HEC MONTRÉAL

Debt Renegotiations and the Pricing of Sovereign Debt

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Résumé

Ce mémoire étudie la tarification de la dette souveraine en intégrant les leviers stratégiques disponibles pour le souverain, à savoir la restructuration de la dette ou le défaut de paiement. Les marchés financiers prennent-ils en compte, dans la tarification de la dette souveraine, l'option du souverain de restructurer ou de faire défaut sur sa dette? Nous constatons que les obligations souveraines sont des créances contingentes sur la « santé » financière des souverains, mesurée par des variables macroéconomiques, auxquelles les termes de renégociation de la dette et le moment du défaut anticipé par les créanciers se sont révélés sensibles. L'analyse corrobore également l'observation récente selon laquelle les approches par menus, où les souverains bénéficient d'un allègement de la dette sous la forme d'un mélange de rééchelonnement et d'annulation de la dette, sont la norme plutôt que l'exception.

Mots clés: créances contingentes, dette souveraine, processus de diffusion

Abstract

This thesis studies sovereign debt pricing by incorporating strategic leverages available to the sovereign, namely debt restructuration or default. Do the financial markets factor in the sovereign debt pricing, the sovereign's option of future debt restructuration or default? We find that sovereign bonds are contingent claims on the sovereign's financial "health", measured by macroeconomic variables, to which debt renegotiation terms and timing of default anticipated by the creditors have proven sensitive. The analysis also corroborates the late observation that menu approaches, where the sovereigns are granted debt relief in form of a mix of debt rescheduling and debt write-off, are the norm rather than exception.

Keywords: contingent claims, sovereign debt, diffusion process

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1 Introduction

One of the main financing tools for sovereigns is debt. This sovereign debt can be from other governments, commercial financial institutions and international financial institutions such as the World Bank, the International Monetary Fund or African Development Bank.

When a sovereign has trouble fulfilling its external obligations, unlike for private creditors, there is no legal procedure for bankruptcy. Then, the struggling country needs to enter negotiations with its creditors to be granted debt relief. When an agreement is reached, the sovereign can usually benefit from three types of debt restructuration: debt rescheduling, debt reduction and a package deal; the "package deal" being the mix of debt rescheduling and debt reduction.

The former occurs when creditors grant relief to the sovereign by changing the debt maturity. The sovereign is given a grace period of T units of time: during this T units of time, the sovereign benefits of a debt service suspension. In case of debt reduction, the initial debt service of the sovereign is reduced by a fraction β, either by reducing the principal amount or the interest rate or both.

When a debt restructuration agreement cannot be reached then, the sovereign opts for an exit plan. This alternative results in the sovereign incurring sanctions, which can affect its economical outlooks. In fact, the sovereign can be excluded from the loans or exports market. It will likely suffer from a reputational perspective as well.

In this thesis, we leverage on François (2006) debt valuation model and aim to infer from the sovereign debt yield spreads, measured by the J.P. Morgan EMBI+[1] Index, observed in the financial markets, the debt renegotiation terms embedded in the sovereign's debt instrument. The Paris Club additionally provides a framework defining common debt renegotiation agreements or terms.

The Paris Club is an informal group of official creditors whose role is to find coordinated and sustainable solutions to the payment difficulties experienced by debtor countries. As debtor countries undertake reforms to stabilize and restore their macroeconomic and financial situation, Paris Club creditors provide an appropriate debt treatment, both to the "Official Development Assistance" ("ODA") and non-ODA portions of the debt stock. The ODA portion of the debt (or ODA credits) is defined by the OECD as credits with a low interest rate and aimed at development. Paris Club creditors provide debt treatments to debtor countries in the form of rescheduling, which is debt relief by postponement or, in

[1] JP Morgan EMBI+ Index stands for J.P. Morgan Emerging Markets Bond Index Plus. It tracks liquid, US Dollar emerging market fixed and floating-rate debt instruments issued by sovereign entities only.

the case of concessional rescheduling, reduction in debt service obligations during a defined period or as of a set date.

Since its inception in 1956, the Paris Club, has reached 478 agreements with 102 different debtor countries, for a total debt treatment of $614 billions.[2] Debtor countries can benefit from standard renegotiation terms, defined by the Paris Club: Classic, Houston, Naples, or Cologne terms. The breakdown of the agreements is as follows: 174 "Classic Terms", 107 "Naples Terms", 35 "Houston Terms" and 48 "Cologne Terms" for the standard terms of treatment, along with 36 "Heavily Indebted Poor Countries (HIPC) Initiative" and 78 Ad Hoc renegotiations. 61 countries benefited from "Classic Terms", 21 benefited from "Houston Terms", 33 countries for "Naples Terms" and 37 countries in the case of "Cologne Terms".

The **Classic Terms** consist of a rescheduling of the ODA and non-ODA credits at an appropriate interest rate, defined by the Paris Club. In the case of the **Houston Terms**, non-ODA repayment periods are lengthened to or beyond 15 years (with 2-3 years grace) and ODA repayment periods are lengthened up to 20 years with a maximum of 10-year grace, along with a rescheduling of the ODA credits at a concessional rate and a potential debt swap. In the case of **Naples Terms** (previously London or Toronto terms), the non-ODA portion of the debt is cancelled to a 67% level, and the outstanding amount being rescheduled for either 23 years with a 6-year grace or 33 years; the ODA credits are rescheduled at a concessional rate for 40 years with 16-year grace. In addition, both the ODA and non-ODA credits might be subject to debt swaps. Regarding the **Cologne Terms** (previously Lyon terms), the non-ODA credits are cancelled up to 90%, and the outstanding amount being rescheduled for 23 years with a 6-year grace meanwhile the ODA-credits are rescheduled for 40 years with a 16-year grace, with a potential debt swap for both portions of the debt. Table 1 summarizes the renegotiation terms.

Table 1. Paris Club renegotiation deals

The table illustrates the standard renegotiation terms of the Paris Club, and the corresponding grace period and cancellation level granted to the debtor countries, for both portions of their debt: Official Development Assistance (ODA) credits, and non-ODA credits.

Standard Renegotiation Terms	**Houston**	**Naples**	**Cologne**	**Classic**
	Grace Period			
Non-ODA Credits	2 or 3 years	Up to 6 years	6 years	Case-by-Case
ODA Credits	Up to 10 years	16 years	16 years	
	Cancellation Level			
Non-ODA Credits		Up to 67%	Up to 90%	
ODA Credits				

[2] Statistics as of February 4, 2024.

Furthermore, debtor countries can benefit from some other case-by-case agreements, either HIPC initiative for HIPC countries or Ad Hoc renegotiations for non-HIPC countries. These renegotiations often result in a mix of debt cancellation up to a certain percentage and debt rescheduling.

Without mentioning any causality, Yeyati and Panizza (2008) highlighted the empirical evidence that prior to the default time, the slope of GDP's growth rate tends to downward. After default, the growth rate continues to decrease but at a slower pace, when using annual GDP data. However, after the official date of default, this slope tends to follow an upward trend when using quarterly GDP data. Hence, "*most of the financial distress that precedes the default decision may be due to its anticipation*". It is then logical to price sovereign's bonds spread by incorporating the eventuality of a debt restructuration with its two outcomes: sovereign and lender reach a mutually accepted renegotiation deal or not.
With a study case of Argentina, Yue (2010) studies the interaction between bond yield spreads and debt renegotiations, in a framework where the debt renegotiation process is endogenous and takes the form of a generalized Nash bargaining game. The author finds that that the pricing of sovereign bonds encompasses the risk of default as well as the risk of debt restructuring.

Yet, to the best of our knowledge, no analytical unified framework has been developed to price sovereign debt across different set of debt renegotiation terms: pure debt reduction, pure debt rescheduling or package deal. François (2006) pioneers such an analytical formulation. Similar to the strategic debt service model developed by Mella-Barral and Perraudin (1997) on corporate debt, his debt valuation model prices the sovereign debt in continuous time, where the debt issued by the sovereign is perpetual. As is customary in the literature, in continuous time, the sovereign decides dynamically when to initiate its debt renegotiation, in order to alleviate a pressure that a debt service might put on its economy. The sovereign could also decide when to "default" or stop servicing its debt.

The remainder of the thesis is organized as follows. Section 2 presents the debt valuation model and the debt renegotiation process assumed. Section 3 presents the data used to estimate the sovereign's debt renegotiation terms, along with a sensitivity analysis of the debt valuation model. Section 4 presents the methodology pertaining to the estimation of sovereigns' debt renegotiation terms and discusses the results. Sections 5 and 6 offer concluding remarks, in English and French respectively.

2 The Sovereign Debt Valuation Model

In Subsection 2.1, we introduce the pricing of the sovereign's debt for arbitrary decision thresholds. Subsection 2.2 presents the sovereign's equity calculation. Subsection 2.3 concludes with the determination of the optimal endogenous thresholds for the sovereign's actions: "renegotiate" and "default".

2.1 Debt Price

The debt pricing is done following the model of François (2006). In the spirit of Hayri (2000) or Gibson and Sundaresan (1999), François (2006) considers a sovereign whose rate of time preference ρ is greater than the world interest rate r.

The sovereign revenue flows x follow a geometric Brownian motion:

$$\frac{dx_t}{x_t} = \mu dt + \sigma dz_t,$$

where μ and σ are constant, and $(z_t)_{t\geq 0}$ is a standard Brownian.

The sovereign raises external funds from a debt contract that implies a continuous and perpetual debt service s. The sovereign services the coupon s in full as long as the underlying variable state x remains above a certain threshold x_b. At a time $t = \tau_b$, where the threshold x_b is reached from above for the first time, the sovereign initiates a renegotiation process with its creditors. We assume that the sovereign has all bargaining power, and the renegotiation threshold x_b is determined endogenously by the sovereign such that it maximizes its equity claim value. However, the renegotiations are costly. These costs are a priori incurred by the sovereign and its creditors. The costs can be either reputational or administrative. Indeed, the sovereign access to international capital markets could be jeopardized by a bad reputation, as well as it could face some diplomatic or legal expenses. The erstwhile charges might apply to the creditors as well. Moreover, the creditors might also face a reputational cost in the sense that they would be pictured as renegotiation-ready (or malleable) by some other debtors, that could ask for concessions on their debt later as well. However, following Moraux and Silaghi (2014), we assume that, in the spirit of Mella-Barral and Perraudin (1997) and Garleanu and Zwiebel (2009), the sovereign would single-handedly pay for the costs, as the sovereign has all bargaining power and is knowledgeable of the bilateral benefits of the renegotiations. Further, we impose that the renegotiations could only occur once in the lifetime of the debt. It is in the best interest of both parties, the sovereign and the creditors, to renegotiate the debt service; otherwise, the sovereign might default immediately, and the creditors could not recover any amount of the debt stock.

In the spirit of the Paris Club, three outcomes for the renegotiations are made available to the sovereign and creditors: debt reduction, debt rescheduling and a package deal.

The debt reduction entails a permanent debt service reduction. A fraction β of the debt is written off; and the sovereign pays a debt service of $(1-\beta)s$ from the renegotiations onwards.

In the case of a debt rescheduling, the sovereign benefits from a grace period of T, during which it will pay no debt service. The payment will resume at the end of the debt moratorium.

Finally, the package deal consists of a mix of debt reduction and debt rescheduling. Upon agreement for a package deal, the sovereign will be granted a debt moratorium T, during which the debt service will be equal to zero. At the end of the debt moratorium, the debt service will be $(1-\beta)s$ from the renegotiations onwards.

We capture the costs of the renegotiations and default as a reduction in the sovereign's revenue growth rate for a short period of time.
Trebesch and Zabel (2017) shed light on the costs of a sovereign default, with a distinction between "hard" default or "soft" default. "Hard" default, referred to as default or exit in this thesis, occurs when the sovereign is unwilling to continue paying its debt service. On the other hand, "soft" default, referred to as renegotiation in this thesis, corresponds to a case "*in which the government opted for a consensual stance towards creditors*". The authors stress the effect of both types of default on the sovereign's revenues growth. The "hard" default results in decrease in the drift of the sovereign's revenue flows over the two years consecutive to the default, meanwhile the decrease is smaller and can be captured in the first year in the case of "soft" default. Zymek (2012) unveils the impact of "soft" default on exports and find its results to be similar to two prior papers: Rose (2005) and Martinez and Sandleris (2011). Zymek (2012) finds debt renegotiations to result in 0.8 to 1.5 years of exports losses. Borensztein and Panizza (2008, 2010) provide evidence that both "soft" and "hard" defaults have a negative effect on the sovereign's trade, but this effect is short lived and can be significantly captured within the first two years of the renegotiation episode.
Hence, we assume the sovereign to lose one year of exports in the event of a debt renegotiation, while an exit results in a loss of two years of exports.
Therefore, denoting α the sovereign's trade, the drift m_1 of the sovereign's revenue flows after renegotiations is such that

$$\frac{x}{\rho - m_1} = \int_0^{+\infty} x e^{-(\rho-\mu)t} dt - \int_0^1 \alpha x e^{-(\rho-\mu)t} dt , \tag{2.1}$$

which yields

$$m_1 = \rho - \frac{\rho - \mu}{1 - \alpha[1 + e^{-(\rho-\mu)}]} . \tag{2.2}$$

Following the renegotiations, the sovereign holds an option to exit (or default), which means that the sovereign could unilaterally choose to put an end to servicing the debt and incurring additional economic sanctions, along with an access denial to the international capital and trade markets. Let's denote x_e the exit threshold, and τ_e the first time the underlying state variable is at or below the latter threshold after the debt moratorium, if any. If the sovereign was living is autarky or denied access to international borrowing, drift of the sovereign's revenue flows after exit would be equal to $m_2 \leq m_1$. Consequently, in the case the sovereign chooses to exercise its exit option, it incurs sanctions (i.e., loss of two years of exports) that reduce the growth rate of its revenue flows from m_1 to m_2. Therefore, the drift m_2 of the sovereign's revenue flows after exit is such that

$$\frac{x}{\rho - m_2} = \int_0^{+\infty} x e^{-(\rho - m_1)t} dt - \int_0^2 \alpha x e^{-(\rho - m_1)t} dt\,, \tag{2.3}$$

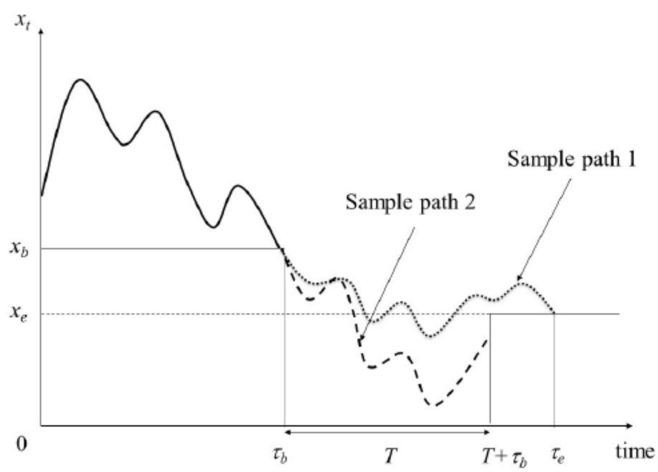
which yields

$$m_2 = \rho - \frac{\rho - m_1}{1 - \alpha[1 + e^{-2(\rho - m_1)}]}. \tag{2.4}$$

Under this setup, following François (2006), the initial price of sovereign debt subject to a package restructuring is the present value of the debt service from the time of issuance to the time of exit. Figure 1 provides a graphic representation of the possible scenarios regarding sovereign debt renegotiations. As can be seen from Figure 1, sovereign debt after renegotiations (after date τ_b) is akin to a *partial* barrier option.

Figure 1. Possible scenarios for a package restructuring

The figure plots the two possible outcomes for a package restructuring. At date τ_b, the sovereign is granted a moratorium on debt payments for *T* units of time. Sample path 1 represents a sovereign who resumes paying the debt service after the end of the moratorium *T* and who eventually exits at date τ_e. Sample path 2 represents a sovereign who immediately exits at the end of the moratorium.



For arbitrary thresholds x_b and x_e (with $x_b > x_e$), the debt value is given by:

$$Debt(x, x_b, x_e, s) = s\mathbb{E}\left(\int_0^{\tau_b} e^{-rt} dt\right)$$

$$+(1-\beta)s\mathbb{E}(e^{-r\tau_b})\mathbb{E}_{x_b}\left(1_{x_T > x_e}\int_T^{\tau_e} e^{-rt} dt\right), \tag{2.5}$$

where $\mathbb{E}_{x_b}(.)$ denotes the expectation operator conditional on $\{x_t\}_{t\geq 0}$ starting at x_b.
The solution in closed form (see Appendix A, for the detailed procedure) of (2.5) is:

$$Debt(x, x_b, x_e, s) = \frac{s}{r}\left[1 - \left(\frac{x_b}{x}\right)^{\lambda_1}\right]$$

$$+(1-\beta)\frac{s}{r}\left(\frac{x_b}{x}\right)^{\lambda_1}\left[e^{-rT}\Phi(z_1) - \left(\frac{x_e}{x_b}\right)^{\lambda_2}\Phi(z_2)\right], \tag{2.6}$$

with

$$\lambda_1 = \frac{1}{\sigma}\left(\frac{\mu}{\sigma} - \frac{\sigma}{2} + \sqrt{\left(\frac{\mu}{\sigma} - \frac{\sigma}{2}\right)^2 + 2r}\right),$$

$$\lambda_2 = \frac{1}{\sigma}\left(\frac{m_1}{\sigma} - \frac{\sigma}{2} + \sqrt{\left(\frac{m_1}{\sigma} - \frac{\sigma}{2}\right)^2 + 2r}\right),$$

$$z_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\frac{x_b}{x_e} + \left(m_1 - \frac{\sigma^2}{2}\right)T\right],$$

$$z_2 = \frac{1}{\sigma\sqrt{T}}\left[\ln\frac{x_b}{x_e} - \sigma\left(\sqrt{\left(\frac{m_1}{\sigma} - \frac{\sigma}{2}\right)^2 + 2r}\right)T\right],$$

where $\Phi(.)$ stands for the standard normal cumulative distribution function.

The two polar cases of the package restructuring, $\beta = 0$ or $T = 0$ yield the debt rescheduling and debt reduction deals, respectively.
Setting $T = 0$, we have that $\Phi(z_1) = 1$ and $\Phi(z_2) = 1$, and we obtain the pure debt reduction case, studied in particular by Cohen (1993) and Hayri (2000).

$$Debt^{Red}(x, x_b, x_e) = \frac{s}{r}\left[1 - \left(\frac{x_b}{x}\right)^{\lambda}\right] + (1-\beta)\frac{s}{r}\left(\frac{x_b}{x}\right)^{\lambda}\left[1 - \left(\frac{x_e}{x_b}\right)^{\lambda}\right]. \tag{2.7}$$

On the other hand, setting $\beta = 0$ in equation (2.6), we get a pricing formula for the pure debt rescheduling case.

2.2 Sovereign Equity

The sovereign's net equity is given by the difference between the sovereign's wealth and debt value.

Before the renegotiations, the sovereign's wealth is the expected present value of the sovereign future revenue flows, considering that the sovereign holds two options: renegotiation and exit options. The sovereign's wealth is given by

$$W(x, x_b, x_e) = \mathbb{E}\left(\int_0^{\tau_b} e^{-\rho t} {x_t}^{(\mu)} dt\right) + \mathbb{E}\left(\int_{\tau_b}^{T} e^{-\rho t} {x_t}^{(m_1)} dt\right)$$
$$+\mathbb{E}\left(1_{x_T > x_e} \int_T^{\tau_e} e^{-\rho t} {x_t}^{(m_1)} dt\right) + \mathbb{E}\left(1_{x_T > x_e} \int_{\tau_e}^{\infty} e^{-\rho t} {x_t}^{(m_2)} dt\right)$$
$$+\mathbb{E}\left(1_{x_T \leq x_e} \int_T^{\infty} e^{-\rho t} {x_t}^{(m_2)} dt\right), \qquad (2.8)$$

with ${x_t}^{(\mu)}, {x_t}^{(m_1)}, {x_t}^{(m_2)}$ representing respectively the sovereign revenue flows with drifts before renegotiations, after renegotiations and after exit.

The first term of the equation (2.8) corresponds to the expected present value of the sovereign perpetual revenue flows x growing at rate μ, as long as the sovereign does not initiate a debt renegotiation process. The second term represents the expected present value of the sovereign revenue flows once the renegotiation threshold is hit. In this case, the growth rate becomes m_1, until the end of the debt moratorium. In the event that the sovereign revenue flows x is above the exit threshold x_e after the debt moratorium, its revenue flows continue growing at rate m_1. This is captured by the third term which is the expected present value of the sovereign revenue flows from the end of the moratorium until the state variable x reaches the exit threshold x_e later. The fourth term is the expected present value of the sovereign revenue flows once the exit threshold x_e is hit, in the erstwhile scenario that the state variable x is above the x_e at the end of the debt moratorium but reaches it sometime later. The sovereign "defaults", and its revenue flows grow at rate m_2 henceforward. Finally, the fifth term denotes the expected present value of the sovereign revenue flows in the event that the state variable x is equal or below the exit threshold x_e at the end of the debt moratorium, which implies that the sovereign "defaults", and its revenue flows' drift is m_2 onwards.

Then,

$$W(x, x_b, x_e) = \frac{x}{\rho - \mu} + \left(\frac{x_b}{x}\right)^{\kappa_1} \left(\frac{x_b}{\rho - m_1} - \frac{x_b}{\rho - \mu}\right)$$

$$+ \left(\frac{x_b}{x}\right)^{\kappa_1} \left(\frac{x_e}{x_b}\right)^{\kappa_2} \left(\frac{x_e}{\rho - m_2} - \frac{x_e}{\rho - m_1}\right), \quad (2.9)$$

where

$$\kappa_1 = \frac{1}{\sigma}\left(\frac{\mu}{\sigma} - \frac{\sigma}{2} + \sqrt{\left(\frac{\mu}{\sigma} - \frac{\sigma}{2}\right)^2 + 2\rho}\right),$$

and

$$\kappa_2 = \frac{1}{\sigma}\left(\frac{m_1}{\sigma} - \frac{\sigma}{2} + \sqrt{\left(\frac{m_1}{\sigma} - \frac{\sigma}{2}\right)^2 + 2\rho}\right).$$

Therefore, before renegotiations, the sovereign's net equity is given by

$$n(x, x_b, x_e, s) = W(x, x_b, x_e) - Debt(x, x_b, x_e, s). \quad (2.10)$$

After renegotiations, the sovereign's wealth is the expected present value of its future revenue flows, knowing that the sovereign has already exercised its renegotiation option and now still only holds an exit option. The sovereign wealth after renegotiations is given by

$$W(x, x_e) = \mathbb{E}\left(\int_0^{\tau_e} e^{-\rho t} x_t^{(m_1)} dt\right) + \mathbb{E}\left(\int_{\tau_e}^{\infty} e^{-\rho t} x_t^{(m_2)} dt\right), \quad (2.11)$$

which yields

$$W(x, x_e) = \frac{x}{\rho - m_1} + \left(\frac{x_e}{x}\right)^{\kappa_2} \left(\frac{x_e}{\rho - m_2} - \frac{x_e}{\rho - m_1}\right). \quad (2.12)$$

The first term represents the expected present value of the sovereign future revenue flows from the renegotiation until the exit threshold x_e is hit. The second term corresponds to the expected present value of the sovereign future revenue flows once the exit threshold x_e is hit.

Therefore, the sovereign equity after renegotiations is given by

$$n(x, x_e, (1-\beta)s) = W(x, x_e) - d(x, x_e, (1-\beta)s), \tag{2.13}$$

where $d(x, x_e, (1-\beta)s)$ represents the value of the non-renegotiable debt with continuous debt service $(1-\beta)s$.

The formula for the non-renegotiable debt $d(x, x_e, (1-\beta)s)$ is given by

$$d(x, x_e, (1-\beta)s) = \frac{(1-\beta)s}{r}\left(1 - \left(\frac{x_e}{x}\right)^{\lambda_2}\right). \tag{2.14}$$

2.3 Optimal Decision Thresholds

As outlined in the subsection 2.1, after its debt issuance, the sovereign holds two options: a renegotiation option, and an exit option. The renegotiation option offers the sovereign the right to initiate a renegotiation process and benefits from creditors' concession on its debt service, meanwhile the exit option offers the sovereign the right to unilaterally "default" and stop indefinitely the payment of its debt service.
The renegotiation option and exit option can be exercised subsequently by the sovereign at any time after the debt issuance. These two options can then be viewed as American-style options on the sovereign revenue flows. Conscious of the renegotiation costs and default sanctions, the sovereign should optimally exercise the renegotiation option, and subsequently the exit time at times that maximizes its contingent claim. Dumas (1991) argues that economic agents, acting dynamically in a stochastic environment, under transaction costs or any sort of friction, trigger action when the state of the economic system reaches the boundary of the "*region of no action*". The determination of this region of no action, or equivalently the region of action, is based on smooth pasting or high contact conditions. Smooth pasting conditions are first-order conditions that ensure that the marginal value from continuing to hold an option (i.e., continuation value) matches the marginal value for exercising the option (i.e., exercise value). In Dumas (1991) words, the "*marginal utility should take the same value before and after the action has been taken*".

Shackleton and Sodal (2005) also show an application of the smooth pasting conditions in the context of an optimal early exercise. The authors demonstrate that the optimal early call exercise is associated to a smooth pasting condition, which equalizes the rate of return of the option with the rate of return of the levered payoff. A boundary condition implying the value matching of the option price to its payoff should also be satisfied.

Applying that framework, the sovereign optimal renegotiation and exit thresholds are determined from smooth pasting.

On the one hand, the optimal renegotiation threshold x_b^* is the level at which the "*marginal utility*" or marginal equity of the sovereign is identical before and after the renegotiations. Therefore, x_b^* is the solution of the following smooth pasting condition

$$\left.\frac{\partial n(x, x_b, x_e, s)}{\partial x}\right|_{x=xb} = \frac{\partial n(x_b, x_b, x_e, s)}{\partial x_b}, \qquad (2.15)$$

and is solved numerically. The left-hand term of the smooth pasting condition represents the marginal continuation value meanwhile the right-hand term denotes the marginal exercise value of the renegotiation option.

On the other hand, the optimal exit threshold x_e^* is obtained by matching the present value $CV(x, x_e)$ of future revenues net of the debt service considering the impact of a future exit (i.e., continuation value of the exit option) with the present value $EV(x_e)$ of future revenues after the sovereign is sanctioned for exercising its exit option (i.e., exercise value of the exit option). Therefore, x_e^* is the solution of the following smooth pasting

$$\left.\frac{\partial CV(x, x_e)}{\partial x}\right|_{x=xe} = \frac{\partial EV(x_e)}{\partial x_e}, \qquad (2.16)$$

where

$$CV(x, x_e) = n(x, x_e, (1 - \beta)s),$$

and

$$EV(x_e) = \frac{x_e}{\rho - m_2},$$

which yields

$$x_e^* = \frac{(1 - \beta)s}{r} \frac{\lambda_2}{\kappa_2 + 1} \frac{(\rho - m_1)(\rho - m_2)}{m_1 - m_2}. \qquad (2.17)$$

In addition, under the debt valuation model setting, the optimal renegotiation threshold should be always greater or equal to the optimal exit threshold.

3 Data and Sensitivity Analysis

In subsection 3.1, we present the data used to estimate the debt valuation model and subsequently explain the bond yield spreads observed in the financial markets. In subsection 3.2, we perform various sensitivity analyses to assess the adequacy of the debt valuation model.

3.1 Data

For a sovereign, we measure the bond yield spread by the J.P. Morgan Emerging Markets Bond Index Plus (EMBI+). It tracks liquid, US dollar emerging market fixed, and floating-rate debt instruments issued by sovereign entities only. From the World Bank's World Development Indicators (WDI) Database, we extracted the J.P. Morgan EMBI Index Plus timeseries for 196 countries. Then, we retain the countries with a complete timeseries from 1998-01 until 2020-07. Our final sample is set of eleven countries: Argentina, Brazil, China, Colombia, Ecuador, Mexico, Peru, Philippines, Russian Federation, South Africa, and Turkey. The data is monthly, which leaves us with 271 observations for each country. The descriptive statistics of the J.P. Morgan EMBI Index Plus for our set of countries is the following:

Table 2. Descriptive statistics of the J.P. Morgan EMBI Index Plus

The table presents the descriptive statistics of the bond yield spreads, for each of the 11 countries of our final sample. The bond yield spreads are measured by the J.P. Morgan Emerging Markets Bond Index Plus (EMBI+) and are expressed in basis points. The table is produced based on monthly observations of the Index, from 1998-01 to 2020-07. Data are extracted from the World Bank's WDI Database.

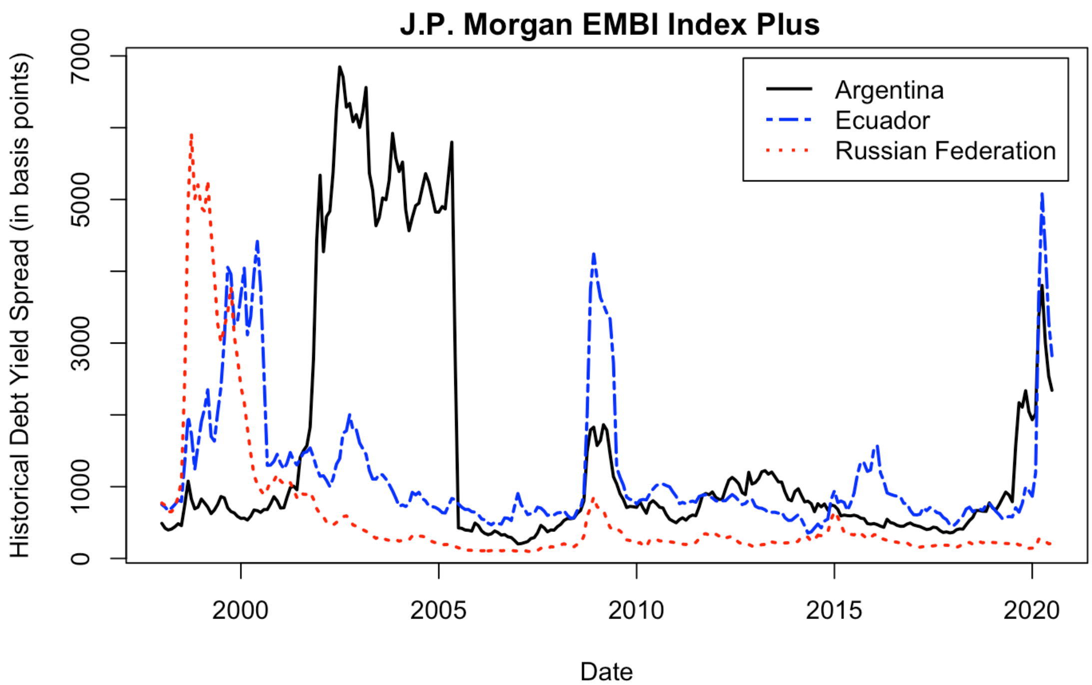
Countries	Mean	St. Dev	Min	1st Quart.	Median	3rd Quart.	Max
Argentina	1517.52	1747.11	202.52	495.55	732.20	1314.78	6847.00
Brazil	458.67	358.84	143.33	225.63	295.82	600.52	2057.36
China	135.71	56.43	37.50	83.92	142.71	174.58	288.24
Colombia	325.16	194.19	108.38	179.64	230.55	456.05	985.95
Ecuador	1163.75	897.15	354.43	665.18	811.64	1250.38	5078.33
Mexico	276.88	138.59	97.59	183.20	237.81	326.79	944.14
Peru	296.71	192.09	103.95	156.57	201.45	426.05	935.76
Philippines	279.72	171.22	69.50	130.21	228.52	422.89	937.29
Russian Federation	594.20	993.08	95.50	191.90	252.60	468.89	5919.29
South Africa	257.10	121.34	57.77	172.47	251.86	307.61	682.48
Turkey	402.56	202.32	162.41	252.33	322.50	508.45	1048.30

We can notice that the bond yield spreads can vary substantially in levels, as well as in volatility from one country to another. Over the sample, Argentina, Ecuador and the Russian Federation are the top 3 sovereigns that had experienced the most volatile bond yield spreads, as well as the highest bond yield spread levels, as shown in the Figure 2.

The computation of the sovereign debt price involves the estimation of six parameters $\Theta = (\mu, \sigma, \alpha, s, \rho, r)$. As defined in the previous section, μ and σ represent the drift and volatility of the sovereign continuous revenue flows. The sovereign continuous revenue flow is an unobservable variable. Hence, we proxy the sovereign revenue flow by its Gross Domestic Product (GDP), which yields an estimate for μ and σ. As we are valuing a perpetual debt, we use the whole available data of GDP to estimate the two parameters.

Figure 2. Historical bond yield spread

The figure plots the time series of the historical debt yield spreads, measured by the J.P. Morgan EMBI+ Spread, for the top 3 sovereigns with the most volatile spreads: Argentina, Ecuador, Russian Federation.



To estimate μ and σ, the constant drift μ and volatility σ of the state variable x, proxied by the GDP of the sovereign, we use a non-parametric approach, the Kernel Density Estimation (KDE). In econometrics, KDE is widely regarded as a smoothing technique where inferences about the population are made, based on finite data sample. As depicted in the Figure 3, the density curve estimated using KDE exhibits characteristics similar to the empirical density function, with the additional advantage of smoothing the density curve. Therefore, KDE allows for better fit of the data and estimation of the parameters, μ in particular. In addition, although the GDP does not seem to be normally distributed, using KDE, instead of a normal distribution, to estimate the drift μ and volatility σ of the GBM x is not inappropriate, since the GDP only serves as a proxy, and we rely on the assumption that the first two moments of the GDP and the state variable are in the same range.

For μ and σ, we first estimate the density function of the state variable x, the GDP growth rate. Its kernel density estimator is:

$$\hat{f}_h(x) = \frac{1}{bh}\sum_{i=1}^{b} K\left(\frac{x - x_i}{h}\right),$$

where h is the smoothing parameter of the density function (bandwidth), b is the size of the sample and K is the Kernel function.

Here, we use a gaussian Kernel function, which yields:

$$K(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}.$$

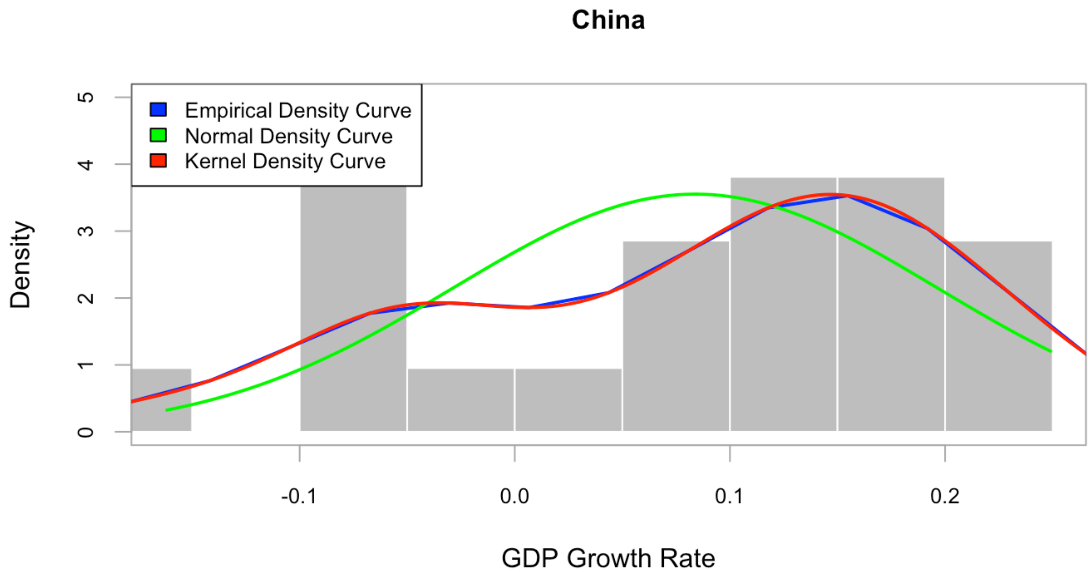
In the same vein, h is determined using Silverman's rule of thumb.

$$h = 0.9\, min\left(\hat{\sigma}, \frac{IQR}{1.34}\right) b^{-\frac{1}{5}},$$

where $\hat{\sigma}$ is the estimator of the standard deviation of the sample and IQR is the interquartile range, the spread between the 3rd and 1st quartiles of the sample.

Figure 3. Kernel density estimation

The figure depicts the growth rate of China's GDP from 1960 to 2020, using the empirical distribution of the sample, a normal-estimated distribution, and a kernel-estimated distribution.



Finally, μ and σ are respectively the mean and standard deviation of the estimated density function. Table 3 provide the results of the estimation of the parameters μ and σ.

Table 3. Estimation of parameters μ and σ

The table summarizes the estimated parameters μ and σ, over the timeframe in bracket for each country. The variables are estimated over the available data until 2020, using a non-parametric estimation: the Kernel distribution. The values in the table are in percentage.

Countries	μ	σ
Argentina (1983 – 2020)	3.26	13.01
Brazil (1989 – 2020)	4.66	16.12

China (1960 – 2020)	14.25	21.19
Colombia (1960 – 2020)	10.17	19.19
Ecuador (1960 – 2020)	5.95	14.53
Mexico (1960 – 2020)	8.25	17.25
Peru (1982 – 2020)	5.87	12.57
Philippines (1960 – 2020)	7.67	18.29
Russian Federation (1989 – 2020)	4.09	18.02
South Africa (1960 – 2020)	6.77	13.33
Turkey (1960 – 2020)	8.42	16.17

The world risk-free rate r is proxied with the 20-year U.S. Treasury bond yield, with an average value of 6% over the available period, which spans from 1962 to 2020, with a discontinuity from December 1986 to October 1993 (Source: Federal Reserve Economic Data).

As is customary in this literature, the sovereign rate of time preference ρ denotes, for the sovereign, the time value of the debt relative to an international risk-free rate r. In other words, ρ is the interest rate paid by the sovereign on its debt. We proxy ρ by the interest payments of the sovereign on its debt. As defined by the World Bank, Interest payments represent the sovereigns' long-term bonds, long-term loans, and other debt instruments vis-à-vis their domestic and foreign residents. The results are shown in Table 4. Although the interest payment rate represents the best proxy of the sovereign rate of time preference, the data collected are sporadic and have missing values.

Table 4. Estimation of parameter ρ

The table presents the interest payments rate of each country, expressed in percentage. The table is produced based on annual observations of the interest payments rate from 1998 to 2020. Data are extracted from the World Bank's World Development Indicators (WDI) Database.

Countries	Mean	SD	Min	1st Quart.	Median	3rd Quart.	Max	N
Argentina	20.42	11.03	8.32	12.34	17.52	25.81	47.91	14
Brazil	27.58	5.41	21.66	23.87	26.01	30.57	39.91	11
China	3.30	0.58	2.20	2.92	3.29	3.47	4.35	16
Colombia	12.96	4.07	8.28	10.88	11.27	14.86	20.62	17
Ecuador	6.02	2.70	2.59	3.58	6.59	7.40	10.17	8
Mexico	13.53	1.96	11.33	11.96	13.31	14.20	18.28	16
Peru	8.48	3.10	4.82	5.79	7.63	11.62	13.50	23
Philippines	23.99	9.12	11.51	16.16	22.41	32.29	39.27	23
Russian Federation	4.08	4.11	1.10	1.81	2.43	3.80	16.06	22
South Africa	12.45	3.87	7.76	9.69	11.29	14.40	21.39	23
Turkey	12.06	11.19	6.15	7.12	8.94	10.71	49.86	14

Hereafter, we consider the median value of each sovereign interest payments as its rate of time preference ρ. However, the median values of China and the Russian Federation interest payment rates are lower than their respective revenue flows growth rate μ. Therefore, we assign the maximal value of the Russian Federation interest payment rates as China and the Russian Federation respective rate of time preference ρ. Table 5 summarizes the final sample's countries rates of time preference.

Table 5. Sovereign rate of time preference ρ

The table summarizes the value of the sovereigns' rate of time preference ρ, estimated from their interest payment rate. The values in the table are expressed in percent (%).

Countries	ρ	Countries	ρ	Countries	ρ
Argentina	17.52	**Ecuador**	6.59	**Russian Federation**	16.06
Brazil	26.01	**Mexico**	13.31	**South Africa**	11.29
China	16.06	**Peru**	7.63	**Turkey**	8.94
Colombia	11.27	**Philippines**	22.41		

In order to estimate the drift of the sovereigns' revenue flows after renegotiations and exit, m_1 and m_2, we rely on the estimation of their respective international trade α. The latter variable is approximated by their respective exports-to-GDP ratio. As defined by the World Bank, the exports-to-GDP ratio represents the value of all goods and other market services provided by the sovereign to the rest of the world. It is therefore a good proxy of the sovereigns' stream flow stemming from its access to the international trade market, which may be jeopardized by renegotiation costs or sanctions in case of debt renegotiations, or exit.

For μ is estimated using the whole data available to us, we extract α over the same timeframe, to keep consistency.

For our final sample of countries, the descriptive statistics of the exports-to-GDP ratio (α), extracted from the World Bank WDI Database, is presented in Table 6.

Table 6. Descriptive statistics of exports-to-GDP ratio (α)

The table presents the descriptive statistics of the exports-to-GDP ratio (α), expressed in percentage (%), over the same timeframe as the GDP data, for each country. Data are extracted from the World Bank WDI Database.

Countries	Mean	SD	Min	1st Quart.	Median	3rd Quart.	Max	N
Argentina	14.06	5.98	6.60	9.73	11.65	18.28	28.40	38
Brazil	11.58	2.77	6.70	9.58	11.65	13.65	16.50	32
China	13.97	9.82	2.50	4.50	12.00	20.30	36.00	61
Colombia	15.16	2.36	10.00	13.30	15.40	16.60	19.30	61
Ecuador	19.47	6.85	8.50	14.70	20.00	23.20	34.20	61

Mexico	18.74	9.91	6.90	8.50	18.10	25.50	39.20	61
Peru	20.26	6.16	11.30	14.80	21.30	25.00	31.50	39
Philippines	28.31	8.81	13.90	20.60	27.45	35.78	43.30	40
Russian Federation	30.71	8.55	13.30	26.10	28.95	34.60	62.30	32
South Africa	24.57	3.33	18.40	22.00	24.40	27.20	32.30	61
Turkey	15.12	8.94	2.10	5.20	15.60	23.00	33.10	61

Equations (2.2) and (2.4) provide with formulas for the determination of m_1 and m_2, the drift of the sovereign's revenue flows after exercising the renegotiation and exit options respectively. The drifts m_1 and m_2 are both function of the parameters μ, ρ and α. Table 7 serves as an illustration of the drifts m_1 and m_2 associated with the average value of the parameters μ, ρ and α, for each sovereign.

Table 7. Illustration of the sovereign's drifts m_1 and m_2

The table presents the sovereign's revenue flows drift after the exercise of the renegotiation and exit options, m_1 and m_2 respectively. For each country of our final sample, the table presents the parameters m_1 and m_2 associated with the average value of their initial drift μ (Table 3). The parameter ρ used in the calculation are the values presented in Table 5. The parameter α used in the calculation is the respective average value, for each country, as presented in the Table 6. The results are expressed in percent (%).

Countries	μ	m_1	m_2
Argentina	3.26	1.46	-0.39
Brazil	4.66	2.06	-0.52
China	14.25	14.00	13.73
Colombia	10.17	10.02	9.85
Ecuador	5.95	5.87	5.77
Mexico	8.25	7.58	6.84
Peru	5.87	5.63	5.37
Philippines	7.67	5.81	3.90
Russian Federation	4.09	2.56	0.97
South Africa	6.77	6.16	5.50
Turkey	8.42	8.35	8.27

For the sovereigns' debt service s, we extract, from the World Bank WDI Database, their respective total debt service. As defined by the World Bank, total debt service represents the sum of principal repayments and interest actually paid in currency, goods, or services on long-term debt, interest paid on short-term debt, and repayments (repurchases and charges) to the International Monetary Fund (IMF). The descriptive statistics of the total debt service is presented in Table 8 below.

Table 8. Descriptive statistics of debt service (s)

The table presents the descriptive statistics of the debt service (s), expressed in billions (Current U.S. billion dollars). The data spans from 1998 to 2020. Data are extracted from the World Bank WDI Database.

Countries	Mean	SD	Min	1st Quart.	Median	3rd Quart.	Max	N
Argentina	20.04	10.58	44.53	13.96	17.02	25.08	43.87	23
Brazil	71.68	28.96	18.43	52.91	59.07	85.93	151.83	23
China	86.37	85.18	4.55	27.53	39.90	104.14	275.79	23
Colombia	11.74	5.74	1.75	7.84	9.97	15.25	24.44	23
Ecuador	4.39	2.46	29.38	2.51	3.92	5.24	9.39	23
Mexico	48.23	14.81	2.18	37.24	41.42	58.95	78.52	23
Peru	5.29	2.40	4.75	3.12	5.51	6.69	11.10	23
Philippines	9.17	2.17	10.81	7.92	9.51	10.60	13.22	23
Russian Federation	56.12	35.30	4.02	20.85	50.08	90.48	110.11	23
South Africa	10.56	8.20	14.94	4.52	6.71	13.95	29.02	23
Turkey	51.74	23.01	0.00	30.50	56.08	63.77	86.68	23

For each sovereign of our final sample, we now have estimates for the parameters $\{\mu, \sigma, \alpha, s, \rho, r\}$ of the debt valuation model. Henceforward, for the calculation, and subsequent estimation of the debt valuation model, we consider the median value of each parameter's estimates as the value of each parameter for the sovereign. The following table summarizes the estimated value of parameters for each sovereign.

Table 9. Summary of estimated parameters

Countries	μ (%)	σ (%)	r (%)	ρ (%)	α (%)	s (billion $US)
Argentina	3.26	13.01	6	17.52	11.65	17.02
Brazil	4.66	16.12	6	26.01	11.65	59.07
China	14.25	21.19	6	16.06	12.00	39.90
Colombia	10.17	19.19	6	11.27	15.40	9.97
Ecuador	5.95	14.53	6	6.59	20.00	3.92
Mexico	8.25	17.25	6	13.31	18.10	41.42
Peru	5.87	12.57	6	7.63	21.30	5.51
Philippines	7.67	18.29	6	22.41	27.45	9.51
Russian Federation	4.09	18.02	6	16.06	28.95	50.08
South Africa	6.77	13.33	6	11.29	24.40	6.71
Turkey	8.42	16.17	6	8.94	15.60	56.08

The above-estimated parameters lay the groundwork for the case study performed for each sovereign of our final sample in section 4.

3.2 Sensitivity Analysis

In this subsection, we perform sensitivity analyses of the debt valuation model over five parameters: μ, σ, α, s, ρ.

We use the historical data of our final sample to define an average value for each of the parameters on the one hand, and a range of values that will be used to perform a sensitivity analysis of the sovereign debt valuation model on the other hand.

Based on our final sample, we define an "average" sovereign whose economy is characterized by the equally weighted average of the parameters estimates across the 11 countries of our final sample outlined in the subsection 3.1. Additionally, we arbitrarily set $x_0 = 100$. These values are summarized in Table 10.

Table 10. Value of the parameters in the baseline case

The table summarizes the "average" sovereign parameters used to perform five sensitivity analyses of the debt valuation model. Each sensitivity analysis is conducted by varying one of these five variables (μ, σ, α, s, ρ), all else held constant.

Parameters	$\boldsymbol{\mu}$ (%)	$\boldsymbol{\sigma}$ (%)	$\boldsymbol{r}$ (%)	$\boldsymbol{\rho}$ (%)	$\boldsymbol{\alpha}$ (%)	$\boldsymbol{s}$ **(billion $US)**
Values	7.21	16.33	6	14.28	18.77	27.20

The sensitivity analyses are performed by varying one parameter at a time, all else equal. The results are reported as debt yield spread: $\frac{s}{Debt} - r$; s, $Debt$ and r represent the debt service-to-GDP, the debt value and the international risk-free rate respectively.
For the purpose of these exercises, we evaluate the sovereign's debt yield spread over the debt restructuring options offered by the Paris Club: Houston (Pure Debt Rescheduling Deal), Naples (Package Deal), Cologne (Package Deal) and Classic Terms (Pure Debt Rescheduling Deal).

Using the baseline parameters, for the pure debt rescheduling deals, we perceive that there is no solution to the smooth pasting condition, to find the optimal renegotiation threshold x_b^*; subsequently, the debt value cannot be calculated. This observation is discussed in section 4.3. As the financial markets anticipate that the pure debt rescheduling would not be an outcome of the debt renegotiation process, their debt pricing would not be done under that parametrization.

Therefore, we rule out the pure debt rescheduling deals and the sensitivity analyses will focus on Naples and Cologne deals. Two Naples deals settings are evaluated: $T = 6$ and $\beta = 0.67$ on the one hand, and $T = 16$ and $\beta = 0.67$ on the other hand. Similarly, two Cologne deals settings are evaluated: $T = 6$ and $\beta = 0.9$ on the one hand, and $T = 16$ and $\beta = 0.9$ on the other hand.

In addition, the range of each parameter studied during the sensitivity analyses is carefully chosen to ensure that there is a solution of the smooth pasting condition to find the optimal renegotiation threshold x_b^*. In other words, we rule out parameter settings that imply an anticipation of creditors of unsuccessful debt renegotiation or an immediate default of the sovereign, without prior debt renegotiation.

In the first sensitivity analysis, we study the parameter μ, the drift of the sovereign's revenue flows.

Figure 4. Sensitivity analysis to sovereign's GDP growth rate (μ)

The figure plots the results of the sensitivity analysis of the debt valuation model vis-à-vis the sovereign's GDP growth rate μ, all else equal. The results are computed as debt yield spread (in basis points), which is the spread between the debt yield ($\frac{s}{Debt}$) and the international risk-free rate r. The spreads are calculated and reported for the three debt restructuring options specified as follows: debt reduction ($T = x_e = 0,\ \beta = 0.5$), debt rescheduling ($T = 5,\ \beta = 0$) and package deal ($T = 5,\ \beta = 0.5$).

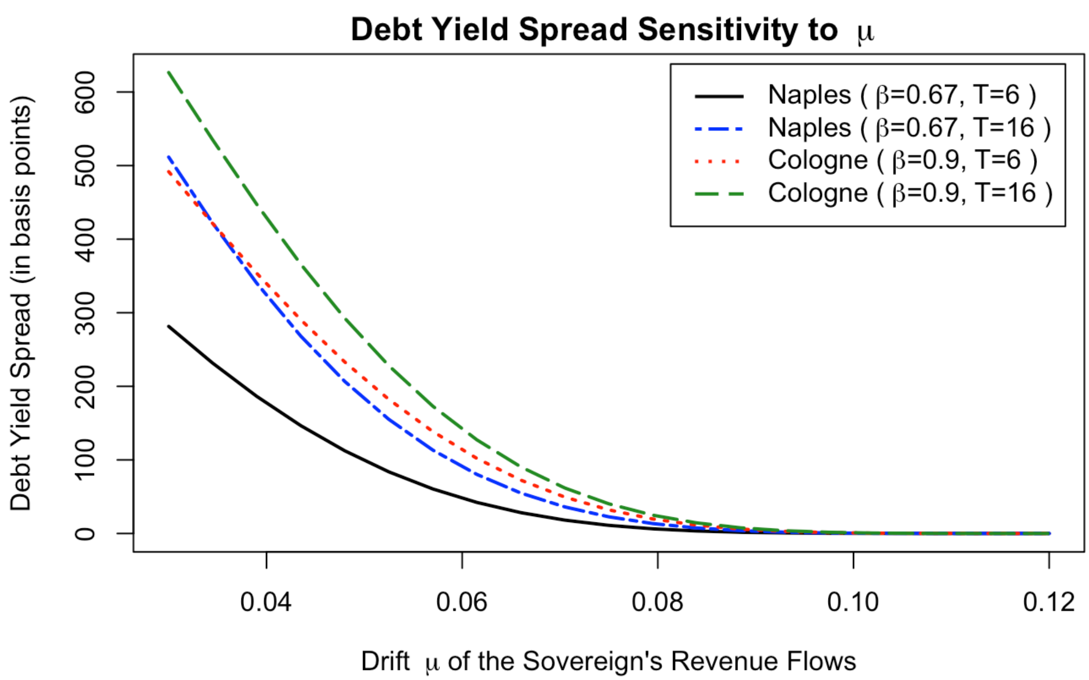


Figure 4 reports the result of this first exercise and illustrates the negative relationship debt yield spread and the GDP growth rate. All things else equal, the likelihood of a sovereign with a high income (or GDP growth rate) to need a debt renegotiation or to default is less than that of a sovereign with a low income. Economic downturns are indeed the norm before default (Reinhart and Rogoff, 2009), or by extension, before debt restructuration. The sovereign's default being negatively related with its income inflow, the risk premium required by the creditors decreases as the sovereign's income growth increases.

Similar to the first sensitivity analysis, we now vary σ, the volatility of the sovereign's revenue flows.

Figure 5. Sensitivity analysis to sovereign's GDP growth rate volatility (σ)

The figure plots the results of the sensitivity analysis of the debt valuation model vis-à-vis the sovereign's GDP growth rate σ, all else equal. The results are computed as debt yield spread (in basis points), which is the spread between the debt yield ($\frac{s}{Debt}$) and the international risk-free rate r. The spreads are calculated and reported for the three debt restructuring options specified as follows: debt reduction ($T = x_e = 0,\ \beta = 0.5$), debt rescheduling ($T = 5,\ \beta = 0$) and package deal ($T = 5,\ \beta = 0.5$).

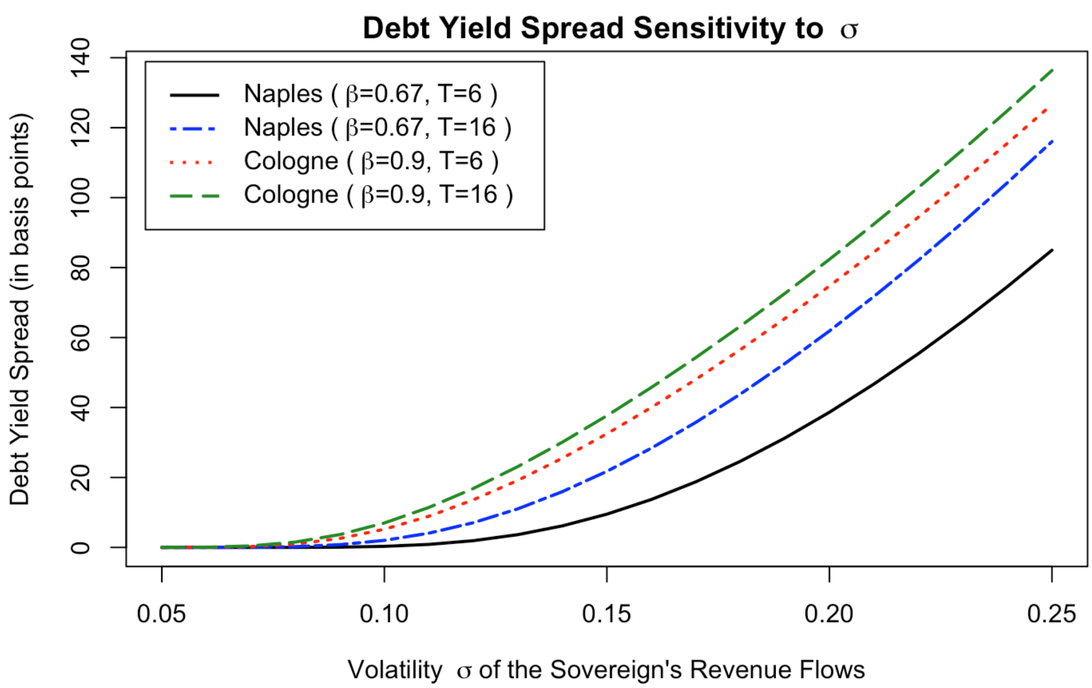


Figure 5 reports the results of this second exercise and shows the positive relationship between the volatility of the GDP growth rate and the debt yield spread. Eaton and Gersovitz (1981) show that the volatility risk indeed increases the state variable ability to reach renegotiation or default thresholds. Hayri (2000) also finds that the stochastic process of a sovereign's income plays in its favor when there is high volatility and trend; in which case, the sovereign has the bargaining power for a debt restructuration. A threat of exit is credible in this case, since the creditors might be willing to avoid repeated default losses that result from the high volatility of the sovereign's incomes and recurrent need for debt restructuring. As the creditors do not have the upper hand in the debt restructuration, their expected loss is high; they counterbalance the expected loss by requiring a higher debt yield. Hence, the debt yield spread is positively related with the sovereign's income volatility.

In the third sensitivity analysis, we study the sovereign's exports-to-GDP ratio (α).

Figure 6. Sensitivity analysis to sovereign's GDP growth rate volatility (α)
The figure plots the results of the sensitivity analysis of the debt valuation model vis-à-vis the sovereign's exports-to-GDP α, all else equal. The results are computed as debt yield spread (in basis points), which is the spread between the debt yield ($\frac{s}{Debt}$) and the international risk-free rate r. The spreads are calculated and reported for the three debt restructuring options specified as follows: debt reduction ($T = x_e = 0,\ \beta = 0.5$), debt rescheduling ($T = 5,\ \beta = 0$) and package deal ($T = 5,\ \beta = 0.5$).

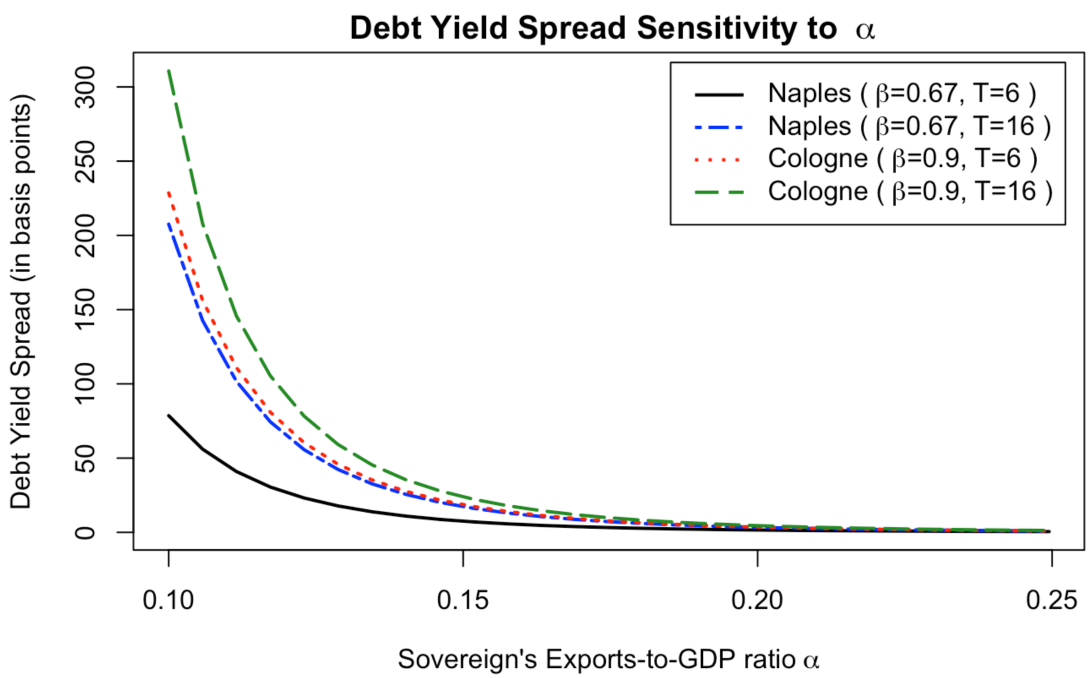


Figure 6 shows the negative relationship between the exports-to-GDP ratio and the debt spread. This observation reflects the fundamental assumption of the debt valuation model, as a sovereign with access to the international debt markets and an open economy is prone to exports' reduction upon choice of the exit option. In fact, the more a sovereign's income relies on exports, the less likely the sovereign is to default on its debt and incur eventual sanctions, which will considerably reduce its income. Then, the risk premium required by creditors is negatively correlated to the sovereign's economy openness. The creditors have the bargaining power, in the case of debt restructuration, the more the sovereign has an open economy.

In the fourth sensitivity analysis, we vary the sovereign's debt service s.

Figure 7. Sensitivity analysis to sovereign's debt service (s)

The figure plots the results of the sensitivity analysis of the debt valuation model vis-à-vis the sovereign's debt service-to-GDP s, all else equal. The results are computed as debt yield spread (in basis points), which is the spread between the debt yield ($\frac{s}{Debt}$) and the international risk-free rate r. The spreads are calculated and reported for the three debt restructuring options specified as follows: debt reduction ($T = x_e = 0,\ \beta = 0.5$), debt rescheduling ($T = 5,\ \beta = 0$) and package deal ($T = 5,\ \beta = 0.5$).

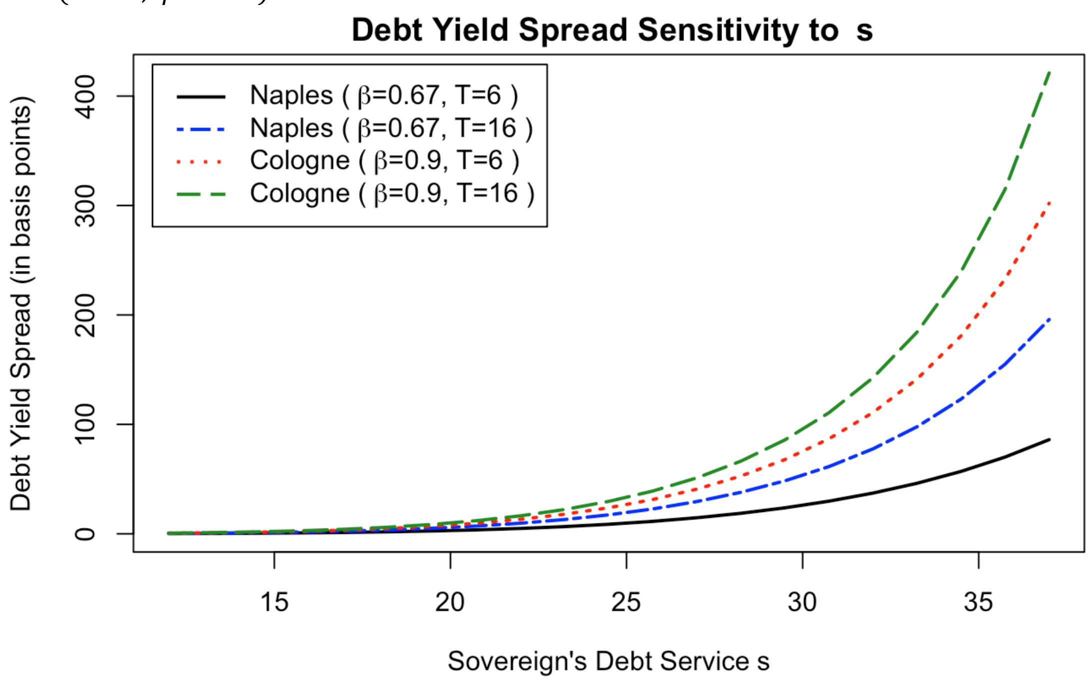


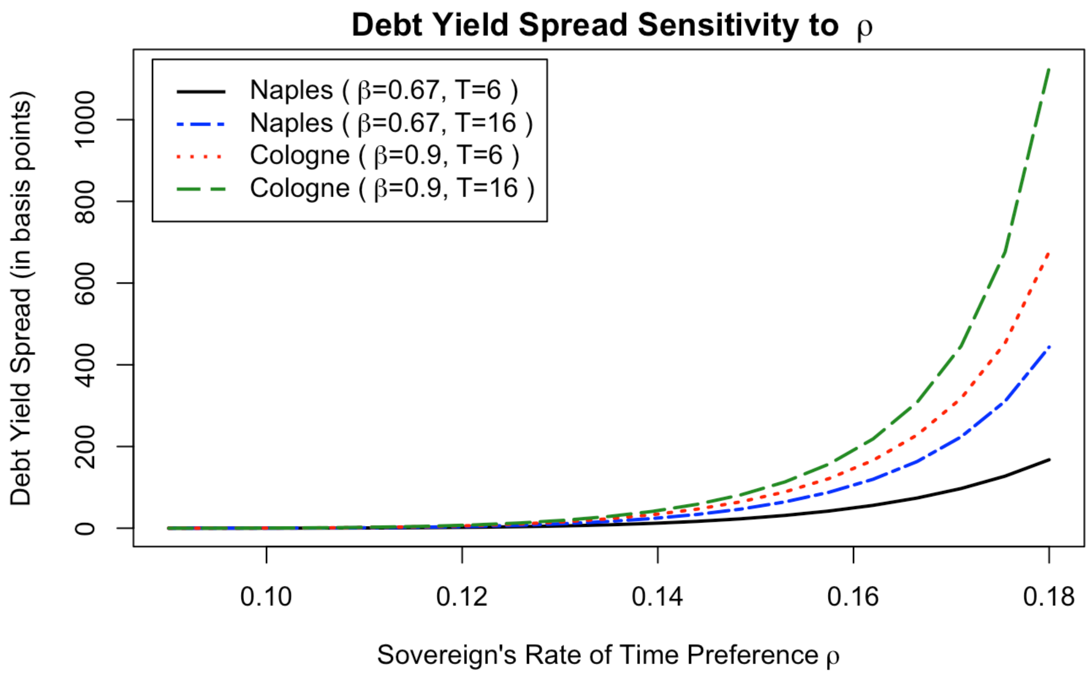
Figure 7 illustrates a positive relationship between the sovereign's debt service and its debt yield spread. Indeed, the more indebted a sovereign is, the higher is its probability of default, or need for debt renegotiation. As the probability of the sovereign's default grows, the risk premium required by the creditors increase as well.

In the fifth sensitivity analysis, we vary the sovereign's rate of time preference ρ.

Figure 8 illustrates the positive relationship between the sovereign's rate of time preference and its debt yield spread. The sovereign's rate of time preference or interest rate, when high, indicates the high borrowing costs associated to the debt, or equivalently a high debt yield.

Figure 8. Sensitivity analysis to sovereign's rate of time preference (ρ)

The figure plots the results of the sensitivity analysis of the debt valuation model vis-à-vis the sovereign's rate of time preference ρ, all else equal. The results are computed as debt yield spread (in basis points), which is the spread between the debt yield ($\frac{s}{Debt}$) and the international risk-free rate r. The spreads are calculated and reported for the three debt restructuring options specified as follows: debt reduction ($T = x_e = 0,\ \beta = 0.5$), debt rescheduling ($T = 5,\ \beta = 0$) and package deal ($T = 5,\ \beta = 0.5$).



Furthermore, the sensitivity analyses unveil that, in general, anticipated higher concessions in case of debt restructuring are associated with higher bond yield spreads. However, the sensitivity analysis with respect to μ shows that there is a trade-off between debt write-off and grace period.

4 Model Estimation and Empirical Results

In subsection 4.1, we explain the methodology employed to estimate the debt renegotiation deals imbedded in the sovereigns' bond pricing, or equivalently sovereigns' debt yield spread. In subsection 4.2, we present the empirical results of that estimation. Subsection 4.3 discusses additional numerical explorations of the debt valuation model.

4.1 Model Estimation

Let y denote a vector of historical debt yield spreads.

Under the debt valuation model, the debt yield spread is a function of the unknown parameters $\mu, \sigma, \rho, \alpha, \beta$ and T,

$$Spread(x;\ \mu,\sigma,\rho,\alpha,\beta,T) \equiv f(x;\ \mu,\sigma,\rho,\alpha,\beta,T) = \frac{s}{Debt(x;\ \mu,\sigma,\rho,\alpha,\beta,T)} - r, \quad (4.1)$$

and

$$x = f^{-1}(\ y;\ \mu,\sigma,\rho,\alpha,\beta,T\ ). \quad (4.2)$$

The historical debt yield spreads y represent an element-by-element transformation of the sovereign revenue flows x.

The debt valuation model relies on the continuous-time stochastic process x, which represents the sovereign's revenue flows. The diffusion process x is latent and could not be directly observed. However, the debt yield spread y could be directly observed in the financial markets. Notwithstanding the fact that x could not be directly observed, the transformation of the diffusion process x by the debt pricing function f (i.e., the debt yield spread y) could be directly observed. Estimating the parameters $\mu,\sigma,\rho,\alpha,\beta$ and T of the debt valuation model could then be done following the transformed-data maximum likelihood estimation methodology, developed by Duan (1994). Indeed, the author studied "*situations in which one needs to estimate the parameters of a postulated model, but the random variate specified in the model is not directly observable*". The benefits of Duan (1994) approach have been documented by Duan, Gauthier, Simonato (2005).

Under the debt valuation model, the state variable x follows a lognormal process. Its one-period log-returns are normally distributed and are therefore characterized by

$$\ln\left(\frac{x_{t+1}}{x_t}\right) \sim N(\mu,\sigma^2).$$

If the state variable x was observable, its log-likelihood function $L_x$would be specified as follows

$$L_x(x;\mu,\sigma) = -\frac{1}{2}\left(T\ \ln 2\pi + T\ln\sigma^2 + \frac{1}{\sigma^2}\sum_{t=1}^{T}\left(\ln\frac{x_{t+1}}{x_t} - \mu\right)^2\right). \quad (4.3)$$

In that case, the log-likelihood function L_x would have only allowed us to estimate the parameters μ and σ, as ρ,α,β and T are not characteristic of the state variable x. Contrarily, the transformed data, i.e., the debt yield spread y, involves μ and σ, along with ρ,α,β and T; hence enabling the estimation by maximum likelihood of the parameters $\mu,\sigma,\rho,\alpha,\beta$ and T.

One could estimate the parameters $\mu, \sigma, \rho, \alpha, \beta$ and T of the debt valuation model. Instead, we decide to reduce the dimensionality of the estimation problem and only estimate the parameters β and T; for the parameters μ, σ, ρ and α, we rely on the parameters estimated in subsection 3.1, as the GDP, interest payment rate and exports-to-GDP could be considered as reliable proxies for x, ρ and α respectively.

Where $g(x)$ represents the density function of the state variable x, the method of transformation (or change of variables formula) yields that the density function of the transformed data y is given by

$$h(y;\ \beta, T) = g(x) / \left|\frac{\partial f(x;\ \beta, T)}{\partial x}\right|, \tag{4.4}$$

with

$$x = f^{-1}(y;\ \beta, T). \tag{4.5}$$

Therefore, the density function of the transformed data y is:

$$h(y;\ \beta, T) = \frac{1}{\sqrt{2\pi\sigma^2}} \left(1 / \left|\frac{\partial f(x;\ \beta, T)}{\partial x}\right|\right) \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right). \tag{4.6}$$

The likelihood function or joint probability density function of $\{1, .., T\}$ observations of the debt yield spread y can then be expressed as:

$$\prod_{t=1}^{T} \frac{1}{\sqrt{2\pi\sigma^2}} \left(1 / \left|\frac{\partial f(x;\ \beta, T)}{\partial x}(\hat{x}_t)\right|\right) \exp\left(-\frac{(\hat{x}_t-\mu)^2}{2\sigma^2}\right), \tag{4.7}$$

where

$$\hat{x}_t = f^{-1}(y_t;\ \beta, T),$$

and

$$\frac{\partial f(x;\ \beta, T)}{\partial x}(\hat{x}_t) = \left.\frac{\partial f(x;\ \beta, T)}{\partial x}\right|_{x=\hat{x}_t}. \tag{4.8}$$

Hence, the log-likelihood function L of the transformed-data y, as specified in the theorem 2.2 of Duan (1994), is given by:

$$L(y; \beta, T) = L_x(\hat{x}; \beta, T) - \sum_{t=1}^{T} \ln\left|\frac{\partial f(x;\ \beta, T)}{\partial x}(\hat{x}_t)\right|, \tag{4.9}$$

where an analytical formulation of the first-order partial derivative $\frac{\partial f(x;\beta,T)}{\partial x}(\hat{x}_t)$ is available.

Finally, we find the estimates of the parameters β and T by maximizing the log-likelihood $L(y;\beta,T)$,

$$(\hat{\beta},\ \hat{T}) = \underset{\beta,T}{\arg\max}[L(y;\beta,T)]. \tag{4.10}$$

4.2 Empirical Results

For each sovereign of our final sample, we proceed to the estimation of the debt renegotiation terms β and T embedded in the sovereign's J.P. Morgan EMBI+ spread (i.e., the sovereign's debt yield spread). The maximum likelihood estimation is done in the realm of the deals available to sovereigns and their creditors, in Paris Club framework. That is, we impose two constraints on the debt renegotiation terms: the debt cancellation ratio β should be less or equal to 0.9, and the debt moratorium or grace period should be less or equal to 16. Results of the estimation are presented in Table 11.

Table 11. MLE results

The table summarizes, over our final sample, the debt renegotiation terms β and T estimated using transformed data maximum likelihood estimation, along with the optimal renegotiation, exit thresholds and log-likelihood associated, x_b^*, x_e^* and $Log-Likelihood$ respectively. For Ecuador, the estimation is unsuccessful; hence, no value has been reported.

Countries	$\hat{\boldsymbol{\beta}}$	$\hat{\boldsymbol{T}}$	$\boldsymbol{x_b^*}$	$\boldsymbol{x_e^*}$	$\boldsymbol{Log-Likelihood}$
Argentina	0.9	16	110.60	10.53	967.01
Brazil	0.71	11.53	130.24	130.24	541.46
China	0.9	16	37.95	5.19	1336.33
Colombia	0.9	16	4.50	0.67	1785.82
Ecuador	N/A	N/A	N/A	N/A	N/A
Mexico	0.9	16	71.84	10.76	846.28
Peru	0.9	16	3.16	0.56	1588.77
Philippines	0.9	16	28.74	2.32	978.66
Russian Federation	0	16	104.47	104.47	470.87
South Africa	0.9	16	8.36	1.43	1357.30
Turkey	0.9	16	12.41	1.87	1508.29

For 8 out of the 11 sovereigns of our final sample, the estimated debt renegotiation outcomes coincide with the Houston Term, with $\hat{\beta} = 0.9$ and $\hat{T} = 16$.

For those sovereigns, where the two constraints of the maximum likelihood estimation are removed, we find that the estimate of the debt cancellation ratio is $\hat{\beta} = 1$; all the debt of the sovereign is written off. The erstwhile observation uncovers a limitation of the debt renegotiation process presented in this thesis. This limitation is the result of a fundamental assumption made vis-à-vis the debt renegotiation process: the sovereign has all bargaining power. Indeed, conceding the sovereign all the bargaining power implies that any default or exit threat made by the sovereign is credible. This advantage plays in the sovereign's favor; the sovereign will initiate the debt renegotiation process with the objective of maximizing its equity. Yue (2010) argues that where the sovereign has all bargaining power, the sovereign can get a complete debt reduction, by making "take-it or leave-it" offers. In the opposite case, where the creditor countries have all bargaining power, they get the full repayment of the debt service. Notwithstanding the renegotiation costs or economic sanctions, the sovereign will always maximize its equity by having totality of its debt written off, especially because the debt renegotiation costs, or economic sanctions are not proportional to the debt renegotiation terms β and T. Indeed, under our debt renegotiation process, a sovereign granted a debt cancellation ratio $\beta = 0.3$ would suffer the same renegotiation costs (i.e., one year of exports loss) or default sanctions (i.e., two years of exports loss) as a sovereign granted $\beta = 0.9$.

4.3 More numerical exploration and Discussion

First, we notice that pure debt rescheduling deals are a marginal proportion of the estimated outcomes of the debt renegotiation process. Therefore, we investigate the attractiveness of pure rescheduling deals.

Over our final sample, the attempt of computation of the pure rescheduling deals' log-likelihood unveils that there is no solution to the smooth pasting condition, to find the optimal renegotiation threshold x_b^* in 82% of the cases. Table 12 presents the results.

We deduce that for most of the sovereigns, in study here, the financial markets are not expecting the sovereign to accept a rescheduling deal, and immediately default after the failing of the renegotiation process. This observation reflects the historical evolution of the renegotiation terms available to debtor countries and their creditors. Indeed, the pure debt rescheduling deals have been the norm for decades. In addition, in the remaining 18% of the cases, where x_b^* exists, the renegotiation threshold is either close or equal to the exit threshold x_e^*. Unlike the package deals, the pure renegotiation deals do not offer enough concessions to the sovereign, so that its exit threshold differs substantially from the renegotiation threshold. Because of the inefficiency of this debt renegotiation deal to offer sufficient debt relief to sovereigns and prevent a potential "default", debt renegotiations have evolved into a "menu" approach, as offered by the Paris Club. Sovereigns could be offered either pure debt rescheduling or package deals.

Table 12. Pure rescheduling deals

The table presents the renegotiation and exit thresholds, x_b^* and x_e^* respectively, for different pure rescheduling terms (i.e., $\beta = 0$).

Countries	$(T = 3)$		$(T = 5)$		$(T = 10)$		$(T = 30)$	
	x_b^*	x_e^*	x_b^*	x_e^*	x_b^*	x_e^*	x_b^*	x_e^*
Argentina	N/A	105.29	N/A	105.29	N/A	105.29	N/A	105.29
Brazil	N/A	442.92	N/A	442.92	N/A	442.92	N/A	442.92
China	N/A	51.91	N/A	51.91	N/A	51.91	N/A	51.91
Colombia	N/A	6.73	N/A	6.73	N/A	6.73	N/A	6.73
Ecuador	N/A	1.46	N/A	1.46	N/A	1.46	N/A	1.46
Mexico	N/A	107.57	N/A	107.57	N/A	107.57	N/A	107.57
Peru	N/A	5.60	N/A	5.60	N/A	5.60	N/A	5.60
Philippines	23.20	23.20	23.20	23.20	23.69	23.20	24.04	23.20
Russian Federation	104.47	104.47	104.47	104.47	104.47	104.47	104.47	104.47
South Africa	N/A	14.27	N/A	14.27	N/A	14.27	N/A	14.27
Turkey	N/A	18.72	N/A	18.72	N/A	18.72	N/A	18.72

Second, we study the case of Ecuador, which is part of the sovereign with the most volatile spreads over our final sample. The first amendment done to the maximum likelihood estimation is to correct the data. We investigate the parameter ρ, as it is the sole parameter to suffer from both low-frequency data and sporadicity altogether. Ecuador has a high-spread profile, similar to Argentina and the Russian Federation; hence we assign Ecuador $\rho = 16.79\%$, which corresponds to the average of those sovereigns' rate of time preference. Where the data is amended, the estimation of the debt renegotiation terms is successful. The results are the following: $\hat{\beta} = 0.9$, $\hat{T} = 16$, $x_b^* = 12.80$, $x_e^* = 1.34$ and $Log - Likelihood = 1736.07$.

Third, we notice that the implied sovereign revenue could not always be determined, when the spreads observed in the markets are too high. Ecuador is a good illustration of this observation. We consider the outcome of the debt renegotiation to specified as Houston Term, with $\beta = 0.9$ and $T = 6$. In this case, the unobserved or implied sovereign revenue flow, associated with observed debt yield spreads, cannot be computed for historical spreads higher than 1786.94 basis points. In addition to the fact that the parameters of the debt valuation model are estimated from low-frequency data, this limitation can be explained by the specification of the stochastic process of the sovereign's revenue flows: singleness of the risk factor and non-stochasticity of the parameters.

Indeed, the diffusion process x assumes volatility risk as the only risk factor of debt valuation model, and equivalently the sovereign's bond yield spread. Among others, Longstaff, Pan, Pedersen and Singleton (2011) prove the sovereign's bond yield spread to encompass both a continuous risk component and a jump risk component. The volatility risk factor of the debt valuation model accounts for the former risk component. One could argue that the latter risk component is implicit to the debt renegotiation process: the anticipated renegotiation and exit thresholds indicate the potential actions of the sovereigns in case of extreme credit events. However, as a matter of fact, in the debt valuation model framework, where the creditors anticipate *ex ante* that the sovereign will not be willing to renegotiate its debt and immediately default, there is no renegotiation threshold anticipated by the creditors, and therefore the debt has no value to them, and they do not enter such transactions. In reality, the creditors could charge the sovereign a high debt yield spread, instead of not entering such transactions to compensate for the default risk. Hence, the jump risk is not properly accounted for in our research. The jump risk could be incorporated by re-defining the dynamic of the diffusion process x, in the debt valuation model. The sovereign revenue flows could then be specified as follows:

$$\frac{dx_t}{x_t} = \mu dt + \sigma dz_t + (\mathrm{J} - 1) dN_t,$$

where

N_t is a Poisson process with intensity λ,

and J is the jump size.

On the other hand, the maximum likelihood estimation is built on the premise that the GDP data represents a good proxy for the sovereign's implied revenue flows, or unobserved variable. Figure 9 (Panel A) compares the sovereign's implied revenue flows or unobserved variable $\hat{x} = f^{-1}(y; \mu, \sigma, \rho, \alpha, \beta, T)$ with the sovereign's GDP, for the estimated renegotiation deals. Table 13 presents the correlation between these two variables.
Except China, the sovereign's implied revenue flows or unobserved variable and the sovereign's GDP are generally positively correlated, although the correlation exhibited can be poor for some sovereigns. The poor correlation between the sovereign's implied revenue flows or unobserved variable $\hat{x}$ and the sovereign's GDP stems from the fact that the parameters of the debt valuation model are constant and does not provide enough flexibility for the model to accurately estimate the sovereign's continuous revenue flows.

Table 13. Correlations

The table presents the correlation between the implied sovereign revenues flows and the historical spread on the one hand, and the correlation between the implied sovereign revenues flows and the GDP on the other hand.

Countries	$Corr(\hat{x}_t, y)$	$Corr(\hat{x}_t, GDP)$
Argentina	-80.29%	39.56%
Brazil	-87.83%	85.92%
China	-95.26%	-58.96%
Colombia	-94.16%	87.48%
Ecuador*	-86.16%	48.28%
Mexico	-91.47%	47.30%
Peru	-94.90%	85.02%
Philippines	-92.53%	95.50%
Russian Federation	-67.60%	54.91%
South Africa	-90.02%	17.94%
Turkey	-94.34%	71.45%

*For Ecuador, the correlation has been calculated, assuming $\rho = 16.79\%$.

Furthermore, Figure 9 (Panel B) depicts the negative correlation between the sovereign's implied revenue flows and the sovereign's bond yield spread, for the estimated renegotiation deals. A high debt yield spread is an indicator of the financial markets pricing sentiment vis-à-vis its "default", or a potential debt renegotiation. Conversely, when the sovereign revenue flows are high, the financial markets price the sovereign's bonds with a low probability assigned to a potential debt renegotiation or exit. Table 13 also presents the correlation of these two variables.

Figure 9. Comparisons

Panel A presents sovereigns' GDP data on the one hand, and sovereigns implied theoretical revenue flows given its debt yield spreads on the other hand. The implied theoretical revenue flow is computed as $\hat{x} = f^{-1}(\,y;\ \mu, \sigma, \rho, \alpha, \beta, T\,)$, where μ and σ are proxied using the GDP data. Panel B presents sovereigns' J.P. Morgan EMBI+ Spread on the one hand, and sovereigns implied theoretical revenue flows (or unobserved variable) given its debt yield spreads on the other hand.

Panel A	Panel B

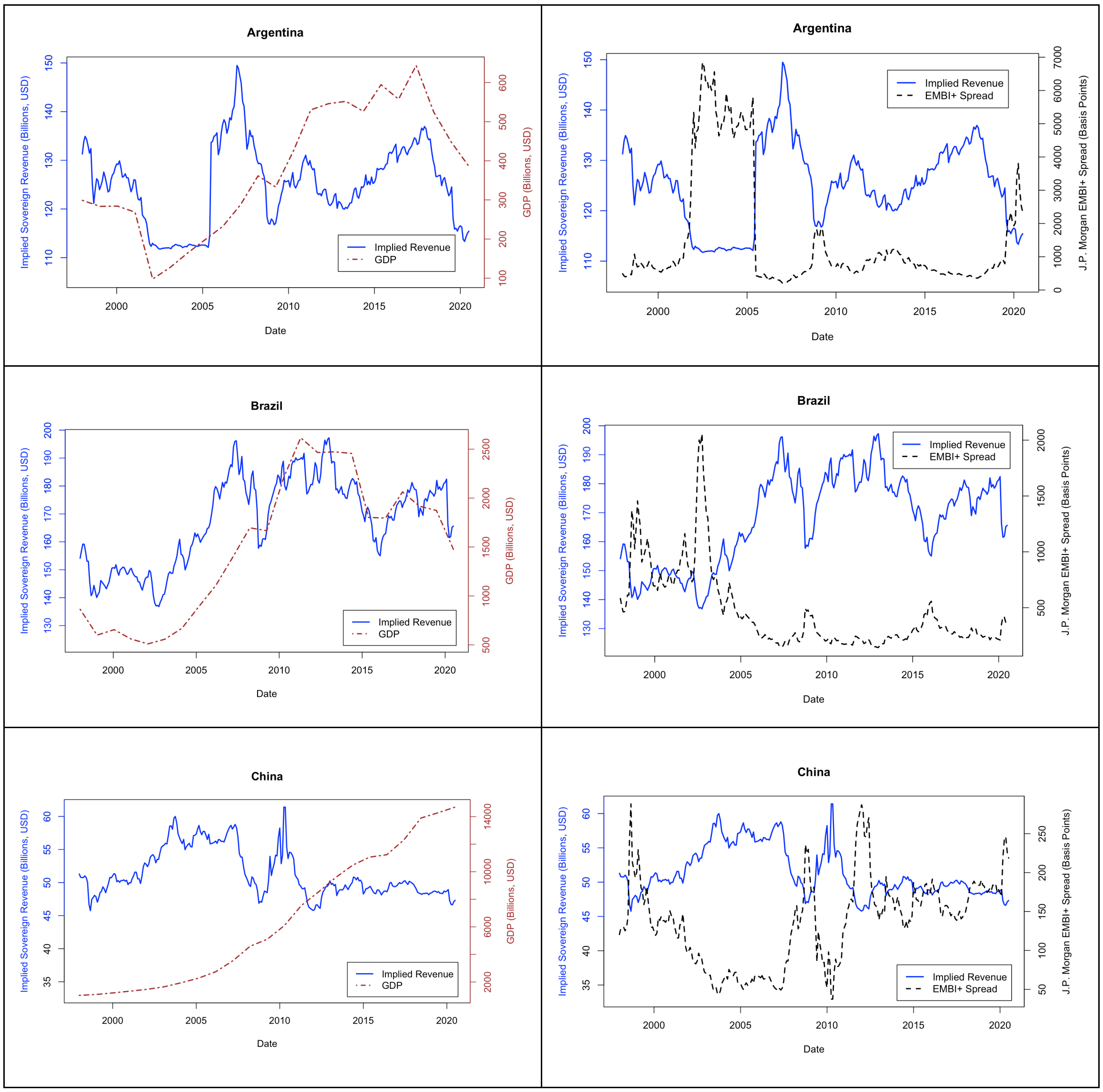


Figure 9 (*Continued*)

Panel A	**Panel B**

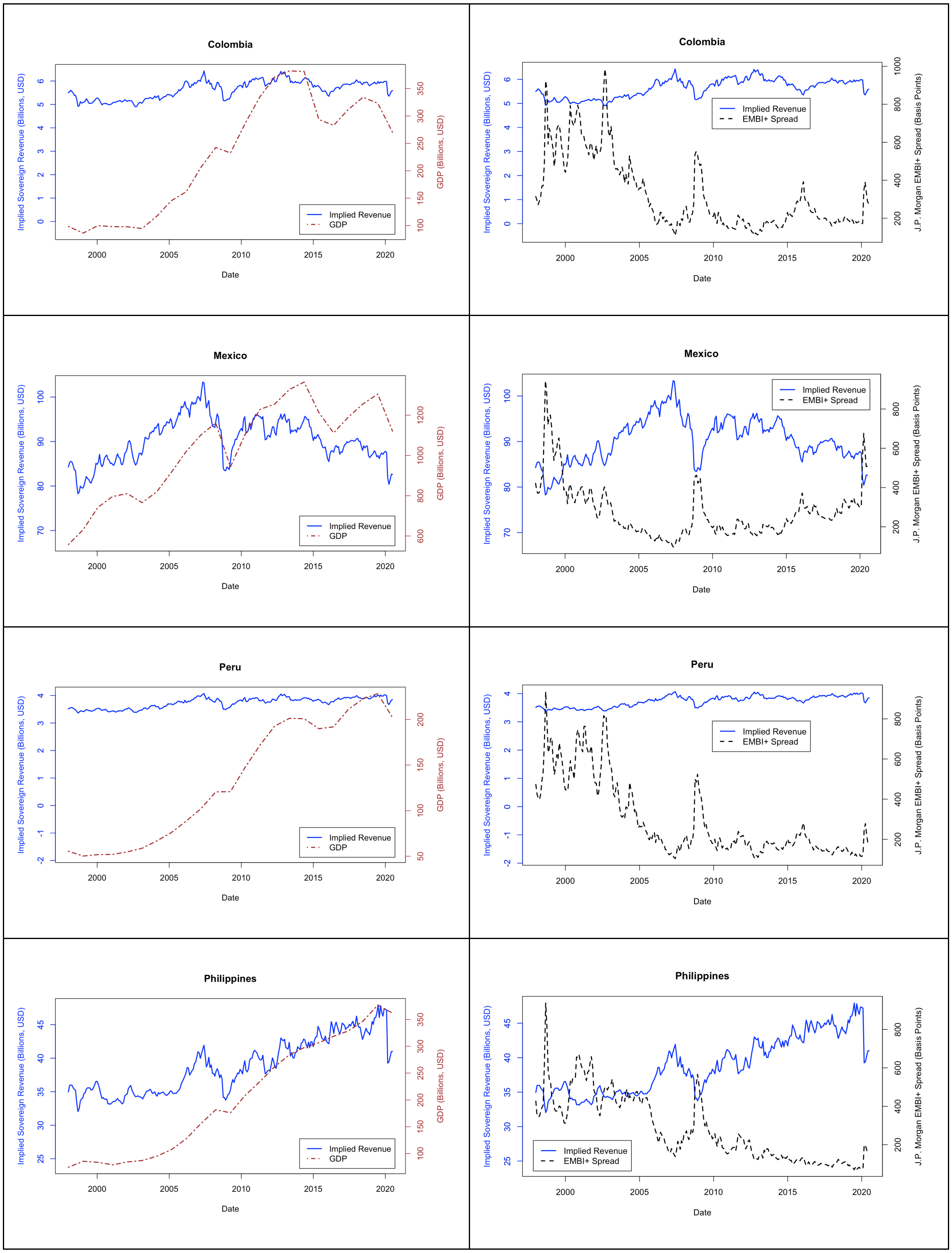
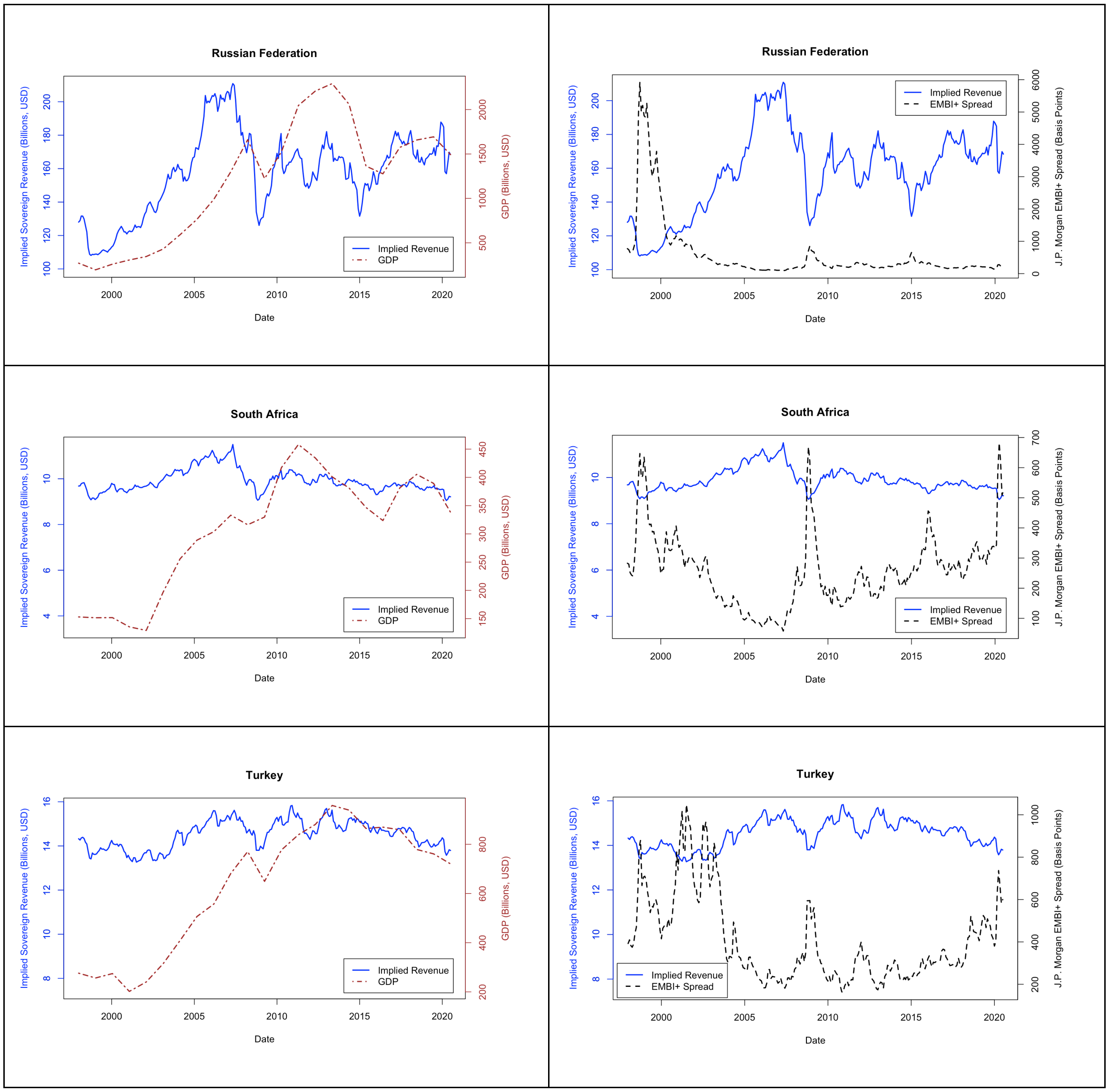


Figure 9 (*Continued*)

Panel A	**Panel B**



5 Conclusion (EN)

In this thesis, we show that the debt valuation model developed by François (2006) effectively allows to price the sovereigns' debt, incorporating their optionality or ability to renegotiate the debt at a future time following the three major debt renegotiation terms: pure debt rescheduling, pure debt reduction, package deals, in a unified framework.

With an inference from the financial markets data (namely the J.P. Morgan EMBI+ Index), we have been able to show that the most common debt renegotiation terms embedded in the sovereigns' debt yield spread corresponds to a package deal, with the most favorable outcome for the sovereigns. This result has been deduced in a framework where the sovereign has been given all the bargaining power vis-à-vis the renegotiation timing. This assumption, although extreme is acceptable as the costs of renegotiations or economic sanctions, following a debt renegotiation or default are short-lived, and are outweighed by the benefits from renegotiations or default.

The result of our empirical analysis justifies the historical evolution of the debt renegotiations, from pure debt rescheduling to "package deals". The former deals have indeed proven inefficient to offer optimal debt relief to the sovereigns, which could prevent them from a default.

However, our research relies on few assumptions: volatility risk as the only risk factor, constancy of the model's parameters and their approximation with observed macroeconomic variables, sovereign's bargaining power.

An important extension of our research would be to explicitly incorporate a jump-to-default to the diffusion process of the state variable (or sovereign revenue flows), in the debt valuation model.

6 Conclusion (FR)

Dans ce mémoire, nous démontrons que le modèle d'évaluation de la dette développée par François (2006) permet de calculer, dans un cadre analytique unifié, le prix de la dette des États, en incorporant l'éventualité de la renégociation de la dette suivant les trois majeurs accords de renégociation : pur rééchelonnement de la dette, pure réduction de la dette et les accords « *package* ».

En se servant des données des marchés financiers (l'index J.P. Morgan EMBI +, dans notre cas), nous avons montré que les accords de renégociation implicitement incorporés dans les écarts de rendement de la dette correspondent à des accords « *package* », les plus avantageux pour l'État en question. Ce résultat a été établi dans un cadre où l'État possède tout le pouvoir de négociation, dans le processus de négociation de la dette souveraine. Quoiqu'extrême, cette hypothèse est acceptable puisque le rapport coûts-bénéfices est avantageux pour le Souverain. En effet, les coûts de renégociation ou sanctions encourues, après une renégociation ou défaut sur la dette, sont éphémères et surpassés par les bénéfices d'une renégociation ou défaut.

Le résultat de notre analyse empirique justifie l'évolution historique des renégociations de la dette, passant du pur rééchelonnement de la dette à des accords "package". Les premiers accords se sont en effet avérés inefficaces pour offrir un allègement optimal de la dette aux souverains, ce qui pourrait les empêcher de faire défaut.

Cependant, notre recherche repose sur quelques hypothèses : le risque de volatilité comme seul facteur de risque, la constance des paramètres du modèle et leur approximation avec les variables macroéconomiques observées, le pouvoir de négociation des souverains.

Une extension importante de notre recherche serait d'incorporer explicitement un saut au défaut dans le processus de diffusion de la variable d'état (ou des flux de revenus souverains), dans le modèle d'évaluation de la dette.

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A Appendix

A.1 Mathematical Formulation

From François (2006):

Using the strong Markov property of Brownian motion, the initial value of sovereign debt subject to a package restructuring is given by the equation (2.5),

$$Debt(x, x_b, x_e, s) = s\mathbb{E}\left(\int_0^{\tau_b} e^{-rt}dt\right) + (1-\beta)s\mathbb{E}(e^{-r\tau_b})\mathbb{E}_{x_b}\left(1_{x_T>x_e}\int_T^{\tau_e} e^{-rt}dt\right)$$

where $\mathbb{E}_{x_b}(.)$ denotes the expectation operator conditional on $\{x_t\}_{t\geq 0}$ starting at x_b.

Standard calculations (see e.g. Karatzas and Shreve (1991) p.197 yield $\mathbb{E}(e^{-r\tau_b}) = \left(\frac{x_b}{x}\right)^{\lambda}$ with $\lambda = \frac{1}{\sigma}\left(\frac{\mu}{\sigma} - \frac{\sigma}{2} + \sqrt{\left(\frac{\mu}{\sigma} - \frac{\sigma}{2}\right)^2 + 2r}\right)$. Developing and rearranging terms, we get

$$Debt(x, x_b, x_e, s) = \frac{s}{r}\left[1 - \left(\frac{x_b}{x}\right)^{\lambda_1}\right]$$

$$+(1-\beta)\frac{s}{r}\left(\frac{x_b}{x}\right)^{\lambda}\left[e^{-rT}\mathbb{E}_{x_b}\left(1_{x_T>x_e}\right) - \mathbb{E}_{x_b}\left(1_{x_T>x_e}e^{-r\tau_e}\right)\right].$$

First, we have that

$$\mathbb{E}_{x_b}\left(1_{x_T>x_e}\right) = \mathbb{P}\left(x_b \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\mathcal{Z}_t\right] > x_e\right),$$

Or, equivalently,

$$\mathbb{E}_{x_b}\left(1_{x_T>x_e}\right) = \Phi(\mathcal{Z}_1).$$

The computation of the second expectation is adapted from the proof derived by Carr (1995). We have that

$$\mathbb{E}_{x_b}\left(1_{x_T>x_e}e^{-r\tau_e}\right) = \mathbb{E}\left(1_{X_T>\ln\frac{x_e}{x_b}}e^{-r\tau_e}\right),$$

Where $X_T = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\mathcal{Z}_t$ is an arithmetic Brownian motion starting from zero. Let $\eta \coloneqq \mu - \frac{\sigma^2}{2}$ and $\tilde{\eta} \coloneqq \sqrt{\eta^2 + 2\sigma^2 r}$. We can write

$$\mathbb{E}_{x_b}\left(1_{x_T>x_e}e^{-r\tau_e}\right) = \mathbb{E}\left(1_{X_T>\ln\frac{x_e}{x_b}}\exp\left[-\frac{\tilde{\eta}^2-\eta^2}{2\sigma^2}\tau_e\right]\right).$$

Some algebra then yields (with $X_{\tau_e} = \ln\frac{x_e}{x_b}$)

$$\mathbb{E}_{x_b}\left(1_{x_T>x_e}e^{-r\tau_e}\right)$$
$$= \exp\left[-\frac{\tilde{\eta} - \eta}{\sigma^2}\ln\frac{x_e}{x_b}\right]$$
$$\times \mathbb{E}\left(1_{X_T>\ln\frac{x_e}{x_b}}\exp\left[\frac{\tilde{\eta} - \eta}{\sigma^2}X_{\tau_e} - \left(\eta\frac{\tilde{\eta} - \eta}{\sigma^2} + \frac{(\tilde{\eta} - \eta)^2}{2\sigma^2}\right)\tau_e\right]\right).$$

Let $\gamma \coloneqq \frac{\tilde{\eta}-\eta}{\sigma^2}$ and $g(\gamma) \coloneqq \eta\gamma + \frac{\sigma^2}{2}\gamma^2$. Using this notation and simplifying, we get

$$\mathbb{E}_{x_b}\left(1_{x_T>x_e}e^{-r\tau_e}\right) = \left(\frac{x_e}{x_b}\right)^{\lambda}\mathbb{E}\left(1_{X_T>\ln\frac{x_e}{x_b}}\exp\left[\gamma X_{\tau_e} - g(\gamma)\tau_e\right]\right).$$

From Girsanov theorem, we obtain that

$$\mathbb{E}_{x_b}\left(1_{x_T>x_e}e^{-r\tau_e}\right) = \left(\frac{x_e}{x_b}\right)^{\lambda}\widetilde{\mathbb{P}}\left(X_T > \ln\frac{x_e}{x_b}\right),$$

Where $\widetilde{\mathbb{P}}$ is a probability measure, equivalent to $\mathbb{P}$, under which the drift of the arithmetic Brownian motion X_T is $-\eta$. Simplifying this expression yields equation (2.6).