k^2}{2}\int_0^T(\nu_t^i)^2dt + \delta_k\int_0^T \nu_t^i dW_t + C_i^*(T) + \Phi_T^i. \tag{2.92}$$

To this purpose, we apply Itô's lemma to the following process

$$\Theta_t = \frac{\delta_k}{2}\big[(X_t^i)^\top\Pi_k(t)X_t^i + 2(s_k(t))^\top X_t^i\big], \tag{2.93}$$

which leads to

$$
\begin{aligned}
d\Theta_t = \frac{\delta_k}{2}\Big(& 2(X_t^i)^\top \Pi_k(t)\left[(\mathbb{A}_k X_t^i + \mathbb{B}_k u_t^i + \mathbb{M}_k)dt + \Sigma_k dW_t^i\right] \\
& + tr\left\{ \left[\sigma_k dw_t^i, \sigma_0 dw_t^0, 0\right]\Pi_k(t) \begin{bmatrix} \sigma_k dw_t^i \\ \sigma_0 dw_t^0 \\ 0 \end{bmatrix} \right\} + (X_t^i)^\top \dot{\Pi}_k(t) X_t^i dt \\
& + 2(s_k(t))^\top \left[(\mathbb{A}_k X_t^i + \mathbb{B}_k u_t^i + \mathbb{M}_k)dt + \Sigma_k dW_t^i\right] + 2(\dot{s_k}(t))^\top X_t^i dt \Big).
\end{aligned} \tag{2.94}
$$

Then, from the fundamental theorem of calculus we have

$$
\begin{aligned}
\int_0^T d\Theta_t &= \Theta_T - \Theta_0 \\
&= \frac{\delta_k}{2}\int_0^T \Big[(X_t^i)^\top \dot{\Pi}_k(t) X_t^i + 2(X_t^i)^\top \Pi_k(t)\mathbb{A}_k X_t^i + 2(X_t^i)^\top \Pi_k(t)\mathbb{B}_k u_t^i \\
&\quad + 2(X_t^i)^\top \Pi_k(t)\mathbb{M}_k + 2(\dot{s_k}(t))^\top X_t^i + 2(s_k(t))^\top(\mathbb{A}_k X_t^i + \mathbb{B}_k u_t^i + \mathbb{M}_k) \\
&\quad + \Pi_{11}^k \sigma_k^2 + \Pi_{22}^k \sigma_0^2)\Big] dt + \delta_k \int_0^T \left[(X_t^i)^\top \Pi_k(t) + (s_k(t))^\top\right]\Sigma_k dW_t^i.
\end{aligned} \tag{2.95}
$$

We then add $G(u^i)$ to both side of (2.95) and obtain $G(u^i) = G(u^i) - \Theta_T + \Theta_0 + \int_0^T d\Theta_t$. We substitute $G(u^i)$ with (2.54) and u_t^i with the control action in (2.60). With some simplifications, we express $G(u^i)$ as

$$
\begin{aligned}
G(u^i) = & \frac{\delta_k}{2}(X_T^i)^\top \mathbb{G}_k X_T^i - \frac{\delta_k}{2}(X_T^i)^\top \Pi_k(T) X_T^i - \delta_k (s_k(T))^\top X_T^i + \frac{\delta_k}{2}(X_0^i)^\top \Pi_k(0) X_0^i \\
& + \delta_k (s_k(0))^\top X_0^i + \frac{\delta_k}{2}\int_0^T \Big\{ (X_t^i)^\top \mathbb{Q}_k X_t^i - (X_t^i)^\top \mathbb{N}_k \mathbb{N}_k^\top X_t^i \\
& - (X_t^i)^\top \Pi_k(t)\mathbb{B}_k\mathbb{B}_k^\top \Pi_k(t) X_t^i - (s_k(t))^\top \mathbb{B}_k \mathbb{B}_k^\top s_k(t) + (X_t^i)^\top \dot{\Pi}_k(t) X_t^i \\
& + 2(X_t^i)^\top \Pi_k(t)\mathbb{A}_k X_t^i - 2(X_t^i)^\top \Pi_k(t)\mathbb{B}_k \mathbb{N}_k^\top X_t^i + 2(X_t^i)^\top \Pi_k(t)\mathbb{M}_k + 2(\dot{s_k}(t))^\top X_t^i \\
& + 2(s_k(t))^\top \mathbb{A}_k X_t^i - 2(s_k(t))^\top \mathbb{B}_k \mathbb{N}_k^\top X_t^i - 2(s_k(t))^\top \mathbb{B}_k\mathbb{B}_k^\top \Pi_k(t) X_t^i + 2(s_k(t))^\top \mathbb{M}_k \\
& + \Pi_{11}^k \sigma_k^2 + \Pi_{22}^k \sigma_0^2 \Big\} dt + \delta_k \int_0^T \left[(X_t^i)^\top \Pi_k(t) + (s_k(t))^\top\right]\Sigma_k dW_t^i \\
& - \frac{\delta_k^2}{2}\int_0^T \Big(\left[(X_t^i)^\top \Pi_k(t) + (s_k(t))^\top\right]\Sigma_k\Big)^2 dt \\
& + \frac{\delta_k^2}{2}\int_0^T \Big(\left[(X_t^i)^\top \Pi_k(t) + (s_k(t))^\top\right]\Sigma_k\Big)^2 dt.
\end{aligned} \tag{2.96}
$$

For $G(u^i)$ to follow the form of (2.92), we need

$$\nu_t^i = \big[(X_t^i)^\top \Pi_k(t) + (s_k(t))^\top\big]\Sigma_k, \tag{2.97}$$

which is integrant of the stochastic integral. We also need

$$\begin{aligned} C_i^*(T) = \frac{\delta_k}{2}(X_0^i)^\top \Pi_k(0) X_0^i + \delta_k (s_k(0))^\top X_0^i + \frac{\delta_k}{2}\int_0^T \big\{ -(s_k(t))^\top \mathbb{B}_k \mathbb{B}_k^\top s_k(t) \\ + 2(s_k(t))^\top \mathbb{M}_k + \Pi_{11}^k \sigma_k^2 + \Pi_{22}^k \sigma_0^2 + \delta_k \Sigma_k^\top s_k(t)(s_k(t))^\top \Sigma_k \big\} dt. \end{aligned} \tag{2.98}$$

Furthermore, we have the random variable Φ_T^i given by

$$\begin{aligned} \Phi_T^i = &\frac{\delta_k}{2}(X_T^i)^\top \mathbb{G}_k X_T^i - \frac{\delta_k}{2}(X_T^i)^\top \Pi_k(T) X_T^i - \delta_k (s_k(T))^\top X_T^i \\ &+ \frac{\delta_k}{2}\int_0^T \Big\{ \Big\{ (X_t^i)^\top \mathbb{Q}_k X_t^i - (X_t^i)^\top \mathbb{N}_k \mathbb{N}_k^\top X_t^i - (X_t^i)^\top \Pi_k(t) \mathbb{B}_k \mathbb{B}_k^\top \Pi_k(t) X_t^i \\ &+ (X_t^i)^\top \dot{\Pi}_k(t) X_t^i + 2(X_t^i)^\top \Pi_k(t) \mathbb{A}_k X_t^i - 2(X_t^i)^\top \Pi_k(t) \mathbb{B}_k \mathbb{N}_k^\top X_t^i \\ &+ 2(X_t^i)^\top \Pi_k(t) \mathbb{M}_k + 2(\dot{s_k}(t))^\top X_t^i + 2(s_k(t))^\top \mathbb{A}_k X_t^i - 2(s_k(t))^\top \mathbb{B}_k \mathbb{N}_k^\top X_t^i \\ &- 2(s_k(t))^\top \mathbb{B}_k \mathbb{B}_k^\top \Pi_k(t) X_t^i \Big\} + \delta_k \Pi_k(t) \Sigma_k \Sigma_k^\top \Pi_k(t) X_t^i \\ &+ 2\delta_k \Pi_k(t) \Sigma_k \Sigma_k^\top s_k(t) X_t^i \Big\} dt. \end{aligned} \tag{2.99}$$

Following Lemma 2 in Liu et al. (2023), we need $\Phi_T^i = 0$ for $\exp(G(u^i) - C_i^*(T))$ to define a Radon-Nikodym derivative. To fulfill this requirement, we need the terminal conditions

$$\Pi_k(T) = \mathbb{G}_k, \tag{2.100a}$$

$$s_k(T) = 0. \tag{2.100b}$$

and

$$\begin{aligned} \dot{\Pi}_k(t) + \Pi_k(t)\mathbb{A}_k + \mathbb{A}_k^\top \Pi_k(t) - (\Pi_k(t)\mathbb{B}_k + \mathbb{N}_k)(\mathbb{B}_k^\top \Pi_k(t) + \mathbb{N}_k^\top) + \mathbb{Q}_k \\ + \delta_k \Pi_k(t) \Sigma_k \Sigma_k^\top \Pi_k(t) = 0 \end{aligned} \tag{2.101}$$

$$\dot{s_k}(t) + \big((\mathbb{A}_k - \mathbb{B}_k \mathbb{N}_k^\top)^\top - \Pi_k(t)\mathbb{B}_k \mathbb{B}_k^\top\big) s_k(t) + \Pi_k \mathbb{M}_k + \delta_k \Pi_k(t) \Sigma_k \Sigma_k^\top s_k(t) = 0 \tag{2.102}$$

Therefore, $G(u^i)$ satisfies (2.92) and $\exp(G(u^i) - C_i^*(T))$ is a Radon-Nikodym derivative that defines a probability measure $\hat{\mathbb{P}}^i$ equivalent to $\mathbb{P}^i$ according to Lemma 2 in Liu et al. (2023).

We can then define

$$\begin{aligned}\hat{M}_t^i &= \frac{M_{2,t}^i}{M_{1,t}^i} \\ &= \frac{\mathbb{E}[\mathscr{L}^i(T)\exp(-\frac{\delta_k^2}{2}\int_0^T(\nu_t^i)^2dt + \delta_k\int_0^T \nu_t^i dW_t)|\mathscr{F}_t^i]}{\mathbb{E}[\exp(-\frac{\delta_k^2}{2}\int_0^T(\nu_t^i)^2dt + \delta_k\int_0^T \nu_t^i dW_t)|\mathscr{F}_t^i]},\end{aligned} \tag{2.103}$$

where

$$\mathscr{L}^i(T) = e^{\mathbb{A}_k^\top T}\mathbb{G}_k X_T^i + \int_0^T e^{\mathbb{A}_k^\top s}(\mathbb{Q}_k X_s^i + \mathbb{N}_k u_s^i)ds, \tag{2.104}$$

and $\hat{M}_t^i = \hat{\mathbb{E}}[\mathscr{L}^i(T)|\mathscr{F}_t^i]$ behaves like a martingale under the new measure $\hat{\mathbb{P}}^i$. Moreover, we can express $\hat{M}_t^i$ as

$$\hat{M}_t^i = \hat{M}_0^i + \int_0^t Z_s^i d\hat{W}_s^i, \qquad d\hat{M}_t^i = Z_t^i d\hat{W}_t^i, \tag{2.105}$$

using the martingale representation theorem where

$$d\hat{W}_t^i = -\delta_k\big([(X_t^i)^\top \Pi_k(t) + (s_k(t))^\top]\Sigma_k\big)dt + dW_t^i \tag{2.106}$$

is a Brownian motion under the new measure $\hat{\mathbb{P}}^i$. Therefore, we can rewrite the control action in (2.57) as

$$u_t^i = -\Big[\mathbb{N}_k^\top X_t^i + \mathbb{B}_k^\top\big(e^{-\mathbb{A}_k^\top t}\hat{M}_t^i - \int_0^t e^{\mathbb{A}_k^\top(s-t)}(\mathbb{Q}_k X_s^i + \mathbb{N}_k u_s^i)ds\big)\Big]. \tag{2.107}$$

We then define the adjoint process under measure $\hat{\mathbb{P}}^i$ as

$$p_t^i = e^{-\mathbb{A}_k^\top t}\hat{M}_t^i - \int_0^t e^{\mathbb{A}_k^\top(s-t)}(\mathbb{Q}_k X_s^i + \mathbb{N}_k u_s^i)ds. \tag{2.108}$$

Applying Itô's lemma to the adjoint process and we obtain

$$\begin{aligned}dp_t^i = &\big[(-\mathbb{Q}_k - \mathbb{A}_k^\top\Pi_k(t) + \mathbb{N}_k\mathbb{N}_k^\top + \mathbb{N}_k\mathbb{B}_k^\top\Pi_k(t))X_t^i - \mathbb{A}_k^\top s_k(t) + \mathbb{N}_k\mathbb{B}_k^\top s_k(t)\big]dt \\ &+ e^{-\mathbb{A}_k^\top t}Z_t^i d\hat{W}_t^i.\end{aligned} \tag{2.109}$$

We also apply Ito's lemma on the ansatz (2.58) which leads to

$$\begin{aligned}
dp_t^i &= d(\Pi_k(t)X_t^i + s_k(t)) \\
&= \big(\dot{\Pi}_k(t)X_t^i + \Pi_k(t)\mathbb{A}_k X_t^i - \Pi_k(t)\mathbb{B}_k\big(\mathbb{N}_k^\top X_t^i + \mathbb{B}_k^\top \Pi_k(t)X_t^i + \mathbb{B}_k^\top s_k(t)\big) \\
&\quad + \Pi_k(t)\mathbb{M}_k + \delta_k \Pi_k(t)\Sigma_k (X_t^i)^\top \Pi_k(t)\Sigma_k + \delta_k \Pi_k(t)\Sigma_k (s_k(t))^\top \Sigma_k + \dot{s}_k(t)\big)dt \\
&\quad + \Pi_k(t)\Sigma_k d\hat{W}_t^i.
\end{aligned} \tag{2.110}$$

The drift and diffusion coefficients of both SDEs (2.109) and (2.110) are equal at each time t following the conditions (2.101) and (2.102). Hence, based on Theorem 3 in Liu et al. (2023), we have

$$\Pi_k(t)X_t^i + s_k(t) = e^{-\mathbb{A}_k^\top t}\hat{M}_t^i - \int_0^t e^{\mathbb{A}_k^\top (s-t)}(\mathbb{Q}_k X_s^i + \mathbb{N}_k u_s^i)ds, \tag{2.111}$$

under measure $\hat{\mathbb{P}}^i$.

Again, following Theorem 3 in Firoozi et al. (2020) and Corollary 4 in Liu et al. (2023), we have that $u_t^{i,*}$ given by (2.60) with $\Pi_k(t)$ and $s_k(t)$ satisfying the conditions (2.101) - (2.102) is the optimal transaction strategy for a representative minor bank under measure $\mathbb{P}^i$ in a risk-sensitive interbank setting defined by (1.4) and (1.5).

2.3.1 Simplification of Best-Response Strategies

In this section, we want to obtain a reduced expression of the optimal strategies of the major bank and a representative minor bank given by $u_t^{0,*}$ and $u_t^{i,*}$, following the forms (2.44) and (2.60) respectively. In the works of Carmona et al. (2015) and Chang et al. (2023), a reduction of the optimal strategy in the form of mean reversion can be achieved in the risk-neutral setting. By obtaining the reduced form, it is more straightforward to observe the effect each variable has on the optimal trading rate and dynamic of an agent. In the following subsections, we attempt to derive the simplified forms of the best-response strategies of the major bank and a minor bank for our model, and verify their validity in the risk-sensitive case.

2.3.1.1 Simplification for the Major Agent

Building upon prior studies in Chang et al. (2023), we adopt the following ansatz for the major bank's optimal transaction strategy $u_t^{0,*}$

$$u_t^{0,*} = -\left[\mathbb{N}_0^\top X_t^0 - \frac{1}{q_0}\mathbb{B}_0^\top \Phi_t^0 \mathbb{N}_0^\top X_t^0 + \mathbb{B}_0^\top s_0(t)\right], \tag{2.112}$$

and obtain

$$\begin{aligned} u_t^{0,*} &= [1, H_0^\pi] q_0(\theta_0\lambda - 1) \begin{bmatrix} x_t^0 \\ \bar{x}_t \end{bmatrix} (1 - \frac{1}{q_0}[1, 0] \begin{bmatrix} \phi_t^0 \\ \psi_t^0 \end{bmatrix}) - \mathbb{B}_0^\top s_0(t) \\ &= (q_0 - \phi_t^0)\left[(\theta_0(1-\lambda)\otimes\pi)\bar{x}_t + (\theta_0\lambda - 1)x_t^0\right] - \mathbb{B}_0^\top s_0(t), \end{aligned} \tag{2.113}$$

which is a reduced representation of major bank's optimal borrowing/lending rate. This dictates that $\Pi_0(t)$ follows the form

$$\Pi_0(t) = \Phi_t^0[\theta_0\lambda - 1, \theta_0(1-\lambda)\otimes\pi], \quad \Phi_t^0 = [\phi_t^0, \psi_t^0]^\top. \tag{2.114}$$

We can then write (2.42) as

$$p_t^0 = \Phi_t^0(-\frac{1}{q_0})\mathbb{N}_0^\top X_t^0 + s_0(t). \tag{2.115}$$

Moreover, we want to simplify the matrix form of the Riccati equation (2.79) and obtain an ODE for the components in $\Pi_0(t)$ that explicitly appear in the reduced control action (2.113). With the ansatz (2.114) we have

$$\Pi_0(t) = \begin{bmatrix} \Pi_{11}^0 & \Pi_{12}^0 \\ \Pi_{21}^0 & \Pi_{22}^0 \end{bmatrix} = \begin{bmatrix} \phi_t^0(\theta_0\lambda - 1) & \phi_t^0\theta_0(1-\lambda)\otimes\pi \\ \phi_t^0\theta_0(1-\lambda)\otimes\pi & \psi_t^0\theta_0(1-\lambda)\otimes\pi \end{bmatrix} \tag{2.116}$$

which is a symmetric matrix. We then expand all the matrices in equation (2.79), leading to the following expression of entry Π_{11}^0

$$\begin{aligned} -\dot{\Pi}_{11}^0 = 2(\Pi_{11}^0 a_0(\theta_0\lambda - 1) + \Pi_{12}^0\bar{G}) - (\Pi_{11}^0 - q_0(\theta_0\lambda - 1))^2 + \varepsilon_0(\theta_0\lambda - 1)^2 \\ + \delta_0(\Pi_{11}^0)^2(\sigma_0)^2, \end{aligned} \tag{2.117}$$

where Π_{12}^0 is also characterized by ϕ_t^0 as defined in (2.116). Multiplying the matrices in the term that includes the degree of risk sensitivity in (2.79) leads to

$$\delta_0 \Pi_0(t) \Sigma_0 \Sigma_0^\top \Pi_0(t) = \delta_0 \begin{bmatrix} (\Pi_{11}^0)^2 (\sigma_0)^2 & \Pi_{11}^0 \Pi_{12}^0 (\sigma_0)^2 \\ \Pi_{21}^0 \Pi_{11}^0 (\sigma_0)^2 & \Pi_{21}^0 \Pi_{12}^0 (\sigma_0)^2 \end{bmatrix}. \tag{2.118}$$

We can see that ϕ_t^0 is the only variable in the ansatz that is included in the term $\delta_0 \Pi_0(t) \Sigma_0 \Sigma_0^\top \Pi_0(t)$. Therefore, we can reduce (2.117) to an ODE that contains ϕ_t^0 for the risk-sensitive case. Substituting Π_{11}^0 and Π_{12}^0 with $\phi_t^0(\theta_0\lambda - 1)$ and $\phi_t^0 \theta_0 (1-\lambda) \otimes \pi$ respectively in (2.117) and we have

$$-\dot{\phi}_t^0 = 2\phi_t^0 \big(a_0(\theta_0\lambda - 1) + H_0^\pi \bar{G}\big) - (\phi_t^0 - q_0)^2 (\theta_0\lambda - 1) + \varepsilon_0(\theta_0\lambda - 1) \\ + \delta_0 (\phi_t^0)^2 (\sigma_0)^2 (\theta_0\lambda - 1), \tag{2.119}$$

with the terminal condition

$$\phi_T^0 = c_0(\theta_0\lambda - 1). \tag{2.120}$$

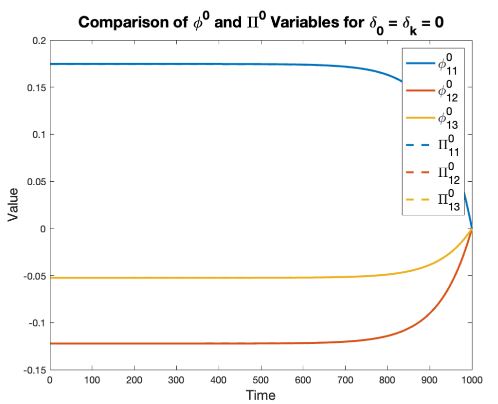


Figure 2.1: Comparison of $\phi^0(t)$ and $\Pi^0(t)$ variables for $\delta_0 = \delta_k = 0$.

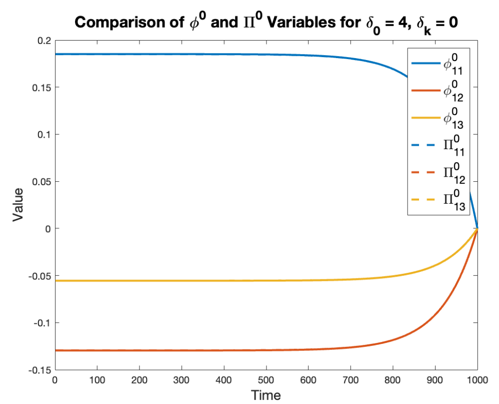


Figure 2.2: Comparison of $\phi^0(t)$ and $\Pi^0(t)$ variables for $\delta_0 = 4, \delta_k = 0$.

We also verify the ansatz by plotting the entries of $\Pi_0(t)$ in both the original form and ansatz form in (2.116). From Fig. 2.1 and Fig. 2.2 we can see that the values of entries $(1,1)$, $(1,2)$, and $(1,3)$ under both forms overlap perfectly at each time t when the major bank is risk-neutral and when it is risk-sensitive, while minor banks remain risk-neutral. This indicates that the ansatz in (2.112) is valid for the risk-sensitive setting and we proceed to express the major bank's optimal trading strategy as in (2.113) where ϕ_t^0 satisfies the ODE (2.119).

2.3.1.2 Simplification for a Representative Minor Agent

We now proceed to obtain the reduced form of the optimal trading strategy for a representative minor bank. We adopt the ansatz

$$u_t^{i,*} = -[\mathbb{N}_k^\top X_t^i - \frac{1}{q_k}\mathbb{B}_k^\top \Phi_t^k \mathbb{N}_k^\top X_t^i + \mathbb{B}_k^\top s_k(t)]. \tag{2.121}$$

With some simplifications, we obtain

$$\begin{aligned} u_t^{i,*} =& -[\mathbb{I}_n, -H_k, -\hat{H}_k^\pi] q_k \begin{bmatrix} x_t^i \\ x_t^0 \\ \bar{x}_t \end{bmatrix} (1 - \frac{1}{q_k}[1, 0, 0] \begin{bmatrix} \phi_t^k \\ \psi_t^k \\ \eta_t^k \end{bmatrix}) - \mathbb{B}_k^\top s_k(t) \\ =& (q_k - \phi_t^k)\big[(\theta_k(1-\lambda)\otimes\pi)\bar{x}_t + \theta_k\lambda x_t^0 - x_t^i\big] - \mathbb{B}_k^\top s_k(t), \end{aligned} \tag{2.122}$$

which implies that $\Pi_k(t)$ follows the expression

$$\Pi_k(t) = \Phi_t^k[-1, \theta_k\lambda, \theta_k(1-\lambda)\otimes\pi], \quad \Phi_t^k = [\phi_t^k, \psi_t^k, \eta_t^k]^\top. \tag{2.123}$$

With some algebraic manipulation, we can rewrite ansatz (2.58) as

$$p_t^i = \Phi_t^k(-\frac{1}{q_k})\mathbb{N}_k^\top X_t^i + s_k(t). \tag{2.124}$$

Furthermore, as the number of minor banks in a representative subgroup k goes to infinity, the control mean-field of the subgroup is given by

$$\bar{u}_t^k = (q_k - \phi_t^k)\big[(\theta_k(1-\lambda)\otimes\pi)\bar{x}_t + \theta_k\lambda x_t^0 - \bar{x}_t^k\big] - \mathbb{B}_k^\top s_k(t). \tag{2.125}$$

Substituting this control mean-field into the log-monetary reserves mean-field dynamics (2.11) leads to

$$d\bar{x}_t = (\bar{A}\bar{x}_t + \bar{G}x_t^0 + \bar{m})dt \tag{2.126}$$

where

$$\bar{A} = \begin{bmatrix} \bar{A}_1 \\ \vdots \\ \bar{A}_K \end{bmatrix}, \quad \bar{G} = \begin{bmatrix} \bar{G}_1 \\ \vdots \\ \bar{G}_K \end{bmatrix}, \quad \bar{m} = \begin{bmatrix} \bar{m}_1 \\ \vdots \\ \bar{m}_K \end{bmatrix}, \tag{2.127}$$

and

$$\bar{A}_k = (\theta_k(1-\lambda)\otimes\pi - \mathbf{e}_k)(a+q_k-\phi_t^k), \tag{2.128a}$$

$$\bar{G}_k = \theta_k\lambda(a+q_k-\phi_t^k), \tag{2.128b}$$

$$\bar{m}_k = \gamma^k(t) - \mathbb{B}_k^\top s_k(t). \tag{2.128c}$$

We then proceed to simplify the ODE (2.101) and obtain an expression for the element Π_{11}^k, where $\Pi_k(t)$ admits the following structure according to the adopted anasatz

$$\Pi_k(t) = \begin{bmatrix} \Pi_{11}^k & \Pi_{12}^k & \Pi_{13}^k \\ \Pi_{21}^k & \Pi_{22}^k & \Pi_{23}^k \\ \Pi_{31}^k & \Pi_{32}^k & \Pi_{33}^k \end{bmatrix} = \begin{bmatrix} -\phi_t^k & \phi_t^k\theta_k\lambda & \phi_t^k\theta_k(1-\lambda)\otimes\pi \\ \phi_t^k\theta_k\lambda & \psi_t^k\theta_k\lambda & \psi_t^k\theta_k(1-\lambda)\otimes\pi \\ \phi_t^k\theta_k(1-\lambda)\otimes\pi & \psi_t^k\theta_k(1-\lambda)\otimes\pi & \eta_t^k\theta_k(1-\lambda)\otimes\pi \end{bmatrix}. \tag{2.129}$$

With some calculations, we derive

$$\dot{\phi}_t^k = 2(a+q_k)\phi_t^k + (\delta_k(\sigma_k)^2 + \delta_k(\sigma_0)^2(\theta_k)^2\lambda^2 - 1)(\phi_t^k)^2 + \varepsilon_k - (q_k)^2. \tag{2.130}$$

If we expand the matrices in the term that includes the degree of risk sensitivity in (2.101), we obtain

$$\begin{aligned} &\delta_k\Pi_k(t)\Sigma_k\Sigma_k^\top\Pi_k(t) \\ &= \delta_k \begin{bmatrix} (\Pi_{11}^k)^2(\sigma_k)^2 + (\Pi_{12}^k)^2(\sigma_0)^2 & \Pi_{11}^k\Pi_{12}^k(\sigma_k)^2 + \Pi_{12}^k\Pi_{22}^k(\sigma_0)^2 & \Pi_{11}^k\Pi_{13}^k(\sigma_k)^2 + \Pi_{12}^k\Pi_{23}^k(\sigma_0)^2 \\ \Pi_{21}^k\Pi_{11}^k(\sigma_k)^2 + \Pi_{22}^k\Pi_{21}^k(\sigma_0)^2 & (\Pi_{21}^k)^2(\sigma_k)^2 + (\Pi_{22}^k)^2(\sigma_0)^2 & \Pi_{21}^k\Pi_{13}^k(\sigma_k)^2 + \Pi_{22}^k\Pi_{23}^k(\sigma_0)^2 \\ \Pi_{31}^k\Pi_{11}^k(\sigma_k)^2 + \Pi_{32}^k\Pi_{21}^k(\sigma_0)^2 & \Pi_{31}^k\Pi_{12}^k(\sigma_k)^2 + \Pi_{32}^k\Pi_{22}^k(\sigma_0)^2 & (\Pi_{31}^k)^2(\sigma_k)^2 + (\Pi_{32}^k)^2(\sigma_0)^2 \end{bmatrix}. \end{aligned} \tag{2.131}$$

From (2.131) we can see that the ODE for entry Π_{12}^k contains Π_{22}^k when the minor agent is risk-sensitive, i.e., when $\delta_k \neq 0$, as it is impacted by the term $\Pi_{11}^k\Pi_{12}^k(\sigma_k)^2 +$

$\Pi^k_{12}\Pi^k_{22}(\sigma_0)^2$. However, for the structure (2.123) resulting from the ansatz (2.122), entry $(1,2)$, $\phi^k_t\theta_k\lambda$, does not capture entry $(2,2)$, $\psi^k_t\theta_k\lambda$. Therefore, this ansatz is not valid for simplifying the Riccati equation (2.101) when the minor agent is risk-sensitive.

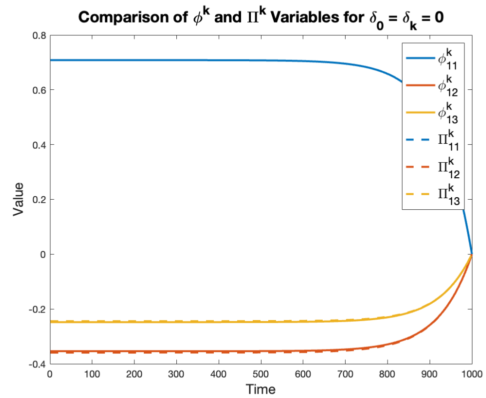


Figure 2.3: Comparison of $\phi^k(t)$ and $\Pi^k(t)$ variables for $\delta_0 = \delta_k = 0$.

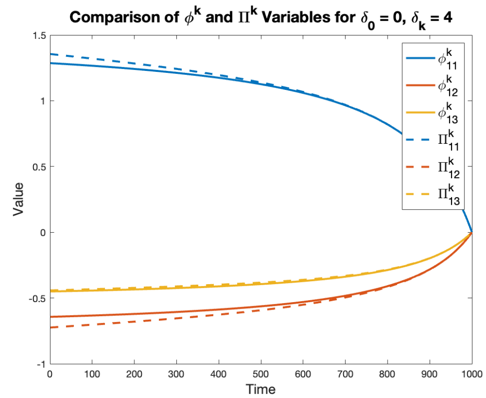


Figure 2.4: Comparison of $\phi^k(t)$ and $\Pi^k(t)$ variables for $\delta_0 = 0, \delta_k = 4$.

Fig. 2.3 and Fig. 2.4 further confirm this conclusion. We plot the values of the entries $(1,1)$, $(1,2)$, and $(1,3)$ at each time t under both the original form and ansatz form in (2.129) for a generic minor bank. As we can see from the graphs when the minor bank is risk-neutral, the values of Π^k_{11} and $-\phi^k_t$ match perfectly. However, there are discrepancies between the values of Π^k_{11} Π^k_{12} and Π^k_{13} and their respective expressions using the ansatz when the minor bank behaves in a risk-sensitive fashion, as shown in Fig. 2.4. Therefore, the ansatz in (2.121) is not valid in the risk-sensitive setting, and we use the general form of the optimal trading rate (2.60), as well as the general solutions (2.101) and (2.102) for all minor agents in our model.

Other forms of ansatz may be used to capture the interaction explained above. However, several ODEs would be involved to characterize the optimal trading strategies of banks. Therefore, this would not lead to a useful simplification that we desire.

Chapter 3

Numerical Experiments

As previously discussed in the literature review, due to the connected nature of the interbank market, the effect of the failure of one large institute may quickly spread across the system and lead to more bank defaults. Hence, it is crucial to analyze and understand the factors causing banks to default so that the disastrous aftermath might be prevented. In this chapter, we first investigate the impact of volatility and risk sensitivity of the major and minor banks on their mean reversion rates and optimal transaction rates. We then define mathematically the total and conditional probabilities of default for each bank. We also study the probability of the systemic event, which occurs when the market state of monetary reserves falls below a certain default threshold (Carmona et al., 2015). To observe the effect of specific components more directly, such as the major bank's size and the degree of risk sensitivity of banks we then perform Monte Carlo simulations on banks' log-monetary reserves dynamics and optimal transaction rates and acquire probabilities of default and systemic risk in different scenarios.

3.1 Impact of Volatility and Risk Sensitivity on Optimal Transaction Rates

In this section, we study the effect of changes in the major bank and minor banks' volatility level and degree of risk sensitivity on the mean reversion rates and optimal transaction rates of banks, inspired by Chang et al. (2023). We assume there is one type of minor banks for this purpose, where the same parameters characterize all minor banks. As shown in Section 2.3.1, the major bank's optimal transaction rate admits a reduced expression given by (2.13) in the form of mean reversion where ϕ_t^0 follows the ordinary differential equation (ODE) given by (2.15). The mean reversion rate of the major bank can be expressed as $a_0 + q_0 - \phi_t^0$ as seen in the closed-loop dynamics (2.26) in Corollary 1. Therefore, the magnitude of ϕ_t^0 describes the major bank's mean reversion rate to the market state, where a greater magnitude of ϕ_t^0 indicates a higher mean reversion rate. For the following simulations, we plot the trajectories of ϕ_t^0 and set $\lambda = 0.5, a_0 = a = 5, c_0 = c_k = 0, \varepsilon_0 = \varepsilon_k = 10, \theta_0 = \theta_k = 1, \gamma_0 = \gamma_k = 0.3$ and $q_0 = q_k = 1$.

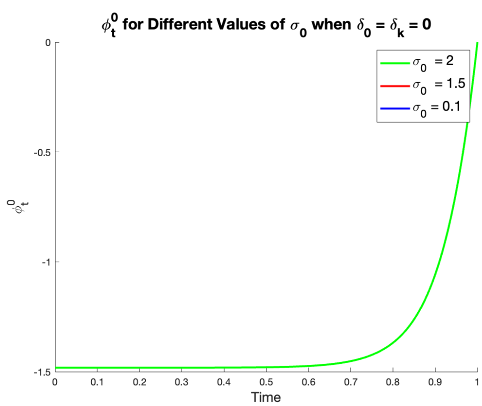


Figure 3.1: ϕ_t^0 for various values of σ_0 when $\delta_0 = \delta_k = 0, \sigma_k = 1$.

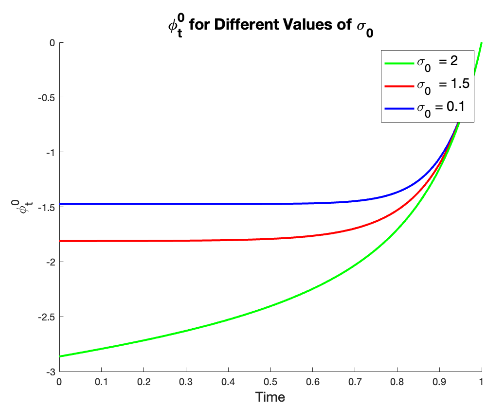


Figure 3.2: ϕ_t^0 for various values of σ_0 when $\delta_0 = \delta_k = \sigma_k = 1$.

The parameter σ_0 describes the volatility of the major bank's log-monetary reserves, which stems from the daily transactions of the bank's retail customers. A higher value of σ_0 indicates more changes in the bank's monetary reserves, whether positive or negative. As seen in Fig. 3.1, when all banks in the interbank system are risk-neutral, changes in the major bank's volatility level, σ_0, do not impact the trajectory of ϕ_t^0. However, when we perform our simulations in a risk-sensitive setting, we observe a more significant impact. As shown in Fig. 3.2, a risk-sensitive major bank's mean reversion rate would increase as its volatility level increases, while the minor banks' volatility level remains constant. This indicates that as there are more uncertainties in the major bank's monetary reserves, the bank adjusts its strategies by trading more frequently with other banks to return to its target reserves.

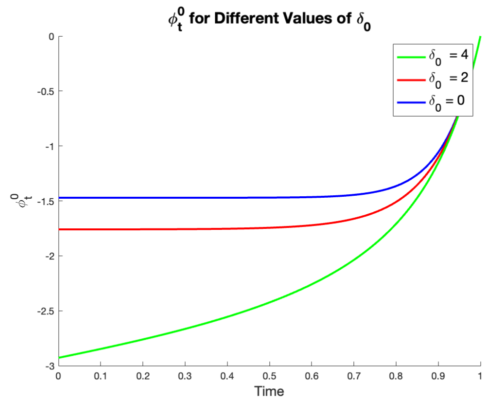


Figure 3.3: ϕ_t^0 for various values of δ_0 when $\delta_k = \sigma_0 = \sigma_k = 1$.

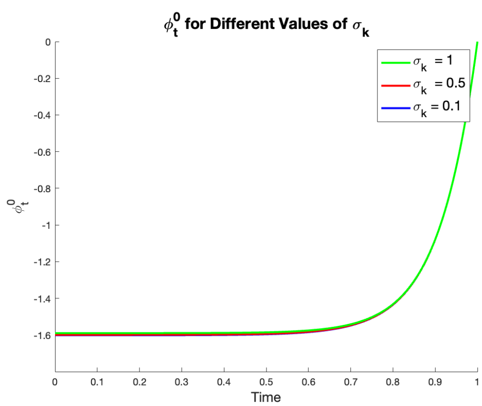


Figure 3.4: ϕ_t^0 for various values of σ_k when $\delta_0 = \delta_k = \sigma_0 = 1$.

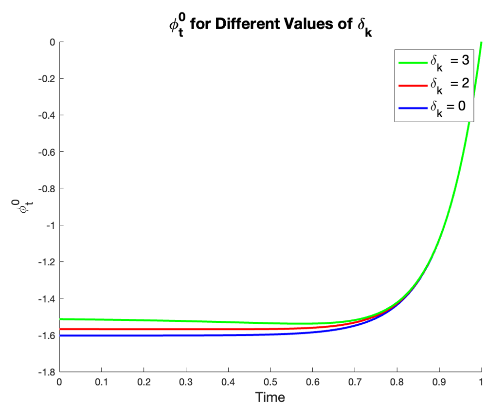


Figure 3.5: ϕ_t^0 for various values of δ_k when $\delta_0 = \sigma_0 = \sigma_k = 1$.

Moreover, Fig. 3.3 illustrates that as the risk sensitivity degree of the major bank increases, meaning that it is more risk-averse, the bank reverts to its target reserves level more quickly. Since the major bank is more cautious about the uncertainties in the interbank market, it aims to stay close to the target level to avoid risk. We can also see from Fig. 3.4 and Fig. 3.5 that the magnitude of ϕ_t^0 decreases slightly as the volatility level, σ_k, and degree of risk sensitivity, δ_k, of minor banks increase, while the volatility and risk sensitivity level of the major bank remain the same. In an interbank system, minor banks are more conservative with their lending/borrowing activities as they become more risk-averse, which could lead to a healthier market. Since the major bank's risk sensitivity degree remains constant, it can reduce its mean reversion rate in a more stable environment. Additionally, as indicated by the ODE in (2.15), ϕ_t^0 depends on the variable $\bar{G}$, which is defined in (2.23) and includes the term Π_{12}^k. More precisely, from (2.131), we know that Π_{12}^k depends on the parameters σ_k and δ_k. Therefore, the changes in these parameters of the minor banks have a slight effect on the mean reversion rate of the major bank.

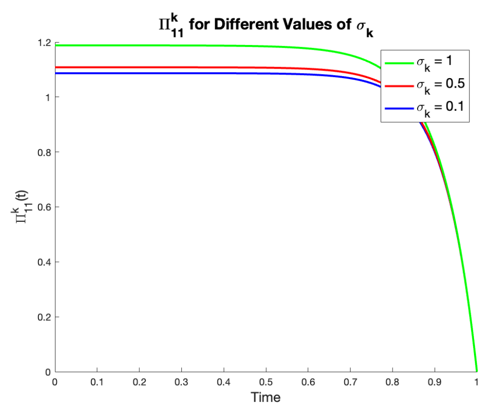


Figure 3.6: Value of Π_{11}^k for various values of σ_k when $\delta_0 = \delta_k = \sigma_0 = 1$.

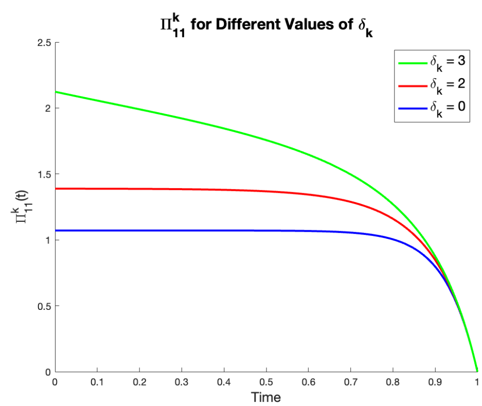


Figure 3.7: Value of Π_{11}^k for various values of δ_k when $\delta_0 = \sigma_0 = \sigma_k = 1$.

Furthermore, we can see in Corollary 1 that the variables Π_{11}^k, Π_{12}^k and Π_{13}^k appear in a minor bank's dynamics and affect its optimal transaction rate. Therefore, we proceed to analyze the effect σ_k, δ_k, σ_0 and δ_0 have on Π_{11}^k, Π_{12}^k, and Π_{13}^k for a minor bank. Similar to the major bank's case, we do not observe any impacts of the changes in the parameters in a risk-neutral setting. Hence, we conduct the simulations with risk-sensitive banks. As shown in Fig. 3.6 and Fig. 3.7, Π_{11}^k of a generic minor bank increases as its volatility

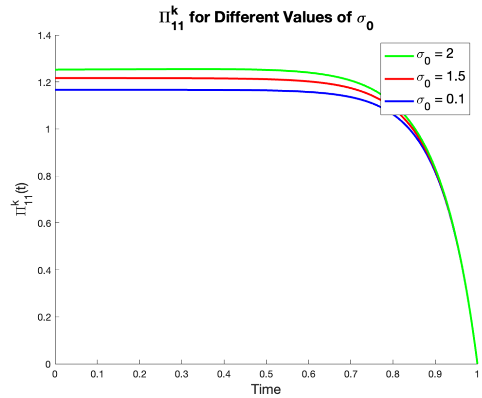


Figure 3.8: Value of Π^k_{11} for various values of σ_0 when $\delta_0 = \delta_k = \sigma_k = 1$.

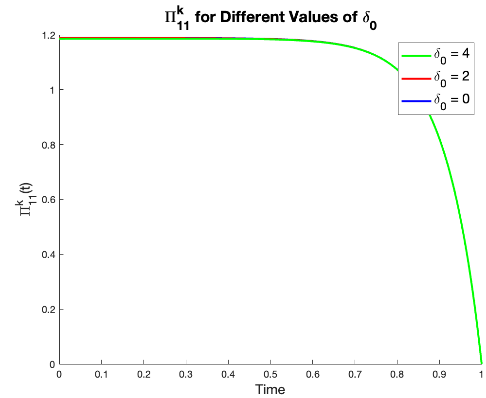


Figure 3.9: Value of Π^k_{11} for various values of δ_0 when $\delta_k = \sigma_0 = \sigma_k = 1$.

level of monetary reserves and degree of risk sensitivity increase, which exhibits a similar trend as that of the major bank's mean reversion rate in Fig. 3.2 and Fig. 3.3. However, Fig. 3.8 illustrates that the major bank's volatility level has a more significant impact on Π^k_{11} compared to the effect σ_k has on major bank's mean reversion rate, $a_0 + q_0 - \phi^0_t$, in Fig. 3.4. This can be explained by the setting of our model, which states that the major bank's actions affect each minor bank, whereas an individual minor bank's effect on the major bank is negligible. Furthermore, the changes in the major bank's risk sensitivity degree have almost no effect on Π^k_{11} as shown in Fig. 3.9, since δ_0 does not explicitly show up in the expression for Π^k_{11} as outlined by (2.131).

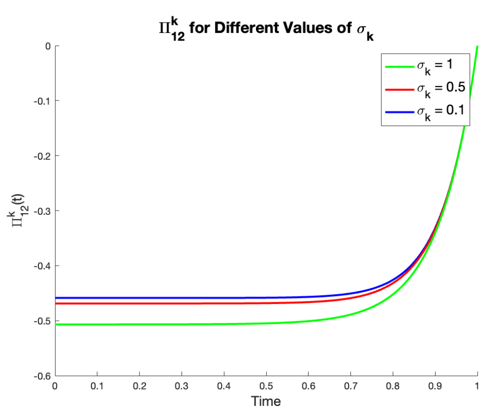


Figure 3.10: Value of Π^k_{12} for various values of σ_k when $\delta_0 = \delta_k = \sigma_0 = 1$.

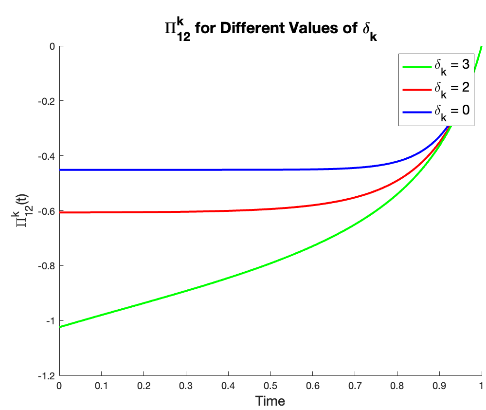


Figure 3.11: Value of Π^k_{12} for various values of δ_k when $\delta_0 = \sigma_0 = \sigma_k = 1$.

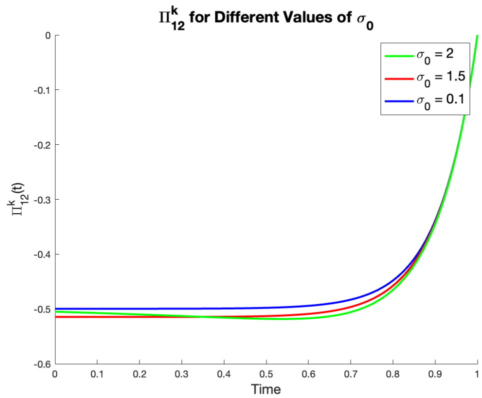


Figure 3.12: Value of Π^k_{12} for various values of σ_0 when $\delta_0 = \delta_k = \sigma_k = 1$.

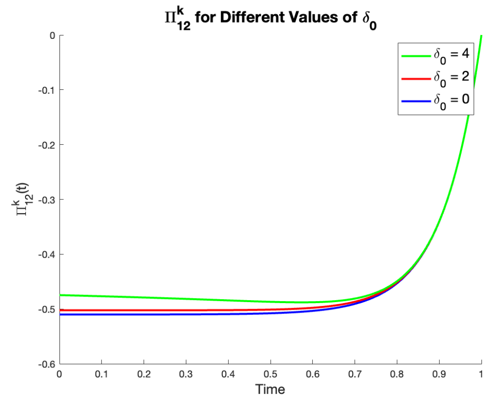


Figure 3.13: Value of Π^k_{12} for various values of δ_0 when $\delta_k = \sigma_0 = \sigma_k = 1$.

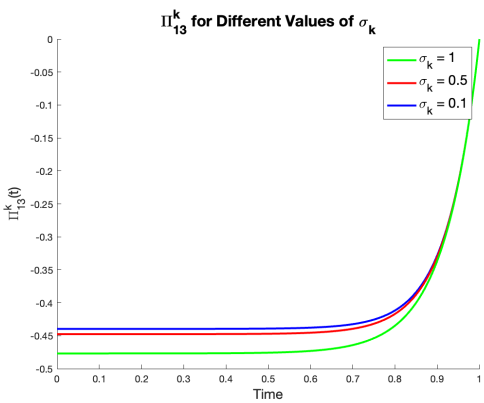


Figure 3.14: Value of Π^k_{13} for various values of σ_k when $\delta_0 = \delta_k = \sigma_0 = 1$.

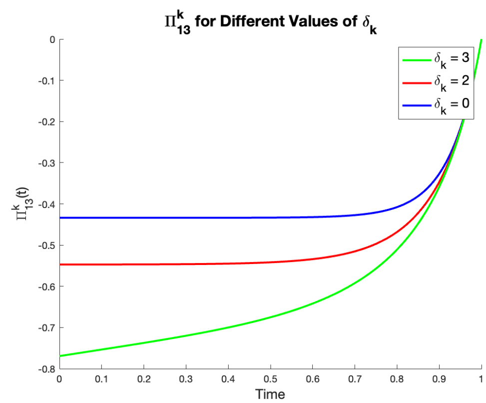


Figure 3.15: Value of Π^k_{13} for various values of δ_k when $\delta_0 = \sigma_0 = \sigma_k = 1$.

The effects of a minor bank's volatility level and risk sensitivity degree on the values of Π^k_{12} and Π^k_{13} show similar patterns as observed in Fig. 3.10 - Fig. 3.11 and Fig. 3.14 - Fig. 3.15. The magnitudes of both variables increase as a minor bank's monetary reserves face more uncertainty and as the bank becomes more risk-averse. Furthermore, we do not see a clear trend in the change of the value of Π^k_{12} as the major bank's volatility level, σ_0, rises as shown in Fig. 3.12, whereas an increase in the major bank's risk sensitivity degree has opposite effects on the magnitudes of Π^k_{12} and Π^k_{13}, as illustrated in Fig. 3.13 and Fig. 3.17 respectively.

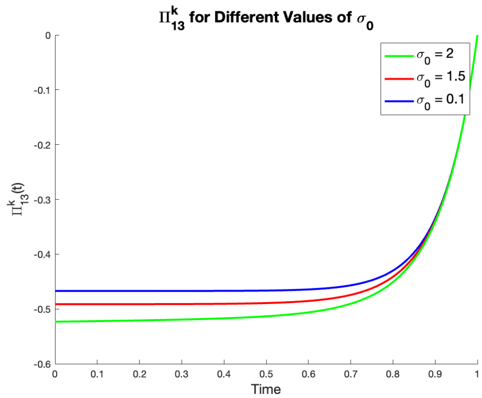


Figure 3.16: Value of Π^k_{13} for various values of σ_0 when $\delta_0 = \delta_k = \sigma_k = 1$.

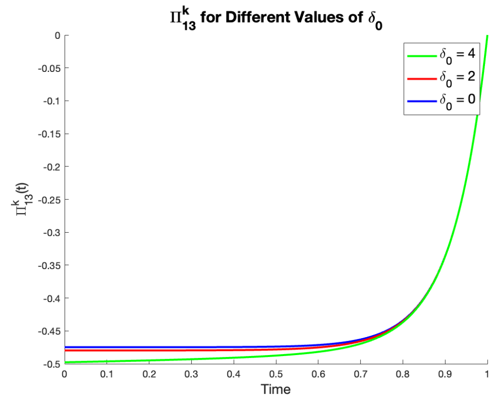


Figure 3.17: Value of Π^k_{13} for various values of δ_0 when $\delta_k = \sigma_0 = \sigma_k = 1$.

3.2 Default Probability and Systemic Risk

We now introduce the mathematical expression for the default probability of the major bank, which is the likelihood that the major bank's monetary reserves level, x_t^0, hits and falls below a certain threshold D within a specific time period

$$P(MD) = \mathbb{P}(\text{default of major bank on } [0,T]) = \mathbb{P}\left(\min_{t\in[0,T]} (x_t^0) \leq D\right). \tag{3.1}$$

For a representative minor bank i in subgroup k, the probability of default is defined as

$$P(i_k D) = \mathbb{P}(\text{default of minor bank } i \text{ in type } k \text{ on } [0,T]) = \mathbb{P}\left(\min_{t\in[0,T]} (x_t^{i,k}) \leq D\right), \tag{3.2}$$

where $x_t^{i,k}$ is the log-monetary reserves of the minor bank. Furthermore, inspired by Carmona et al. (2015) and Chang et al. (2023), we assume that systemic risk occurs when the market average state of the log-monetary reserves, $\lambda x_t^0 + (1-\lambda)x_t^{(N)}$, reaches the default threshold before time T, i.e.

$$P(SR) = \mathbb{P}(\text{systemic risk on } [0,T]) = \mathbb{P}\left(\min_{t\in[0,T]} \left(\lambda x_t^0 + (1-\lambda)x_t^{(N)}\right) \leq D\right). \tag{3.3}$$

Apart from analyzing how different parameters in the dynamics and cost functionals contribute to the failure of banks and the system, we also want to study the effect of the default of a large bank on the interbank market. Therefore, we define the default probability of a representative minor bank i and systemic risk conditioned on whether the major bank defaults or not within the time period

$$P(i_kD|MD) = \mathbb{P}\left(\min_{t\in[0,T]} (x_t^{i,k}) \le D|\text{major bank defaults on } [0,T]\right), \tag{3.4}$$

$$P(i_kD|MS) = \mathbb{P}\left(\min_{t\in[0,T]} (x_t^{i,k}) \le D|\text{major bank survives on } [0,T]\right), \tag{3.5}$$

$$P(SR|MD) = \mathbb{P}\left(\min_{t\in[0,T]} \left(\lambda x_t^0 + (1-\lambda)x_t^{(N)}\right) \le D|\text{major bank defaults on } [0,T]\right), \tag{3.6}$$

$$P(SR|MS) = \mathbb{P}\left(\min_{t\in[0,T]} \left(\lambda x_t^0 + (1-\lambda)x_t^{(N)}\right) \le D|\text{major bank survives on } [0,T]\right), \tag{3.7}$$

where MD stands for "major bank defaults" and MS stands for "major bank survives". According to the law of total probability, we obtain

$$\begin{aligned} P(i_kD) = P(i_kD\cap MD) + P(i_kD\cap MS) &= P(i_kD|MD)\times P(MD) + P(i_kD|MS)\times P(MS) \\ &= P(i_kD|MD)\times P(MD) + P(i_kD|MS)\times\left(1-P(MD)\right) \\ &= \left(P(i_kD|MD) - P(i_kD|MS)\right)\times P(MD) + P(i_kD|MS). \end{aligned} \tag{3.8}$$

Similarly, we have

$$P(SR) = \left(P(SR|MD) - P(SR|MS)\right)\times P(MD) + P(SR|MS). \tag{3.9}$$

Additionally, minor banks in the same subgroup share the same features, and their dynamics and optimal transaction strategies differ only by the Brownian motions. While the impact of an individual minor bank on the interbank system is minimal, that of all small banks in a subgroup together may be significant. Hence, the size of a subgroup of minor banks together can be similar to that of a major bank. Therefore, it would also be interesting to examine the consequence when a subgroup k defaults. This is the case where the average level of the subpopulation's reserves, $x_t^{(N),k} = \frac{1}{N_k}\sum_{i\in I_k} x_t^i$, falls below the default threshold. We hereby define some conditional probabilities depending on whether a subgroup of minor banks defaults or not

$$P(MD|TkD) = \mathbb{P}\left(\min_{t\in[0,T]} \left(\text{major bank defaults on } [0,T]\ \right)|\text{subgroup } k \text{ defaults on } [0,T]\ \right), \tag{3.10}$$

$$P(MD|TkS) = \mathbb{P}\left(\min_{t\in[0,T]} \left(\text{major bank defaults on } [0,T]\right)|\text{subgroup } k \text{ survives on } [0,T]\right), \tag{3.11}$$

$$P(i_jD|TkD) = \mathbb{P}\left(\min_{t\in[0,T]} (x_t^{i,j}) \ \le\ D|\text{subgroup } k \text{ defaults on } [0,T]\ \right), \tag{3.12}$$

$$P(i_jD|TkS) = \mathbb{P}\left(\min_{t\in[0,T]} (x_t^{i,j}) \ \le\ D|\text{subgroup } k \text{ survives on } [0,T]\ \right), \tag{3.13}$$

$$P(SR|TkD) = \mathbb{P}\left(\min_{t\in[0,T]} \left(\lambda x_t^0 + (1-\lambda)x_t^{(N)}\right) \ \le\ D|\text{subgroup } k \text{ defaults on } [0,T]\ \right), \tag{3.14}$$

$$P(SR|TkS) = \mathbb{P}\left(\min_{t\in[0,T]} \left(\lambda x_t^0 + (1-\lambda)x_t^{(N)}\right) \ \le\ D|\text{subgroup } k \text{ survives on } [0,T]\right), \tag{3.15}$$

where TkD stands for when subgroup k defaults and TkS stands for when subgroup k survives. We study the probability of a minor bank $x_t^{i,j}$ defaulting in scenarios when it belongs to a non-defaulting subgroup j that is different from the defaulting subgroup k and when it belongs to the defaulting subgroup k.

In the following sections, we implement some numerical experiments to better understand and analyze the different default probabilities defined earlier under various scenarios. In Chapter 2, we present the optimal transaction strategies of the major bank and a representative minor bank in the infinite population in which the system achieves a Nash equilibrium. Previous studies have shown that in a Monte Carlo simulation setting, the system's behaviour with an infinite population is a reliable approximation of the behaviour observed in a finite population with a small number of minor banks (Chang et al., 2023). Therefore, we apply the dynamics (1.1) and (1.4) to simulate an interbank market with one major bank and 20 minor banks. We also apply the optimal transaction rates (2.13) and (2.14) obtained in Chapter 2 in a finite population, where we use $x_t^{(N)}$ as the average reserves of all minor banks instead of the state mean-field, $\bar{x}_t$. We then employ the ODEs in (2.15) and (2.16) to solve the optimal transaction strategies of the major bank and a generic minor bank. The interbank system defaults when the market state, $\lambda x_t^0 + (1-\lambda)x_t^{(N)}$, falls below the default threshold. The parameter values used in the simulations are chosen according to research that has been done on similar subjects so we can compare our results with existing studies.

3.2.1 Impact of Major Bank's Risk Sensitivity Degree

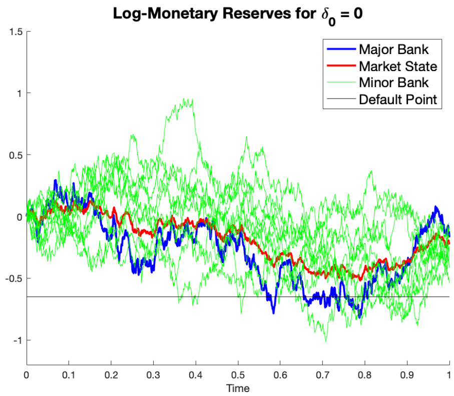
As discussed in the literature review, a large bank may contribute more to systemic risk due to its complex structure (Laeven et al., 2014). Therefore, it is essential to study how different factors impact the dynamics of the major bank so we can construct an interbank system that is less likely to fail. We first examine how different degrees of risk sensitivity of the major bank affect its stability, a representative minor bank i, and the system. We assume all 20 minor banks are homogeneous and share the same parameters. Therefore, there is one type of minor banks in the system and $K = 1$. We select a range of values for δ_0 from 0 to 4, with 0 meaning the bank is risk-neutral and non-zero values indicating the bank is risk-averse; the larger the value of risk sensitivity degree, the more risk-averse the bank is. A risk-averse bank demonstrates a conservative approach in its decision-making process and is willing to tolerate a lower level of profitability in order to mitigate potential risks.

We first simulate the system assuming the major bank's size λ is 0.4, meaning it takes up 40% of the interbank market. We also assume that all minor banks are risk-averse, with a risk sensitivity degree δ_k of 1. As shown in Table 3.1, the default probability of the major bank, $P(MD)$, decreases from 0.2664 to 0.2440 as δ_0 increases from 0 to 4. This suggests that as the major bank becomes more risk-averse, its likelihood of default decreases. Furthermore, the total default probability of a representative minor bank, $P(i_kD)$, decreases as the major bank's risk sensitivity degree increases, indicating that a risk-averse major bank improves the stability of minor banks. However, $P(i_kD|MS)$, the probability of a minor bank defaulting conditioned on the major bank surviving, shows a slightly increasing trend from 0.0837 to 0.0895, while the default probability of a minor bank conditioned on the major bank defaulting, $P(i_kD|MD)$, decreases from 0.4985 to 0.4831 as δ_0 increases. These results indicate that if the major bank is successful, its risk aversion does not necessarily benefit the minor banks. However, when the major bank defaults, its being risk-averse reduces the severity of a minor bank defaulting. As the major bank becomes less inclined to take risks, it may adopt more careful lending practices. This could lead to less liquidity in the market, especially when the major bank survives. Smaller banks, which often rely on the liquidity provided by the large bank, may find it harder to obtain the necessary funds, increasing their default risks. Moreover, $P(i_kD|MD)$ values are much higher than $P(i_kD|MS)$. This indicates that the default of a major bank can trigger a domino effect, impacting the liquidity and solvency of minor banks and, therefore, leading to a higher probability of minor banks defaulting.

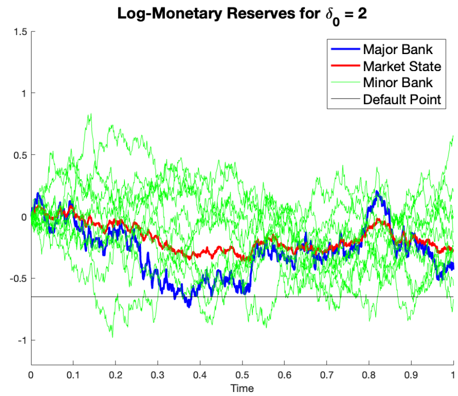
δ_0	$P(MD)$	$P(i_kD)$	$P(i_kD\|MS)$	$P(i_kD\|MD)$	$P(SR)$	$P(SR\|MS)$	$P(SR\|MD)$
0	0.2664	0.1942	0.0837	0.4985	0.0490	0	0.1839
2	0.2608	0.1911	0.0842	0.4940	0.0458	0	0.1757
4	0.2440	0.1855	0.0895	0.4831	0.0386	0	0.1581

Table 3.1: Default probabilities of the major bank, a representative minor bank, and systemic risk for different values of δ_0 ($\lambda = 0.4; a_0 = 3.33; a = 5; \varepsilon_0 = 30; \varepsilon_k = 15; \gamma^0(t) = \gamma^k(t) = 0.3; \theta_0 = \theta_k = 1; \delta_k = 1$).

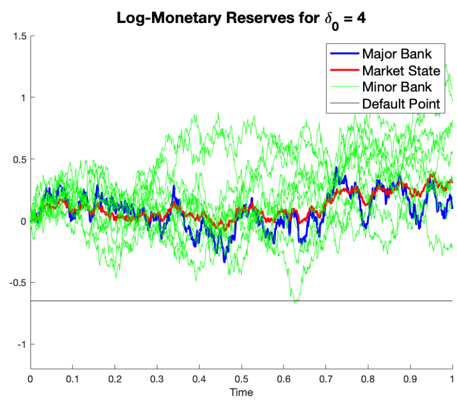
Furthermore, the probability of systemic risk reflects the interbank market's overall health. As shown in Table 3.1, the total probability of systemic risk, $P(SR)$, decreases from 0.0490 to 0.0386 as δ_0 increases. This implies that a more risk-averse major bank reduces systemic risk, leading to more prudent financial practices. Additionally, the conditional probabilities illustrate the system's vulnerability to the status of the major bank. The fact that $P(SR|MS)$, the conditional probability of systemic risk when the major bank survives, is zero for all values of δ_0 suggests that the system is relatively stable when the major bank does not default. However, the significantly higher systemic risk conditioned on major bank defaulting, $P(SR|MD)$, indicates that the system is very vulnerable to the major bank's default. Nonetheless, as the major bank becomes more risk-averse, it re-



(a) when $\delta_0 = 0$



(b) when $\delta_0 = 2$



(c) when $\delta_0 = 4$

Figure 3.18: Trajectories of log-monetary reserves for the major bank, 10 representative minor banks, the major bank and the market state for different values of δ_0.

sults in a less severe market default, as seen in the decrease of the $P(SR|MD)$ values from 0.1839 to 0.1581. This suggests that a risk-sensitive major bank improves the financial stability of the interbank transactions market.

Fig. 3.18 further illustrates the effect of a major bank's risk sensitivity degree. We plot the trajectories of monetary reserves of the major bank, the market, and ten minor banks with the same parameters as in Table 3.1. As we can see from the plots, the major bank and several minor banks default when the major bank is risk-neutral. The levels of log-monetary reserves of banks and the market state increase slightly when $\delta_0 = 2$. However, the monetary reserves of the major bank and one minor bank reach the default threshold. When the degree of risk sensitivity is 4, meaning the major bank is more risk-averse, we can see a significant improvement in the stability of banks and the market as shown in Fig. 3.18c. All trajectories show an upward trend, while only one minor bank's monetary reserves touch the default threshold. The monetary reserves of the major bank also stay closer to the market state. Therefore, having a risk-averse major bank in the interbank market enhances the financial health of itself, other small banks and the entire system.

We now analyze the effect of the major bank's risk sensitivity degree when its size takes up a more significant portion of the interbank market. Although a large bank operates more efficiently due to economies of scale and scope, other issues emerge with its size. With the same setup, we first simulate the system with $\lambda = 0.5$, indicating that the major bank's size is half the market size. The mean reversion rate of the major bank equals that of the minor banks according to the market clearing condition (1.11) we defined in Section 1.3, without considering the optimal transaction rate of banks.

δ_0	$P(MD)$	$P(i_kD)$	$P(i_kD\|MS)$	$P(i_kD\|MD)$	$P(SR)$	$P(SR\|MS)$	$P(SR\|MD)$
0	0.2738	0.2236	0.0857	0.5895	0.0878	0	0.3207
2	0.2694	0.2184	0.0881	0.5716	0.0826	0	0.3066
4	0.2634	0.2091	0.0900	0.5421	0.0767	0	0.2977

Table 3.2: Default probabilities of the major bank, a representative minor bank, and systemic risk for different values of δ_0 ($\lambda = 0.5; a_0 = a = 5; \varepsilon_0 = 30; \varepsilon_k = 15; \gamma^0(t) = \gamma^k(t) = 0.3; \theta_0 = \theta_k = 1; \delta_k = 1$).

The results are presented in Table 3.2. We can see that the probability trends are consistent with those in Table 3.1 when $\lambda = 0.4$. The default probabilities of the major bank, a generic minor bank, and systemic risk decrease as the major bank becomes more averse to risks, confirming that a risk-sensitive major bank enhances the interbank system's stability. However, the magnitudes of these probabilities increase as the size of major bank grows. The substantial increase in $P(SR|MD)$ across different risk sensitivity degrees is consistent with the notion that larger banks are more integral to the financial system's stability, and their problems can lead to more significant systemic issues.

Next, we consider the scenario where the major bank's size, λ, is 0.7 while keeping the other parameters constant. As shown in Table 3.3, the results do not show a linear trend when the major bank's risk sensitivity increases. The major bank's default probability varies around 0.2720, while the total default probability of a representative minor bank fluctuates around 0.2490. However, we see a significant increase in the values of $P(i_kD|MD)$ and the total and conditional probabilities of systemic risk compared to when the major bank's size is smaller. The zero values for $P(SR|MS)$ across all sizes again suggest that the system remains stable as long as the major bank does not default. However, the massive jump in the systemic risk indicates that the system is incredibly fragile when the major bank possesses too much power in the market, particularly in scenarios where the major bank defaults. Therefore, depending on how minor banks interact with the altered market dynamics, the large size of the major bank can offset the positive impact of its risk aversion.

δ_0	$P(MD)$	$P(i_kD)$	$P(i_kD\|MS)$	$P(i_kD\|MD)$	$P(SR)$	$P(SR\|MS)$	$P(SR\|MD)$
0	0.2721	0.2490	0.0980	0.6539	0.1576	0	0.5791
2	0.2708	0.2496	0.0981	0.6575	0.1597	0	0.5895
4	0.2720	0.2488	0.0980	0.6524	0.1574	0	0.5787

Table 3.3: Default probabilities of the major bank, a representative minor bank, and systemic risk for different values of δ_0 ($\lambda = 0.7; a_0 = 11.67; a = 5; \varepsilon_0 = 30; \varepsilon_k = 15; \gamma^0(t) = \gamma^k(t) = 0.3; \theta_0 = \theta_k = 1; \delta_k = 1$).

Overall, our results indicate that a risk-averse major bank's presence decreases the default probabilities of itself, a generic minor bank, and systemic risk when its size is less than or equal to half of the market size. A more risk-averse stance by the major bank can lead to stabilizing effects as it employs more cautious lending approaches and liquidity conditions. The severity of default events when they occur is also reduced when the major bank becomes more risk-averse. However, the benefits of a risk-averse major bank may diminish when the bank's size gets too large. A larger major bank is associated with higher default probabilities across the interbank system. Comparing the three scenarios underscores the importance of monitoring and managing the size of large financial institutions due to their significant impact on overall financial stability.

3.2.2 Impact of Percentage of the Market State that Major Bank Follows

In this section, we study the effect of θ_0, which is the parameter representing the extent to which the major bank aligns its monetary reserves with the market state, $\lambda x_t^0 + (1-\lambda)x_t^{(N)}$. It is an essential factor since it reflects the major bank's strategies regarding reserves management in response to market conditions. The value of θ_0 ranges from 0.2 to 1, with 0.2 meaning that the major bank will only track 20% of the market state and 1 meaning that the major bank intends to completely match its reserves with the market state. We assume all banks are risk-neutral, and minor banks try to track the market state fully.

θ_0	$P(MD)$	$P(i_kD)$	$P(i_kD\|MS)$	$P(i_kD\|MD)$	$P(SR)$	$P(SR\|MS)$	$P(SR\|MD)$
0.2	0.2038	0.1746	0.1354	0.3278	0.0034	0	0.0167
0.4	0.2218	0.1754	0.1298	0.3354	0.0065	0	0.0293
0.6	0.2428	0.1769	0.1072	0.3943	0.0114	0	0.0470
0.8	0.2468	0.1794	0.0943	0.4392	0.0236	0	0.0956
1	0.2684	0.1988	0.0908	0.4933	0.0462	0	0.1721

Table 3.4: Default probabilities of the major bank, a representative minor bank, and systemic risk for different values of θ_0 ($\lambda = 0.4; a_0 = 3.33; a = 5; \varepsilon_0 = 30; \varepsilon_k = 15; \gamma^0(t) = \gamma^k(t) = 0.3; \theta_k = 1; \delta_0 = \delta_k = 0$).

As summarized in Table 3.4, the default probability of the major bank, $P(MD)$, increases steadily from 0.2038 to 0.2684 as θ_0 increases. This suggests that closely aligning its reserves with the market reserves increases the major bank's default risk, which can be explained by the higher exposure to market fluctuations. The default probability of a representative minor bank i also shows an upward pattern as θ_0 rises, although the increase is not as pronounced as in the major bank's case. We also see a flocking effect in the conditional probabilities of default of the minor bank. Since the major bank has more impact on the market than individual minor banks, they transmit the market trend and policies to minor banks through their actions, and minor banks make decisions accordingly. Therefore, the decrease in values of $P(i_kD|MS)$ as θ_0 increases suggests that if the major bank is stable, strictly following the market state improves the stability of minor banks. In contrast, the default of a major bank that closely tracks the market average reserves negatively impacts minor banks, given the notable increase in $P(i_kD|MD)$. The more significant increase in $P(i_kD|MD)$ values compared to the decrease in $P(i_kD|MS)$ leads to an increase in the minor bank's total default probability.

Furthermore, $P(SR)$ shows a notable increase with higher values of θ_0, indicating that the system's stability is increasingly compromised as the major bank's reserves align more with the market state. The conditional probability $P(SR|MS)$ remains constant at 0, suggesting that as long as the major bank does not default, the system remains stable regardless of its reserve strategy. However, $P(SR|MD)$ increases significantly from 0.0167 to 0.1721 with θ_0. Suppose there is negative news shocking the market. In that case, the major banks will react poorly by completely and recklessly following the market reserves level, which in turn causes more chaos in the market and ultimately leads to a vicious cycle. Therefore, the more pronounced negative impact in $P(SR|MD)$ with no additional positive impact from $P(SR|MS)$ leads to higher values of $P(SR)$ as θ_0 becomes larger.

In summary, the parameter θ_0 plays a significant role in influencing the default probabilities of the major and minor banks and the overall system. Higher values of θ_0, indicating a closer alignment of the major bank's reserves with the market state, tend to increase the default risks across the market, especially when the major bank crashes. This highlights the potential risks associated with reserve management strategies, particularly for large banks with significant market influence, and the crucial role they play in maintaining financial stability.

3.2.3 Impact of Growth Rate of Major Bank

Our model also incorporates a growth rate, γ_0, which is the extra income added to the bank's monetary reserves from customer deposits and other banking activities. This parameter can also be interpreted as an intervention from the central bank for the major bank. As mentioned in the introduction, we can consider a major bank as a systemically important financial institution that is "too big to fail". Therefore, governments and central banks take extra precautions to ensure the major bank does not default during a crisis by bringing in extra liquidity. Hence, it is worth studying the effect of this factor on banks' default probabilities and systemic risk.

γ_0	$P(MD)$	$P(i_kD)$	$P(i_kD\|MS)$	$P(i_kD\|MD)$	$P(SR)$	$P(SR\|MS)$	$P(SR\|MD)$
0.2	0.3380	0.2988	0.1281	0.6331	0.1270	0	0.3757
0.4	0.2718	0.2620	0.1200	0.6424	0.0990	0	0.3642
0.6	0.2272	0.2282	0.1102	0.6294	0.0642	0	0.2826
0.8	0.1830	0.1902	0.0940	0.6197	0.0512	0	0.2798
1	0.1462	0.1628	0.0874	0.6033	0.0396	0	0.2709

Table 3.5: Default probabilities of the major bank, a representative minor bank, and systemic risk for different values of γ_0 ($\lambda = 0.4; a_0 = 3.33; a = 5; \varepsilon_0 = 30; \varepsilon_k = 15; \gamma^k(t) = 0.1; \theta_0 = \theta_k = 1; \delta_0 = \delta_k = 0$).

As seen from the results presented in Table 3.5, as γ_0 increases from 0 to 1, there is a marked decrease in the major bank's default probability, from 0.3380 to 0.1462. This suggests that a higher stream of additional income significantly enhances the major bank's financial stability, reducing its default risk. We also see a declining trend in the default probability of a generic minor bank from 0.2988 to 0.1628 as γ_0 increases. This decrease indicates that the major bank's financial health positively influences minor banks' stability due to improved lending conditions and more available funds. Moreover, the conditional probabilities of a minor bank defaulting all show a downward trend, whether the major bank defaults or not. The decrease in $P(i_kD|MD)$ with higher γ_0 shows that the negative impact of the major bank's default on minor banks is less severe when the major bank is financially more robust. Additionally, the systemic risk decreases substantially as γ_0 increases, with $P(SR|MS)$ remains 0 with different values of growth rate. This trend again confirms the critical role of the major bank's financial performance in the overall stability of the financial system.

Overall, the income/growth rate of the major bank, γ_0, profoundly impacts the default probabilities of the major bank itself, the minor banks, and the whole system. The analysis highlights the interconnected nature of financial institutions and the ripple effects that the financial performance of a major bank can have on the broader financial system.

δ_k	$P(MD)$	$P(MD\|TkS)$	$P(MD\|TkD)$	$P(i_kD)$	$P(i_kD\|TkS)$	$P(i_kD\|TkD)$
0	0.2633	0.2486	1	0.1963	0.1813	0.9475
2	0.2648	0.2477	1	0.1975	0.1796	0.9659
3	0.2662	0.2472	1	0.1979	0.1764	0.9667

Table 3.6: Default probabilities of the major bank and a minor bank for different values of δ_k ($\lambda = 0.4; a_0 = 3.33; a = 5; \varepsilon_0 = 30; \varepsilon_k = 15; q_k = 1; \gamma^0(t) = \gamma^k(t) = 0.3; \theta_0 = \theta_k = 1; \delta_0 = 1$).

3.2.4 Impact of Risk Sensitivity Degree of a Subgroup of Minor Banks

We have introduced different types of minor banks in our system. In this section, we first assume there are 20 minor banks in the interbank market, where all banks belong to the same subgroup. Minor banks of the same type operate in similar sectors in the banking industry and, therefore, are exposed to similar risks and share the same parameters in their dynamics and cost functionals. While an individual minor bank has a negligible impact, the collective impact of a subgroup of minor banks is significant on the interbank market. It is interesting to study the effects of the degree of risk sensitivity and the default of a subgroup on the major bank, minor banks, and the interbank system as a whole. As mentioned previously in the section, we define the default of a subgroup of minor banks as the event of the average monetary reserves of the group fall below a certain threshold.

We assume the major bank's size, λ, is 0.4, and it has a risk sensitivity degree of 1. Therefore, the size of the subgroup of minor banks is 0.6, which is larger than the major bank. As we can see in Table 3.6, the total default probability of the major bank increases from 0.2633 to 0.2662 as the minor banks become more risk-averse. However, the probability of the major bank defaulting conditioned on the subgroup surviving, $P(MD|TkS)$, shows a slight decreasing trend from 0.2486 to 0.2474, while the default probability of the major bank conditioned on the subgroup defaulting remains at 1 across different values of δ_k. The effect of the distress of a large subgroup exceeds the benefits a stable subgroup has on the major bank, leading to an increasing total default probability of the major bank. Furthermore, we observe similar trends in the total and conditional default probabilities of a representative minor bank in the subgroup. There is a decrease in the default probability of the minor bank when the subgroup survives, $P(i_kD|TkS)$, from 0.1813 to 0.1764. However, the more significant increase in the conditional default probability when the subgroup defaults, $P(i_kD|TkD)$, leads to an increase in the total probability of default of the minor bank.

δ_k	$P(SR)$	$P(SR\|TkS)$	$P(SR\|TkD)$
0	0.0463	0.0274	0.9949
2	0.0506	0.0287	0.9934
3	0.0559	0.0309	0.9914

Table 3.7: Default probabilities of systemic risk for different values of δ_k ($\lambda = 0.4; a_0 = 3.33; a = 5; \varepsilon_0 = 30; \varepsilon_k = 15; q_k = 1; \gamma^0(t) = \gamma^k(t) = 0.3; \theta_0 = \theta_k = 1; \delta_0 = 1$).

Last but not least, Table 3.7 shows that the total probability of a systemic event, $P(SR)$, increases from 0.0463 to 0.0559 as the risk sensitivity degree of the subgroup increases from 0 to 3. We can see from the values of $P(SR|TkS)$ that the success of a more risk-averse subgroup might not necessarily benefit the stability of the interbank market. Risk-averse minor banks are less likely to take risks and tend to adopt stricter lending policies. Therefore, being risk-averse, minor banks in the interbank market could lead to less liquidity in the system, increasing systemic risk. However, an increase in δ_k reduces the extremity of systemic default events, as shown by the slight decrease in the conditional probability, $P(SR|TkD)$.

Additionally, we plot the loss distribution of minor banks in Fig. 3.19, illustrating the frequency distribution of different numbers of defaults. We can see that all curves have a downward trend regardless of the subgroup's degree of risk sensitivity, with the

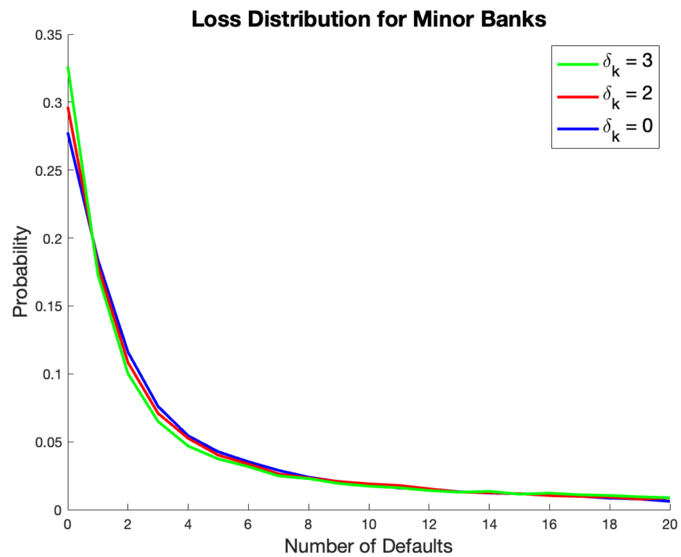


Figure 3.19: Loss distribution for minor banks for various values of δ_k.

probability of all 20 minor banks defaulting close to 0. The probability of no banks defaulting is the highest when $\delta_k = 3$, compared to when the subgroup of minor banks is risk-neutral. The loss distribution for when $\delta_k = 3$ then falls below those for when $\delta_k = 0$ and $\delta_k = 2$, indicating that it is less likely for minor banks in the subgroup to default when the entire subgroup behaves in a more risk-averse fashion. However, the loss distribution of a risk-sensitive subgroup becomes higher when more than 13 minor banks default. The probability of all banks going into default is lowest when the subgroup is risk-neutral. This indicates that even though having a risk-averse subgroup of minor banks increases the stability of banks, it also increases the probability of extreme events. When banks are in distress in a risk-sensitive setting, being cautious with their trading policies might lead to less liquidity in the interbank market, resulting in more banks' monetary reserves reaching the default threshold.

We then analyze the case when two subgroups exist in the interbank market. We again assume there are 20 minor banks, 14 of which belong to subgroup one and 6 of which belong to subgroup two. We first study the risk sensitivity of subgroup one with 14 minor banks, which accounts for 70% of all the minor banks in the market. We also assume the major bank and minor banks under subgroup two all have risk sensitivity degrees of 1, and

δ_1	$P(MD)$	$P(MD\|T1S)$	$P(MD\|T1D)$	$P(SR)$	$P(SR\|T1S)$	$P(SR\|T1D)$
0	0.2614	0.2472	1	0.0461	0.0279	0.9947
2	0.2646	0.2477	1	0.0500	0.0284	0.9933
3	0.2670	0.2480	1	0.0537	0.0293	0.9901

Table 3.8: Default probabilities of the major bank and systemic risk for different values of δ_1 ($\lambda = 0.4; a_0 = 3.33; a = 5; \varepsilon_0 = 30; \varepsilon_k = 15; \gamma^0(t) = \gamma^k(t) = 0.3; \theta_0 = \theta_k = 1; \delta_0 = \delta_2 = 1$).

the major bank's size is 0.4. As presented in Table 3.8 and Table 3.9, we can see an overall increasing trend in the default probabilities as type one minor banks become more risk-sensitive, except for $P(SR|T1D)$. The total default probability of the major bank increases from 0.2614 to 0.2670 as δ_1 increases. Furthermore, the probability of the major bank defaulting conditioned on subgroup one defaulting remains at 1 across all values of δ_1, indicating that the crash of the more significant subgroup almost ensures the failure of the major bank. Systemic risk also rises as δ_1 increases from 0 to 3. We see a slight decreasing trend in the conditional probability of systemic risk when subgroup one defaults, from 0.9947 to 0.9901, suggesting that a large risk-sensitive subgroup reduces the severity of systemic events. However, these high values indicate that the entire system would collapse with almost certainty if subgroup one defaults, regardless of the risk sensitivity.

δ_1	$P(i_1D)$	$P(i_2D)$	$P(i_2D\|T1S)$	$P(i_2D\|T1D)$
0	0.1927	0.1936	0.1791	0.9469
2	0.1954	0.1988	0.1813	0.9615
3	0.1976	0.2029	0.1831	0.9621

Table 3.9: Default probabilities representative minor banks in different subgroups for different values of δ_1 ($\lambda = 0.4; a_0 = 3.33; a = 5; \varepsilon_0 = 30; \varepsilon_k = 15; \gamma^0(t) = \gamma^k(t) = 0.3; \theta_0 = \theta_k = 1; \delta_0 = \delta_2 = 1$).

The default probabilities of representative minor banks from both subgroups show an increasing trend as subgroup one minor banks become more risk-averse, where the probability of default of a type one minor bank is overall lower than that of a type two minor bank for the same degree of risk sensitivity. The values of $P(i_2D|T1D)$ remain above 0.9, indicating a high probability that a minor bank in the other subgroup would default if subgroup one collapses. An increase in δ_1 indicates that minor banks in subgroup one are more averse to risks and are less likely to issue loans as they want to maintain their liquidity in case of emergencies. Since there are 14 banks in subgroup one for our model, their being risk-averse makes it much harder for the major bank and other minor banks in subgroup two to conduct monetary transactions, explaining the increase in the default probabilities of banks.

The above observations are important indicators of the connectivity of the interbank market structure. The crash of an entire subgroup indicates that most minor banks in that subgroup go to default. This sequentially affects the stability of the major bank as many small banks fail to pay their debt. These events further trigger the defaults of more minor banks in other subgroups, ultimately leading to the entire system's collapse.

δ_2	$P(MD)$	$P(MD\|T2S)$	$P(MD\|T2D)$	$P(SR)$	$P(SR\|T2S)$	$P(SR\|T2D)$
0	0.2639	0.2421	0.9991	0.0473	0.0218	0.9054
2	0.2646	0.2418	0.9992	0.0500	0.0214	0.9033
3	0.2650	0.2390	1	0.0505	0.0203	0.9034

Table 3.10: Default probabilities of the major bank and systemic risk for different values of δ_2 ($\lambda = 0.4; a_0 = 3.33; a = 5; \varepsilon_0 = 30; \varepsilon_k = 15; \gamma^0(t) = \gamma^k(t) = 0.3; \theta_0 = \theta_k = 1; \delta_0 = \delta_1 = 1$).

We now study the impact of the degree of risk sensitivity of subgroup two, which is smaller than subgroup one (30% vs. 70%) and contains 6 minor banks in our setting. As we can see in Table 3.10 and Table 3.11, there is again a general increasing pattern in the default probabilities as minor banks in subgroup two become more risk-averse, except for a decrease in the values of $P(MD|T2S)$, $P(SR|T2S)$ and the default probability of a subgroup two minor bank. Even though a more risk-averse subgroup two leads to tighter lending policies and more reserved investment strategies by the minor banks in the group, the major bank can rely on other minor banks in subgroup one to raise funds since subgroup two only takes up a small portion of the market. Therefore, the success of subgroup two improves the stability of major bank as it becomes more risk-sensitive since the minor banks trade less recklessly and enhance their financial health. Although increasing, the probabilities of major bank defaults conditioned on subgroup two defaulting are lower than the values in Table 3.8 as δ_2 increases, ranging from 0.9991 to 1. The default probabilities of a generic type one minor bank conditioned on type two minor banks defaulting are also lower than those under the opposite scenario.

δ_2	$P(i_2D)$	$P(i_1D)$	$P(i_1D\|T2S)$	$P(i_1D\|T2D)$
0	0.1971	0.1924	0.1710	0.9158
2	0.1943	0.1937	0.1712	0.9165
3	0.1926	0.1984	0.1730	0.9178

Table 3.11: Default probabilities representative minor banks in different subgroups for different values of δ_2 ($\lambda = 0.4; a_0 = 3.33; a = 5; \varepsilon_0 = 30; \varepsilon_k = 15; \gamma^0(t) = \gamma^k(t) = 0.3; \theta_0 = \theta_k = 1; \delta_0 = \delta_1 = 1$).

Furthermore, an increase in the degree of risk sensitivity of the smaller subgroup positively impacts the conditional probabilities of the market state falling below the default threshold. By comparing the results in Table 3.8, Table 3.9 and Table 3.10, Table 3.11, we can see that while the risk sensitivity degrees of both subgroups influence the default probabilities of the major bank, the other subgroup of minor banks and the overall system, the impact of the larger subgroup one is generally more pronounced. This is likely a result of its more significant market share, leading to a more substantial influence on the interbank market dynamics and systemic risk.

δ_1	$P(MD)$	$P(MD\|T1S)$	$P(MD\|T1D)$	$P(SR)$	$P(SR\|T1S)$	$P(SR\|T1D)$
0	0.2648	0.2518	1	0.0410	0.0260	1
2	0.2700	0.2533	1	0.0496	0.0303	0.9919
3	0.2743	0.2562	1	0.0540	0.0318	0.9892

Table 3.12: Default probabilities of the major bank and systemic risk for different values of δ_1 ($\lambda = 0.4; a_0 = 3.33; a = 5; \varepsilon_0 = 30; \varepsilon_1 = 15; \varepsilon_2 = 10; q_1 = 0.8; q_2 = 1; \gamma^0(t) = \gamma^k(t) = 0.3; \theta_0 = \theta_k = 1; \delta_0 = 1; \delta_2 = 3$).

δ_1	$P(i_1D)$	$P(i_2D)$	$P(i_2D\|T1S)$	$P(i_2D\|T1D)$
0	0.1914	0.1858	0.1739	0.9481
2	0.1980	0.2096	0.1909	0.9512
3	0.2000	0.2122	0.1979	0.9677

Table 3.13: Default probabilities representative minor banks in different subgroups for different values of δ_1 ($\lambda = 0.4; a_0 = 3.33; a = 5; \varepsilon_0 = 30; \varepsilon_1 = 15; \varepsilon_2 = 10; q_1 = 0.8; q_2 = 1; \gamma^0(t) = \gamma^k(t) = 0.3; \theta_0 = \theta_k = 1; \delta_0 = 1; \delta_2 = 3$).

Additionally, we conduct simulations on the interbank lending market when the two subgroups have different parameters, such as different values for running cost and trading incentive coefficients, ε_k and q_k. In our setting, a higher value of ε_k and a lower value of q_k indicate that the cost of borrowing/lending with the central bank is higher for a minor bank. Therefore, it is more difficult for type one minor banks to trade with the central bank than type two minor banks with the parameters in Table 3.12 and Table 3.13. Comparing to the default probabilities in Table 3.8 and Table 3.9, the values are generally higher in Table 3.12 and Table 3.13 across different degrees of risk sensitivity of subgroup one minor banks. However, we observe the same trends in these values where there is an increasing pattern in the default probabilities across the system as type one minor banks become more risk-averse, except for the values of $P(SR|T1D)$. These simulation results indicate that it is not necessarily beneficial when most banks in the interbank system become more risk-averse, especially when conducting transactions with the central bank is harder. Carmona et al. (2015) conclude that lending/borrowing activities in an interbank market create stability. However, banks being more risk-averse means they will be more cautious about borrowing/lending with other banks, leading to fewer trading opportunities and a less robust interbank market, as indicated by the results presented in this section.

Conclusion

In this thesis, we present a comprehensive study of interbank transactions in a major-minor mean-field game (MFG) framework. The primary objective is to understand how risk sensitivity and the presence of a large bank influence the interbank market, particularly in terms of banks' default probabilities and systemic risk.

Our study begins by developing models for major and minor banks, incorporating the dynamics of log-monetary reserves and cost functionals in a risk-sensitive setting. We then derive optimal transaction strategies and establish Nash equilibria for these interactions in an infinite-population setting, leveraging the convex analysis and variational analysis techniques. Through extensive numerical experiments, we analyze various scenarios in a finite-population setting to assess the impact of different parameters on the interbank market. We observe that the presence of a major bank whose size is less than or equal to half of the market size decreases the default probabilities of itself, a representative minor bank, and the interbank system as it becomes more risk-averse. When a major bank is distressed, its being risk-sensitive reduces the severity of minor banks' defaults and the systemic event. However, when the major bank takes up more than half of the market, the benefits of it being risk averse are compromised by its large size. We also conclude that an increase in the percentage of the market state that the major bank follows has a flocking effect on the default probabilities conditioned on whether the major bank defaults or not, whereas an increase in the major bank's growth rate decreases the likelihood of defaults across the interbank market.

Furthermore, we conduct simulations to analyze the effects of the risk sensitivity degree of a subgroup of minor banks. When there is only one subgroup of minor banks in the market whose size is larger than the major bank, its being risk-averse increases the stability of banks and the market when it survives. However, the total probabilities of default across the system increase as the subgroup of minor banks becomes more risk-averse due to the significant rise in the default probabilities conditioned on the subgroup defaulting. Similar increasing trends can be observed in the default probabilities when there are two subgroups of minor banks. Since higher degrees of risk sensitivity of banks generally lead to increased caution in borrowing and lending activities, a subgroup of minor banks being risk-averse can lead to fewer transactions and less liquidity in the interbank market. Therefore, it is more likely for the monetary reserves of banks and the market state to go below the default threshold due to the interconnectedness of the system.

Overall, this thesis contributes to a deeper understanding of the interactions among major and minor banks in a risk-sensitive interbank market. The findings emphasize the critical role of risk sensitivity, the major bank, and a large subgroup of minor banks in shaping market dynamics and provide valuable insights for policymakers aiming to enhance financial stability. However, a larger amount of simulations and real-world data can be employed for the numerical experiments to better calibrate the parameters for more perceptive results. Future research can extend the models by exploring the effects of additional factors such as regulatory changes and technological advancements in the interbank market.

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