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Systemic risk in an interbank market with a large bank and many small banks

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Résumé

Les transactions interbancaires ont fait l'objet de nombreuses études classiques sur le risque systémique, lorsqu'il existe un grand nombre de petites banques. Deux types d'interactions sont considérés : les transactions interbancaires entre elles et les transactions avec la banque centrale. Cette thèse développe une formulation de jeu pour décrire un système interbancaire qui comprend une grande banque et de nombreuses petites banques. À cette fin, des systèmes de jeu à champ moyen linéaire-quadratique-gaussien (LQG) sont utilisés pour modéliser le logarithme des fonds inversés et le coût d'emprunt ou de prêt encouru par chaque banque. Une analyse variationnelle est ensuite appliquée pour dériver les meilleures stratégies de réponse des banques qui, ensemble, produisent un équilibre pour le système interbancaire. En outre, le risque systémique est caractérisé par la probabilité que l'état du marché mondial passe sous un seuil spécifique à un horizon temporel donné. Par la suite, des simulations de Monte Carlo sont utilisées pour étudier la probabilité de défaut d'une petite banque représentative et le risque systémique. Sur la base des expériences numériques, nous concluons que la présence d'une grande banque sur le marché peut avoir deux effets opposés sur le système selon qu'elle fait défaut ou non. Dans le cas où la grande banque ne fait pas défaut, un agent mineur représentatif est moins susceptible de se retrouver en défaut. De plus, le risque systémique diminue. Cet impact positif augmente avec la taille relative et le taux de réversion moyen de l'agent majeur (équivalent du taux de réversion moyen de l'agent mineur). Cependant, dans un marché interbancaire où la banque principale fait défaut, une banque mineure représentative a un risque de défaut plus élevé. De plus, le risque systémique augmente de manière

significative. Ainsi, la défaillance d'une grande banque est beaucoup plus susceptible d'entraîner l'effondrement de l'ensemble du système. Cet impact négatif devient plus important avec la taille relative et le taux de réversion moyen de l'agent majeur. Nos résultats montrent que l'impact négatif d'une grande banque l'emporte sur son impact positif. Nous observons que la probabilité de défaut totale d'une banque mineure représentative et le risque systémique sont plus élevés dans un marché où il existe une grande banque par rapport au cas où il n'y a pas de grande banque. De toute évidence, plus la taille relative de l'agent majeur et le taux de réversion moyen sont élevés sur le marché, plus le risque systémique est important.

Ces résultats ont des implications importantes pour comprendre comment une grande banque peut affecter le marché interbancaire. Plus précisément, elles pourraient être utilisées pour élaborer des politiques et des réglementations qui améliorent la stabilité de ces marchés. En particulier, il n'est pas sain pour l'économie d'avoir une très grande banque en raison de son externalité négative extrême. Les décideurs politiques peuvent établir des réglementations pour empêcher les banques de devenir trop grandes ou imposer des exigences de capital plus élevées et des réglementations plus strictes aux grandes banques pour s'assurer qu'elles ne se retrouvent pas en défaillance.

Mots-clés

Marché interbancaire ; risque systémique ; jeux à champ moyen linéaire-quadratique-gaussien majeur-mineur ; jeux stochastiques.

Contents

List of Tables

List of Figures

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Abstract

Interbank transactions have been the subject of many classic studies in systemic risk, where there exist a large number of small banks. There are two types of interactions considered: interbank transactions with one another and transactions with the central bank. This thesis develops a game formulation to describe an interbank system which includes a large bank and many small banks. To this end, linear-quadratic-Gaussian (LQG) mean-field game systems are used to model the log reverse funds and the cost of borrowing or lending incurred to each bank. A variational analysis is then applied to derive the best response strategies for banks which together yield an equilibrium for the interbank system. Furthermore, systemic risk is characterized by the probability that the global market state falls below a specific threshold by a given time horizon.

Subsequently, Monte Carlo Simulations are used to study the default probability of a representative small bank and the systemic risk. Based on the numerical experiments, we conclude that having a major bank in the market may have two opposite effects on the system depending on whether it defaults or not. In the case where the major bank does not default, a representative minor agent is less likely to end up in default. Moreover, the systemic risk decreases. This positive impact increases with the relative size and the mean-reversion rate of the major agent (equivalently the mean reversion rate of the minor agent). However, in an interbank market where the major bank defaults, a representative minor bank has a higher default risk. Moreover, the systemic risk increases significantly. Hence the failure of a large bank is highly likely to bring the whole system down. This negative impact becomes larger with the relative size and the mean-reversion rate of the major agent. Our results show that the negative impact of a major bank prevails its positive impact. We observe that the total default probability of a representative minor bank and the systemic risk are higher in a market where there exists a major bank compared to the case where there is no major bank. Obviously, the higher the relative size of the major agent and the mean reversion rate in the market, the higher is the systemic risk.

These findings have significant implications for the understanding how a major bank may affect the interbank market. Specifically, they could be used to develop policies and regulations that improve the stability of such markets. In particular, it is not healthy for the economy to have a very large bank due to its extreme negative externality. Policymakers may set regulations to prevent banks from becoming too large or impose higher capital requirement and stricter regulations on large banks to ensure that they would not end up in default.

Keywords Interbank Market; Systemic Risk; Major-Minor Linear-Quadratic-Gaussian Mean Field Games; Stochastic Games

Chapter 1

Systemic risk in an interbank market with a large bank and many small banks

Abstract

1.1 Introduction

Banking is an essential part of the modern economy and plays an increasingly important role as a society develops. The core business activity of the sector is to absorb deposits and provide credit. Banks connect those financial participants who need to borrow funds with those who are willing to lend out monetary resources. They profit from the difference in the interest rates at various maturities which is a strategy encouraging them to have an asset-liability mismatch. At the same time, the introduction of a deposit insurance system worldwide aggravates the reckless expansion in business lines in the bank industry. Providing loans without due diligence is an example of such expansions, which exacerbates the imbalance between the asset side and liability side.

Therefore, one of the most significant risks faced by a bank is the liquidity risk, which means that they do not have sufficient eligible assets that can be converted rapidly into cash or cash equivalent to meet the demand for withdrawal or to cover day-to-day operating costs. For banks, there are many different ways to raise short-term funds to cover the liquidity gap, for instance selling tradable assets, using credit lines given by other financial institutions, securitizing non-liquid assets on their balance sheet, lending from the central bank, etc. There is no doubt that the interbank market is the most important short-term financial resource provider.

In an interbank market, banks with deficits in their settlement operations can borrow money from those with surplus funds. The interbank market was proven to be able to alleviate a part of the liquidity burden due to a risk-sharing scheme. A distressed bank could use the extra liquidity in the interbank market in an efficient fashion to distribute the losses to more counterparties to avoid ending up in default

Scholars search for explanations from many different aspects, such as behavioral finance, counterparty credit risk and statistics, to get a deeper understanding of systemic risk to take more efficient measures for avoiding future undesirable events. In this thesis, we describe the complicated transactions between banks and an interbank market by means of a mathematical model. In order to model the competition between banks in the market, we consider a game framework where a large number of small banks and a relatively influential bank exist. To the best of our knowledge, this is the first time where the presence of a large bank in an interbank market is dynamically modelled. In the following we motivate the inclusion of such a bank in our model. Due to a variety of features, such as the bank's grade granted by the rating agency and the bank size, not every individual bank has the same influence on the market. In particular, in the wake of the Global Financial Crisis in 2008, a new term was created by regulators worldwide, known as "Systemic Important Banks". More specifically, the Financial Stability Board (FSB) has issued a list of "Global Systemically Important Banks" (G-SIBs), and several national regulatory authorities have also designated "Domestic Systemically Important Banks". When such banks are in trouble, their impact on financial stability and economy is rather great (Huang et al., 2012; Tarashev et al., 2009). The major bank or large bank is used to describe this type of important banks in the system. The log-monetary reserve of this bank affects the market state. Furthermore, in our model there exist a large number of minor banks that individually have a negligible impact on the status of the whole market and the influence of each minor bank diminishes as the number of them increases. In the financial system that we consider, all small banks have the same statistical properties and are of the same size and influence on the system. They affect the market state through the log-monetary reserve empirical average of their ensemble.

Each bank keeps an efficient operation through transactions with other banks and by controlling its borrowing and lending rates with the central bank. It aims to have sufficient liquidity to avoid ending up in default. This is while the bank does not wish to hold a lot of cash, which is a waste of resources. The trade-off is based on the cost of doing business in the interbank market (Denbee et al., 2021), for example, transaction charges and other fees due to the absence of market opportunities. To do this, we assume that each small bank aims to hold their log-monetary reserves as much close as possible to the market average state, a weighted average of the large bank's and the small banks' average log-monetary reserves. Therefore, the target of each bank is to minimize their cost by choosing the optimal strategy to track the market average state. We show that our model in the limiting case, where the number of small banks tends to infinity, is an example of a stochastic Linear-Quadratic (LQG) mean-field games with explicit solutions. We apply the convex analysis approach (Firoozi et al., 2020) to derive the best-response strategy of each bank. The collection of these strategies leads to an ε-Nash equilibrium for the market with a finite number of small banks.

Then, we define the default of a bank and the systemic event as the situation where, respectively, the bank's log-monetary reserve and the market average state end up in the default zone, i.e. fall below the default threshold. Subsequently, through Monte Carlo experiments, we investigate the impact of the large bank on the stability of the interbank market. More specifically, we examine its impact on a representative small bank's default probability and on the systemic risk. We perform this analysis in various scenarios where the relative size and the mean-reversion rate of the major bank change. Based on the numerical experiments, we conclude that having a non-defaulting major bank in the market may have two opposite effects on the system depending on whether it defaults or not. In the case where the major bank does not default, a representative minor agent is less likely to end up in default. Moreover, the systemic risk decreases. This positive impact increases with the relative size and the mean-reversion rate of the major agent (equivalently the mean reversion rate of the minor agent). However, in an interbank market where the major bank defaults, a representative minor bank has a higher default risk. Moreover, the systemic risk increases significantly. Hence the failure of a large bank is far more likely to bring the whole system down. This negative impact becomes larger with the relative size and the mean-reversion rate of the major agent. As a result the total default probability of a representative minor bank and the systemic risk are higher in a market where there exists a major than in the one without a major agent. Obviously, the higher the relative size of the major agent and the mean reversion rate in the market, the higher is the systemic risk. These findings have significant implications for the understanding how a major bank may affect the interbank market. The contributions of this thesis are summarized as follows:

- This is the first time where the presence of a large bank in an interbank market is dynamically modelled in a game formulation of the market.
- We used the mean field game methodology to solve for the optimal strategies of the large bank and minor banks yielding an equilibrium for the interbank market. This required mathematical developments and we were not able to use the existing results in the given form. This was in particular due to our interest (for interpretation purposes) in a specific representation of transaction rates in terms of the difference between the log-monetary reserve of a generic bank and the market state.
- We implemented numerical experiments using Monte Carlo simulations for various scenarios to examine how the default probability of minor banks and the stability of the entire financial system are impacted by
  - the presence of a major bank,
  - the success or failure of the major bank,
  - the relative size of the major bank in the market, and

- the mean-reversion rate of the major bank to its long term average log-monetary reserve.

- Given our results, we make some policy recommendations.

The remainder of this thesis is organized as follows. Section 1.2 reviews the previous works in the literature on systemic risk and mean field game methodology. Section 1.3 introduces the model we use to describe the interbank transactions. Section 1.4 presents the limiting model as the number of bank goes to infinity in the mean field game framework. Section 1.5 details the solution methodology for deriving the optimal strategies of individual banks and an equilibrium for the market. Section 1.6 gives the definition of the default probability of an individual bank and the systemic risk. Section 1.7 performs numerical experiments to investigate the impact of the major bank on the behavior of different banks, their default probability and systemic risk. Section 1.8 concludes our work and discusses potential future work.

1.2 Literature Review

As elaborated in the Introduction, banks make margin by absorbing short-term deposits and making long-term loans. The "riding the curve" strategy makes the mismatch between the asset side and the liability side more serious. If the internal liquidity is dried up due to the request of depositors' withdrawal, it is very likely to lead to a bank run (Diamond and Dybvig, 1983). Denbee et al. (2021) proposed each bank needs to choose the appropriate level of monetary reserves based on the cost-benefit analysis. They need to make a trade-off between undertaking more liquidity risk with a lower level of reserves and paying for a higher opportunity cost tied with forgoing other available investments. The interbank market provides banks with a tool to minimize the amount of capital held in low-return liquid assets (Freixas et al., 2000).

The interbank market plays a crucial role in the financial system. Banks have many different incentives to participate in the interbank market. Wiemers and Neyer (2003) came up with three reasons for banks to trade in this market. The first one is the speculative purpose. They could benefit from the price inconsistency of the same products across different markets. The heterogeneous borrowing cost is another driver for the participation of individual bank in the interbank market. Those banks with relatively low marginal costs of borrowing funds have comparative advantages and they could act as an intermediary agent to grant credits to other banks. The most important motivation for banks to trade in this market is to deal with the daily liquidity fluctuation (Bhattacharya and Gale, 1987). Indeed, the interbank market provides a channel for banks to cover the liquidity gap. Banks with reserves surplus grant credits to those with a deficit in monetary reserves (Freixas and Jorge, 2008).

However, the interbank market is a double-edged sword. On the one hand, it is widely accepted that the interbank market provides a co-insurance against uncertainty in the daily liquidity fluctuations. Furfine (2002) pointed out that a well-operating interbank market could make the liquidity exchange more effective. The diversification in the banking sector also makes the risk-sharing among banks more efficient(Steinbacher and Steinbacher, 2015). The results of Acemoglu et al. (2015) show that a distressed bank could use the extra liquidity in the system in an efficient fashion to distribute the losses to more counterparties to avoid ending up in default. Hence, the highly connected structure of the interbank market could make the system more resilient to prevent an individual bank from insolvency (Franklin and Douglas, 2000).

On the other hand, financial contagion is the price that has been paid for the benefit of risk sharing in an interbank market (ECB, 2009). The high interdependence in the banking industry exacerbates the potential risk spillovers (Hautsch et al., 2015).Siebenbrunner et al. (2017) summarized the four channels of the contagion in the banking sector: first-round, correlated losses, a fire sale of assets, and mark-to-market losses. The "first-round" channel means the direct losses obtained from acting as the counterparty of the failure of a specific bank. Due to the highly interdependent relationships in the banking industry, the idiosyncratic shock of one bank would be propagated quickly throughout the whole network. The empirical analysis of Siebenbrunner et al. (2017) displays that the "first-round" channel caused the highest loss contribution among all other channels. Another contagion channel is the overlapping of the portfolio holdings. A specific factor could shock a set of correlated assets and translate losses to changes in the balance sheet of banks. Elsinger et al. (2006) finds that the loss from correlated losses is the main source of systemic risk. The information about the insolvency of a bank is very likely to cause panic in the market leading to the liquidation of the related assets of this bank at a very low price. The losses caused by the fire-sale decrease the recovery value of the loan made to the insolvent bank and trigger a larger effect in the banking network (Caccioli et al., 2015). Some shocks caused by the failure of a bank are spread by the mark-to-market accounting rules. This has been recognized as an accelerator in the 2008 Global Financial Crisis (Georgescu, 2015). In fact, the valuation based on the balance sheet information will establish the expectation of future insolvency which will affect the availability of the liquidity required by the bank.

Due to the multiple contagion channel in the banking system, there has been an increasing amount of literature on the relationship between the network characteristics of the bank system and systemic risk. Learning from the terminologies in topology, the interbank market network is characterised principally by nodes (individual bank), links (the interbank credit exposure), size (the number of banks in the whole system), connectivity (fraction of links present in the network). Krause and Giansante (2012) propose that the structure of the banking system plays a key role in the scale of contagion from a bank failure in the system. The impact from the connectivity of the interbank market network on the systemic risk has been receiving much attention but it is still inconclusive. The majority of researchers think that connectivity is positively correlated with system risk. Caccioli et al. (2012) shows, even in a scale free network characterized by fewer nodes, the probability of contagion which is led by a more connected individual bank is significantly higher.Kanno (2015) implement simulations and draw the conclusion that the contagion phenomena driven by the failure of a highly interconnected bank is more significant. From the empirical analysis from different countries (Anand et al., 2015; Fan et al., 2018; Diem et al., 2020), the positive relationship between the connectivity and systemic risk has been verified. However, some researchers argue that additional connections will improve the risk-sharing scheme within the network. Allen and Gale (2000), Duffy et al. (2019) and Li et al. (2019) claim that the more complete banking network is the least likely it is to be affected by the idiosyncratic shock caused by the default of an individual bank. They propose the connected linkage between banks could provide a buffer to absorb the loss and reduce the contagion risk. The network concentration has been identified as another influential risk factor. A number of studies have found the the probability of systemic risk is positively related to the concentration of the network. Gai et al. (2011), Fei et al. (2015) and Zedda and Sbaraglia (2020) also conclude that the greater network concentration level amplifies the spread of shock in the system and increases the probability of systemic risk. There are also some other network characteristics considered in the literature such as the first failed bank's centrality in the system (Degryse and Nguyen, 2004; Lee, 2013), and the number and size of the exposure in the network (LI and HE, 2012; Memmel and Sachs, 2013; Bucher et al., 2014).

After the 2008 Global Financial Crisis, much of the literature on drivers of systemic risk has emphasized the importance of a relatively large bank. Steinbacher and Steinbacher (2015) argues that the large and well-capitalized banks are less likely to get in trouble and De Jonghe et al. (2015) claims that the universal banking services provided by a major bank could generate non-interest income which could help large banks to reduce their exposure to systemic risk. Benefiting from the advanced risk management technology and the rigorous supervision of the shareholders in the major banks, they have the advantage of economies of scale in mitigating the financial systemic risk. On the contrary, Huang et al. (2012) applies the decomposition analysis to show that the bank size largely determined the marginal contribution of a single bank in the systemic risk. Tarashev et al. (2009) provides evidence that the systemic significance of the biggest bank is almost ten times greater than that of small banks, even though its size is only five times as much the small banks. This is consistent with the conclusion drawn by Laeven et al. (2016) that the systemic risk increases with the bank size. The concern about the "too-big-to-fail" problem also gives major banks additional incentives to assume more risk than small banks. This is because they can search for a bail-out by the government due to their importance in keeping the stability of the whole system (Altunbas et al., 2017). Anand et al. (2015) uses stress test to show that a large bank has a higher potential to cause a contagious default than relatively small banks. Our work provides additional evidence that a major bank may increase the systemic risk using a game formulation of the market and perfoming extensive Monte Carlo simulations.

More specifically, due to the importance of large banks in the system, in this thesis we aim to dynamically model the presence of a major bank in an interbank market. To the best of our knowledge, this is the first time a major bank is modelled dynamically in a game formulation of interbank markets in order to investigate its impact on systemic risk. We use the mean field game (MFG) methodology to describe the borrowing and lending activities between a large number of small banks, the major bank, and the central bank. Then we investigate the systemic risk in such a financial environment. In the following we briefly review the literature on MFGs and their applications in the domain of interbank systems and systemic risk.

Mean field game (MFG) theory has been developed in the early 21st century to model the interactions between a large number of agents (Lasry and Lions, 2007; Huang et al., 2006, 2007; Gamito García, 2017; Carmona and Delarue, 2018). In such games each agent is not only impacted by its own behaviour but also by the mass behaviour of all other agents. The mathematical limit of this mass effect as the number of agents goes to infinity is called the mean field. MFG theory establishes the existence of approximate Nash equilibria in such games and can be used to obtain the corresponding optimal strategies for each player in the system. Huang et al. (2006, 2007) and Lasry and Lions (2007) independently developed an analytical approach to solve the problem of constructing approximate Nash equilibria for MFGs. In this approach, the system equilibrium solutions are formulated as the solutions to a set of coupled PDEs consisting of the Hamilton-Jacobi-Bellman equation and Kolmogorov equation. Moreover, Carmona and Delarue (2013) developed a probabilistic approach for a similar problem. Due to special characteristics of some agents in reality, such as market power and relatively large size, a significant part of the

MFG literature is devoted to including a major agent to model the special impact of a relatively influential agent in the game (Huang, 2010; Nguyen and Huang, 2012; Nourian and Caines, 2013; Carmona and Zhu, 2016; Carmona and Wang, 2017, 2016; Şen and Caines, 2016; Firoozi and Caines, 2021; Firoozi et al., 2022a; Lasry and Lions, 2018; Bensoussan et al., 2017; Moon and Başar, 2018). A few recent studies investigate the equivalency of solutions to MFG systems with major and minor agents obtained via different approaches (Firoozi et al., 2020; Huang, 2021; Firoozi, 2022)In particular, in this thesis we use the convex analysis method developed in (Firoozi et al., 2020) to derive the best-response transaction strategies in an interbank market with a major agent.

The MFG methodology has been used to solve problems in an extensive range of applications, for example, equilibrium pricing (Shrivats et al., 2022; Gomes and Saúde, 2021; Fujii and Takahashi, 2022), optimal execution problems (Casgrain and Jaimungal, 2020; Firoozi and Caines, 2015; Wu and Liu, 2017; Cardaliaguet and Lehalle, 2018; Lehalle and Mouzouni, 2019), compliance market design (Firoozi et al., 2022b), games of timing (Guéant et al., 2011), and production of exhaustible resources. It is not surprising that the MFG methodology has been used extensively to obtain a deeper understanding of systemic risk. The first interbank model using MFGs was proposed in (Carmona et al., 2015) with N small banks borrowing or lending to each other and to the central bank. In this model the log-monetary reserves of the banks are modelled as a system of mean-reverting controlled diffusion processes coupled in the drift with the average log-monetary reserve of all banks and subject to correlated noise processes. It is concluded that interbank transaction improve the stability in the interbank market. Fouque and Ichiba (2013) use a set of interacting Feller diffusion processes to model the monetary reserves of banks and quantify the relationship between the lending preference of a bank and its bankruptcy. They conclude that the growth rate and lending preferences are important for understanding the systemic risk in interbank lending. Furthermore, Fouque and Sun (2013) investigate different types of coupled systems: uncontrolled systems coupled through the drift term and correlated Brownian motions, and controlled systems with mean field interaction. They show that the interbank borrowing and lending activities increase both the stability and the likelihood of the systemic event. These results are consistent with those from the numerical experiments of Garnier et al. (2013). Some other authors advanced these models by taking into account more factors. The Cox–Ingersoll–Ross (CIR) process was used to replace the Ornstein–Uhlenbeck (OU) processes used in Carmona et al. (2015) to model the evolution of the log-monetary reserves (Sun, 2018). Sun (2019) extended the interbank model by considering heterogeneous borrowing and lending among banks.

1.3 Interbank Transactions Model Model

In this section, we introduce the model we use to demonstrate the transactions in an interbank system consisting of a major bank and a large number of small banks. In our model, an individual bank trades with other banks and also choose an optimal strategy to borrow from or to lend to the central bank to minimize its cost. We note that in reality banks never lend money to the central bank. Hence, we need to give a broader definition of this lending activity. A key instrument of monetary policy by the central bank is the open market operation that adjusts the market liquidity through the sale or purchase of financial assets on the market. Hence, in this work "Lending to the Central Bank" refers to an individual bank activity when it buys a Treasury Bond or acts as a Repo buyer. Under these circumstances, liquidity in the market decreases.

1.3.1 Major Bank Model

As mentioned in the Introduction, systemically important banks were identified by the regulators given their special position in the financial system. In our setting, we use the "major bank" to model the behavior of a relatively large influential bank, which is to be differentiated from a small or minor bank.

We denote the major bank by $\mathscr{A}^0$ and its logarithm of monetary reserves (log-monetary reserves) at time t by x_t^0. The major bank in our hypothesized economic environment borrows or lends to other banks when its log-monetary reserve is respectively lower or higher than the average log-monetary reserve across small banks. These transaction rates are proportional to the distance of the major bank from the average log-monetary reserve. Therefore the log-monetary reserve of the major bank is assumed to satisfy

$$dx_t^0 = a_0 \left[F_0 x_t^{(N)} - x_t^0 \right] dt + u_t^0 dt + \sigma_0 dW_t^0, \tag{1.1}$$

where

$$x_t^{(N)} = \frac{1}{N} \sum_{i=1}^{N} x_t^i. \tag{1.2}$$

In the above SDE, $x_t^{(N)}$ represents the average log-monetary reserves of the large population of minor banks. u_t^0 models the borrowing and lending activities of the major bank with the central bank. Moreover, $a_0 \left[F_0 x_t^{(N)} - x_t^0 \right]$ models transactions of the major bank with other banks in the market. As a_0 increases, the major bank tends to trade more frequently with other banks and to mean-revert more quickly to a fraction F_0 of the market mean log-monetary reserves. F_0 quantifies the relative power the major bank aims to have with respect to the aggregate log-reserve of minor banks. The parameter σ_0 represents the volatility of its log-monetary reserve which is coming from the depositing and withdrawing activity of retail customers modelled by the Brownian motion W_t^0 at each point t in time.

From the perspective of a bank, it is natural to optimize the use of their deposits. Hence, a bank is motivated to keep a minimum required amount of money in its accounts and to borrow from other banks when it needs more money to cover the liquidity gap. Therefore the operational target for the major bank is to control its rate (the amount per unit time) of borrowing and lending reserves with the central bank and keep its log-monetary reserve as close as possible to the average log-monetary reserve of the minor banks. Mathematically, the objective of the major bank is to minimize its cost functional

$$\begin{aligned} J_0^N(u^0, u^{-0}) = \mathbb{E}\Big[\int_0^T \Big\{ \frac{1}{2}\big(u_t^0\big)^2 - q_0 u_t^0 \big(F_0 x_t^{(N)} - x_t^0\big) \\ + \frac{\varepsilon_0}{2}\big(F_0 x_t^{(N)} - x_t^0\big)^2 \Big\} dt + \frac{c_0}{2}\big(F_0 x_T^{(N)} - x_T^0\big)^2 \Big] \end{aligned} \tag{1.3}$$

The optimal strategy chosen by the major bank is represented by u^0, and u^{-0} is the collection of the optimal controls of all other banks besides the major bank $u^{-0} = (u^1, \dots, u^N)$. From the cost functional above, the parameter q_0 quantifies the incentive to participate in borrowing and lending activity and a higher q_0 is akin to the regulator having low fees. ε_0 measures the penalization posed on the major bank when its log-monetary reserve deviates from the average log-monetary reserves of minor banks during the considered period. The parameter c_0 penalize the major bank if there exist a difference between its log-monetary reserves and the average log-monetary reserves of minor banks in the terminal time.

The information set of the major bank is denoted by $\mathscr{F}^0 = (\mathscr{F}_t^0)_{t\in[0,T]}$. It is generated by the sample paths of the state of the major bank. The admissible set $\mathscr{U}_0$ of control action for the major bank consists of all $\mathscr{F}^0$-adapted $\mathbb{R}$-valued processes such that $\mathbb{E}\left[\int_0^T u_t^2 dt\right] < \infty$.

1.3.2 Minor Bank Model

In our model, we assume there are a large number N of minor banks in the market. Each minor bank represents a small bank that has a negligible impact on the financial system as the number N grows. We assume that all minor agents are homogeneous, i.e. they share the same model parameters and hence are statistically identical. We denote a minor bank by $\mathscr{A}^i, i \in \{1,...,N\}, N < \infty$ and its log-monetary reverse at time t by x_t^i. The log-monetary reserve x_t^i of minor bank $\mathscr{A}^i$ is assumed to satisfy the SDE

$$dx_t^i = a\left[\left(Fx_t^{(N)} + Gx_t^0\right) - x_t^i\right]dt + u_t^i dt + \sigma dW_t^i. \tag{1.4}$$

The difference between the log-monetary reserve dynamics of the major bank and that of a minor bank is that a minor bank is directly influenced by the major bank. A minor bank $\mathscr{A}^i$ is motivated to keep its liquidity as much close as possible to the market average state $\left(Fx_t^{(N)} + Gx_t^0\right)$. The market average state is modeled by a linear combination of the average log-monetary reserve of all minor banks and the log-monetary reserve of the major bank. The parameters F and G indicate the relative size of the major bank and the mass of minor banks and satisfy $F+G=1$. Since the value of these parameters may not be observed by an individual bank in the market, we assume it is provided by the central bank. The term $\left[\left(Fx_t^{(N)}+Gx_t^0\right)-x_t^i\right]$ models the transactions of the minor bank with the major bank and other minor banks. a represents with which the minor bank mean-reverts to the market state through interbank transactions. u_t^i is the control action of each minor bank and it models the borrowing and lending activities of the minor bank with the central bank. The parameter σ_i represents the volatility of its log-monetary reverse arising from the depositing and withdrawing activity from their retail customers modelled by the Brownian motion W_t^i at each point t in time. All Brownian motions in this model $\{W_t^0, W_t^i, i\in\{1,2,\ldots,N\}\}$ are independent.

Each minor bank aims to operate efficiently. Hence, it chooses an optimal strategy to minimize its cost functional as in

$$\begin{aligned}J_i^N(u^i,u^{-i}) = \mathbb{E}\Big[\int_0^T \Big\{\frac{1}{2}\big(u_t^i\big)^2 - qu_t^i\big(Fx_t^{(N)}+Gx_t^0-x_t^i\big)\\ +\frac{\varepsilon}{2}\big(Fx_t^{(N)}+Gx_t^0-x_t^i\big)^2\Big\}dt+\frac{c}{2}\big(Fx_T^{(N)}+Gx_T^0-x_T^i\big)^2\Big].\end{aligned} \quad (1.5)$$

The strategy chosen by a representative minor bank-i is represented by u^i, and the collection of strategies chosen by all other banks is represented by $u^{-i}=(u^0,u^1,\ldots,u^{i-1},u^{i+1},\ldots,u^N)$. The cost functional of a minor bank is similar to that of the major bank. The difference exists in the value of individual parameters and in the level of log-monetary reserve they aim to hold. Due to the small size of a minor bank compared to that of the major agent the value of its cost parameters q, ε, and c could be different from those of the major bank. Moreover, a minor bank wishes to keep its log-monetary reserve as close as possible to the market state $(Fx_t^{(N)}+Gx_t^0)$ described above. Moreover, for the minor bank's cost functional to be convex we assume $q^2\le\varepsilon$.

The information set of a representative minor bank $\mathscr{A}^i$ is denoted by $\mathscr{F}^i=(\mathscr{F}_t^i)_{t\in[0,T]}$. It is generated by the states of the major bank and the minor bank $\mathscr{A}^i$. The admissible set $\mathscr{U}_i$ of control action for the minor agent consists of all $\mathscr{F}^i$-adapted $\mathbb{R}$-valued processes such that $\mathbb{E}\left[\int_0^T(u_t^i)^2dt\right]<\infty$ (this assumption makes sure that the optimal control problem is well defined).

1.3.3 Market Clearing Condition

We build our economy in a closed environment, which means that all banks in the financial system can only transact with the central bank and other banks in the same system. This implicitly proposes a constraint on our model that the sum of all log-monetary reserves traded in the financial environment should sum to zero. In an individual transaction, one bank takes on the role of lending and there must exist another bank acting as its counterparty and borrowing funds to it. Since all interbank transactions take place in a closed economy, the total volume of log-monetary reserve transactions should be zero after all netting activities at each point t in time, i.e.

$$\frac{a}{N}\sum_{i=1}^{N}\left[\left(Fx_t^{(N)}+Gx_t^0\right)-x_t^i\right]+a_0\left[F_0x_t^{(N)}-x_t^0\right]=0, \qquad t\in[0,T]. \tag{1.6}$$

In above equation the terms $\frac{a}{N}\sum_{i=1}^{N}\left[\left(Fx_t^{(N)}+Gx_t^0\right)-x_t^i\right]$ and $a_0\left[F_0x_t^{(N)}-x_t^0\right]$ represent, respectively, the average transactions of minor banks and the transcations of the major bank per unit time with other banks in the market. We perform some algebraic manipulations on (1.6) to get

$$\left(aF-a+a_0F_0\right)x_t^{(N)}+\left(aG-a_0\right)x_t^0=0, \qquad t\in[0,T]. \tag{1.7}$$

In order for the above condition to be satisfied at every time instant $t\in[0,T]$ and for every value of processes $x_t^{(N)}$ and x_t^0, we must have

$$a_0F_0=a-aF, \tag{1.6a}$$

$$a_0=aG. \tag{1.6b}$$

By substituting (1.6b) into (1.6a) we obtain

$$F_0=\frac{1-F}{G}. \tag{1.8}$$

In this model, we interprate the parameters F and G as the relative size of major bank and the mass of minor banks in the market average state, i.e.

$$F+G=1 \tag{1.9}$$

We substitute this constraint into (1.8) and obtain $F_0 = 1$. Hence, in the rest of the thesis we use $F_0 = 1$ in the major bank's model.

Furthermore, from (1.6b), we obtain a relationship between the mean reversion rate of the major bank a_0 and that of the minor banks a. As discussed G is associated with the relative size of the major bank in the market average state and hence its value is smaller than one. This means that the major bank always has a lower mean-reversion rate than minor banks. In fact, the trade flow of a minor bank is divided into the trades with other minor banks and with the major bank. In equilibrium the major bank gets the share aGx_t^0 of trades which corresponds to its trade flow $a_0x_t^0$ and gives rise to (1.6b).

It is worth mentionning that the smaller value of a_0 can be interpreted as a larger market friction for the major bank, which hinder it from trading quickly. This might seem counterintuitive, since it is generally perceived that the major bank has comparative advantages due to its high credit rating or well-capitalization. We note that in our model banks can only remedy their monetary reserves deficit either through interbank transactions or transactions with the central bank. In addition, previous studies show that the major bank might not be as advantaged as we thought in the interbank market. Bucher et al. (2014) and Arce et al. (2017) proposed that interbank frictions mainly exist in the form of transaction cost and regarded it broadly as the search cost. A bank must find an appropriate counterparty that satisfies two conditions: (1) matching the liquidity requirements and (2) willing to make an agreement with it. Hence, the major bank has to split a large amount of liquidity needs into smaller ones and fulfill each part by using the credit facilities in different counterparties. This may increase the search costs for the major bank. At the same time, the major bank is not regarded as the optimal transaction partner for a minor bank. This is because minor banks do not have too much bargaining power over the major bank. Thus, minor banks are more likely to trade together first and then trade with the major bank if they cannot find other minor banks (Colliard et al., 2016). These two considerations might impede the major bank to mean-revert to their target log-monetary reserves as quickly as the minor banks.

1.4 Limiting Interbank Transactions Model: A MFG formulation

As described in Section 1.3, banks in the financial system under study behave in a competitive manner and aim to minimize their cost to achieve an optimal operation. Each individual bank makes its own decision based on the information available at that time to borrow or lend in the market. On the one hand a bank wishes to secure enough liquidity and on the other hand it does not want to hold too much money to lose an excellent investment opportunity. We aim to obtain a collection of optimal borrowing and lending strategies for individual banks yielding an equilibrium for the market. This could be typically challenging even when there are a small number of banks in the financial environment. Therefore we resort to the mean field game (MFG) methodology.

MFG theory analyzes a game environment where exists a large number of players taking strategies to minimize their own cost function. These players act in a non-cooperative fashion and the information they rely on is the empirical distribution of states across population instead of the individual strategy taken by other players in the game. The aggregate effect of the population appears in the optimization problem through the dynamics or the cost functions. The generic idea of the MFG methodology is that some simplifications could be made in the limiting case with an infinite number of agents. The theory establishes the existence of the appropriate equilibria and an asymptotic solution for this class of games when the number N of agents in the system goes to infinity. Moreover, it is shown that a limiting equilibrium yields an approximate equilibrium for the original finite-player game.

Here we introduce the mean field of log-monetary reserves $\bar{x}_t$ and the mean field of transactions with the central bank $\bar{u}_t$ in the limiting case as in

$$\bar{x}_t = \mathbb{E}[x_t^{\cdot} | \mathscr{F}_t^0], \tag{1.10}$$

$$\bar{u}_t = \mathbb{E}[u_t^{\cdot} | \mathscr{F}_t^0], \tag{1.11}$$

where $x_t^{\cdot}$ and $u_t^{\cdot}$ denote, respectively, the log-monetary reserve and the borrowing and lending activities of a representative minor bank. If the limit exists, the mean field terms are equivalent to the mathematical limit of the following empirical averages as the number of banks N goes to infinity.

$$\bar{x}_t = \lim_{N\to\infty} x_t^{(N)} = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} x_t^i, \tag{1.12}$$

$$\bar{u}_t = \lim_{N\to\infty} u_t^{(N)} = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} u_t^i. \tag{1.13}$$

Accordingly, we express the interbank model in the limiting case. This includes (i) the dynamics of log-monetary reserves for bank, (ii) the cost faced by banks, and (iii) the mean field equation as follows.

(i) Major bank

$$dx_t^0 = a_0\left[\bar{x}_t - x_t^0\right]dt + u_t^0 dt + \sigma_0 dW_t^0, \tag{1.14}$$

$$J^0(u) = \mathbb{E}\Big[\int_0^T \Big\{\frac{1}{2}\big(u_t^0\big)^2 - q_0 u_t^0\big(\bar{x}_t - x_t^0\big) + \frac{\varepsilon_0}{2}\big(\bar{x}_t - x_t^0\big)^2\Big\}dt + \frac{c_0}{2}\big(\bar{x}_T - x_T^0\big)^2\Big]. \tag{1.15}$$

(ii) Minor banks

$$dx_t^i = a\left[\big(F\bar{x}_t + Gx_t^0\big) - x_t^i\right]dt + u_t^i dt + \sigma dW_t^i, \tag{1.16}$$

$$\begin{aligned} J^i(u) = \mathbb{E}\Big[\int_0^T &\Big\{\frac{1}{2}\big(u_t^i\big)^2 - q u_t^i\big(F\bar{x}_t + Gx_t^0 - x_t^i\big) \\ &+ \frac{\varepsilon}{2}\big(F\bar{x}_t + Gx_t^0 - x_t^i\big)^2\Big\}dt + \frac{c}{2}\big(F\bar{x}_T + Gx_T^0 - x_T^i\big)^2\Big]. \end{aligned} \tag{1.17}$$

(iii) Mean Field Equation We can derive the equation that the mean field satisfies in terms of $\bar{u}_t$ by taking the conditional expectation of (1.16) conditioned on the information set $\mathscr{F}_t^0$. As a result the diffusion part disappears due to the independence of Brownian motions $\{w_t^0, w_t^i\}$ and the mean field $\bar{x}_t$ satisfies

$$d\bar{x}_t = \left[a(F-1)\bar{x}_t + aGx_t^0 + \bar{u}_t\right]dt. \tag{1.18}$$

1.5 Best-Response Transactions and System Equilibria

There are many different ways to search for the optimal actions of the banks in our financial system, such as the stochastic maximum principle, dynamic programming and calculus of variations. In this thesis, we apply the convex analysis approach developed by the Firoozi et al. (2020) to seek the best response strategies for the major bank and for each individual minor bank.

In Section 1.4, we replaced the average terms of all minor banks in dynamics and cost functional by their infinite population limit to obtain the infinite population version of the stochastic game. In this section we aim to derive the system equilibria for this limiting market model. We note that in the limiting case, each minor bank is interacting with the major bank and the mean field instead of $N-1$ other minor banks. A similar situation holds for the major bank as it is interacting only with the mean field. Moreover, we can derive the equation that the mean field satisfies. All these observations motivate us to extend the dynamics of (i) the major bank to include the mean-field of log-monetary reserves, and (ii) a representative minor bank to include the major bank's log-monetary reserve dynamics, and the mean-field dynamics. In this manner for each individual banks in the system, we obtain a stochastic control problem. These optimal control problems are linked through the shared elements, i.e. the major bank's log-monetary reserve and the mean-field of log-monetary reserves. Then we can solve these stochastic control problems to obtain the best-response strategies and a consistent mean-field. In the last step, we use these limiting best response strategies in the finite population market model which yields and ε-Nash equilibrium for the latter.

1.5.1 Important Results

We first summarize all the important results of our work and then we show the detailed derivation in the following subsection.

Theorem 1 (Best-Response Transactions) *For the interbank market model given by* (1.14) - (1.17)*, the optimal borrowing and lending strategies for the major bank and a representative minor bank, and the mean field equation are given by*

***(i) Major bank**:*

- *Optimal control*

$$u_t^{0,*} = \left(q - \phi_t^0\right)\left(F\bar{x}_t - x_t^0\right) \tag{1.19}$$

- *ODE that the coefficient in the optimal control satisfies*

$$\begin{aligned}\dot{\phi}_t^0 &= 2(a_0 + q_0)\phi_t^0 - \left(\phi_t^0\right)^2 + \left(a + q - \phi_t\right)G\left(\phi_t^0\right) + \varepsilon_0 - q_0^2 \\ \phi_T^0 &= -c_0\end{aligned} \tag{1.20}$$

<u>(ii) Minor banks</u>:

- *Optimal control*

$$u_t^{i,*} = \left(q - \phi_t\right)\left[\left(F\bar{x}_t + Gx_t^0\right) - x_t^i\right] \tag{1.21}$$

- *ODE that the coefficient in the optimal control satisfies*

$$\begin{aligned}\dot{\phi}_t &= 2(a + q)\phi_t - \left(\phi_t\right)^2 + \varepsilon - q^2 \\ \phi_T &= -c\end{aligned} \tag{1.22}$$

<u>(iii) Mean Field Equation</u>:

$$d\bar{x}_t = \left(a + q + \phi_t\right)\left[(F-1)\bar{x}_t + Gx_t^0\right]dt. \tag{1.23}$$

Proof. See Section 1.5.2. □

In our model, each individual bank interacts with the aggregate effect of the population and simultaneously takes an optimal strategy for trading with the central bank. Hence the notion of Nash equilibrium becomes relevant. A Nash equilibrium is characterized by the fact that no individual agent can obtain an additional benefit by just unilaterally changing the strategies it takes. Thus an agent has no motivation to deviate from a Nash strategy while all other agents are following them. In the following we first give the mathematical definition of Nash equilibrium and then show that the set of obtained optimal strategies in Theorem 1 yields a Nash-equilibrium for the limiting interbank market.

Consider a non-cooperative game with N agents. Each agent-i, $i \in \{1, \ldots, N\}$, has a choice of strategy denoted by u^i in the admissible set of strategies $\mathscr{U}$. We denote by

u^{-i} the collection of strategies chosen by all other agents other than agent-i. To be more specific, (u^i, u^{-i}) represents the N-tuple $(u^1, \ldots, u^N)$, and (u, u^{-i}) represents the N-tuple $(u^1, \ldots, u^{i-1}, u, u^{i+1}, \ldots, u^N)$, where for the latter the i-th element u^i in the original tuple is replaced by u.

Definition 1 (Nash Equilibrium) *An N-tuple of strategies $(u^1, \ldots, u^N) \in \mathscr{U}^1 \times \cdots \times \mathscr{U}^N$ is said to be a Nash equilibrium for an N-player non-cooperative game if for every $i \in \{1, \ldots, N\}$ and $u \in \mathscr{U}^i$,*

$$J^i(u^1, \ldots, u^i, \ldots, u^N) \leq J^i(u^1, \ldots, u^{i-1}, u, u^{i+1}, \ldots, u^N), \tag{1.24}$$

or equivalently

$$u^i = \underset{u \in \mathscr{U}^i}{\arg\min}\, J^i(u, u^{-i}). \tag{1.25}$$

Theorem 2 (Nash Equilibrium for Infinite-Population Interbank Market) *For the limiting interbank market* (1.14)-(1.17)*, the N-tuple of best-response strategies $U = (u^0, u^1, \ldots, u^i, \ldots, u^\infty)$, characterized by* (1.19)-(1.22)*, yields a Nash equilibrium.*

Proof. Given that all banks are following the strategies from U, the mean field satisfies (1.23). Now if a minor bank unilaterally deviates from U, as individually it has a negligible impact, this deviation does not affect the mean field value and its characterization. Hence, the minor bank seeks an optimal strategy in response to the same mean field as before. This yields to the strategy specified by (1.52)-(1.22). Hence the minor bank cannot benefit by deviating unilaterally. A similar reasoning can be used for the unilateral deviation of the major bank from U. In this case still the mean field satisfies (1.23), where the value of x_t^0 is updated. This results in the same optimal control law for the major bank. Hence U forms a Nash equilibrium for the limiting interbank market model (1.14)-(1.17) (see also (Huang, 2010; Carmona and Wang, 2016)). □

We note that we are interested in an equilibrium for the original finite-population interbank market model described by (1.1)-(1.5). Now we connect the obtained solutions for the limiting model to the finite-population model through the notion of ε-Nash equilibrium. An ε-Nash equilibrium is an approximation to the Nash equilibrium. The difference exists in that an agent may have small incentives to unilaterally change its strategy in an ε-Nash equilibrium. Hence, the requirement that no agent has any incentive to deviate from its strategy in Nash equilibrium has been weakened. However, the incentive will not be larger than ε, where ε is of small value.

Definition 2 (ε-Nash property) *An N-tuple of strategies $(u^1,\ldots,u^N) \in \mathscr{U}^1 \times \cdots \times \mathscr{U}^N$ is said to be an ε-Nash equilibrium solution for an N-player non-cooperative game if there exists an $\varepsilon \geq 0$ such that for $i \in \{1,\ldots,N\}$ and $u \in \mathscr{U}^i$,*

$$J^i(u^i,u^{-i}) \leq J^i(u,u^{-i}) + \varepsilon, \tag{1.26}$$

where u is an admissible alternative strategy for agent-i.

Theorem 3 (ε-Nash Equilibrium for Finite-Population Interbank Market) *For the finite-population interbank market* (1.1)-(1.5)*, the N-tuple of best-response strategies $U = (u^0,\ldots,u^i,\ldots,u^N)$, specified by* (1.19)-(1.22)*, yields a ε-Nash equilibrium.*

Proof. The interbank model considered is a special case of LQG mean field games with one major agent and a large population of minor agents. Hence, the proof of ε-Nash property follows from the existing results in the literature, see e.g. (Huang, 2010; Carmona and Zhu, 2016). □

1.5.2 Methodology

In this section we obtain the optimal trading strategy for the major bank and a representative minor bank $\mathscr{A}^i$ for the limiting interbank model given by (1.1)-(1.5), and market clearing conddtion (1.6) and (1.9). We explain each step of the solution methodology in detail. First of all, following the MFG methodology with a major agent (Huang, 2010), we extend the dynamics of the log-monetary reserves for the major bank as we described at the very beginning of this section. Then following the convex analysis method developed by Firoozi et al. (2020), we perturb the major bank's control action by a relatively small value and investigate how this disturbance propagates through the whole economy. Subsequently, we write down the Gâteaux Derivative of the major bank and set it equal to zero to derive the major bank's optimal control. We note that we cannot directly use the results derived by Huang (2010) and Firoozi et al. (2020). This is because, for the purpose of our work, we are interested in deriving the optimal transcation rates in terms of the difference between the market state (or the average log-monetary reserve) and the bank's log-monetary reserves (i.e. $(Fx_t^{(N)} - x_t^0)$ or $(Fx_t^{(N)} + Gx_t^0 - x_t^i)$). In order to completely characterize the major bank's optimal strategy we need to characterize the mean field. Thus we investigate a representative minor bank's problem subsequently. We follow a similar variational analysis to solve the minor bank's problem. Then we return to the major bank's problem and complete the analysis to obtain an explicit representation of the optimal control strategy.

(a) Major Bank Problem

Step (i): Perturb the control of the major bank by δ_0 in the direction $\omega^0 \in \mathscr{U}^0$. The dynamics for log-monetary reserves of the major bank is subjected to the perturbed control $u_0^0 + \delta_0 \omega^0$.

$$dx_t^{0,\delta_0} = a_0\big(\bar{x}_t^{\delta_0} - x_t^{0,\delta_0}\big)dt + \big(u_t^0 + \delta_0\omega^0\big)dt + \sigma_0 dW_t^0 \tag{1.27}$$

Step (ii): Follow the effect of the major bank's perturbed control action on its own state and every minor bank's state to obtain the resulting perturbed mean-field $\bar{x}_t^{\delta_0}$.

Perturbed minor banks' states x^{i,δ_0}:

$$dx_t^{i,\delta_0} = a\left[\big(F\bar{x}_t^{\delta_0} + Gx_t^{0,\delta_0}\big) - x_t^{i,\delta_0}\right]dt + u_t^i dt + \sigma dW_t^i \tag{1.28}$$

Perturbed mean-field $\bar{x}_t^{\delta_0}$: Taking the conditional expectation of (1.28) given $\mathscr{F}_t^0$ yields the mean-field equation:

$$d\bar{x}_t^{\delta_0} = \left[a(F-1)\bar{x}_t^{\delta_0} + aGx_t^{0,\delta_0} + \bar{u}_t\right]dt \tag{1.29}$$

From (1.28) and (1.29), we get a clear idea about how a shock of the major bank influence the minor banks and the whole system, which is a benefit obtained by the convex analysis approach (Firoozi et al., 2020). The perturbation in the trading activity of the major bank with the central bank affects its own log-monetary reserves. As its special status in the market, the log-monetary reserves of the major bank directly influence that of minor banks and indirectly affect the mean-field of the market state. In turn, the mean-field influence the major bank through participating in its log-monetary reserves evolvement.

Step (iii): Extend the major bank's state to include the joint dynamics of the major bank's state and the mean-field

$$X_t^{0,\delta_0} = \begin{bmatrix} x_t^{0,\delta_0} \\ \bar{x}_t^{0,\delta_0} \end{bmatrix} \tag{1.30}$$

We substitute the original dynamics for minor banks to the equation of mean-field to get the dynamics of the extended state and the extended cost function:

$$dX_t^{0,\delta_0} = \begin{bmatrix} dx_t^{0,\delta_0} \\ d\bar{x}_t^{0,\delta_0} \end{bmatrix} = \left[\tilde{A}_0 X_t^{0,\delta_0} + \mathbb{B}_0 u_t^0 + \tilde{B}_0 \bar{u}_t + \delta_0 \mathbb{B}_0 \omega^0\right] dt + \Sigma_0 dW_t^0 \tag{1.31}$$

where

$$\tilde{A}_0 = \begin{bmatrix} -a_0 & a_0 \\ aG & a(F-1) \end{bmatrix}, \quad \mathbb{B}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \tilde{B}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \Sigma_0 = \begin{bmatrix} \sigma_0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{1.32}$$

$$\begin{aligned} J^0(u^0 + \delta_0 \omega^0) = \tfrac{1}{2}\mathbb{E}\Bigg[\int_0^T \Big\{ &(X_s^{0,\delta_0})^\intercal \mathbb{Q}_0 X_s^{0,\delta_0} + 2(X_s^{0,\delta_0})^\intercal \mathbb{N}_0 (u_s^0 + \delta_0 \omega_s^0) + \\ &(u_s^0 + \delta_0 \omega_s^0)^2 \Big\} dt + (X_T^{0,\delta_0})^\intercal \mathbb{G}_0 X_T^{0,\delta_0} \Bigg], \end{aligned} \tag{1.33}$$

where

$$\mathbb{Q}_0 = \begin{bmatrix} \varepsilon_0 & -\varepsilon_0 \\ -\varepsilon_0 & \varepsilon_0 \end{bmatrix}, \quad \mathbb{N}_0 = \begin{bmatrix} q_0 \\ -q_0 \end{bmatrix}, \quad \mathbb{G}_0 = \begin{bmatrix} c_0 & -c_0 \\ -c_0 & c_0 \end{bmatrix}. \tag{1.34}$$

Step (iv): Summarize for the unperturbed extended dynamics and cost function for major bank.

- By setting the perturbation in (1.5.2) to zero, we get the unperturbed extended dynamics for major banks:

$$dX_t^0 = \left[\tilde{A_0}X_t^0 + \mathbb{B}_0 u_t^0 + \tilde{B_0}\bar{u}_t\right]dt + \Sigma_0 dW_t^0. \tag{1.35}$$

- Cost function of major bank expressed in extended state:

$$J^0(u^0) = \tfrac{1}{2}\mathbb{E}\left[\int_0^T \left\{(X_s^0)^\intercal \mathbb{Q}_0 X_s^0 + 2(X_s^0)^\intercal \mathbb{N}_0 (u_s^0) + (u_s^0)^2\right\}dt + (X_T^0)^\intercal \mathbb{G}_0 X_T^0\right]. \tag{1.36}$$

Now we aim to characterize the mean field of controls $\bar{u}_t$ appearing in the dynamical model of the major bank. For this purpose, we look into the problem of a representative minor agent.

(b) Minor Bank Problem

Step (i): Perturb a minor bank's control action by δ_i in the direction $\omega^i \in \mathscr{U}^i$.

$$dx_t^{i,\delta_i} = a\left[\left(F\bar{x}_t^{\delta_0} + Gx_t^{0,\delta_i}\right) - x_t^{i,\delta_i}\right]dt + \left(u_t^i + \delta_i\omega^i\right)dt + \sigma dW_t^i \tag{1.37}$$

Step (ii):By taking the average of (1.37) and then the mathematical limit as $N \to \infty$, we get the perturbed mean-field $\bar{x}^{\delta_i}$:

$$d\bar{x}_t^{\delta_i} = \left[a(F-1)\bar{x}_t^{\delta_i} + aGx_t^{0,\delta_i} + \bar{u}_t\right]dt \tag{1.38}$$

The shock on the minor banks propagates throughout the whole system in a different fashion as that of the major bank. From (1.37) and (1.38), a disturbance in a small bank's trading activity with a central bank affects its own log currency reserves. Because of the negligible impact of one minor bank, the mean-field and major bank are not effected by the perturbation of minor banks' control action, so we get $\bar{x}_t^{\delta_i} = \bar{x}_t$ and $x_t^{0,\delta_i} = x_t^0$.

Step (iii): Extend the minor bank's state to include the joint dynamics of the minor banks, major banks and mean-field.

$$X_t^{i,\delta_i} = \begin{bmatrix} x_t^{i,\delta_i} \\ x_t^{0,\delta_i} \\ \bar{x}_t^{0,\delta_0} \end{bmatrix} = \begin{bmatrix} x_t^{i,\delta_i} \\ x_t^0 \\ \bar{x}_t \end{bmatrix} = \begin{bmatrix} x_t^{i,\delta_i} \\ X_t^{0,\delta_0} \end{bmatrix} \tag{1.39}$$

We substitute the equation of the extended dynamics for major bank (1.5.2) and the dynamics for minor (1.37) to get the expression of the extended dynamics for the minor bank-i as in

$$dX_t^{i,\delta_i} = \begin{bmatrix} dx_t^{i,\delta_i} \\ dX_t^{0,\delta_0} \end{bmatrix} = \left[\tilde{A}X_t^{i,\delta_i} + \mathbb{B}u_t^i + \tilde{B}\bar{u}_t + \delta_i \mathbb{B}\omega^i\right] dt + \Sigma dW_t^i \tag{1.40}$$

where

$$\tilde{A} = \begin{bmatrix} -a & [aG, aF] \\ 0 & \tilde{A}_0 - \mathbb{B}_0\mathbb{N}_0^\intercal - \mathbb{B}_0\mathbb{B}_0^\intercal \phi_t^0 \mathbb{B}_0^\intercal \tilde{A}_0 \end{bmatrix}, \quad \mathbb{B} = \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 \\ \tilde{B}_0 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma & 0 \\ 0 & \Sigma_0 \end{bmatrix}. \tag{1.41}$$

Then we obtain the perturbed cost function for minor bank-i:

$$J^i(u^i + \delta_i\omega^i) = \tfrac{1}{2}\mathbb{E}\Big[\int_0^T \Big\{(X_s^{i,\delta_i})^\intercal \mathbb{Q} X_s^{i,\delta_i} + 2(X_s^{i,\delta_i})^\intercal \mathbb{N}\big(u_s^i + \delta_i\omega_s^i\big) + \big(u_s^i + \delta_i\omega_s^i\big)^2\Big\} ds + \big(X_T^{i,\delta_i}\big)^\intercal \widehat{Q} X_T^{i,\delta_i}\Big], \tag{1.42}$$

where

$$\mathbb{Q} = \begin{bmatrix} \varepsilon & -G\varepsilon & -F\varepsilon \\ -G\varepsilon & G^2\varepsilon & FG\varepsilon \\ -F\varepsilon & FG\varepsilon & F^2\varepsilon \end{bmatrix}, \quad \mathbb{N} = \begin{bmatrix} q \\ -qG \\ -qF \end{bmatrix}, \quad \widehat{Q} = \begin{bmatrix} c & -cG & -cF \\ -cG & cG^2 & cFG \\ -cF & cFG & cF^2 \end{bmatrix}. \tag{1.43}$$

<u>**Step (iv)**</u>: We use the theorem 2-4 developed by Firoozi et al. (2020) to obtain the best response strategy for the minor bank. For the LQG system (1.37) and (1.42) we write down the Gâteaux Derivative:

$$\langle \mathscr{D}J_i^\infty(u), \omega^i\rangle = \mathbb{E}\Big[\int_0^T \omega_t^i \Big\{\mathbb{N}^\intercal X_t^i + u_t^i + \mathbb{B}^\intercal \Big(e^{-\tilde{A}^\intercal t} M_t - \int_0^t e^{\tilde{A}^\intercal (s-t)}(\mathbb{Q}X_s^i + \mathbb{N}u_s^i) ds\Big)\Big\}\Big] dt, \tag{1.44}$$

where M_t is a martingale and

$$M_t^i = \mathbb{E}\Big[e^{\tilde{A}^\intercal}\widehat{Q}X_T^i + \int_0^T e^{\tilde{A}s}(\mathbb{Q}X_s^i + \mathbb{N}u_s^i) ds | \mathscr{F}_s\Big], \tag{1.45}$$

$$
\begin{aligned}
M_t^i &= M_0^i + \int_0^t Z_s^i dW_s^i, \\
dW_t^i &= Z_t^i dW_t^i.
\end{aligned}
\tag{1.46}
$$

By setting the perturbation δ_i in (1.40) to zero, we get the unperturbed extended state dynamics for minor banks.

$$
dX_t^i = \left[\tilde{A}X_t^i + \mathbb{B}u_t^i + \tilde{B}\bar{u}_t\right] dt + \Sigma dW_t^i. \tag{1.47}
$$

From the theorem 3 shown by Firoozi et al. (2020), we could derive the minor bank's optimal control action:

$$
u_t^{i,*} = -\left[\mathbb{N}^\intercal X_t^i + \mathbb{B}^\intercal \left(e^{-\tilde{A}^\intercal} M_t^i - \int_0^t e^{\tilde{A}^\intercal (s-t)} (\mathbb{Q}X_s^i + \mathbb{N}u_s^{i,*}) ds\right)\right]. \tag{1.48}
$$

Then we define the minor bank's adjoint process p_t^i by

$$
p_t^i = e^{-\tilde{A}^\intercal} M_t^i - \int_0^t e^{\tilde{A}^\intercal (s-t)} (\mathbb{Q}X_s^i + \mathbb{N}u_s^{i,*}) ds, \tag{1.49}
$$

and adopt the ansatz

$$
p_t^i = -\frac{1}{q}\Phi_t \mathbb{N}^\intercal X_t^i = \Phi_t \left[\left(F\bar{x}_t + Gx_t^0\right) - x_t^i\right], \tag{1.50}
$$

where

$$
\Phi_t = \begin{bmatrix} \phi_t \\ \psi_t \\ \lambda_t \end{bmatrix}. \tag{1.51}
$$

Hence the optimal control action (1.48) can be written as

$$
\begin{aligned}
u_t^{i,*} &= -\left[\mathbb{N}^\intercal X_t^i + \mathbb{B}^\intercal p_t^i\right] \\
&= -\left[\mathbb{N}^\intercal X_t^i - \frac{1}{q}\mathbb{B}^\intercal \Phi_t N^\intercal X_T^i\right] \\
&= \left(-q + \phi_t\right)x_t^i + \left(qG - \phi_t G\right)x_t^0 + \left(qF - \phi_t F\right)\bar{x}_t \\
&= \left(q - \phi_t\right)\left[\left(F\bar{x}_t + Gx_t^0\right) - x_t^i\right].
\end{aligned}
\tag{1.52}
$$

We average the expression (1.52) and then take its limit as $N \to \infty$ to get the mean-field of the optimal control $\bar{u}_t$:

$$
\begin{aligned}
\bar{u}_t &= q\big[Gx_t^0 + (F-1)\bar{x}_t\big] + \phi_t\big[-Gx_t^0 - (F-1)\bar{x}_t\big], \\
&= \frac{1}{a}\big(q-\phi_t\big)\tilde{B}_o^\intercal \tilde{A}_0 X_t^0, \\
&= \big(q - \mathbb{B}^\intercal \Phi_t\big) K^\intercal X_t^i,
\end{aligned} \tag{1.53}
$$

where

$$
K = \begin{bmatrix} 0 \\ G \\ F-1 \end{bmatrix}. \tag{1.54}
$$

We apply Ito's Lemma to (1.49) and use the martingale representation theorem, and we find the minor bank's adjoint process p_t^i satisfied the following SDE:

$$
dp_t^i = \Big[-\tilde{A}^\intercal p_t^i - \big(\mathbb{Q}X_t^i + \mathbb{N}u_t^{i,*}\big)\Big]dt + e^{-\tilde{A}^\intercal t} Z_t^i dW_t^i. \tag{1.55}
$$

Substituting (1.50) into (1.55) results in:

$$
dp_t^i = \Big[\frac{1}{q}\tilde{A}^\intercal \Phi_t \mathbb{N}^\intercal - \mathbb{Q} + \mathbb{N}\mathbb{N}^\intercal - \frac{1}{q}\mathbb{N}\mathbb{B}^\intercal \Phi_t \mathbb{N}^\intercal\Big] X_T^i dt + e^{-\tilde{A}^\intercal t} Z_t^i dW_t^i. \tag{1.56}
$$

Moreover, we apply Itô's Lemma to (1.50) and the p_t^i satisfied another SDE:

$$
dp_t^i = \Big[-\frac{1}{q}\dot{\Phi}_t \mathbb{N}^\intercal X_t^i - \frac{1}{q}\Phi_t \mathbb{N}^\intercal \big(\tilde{A}X_t^i + \mathbb{B}u_t^i + \tilde{B}\bar{u}_t\big)\Big]dt - \frac{1}{q}\Phi_t \mathbb{N}^\intercal \Sigma dW_t^i. \tag{1.57}
$$

Substituting (1.48) and (1.53) into the drift term of (1.57) and results in:

$$
\begin{aligned}
dp_t^i = \Big[&-\frac{1}{q}\dot{\Phi}_t \mathbb{N}^\intercal - \frac{1}{q}\Phi_t \mathbb{N}^\intercal \tilde{A} + \frac{1}{q}\Phi_t \mathbb{N}^\intercal \mathbb{B}\mathbb{N}^\intercal - \frac{1}{q^2}\Phi_t \mathbb{N}^\intercal \mathbb{B}\mathbb{B}^\intercal \Phi_t \mathbb{N}^\intercal \\
&- \frac{1}{q}\big(q - \mathbb{B}^\intercal \phi_t\big)\Phi_t \mathbb{N}^\intercal \tilde{B} K^\intercal\Big] X_t^i dt - \frac{1}{q}\Phi_t \mathbb{N}^\intercal \Sigma dW_t^i.
\end{aligned} \tag{1.58}
$$

Then we match the two SDE (1.56) and (1.58) to get the two conditions that Φ_t satisfies as in

- Diffusion term:

$$
-\frac{1}{q}\Phi_t \mathbb{N}^\intercal \Sigma = e^{-\tilde{A}^\intercal t} Z_t^i. \tag{1.59}
$$

- Drift term:

$$-\frac{1}{q}\dot{\Phi}_t\mathbb{N}^\intercal - \frac{1}{q}\Phi_t\mathbb{N}^\intercal\tilde{A} + \frac{1}{q}\Phi_t\mathbb{N}^\intercal\mathbb{B}\mathbb{N}^\intercal - \frac{1}{q^2}\Phi_t\mathbb{N}^\intercal\mathbb{B}\mathbb{B}^\intercal\Phi_t\mathbb{N}^\intercal - \frac{1}{q}\big(q-\mathbb{B}^\intercal\Phi_t\big)\phi_t\mathbb{N}^\intercal\tilde{B}K^\intercal$$
$$= \frac{1}{q}\tilde{A}^\intercal\Phi_t\mathbb{N}\intercal - \mathbb{Q} + \mathbb{N}\mathbb{N}^\intercal - \frac{1}{q}\mathbb{N}\mathbb{B}^\intercal\Phi_t\mathbb{N}^\intercal. \tag{1.60}$$

In this section, we characterized the optimal control actions of minor banks which are used for deriving the mean-filed of the control actions $\bar{u}_t$. This will be used in the next section to complete the solution of the major bank's problem.

(c) Major Bank Problem

We recall some results from the previous section and do some calculation for the major bank's LQG system. Then we derive the optimal control for the major bank.

- Substitute $\bar{u}_t$ (1.53) into unperturbed extended dX_t^0 (1.35) to to get the extended state dynamics:

$$\begin{aligned} dX_t^0 &= \left\{\left[\tilde{A}_0 + \frac{1}{a}(q-\phi_t)\tilde{B}_0\tilde{B}^\intercal\tilde{A}_0\right]X_t^0 + \mathbb{B}_0u_t^0\right\}dt + \Sigma_0 dW_t^0 \\ &= [\mathbb{A}_0X_t^0 + \mathbb{B}_0u_t^0]dt + \Sigma_0 dW_t^0, \end{aligned} \tag{1.61}$$

  where

$$\mathbb{A}_0 = \tilde{A_0} + \frac{1}{a}(q-\phi_t)\tilde{B}_0\tilde{B}^\intercal\tilde{A}_0. \tag{1.62}$$

- The cost function from the previous section:

$$J^0(u^0) = \tfrac{1}{2}\mathbb{E}\left[\int_0^T \left\{(X_s^0)^\intercal\mathbb{Q}_0X_s^0 + 2(X_s^0)^\intercal\mathbb{N}_0(u_s^0) + (u_s^0)^2\right\}ds + (X_T^0)^\intercal\mathbb{G}_0X_T^0\right], \tag{1.63}$$

<u>Step (i)</u>: Write down the Gâteaux Derivative of major bank:

$$\langle\mathscr{D}J_0^\infty(u),\omega^0\rangle = \mathbb{E}\left[\int_0^T\omega_t^0\left\{\mathbb{N}^\intercal X_t^0 + u_t^0 + \mathbb{B}^\intercal\left(e^{-\mathbb{A}^\intercal t}M_t^0 - \int_0^t e^{\mathbb{A}^\intercal(s-t)}(\mathbb{Q}_0X_s^i + \mathbb{N}_0u_s^i)ds\right)\right\}\right]dt. \tag{1.64}$$

where M_t^0 is a martingale as in

$$M_t^0 = \mathbb{E}\Big[e^{\mathbb{A}_0^\intercal}\mathbb{G}_0 X_T^0 + \int_0^T e^{\mathbb{A}_0^\intercal s}(\mathbb{Q}_0 X_s^0 + \mathbb{N}_0 u_s^0)ds|\mathscr{F}_s\Big], \tag{1.65}$$

and by the martingale representation theorem we have

$$\begin{aligned} M_t^0 &= M_0^0 + \int_0^t Z_s^0 dW_s^0, \\ dW_t^0 &= Z_t^0 dW_t^0. \end{aligned} \tag{1.66}$$

Step (ii): We obtain the optimal control action for the major bank in the infinite-population limit. Since the $\mathscr{D}J_0^\infty(u)$ has the similar structure as $\mathscr{D}J_(u)$ in the Theorem 3 developed by Firoozi et al. (2020), we could directly write out the optimal control action

$$u_t^{0,*} = -\Big[\mathbb{N}_0^\intercal X_t^0 + \mathbb{B}_0^\intercal\Big(e^{-\mathbb{A}_0^\intercal}M_t^0 - \int_0^t e^{\mathbb{A}_0^\intercal(s-t)}(\mathbb{Q}_0 X_s^0 + \mathbb{N}_0 u_s^{0,*})ds\Big)\Big]. \tag{1.67}$$

Step (iii): We derive the State feedback control action for the major bank. To this purpose we define the major bank's adjoint process p_t^0 by

$$p_t^0 = e^{-\mathbb{A}_0^\intercal}M_t^0 - \int_0^t e^{\mathbb{A}_0^\intercal(s-t)}(\mathbb{Q}_0 X_s^0 + \mathbb{N}_0 u_s^{0,*})ds. \tag{1.68}$$

We adopt the ansatz

$$p_t^0 = -\frac{1}{q_0}\Phi_t^0\mathbb{N}_0^\intercal X_t^0 = \Phi_t^0\big(\bar{x}_t - x_t^0\big), \tag{1.69}$$

where

$$\Phi_t^0 = \begin{bmatrix} \phi_t^0 \\ \psi_t^0 \end{bmatrix}. \tag{1.70}$$

Substituting (1.68) and (1.69) into (1.71):

$$\begin{aligned} u_t^{0,*} &= -\Big[\mathbb{N}_0^\intercal X_t^0 + \mathbb{B}_0^\intercal p_t^0\Big] \\ &= -\Big[\mathbb{N}_0^\intercal X_t^0 + \mathbb{B}_0^\intercal\Phi_t^0\big(\bar{x}_t - x_t^0\big)\Big]. \\ &= \big(q - \phi_t^0\big)\big(F\bar{x}_t - x_t^0\big) \end{aligned} \tag{1.71}$$

We apply Ito's lemma to (1.68) and using the martingale representation theorem to get the SDE that the major bank's adjoint process satisfies as in

$$dp_t^0 = \Big[-\mathbb{A}_0^\intercal p_t^0 - \Big(\mathbb{Q}_0 X_t^0 + \mathbb{N}_0 u_t^{0,*}\Big)\Big]dt + e^{-\mathbb{A}_0^\intercal t} Z_t^0 dW_t^0. \tag{1.72}$$

Then we replace (1.69) and (1.71) into the (1.72) to get

$$dp_t^0 = \Big[-\frac{1}{a_0}\mathbb{A}_0^\intercal \Phi_t^0 \mathbb{B}_0^\intercal \tilde{A}_0 X_t^0 - \mathbb{Q}_0 X_t^0 + \mathbb{N}_0 \mathbb{N}_0^\intercal X_t^0 + \frac{1}{a_0}\mathbb{N}_0 \mathbb{B}^\intercal \Phi_t^0 \mathbb{B}_0^\intercal \tilde{A}_0 X_t^0\Big]dt + e^{-\mathbb{A}_0^\intercal t} Z_t^0 dW_t^0. \tag{1.73}$$

Moreover, we apply Ito's Lemma to (1.69) to obtain another SDE that p_t^0 satisfies as in:

$$\begin{aligned} dp_t^0 = \Big\{\dot{\Phi}_t^0\big(\bar{x}_t - x_t^0\big) + \Phi_t^0\big[a(F-1)\bar{x}_t + aGx_t^0 + \bar{u}_t\big] - \Phi_t^0\Big(a_0\bar{x}_t - a_0 x_t^0 + u_t^0\Big)\Big\}dt \\ -\phi_t^0 \sigma_0 dW_t^0. \end{aligned} \tag{1.74}$$

We then rewrite (1.74) in the extended form and replace (1.71) and (1.53) to get

$$\begin{aligned} dp_t^0 = \Big[\big(\frac{1}{a_0}\dot{\Phi}_t^0 - \Phi_t^0 + \frac{1}{a_0}\Phi_t^0 \mathbb{B}^\intercal \phi_t^0\big)\mathbb{B}_0^\intercal \tilde{A}_0 X_t^0 + \Phi_t^0 \tilde{B}_0^\intercal \tilde{A}_0 X_t^0 + \frac{1}{a}\big(q - \phi_t\big)\Phi_t^0 \tilde{B}_0^\intercal \tilde{A}_0 X_t^0 \\ + \Phi_t^0 \mathbb{N}_0^\intercal X_t^0\Big]dt + \big(-\Phi_t^0 \sigma_0\big)dW_t^0. \end{aligned} \tag{1.75}$$

Finally we match the two SDEs (1.73) and (1.75) to get the two conditions that the Φ_t^0 must satisfy, i.e.

- Diffusion term:

$$e^{-\mathbb{A}_0^\intercal t} Z_t^0 = -\Phi_t^0 \sigma_0. \tag{1.76}$$

- Drift term:

$$\begin{aligned} \big(\frac{1}{a_0}\dot{\Phi}_t^0 - \Phi_t^0 + \frac{1}{a_0}\Phi_t^0 \mathbb{B}^\intercal \Phi_t^0\big)\mathbb{B}_0^\intercal \tilde{A}_0 + \Phi_t^0 \tilde{B}_0^\intercal \tilde{A}_0 + \frac{1}{a}\big(q - \phi_t\big)\Phi_t^0 \tilde{B}_0^\intercal \tilde{A}_0 + \Phi_t^0 \mathbb{N}_0^\intercal = \\ -\frac{1}{a_0}\mathbb{A}_0^\intercal \Phi_t^0 \mathbb{B}_0^\intercal \tilde{A}_0 - \mathbb{Q}_0 + \mathbb{N}_0 \mathbb{N}_0^\intercal + \frac{1}{a_0}\mathbb{N}_0 \mathbb{B}^\intercal \Phi_t^0 \mathbb{B}_0^\intercal \tilde{A}_0. \end{aligned} \tag{1.77}$$

To conclude, we derived the optimal trading strategies for the major bank and a representative minor bank $\mathscr{A}_i$ given, respectively, by (1.71), (1.77), and (1.52), (1.60). We can then exploit the structure of system matrices to simplify the optimal strategies and the associated ODEs by performing simple matrix multiplications. This gives rise to a reduced representation of the optimal trading strategies as in

$$\begin{aligned} u_t^{0,*} &= \left(q-\phi_t^0\right)\left(F\bar{x}_t - x_t^0\right) \\ \dot{\phi}_t^0 &= 2(a_0+q_0)\phi_t^0 - \left(\phi_t^0\right)^2 + \left(a+q+\phi_t\right)G\left(\phi_t^0\right) + \varepsilon_0 - q_0^2 \\ \phi_T^0 &= -c_0 \end{aligned} \tag{1.78}$$

$$\begin{aligned} u_t^{i,*} &= \left(q-\phi_t\right)\left[\left(F\bar{x}_t + Gx_t^0\right) - x_t^i\right] \\ \dot{\phi}_t &= 2(a+q)\phi_t - \left(\phi_t\right)^2 + \varepsilon - q^2 \\ \phi_T &= -c, \end{aligned} \tag{1.79}$$

1.6 Individual Default and Systemic Risk

The most basic idea of banking industry is to bridge the counterparties who want to borrow and lend. They provide the channel making the monetary resources in the system more flexible and used in a more efficient way. Out of profitability, it is natural for banks to absorb the short-term funds and make long-term loans which is a strategy helping them to profit from the difference between the interest rates of various maturities. However, this strategy could also put a bank in a dangerous position where it does not hold enough liquidity to pay back the withdrawal requests initiated by its customers. Usually, banks also use some financial instruments in the interbank market such as credit lines to make up the deficit in their balance sheet. However, the liquidity of a bank can dry up when there exist some negative indicators shown in its daily operation . In the event of a liquidity shortage, a bank faces the default risk.

Because of the complex and close interactions of banks in an interbank market, there exists a phenomenon called "Financial Contagion" in the financial system, just like what happened in the 2008 global financial crisis. The default risk faced by one bank may lead to a bank run as the public is more comfortable with holding cash in their hand instead of depositing it in the bank. If the panic quickly spreads to the whole market, it is very likely to lead to liquidity black holes and disorders in the financial system. This in turn means that more banks will get in trouble. Since globalization has been deepening, almost all markets worldwide are connected with each other through monetary and financial systems. Hence the risk in a bank industry may evolve into a world-wide disaster which disrupts the growth of global GDP.

In general the default of a bank in the financial environment is defined as the situation where it is unable to fulfill the timely repayment its liability or to afford it after selling all the assets. In this work, motivated by Carmona et al. (2015), we consider a simpler definition of default as the scenario where the log-monetary reserve of one bank goes below a specific value which is called the default threshold. We regard the systemic event as a circumstance where the market average log-monetary reserve falls in the default region. The market average log-monetary reserve or state, $F\bar{x}_t + Gx_t^0$, is defined as a linear combination of the major bank's log-monetary reserve and the average log-monetary reserve of the mass of minor banks.

Now we present the mathematical definition of the default probabilities of interest. We denote the default threshold by D, to which we assign an exogenous default threshold $D = -0.65$ derived from Section 1.8. We then define the default probability of bank-i, $i \in \{1, \dots\}$, as in

$$p_i = \mathbb{P}\left(\min_{t\in[0,T]} (x_t^i) \;\le\; D\right). \tag{1.80}$$

Subsequently the probability of systemic event, or equivalently, systemic risk, is defined by

$$p_{SE} = \mathbb{P}(\text{systemic event}) = \mathbb{P}\left(\min_{t\in[0,T]} \left(F\bar{x}_t + Gx_t^0\right) \;\le\; D\right). \tag{1.81}$$

In particular we are interested in the difference that the presence of a major bank makes in the default of probability p_i of a representative small bank-i, $i \in \{1, 2, \dots\}$ and in the systemic risk p_{SE}. The results where there is no major bank in the economy is available in the literature (see e.g. (Carmona et al., 2015; Fouque and Ichiba, 2013; Fouque and Sun, 2013)) and permits us to perform a comparison. Furthermore, to better understand the role of the major bank on the resilience of the interbank market, we define the aforementioned default probabilities conditioned on the event that the major bank defaults or otherwise in the market. Mathematically speaking, for $i \in \{1,2,\dots\}$, we are interested in

$$p_{i|MD} = \mathbb{P}\left(\min_{t\in[0,T]} (x_t^i) \;\leq\; D|\text{major bank defaults on } [0,T]\right), \tag{1.82}$$

$$p_{i|MS} = \mathbb{P}\left(\min_{t\in[0,T]} (x_t^i) \;\leq\; D|\text{major bank does not default on } [0,T]\right). \tag{1.83}$$

Similarly we are interested in the conditional systemic risks as in

$$p_{SE|MD} = \mathbb{P}\left(\min_{t\in[0,T]} \left(F\bar{x}_t + Gx_t^0\right) \;\leq\; D|\text{major bank defaults on } [0,T]\right), \tag{1.84}$$

$$p_{SE|MS} = \mathbb{P}\left(\min_{t\in[0,T]} \left(F\bar{x}_t + Gx_t^0\right) \;\leq\; D|\text{major bank does not default on } [0,T]\right). \tag{1.85}$$

Clearly, du to the law of total probability, the introduced probabilities satisfy

$$p_i = \left(p_{i|MD} - p_{i|MS}\right) \times \mathbb{P}(\text{major bank defaults on } [0,T]) + p_{i|MS}, \tag{1.86}$$

$$p_{SE} = \left(p_{SE|MD} - p_{SE|MS}\right) \times \mathbb{P}(\text{major bank defaults on } [0,T]) + p_{SE|MS}. \tag{1.87}$$

We aim to investigate both total and conditional probabilities of a representative minor bank's default and the systemic event in various scenarios. More specifically, we study how these probabilities change with the relative size G and the mean reversion rate a_0 of the major bank using Monte Carlo simulations presented in the following section.

1.7 Numerical experiments

In Section 1.3 we present a game formulation of an interbank market consisting of a large number of minor banks and one major bank. We then aim to find the best-response strategies of banks such that they yield an equilibrium for the market. Due to the presence of a large number of agents and their interactions, solving such a problem is mathematically intractable in general. We use the mean field methodology to address this problem by solving the infinite-population version of the game when the number of agents goes to infinity. Then we show that the limiting strategies yield an ε-Nash equilibrium for the original finite-population problem. What we care about is the presence and position of the major bank on the behavior of the minor banks and the entire system.

In this section first we numerically implement the infinite-population system. Then we estimate the default probabilities introduced in Section 1.6 using Monte Carlo simulations. Moreover, we illustrate the sample trajectories of banks and the market state. Next we implement the finite population system. We show that the infinite-population system (and hence the limiting best-response strategies) provides a good approximation for the finite-population system through some illustrations. Subsequently we look into the default probability of a representative minor bank and systemic risks. Furthermore, we illustrate sample trajectories and depict the loss distribution of minor agents in various scenarios detailed below.

In a more technical level, in both infinite and finite population cases, we set the default threshold $D = -0.65$[1]. We assume all banks still stay in the system and continue to lend and borrow until the end of time period even though they have reached the default threshold. We use regression analysis to estimate $p_i, p_{i|MD}, p_{i|MS}, p_{SE}, p_{SE|MD}, p_{SE|MS}$ respectively by $\bar{p}_i, \bar{p}_{i|MD}, \bar{p}_{i|MS}, \bar{p}_{SE}, \bar{p}_{SE|MD}, \bar{p}_{SE|MS}$ for the cases where the relative size G and the mean reversion rate a_0 of the major bank changes. The details of the regression models used can be found in Section 1.8.

1.7.1 Infinite Population

For the numercial analysis in the following subsections, we simulate 10^4 minor banks in the economy and perform 5000 simulations for each market setting to estimate the discussed default probabilities using Monte Carlo simulations. The results are summarized in three subsections: 1.7.2.1. Impact of Relative Size of Major Bank, 1.7.2.2. Impact of Mean-Reversion Rate, and 1.7.2.3. Mean-Reversion Rate Across Time.

1.7.2.1 Impact of Relative Size of Major Bank

According to the model we described in Section 1.4, each minor bank wishes to track the average log-monetary reserve in the market or the market state $\left(F\bar{x}_t + Gx_t^0\right)$. The

[1]The method used to collect the default threshold is described in Section 1.8.

market state is a weighted average of the major bank's log-monetary reserve and the mean field of log-monetary reserves. The parameters G and F denote, respectively, the relative sizes of major bank and the mean field such that $G+F=1$. We study how changing the relative size G of the major bank in the set $\{0.1, 0.2, 0.3, \ldots, 0.9\}$ affects total and conditional default probabilities $p_i, p_{i|MD}, p_{i|MS}, i \in \{1, 2, \ldots\}$, and systemic risks $\bar{p}_{SE}, \bar{p}_{SE|MD}, \bar{p}_{SE|MS}$.

(a) Default Probability of a Representative minor bank

We use linear regression[2] to examine the variation of the default probability of a representative minor bank-i, $i \in \{1, \ldots, \infty\}$, for different values of $G \in \{0.1, 0.2, \ldots, 0.9\}$. The obtained results are summarized in Table 1.1. We note that the first column in Table 1.1 is the estimated default probability of a representative small bank in an interbank market where the is no major bank (see e.g. (Carmona et al., 2015; Fouque and Ichiba, 2013; Fouque and Sun, 2013)). By comparing this default probability with that of an interbank market with a major bank, we can better understand the impact of the major agent.

From the estimated values shown in Table 1.1, the average default probability for a representative minor bank is around 0.2363 in the absence of a major bank. However, we observe that the total default probability $\bar{p}_i$ of the minor bank generally increases when there exists a major bank in the market. Moreover, the larger the relative size G of the major bank, the larger is the default probability $\bar{p}_i$. Initially, there seems to be no benefit in having a major bank in the interbank market. By looking more closely into the total default probability $\bar{p}_i$, we can disentangle it into the weighted average of the two conditional probabilities $\bar{p}_{i|MD}$, $\bar{p}_{i|MS}$ depending on whether the major bank defaults or not. We find that the default probability of the minor bank in the system with a defaulting major bank, i.e. $\bar{p}_{i|MD}$, is much higher than the case without a major agent. This could be explained by the fact that if the major bank defaults it may drag down minor banks and this impact is stronger when the major bank has a larger size. Hence the small banks are

[2] See the "Regression Model 1" in Section 1.8.

G	No Major Bank	With a Major Bank		
		Total	Conditional	
			Non-defaulting Major	Defaulting Major
		$\bar{p}_i$	$\bar{p}_{i\|MS}$	$\bar{p}_{i\|MD}$
0	0.2363	-	-	-
0.1	-	0.2531	0.2082	0.3336
0.2	-	0.2877	0.1962	0.4484
0.3	-	0.3120	0.1921	0.5389
0.4	-	0.3417	0.1925	0.6206
0.5	-	0.3584	0.1997	0.6558
0.6	-	0.3727	0.1954	0.6974
0.7	-	0.3896	0.2044	0.7423
0.8	-	0.3919	0.2011	0.7601
0.9	-	0.4103	0.2084	0.7839

Table 1.1: Estimated default probability of a representative minor bank in the infinite population limit for the cases (from left to right) with (i) no major bank, (ii) total default probability ($\bar{p}_i$) with a major bank, (iii) conditional default probability ($\bar{p}_{i|MS}$) with a non-defaulting major bank, and (iv) conditional default probability ($\bar{p}_{i|MD}$) with a defaulting major bank

more likely to wind up in the default area. However, we observe that the presence of a non-defaulting major bank does slightly improve the position of the representative small bank. In this case the average default probability $\bar{p}_{i|MS}$ decreases to around 0.2 in the system. This positive impact holds true even when the relative size G of the major bank is 0.1 with respect to the mean field size $F = 0.1$.

From these simulations we can conclude that a major bank has two opposing effects on the default probability of a representative small bank. On the one hand, a successful major bank improves slightly the position of the small bank, as it can provide additional liquidity when the small bank needs money to cover a liquidity shortage. On the other hand, the huge negative externality that would arise from the failure of a major bank puts the small bank in a more precarious situation. The ultimate status of the minor bank is decided by the dominant influence of these two aspects. In the settings we considered, the negative impact dominates the benefits obtained from having a major bank in the system. Hence the total default probability $\bar{p}_i$ of the minor bank (shown in the second column of

G	No Major Bank	With a Major Bank		
		Total	Conditional	
			Non-defaulting Major	Defaulting Major
		$\bar{p}_{SE}$	$\bar{p}_{SE\|MS}$	$\bar{p}_{SE\|MD}$
0	0	-	-	-
0.1	-	0	0	0
0.2	-	0.0014	0	0.0039
0.3	-	0.0254	0	0.0735
0.4	-	0.0690	0	0.1980
0.5	-	0.1094	0	0.3144
0.6	-	0.1680	0	0.4757
0.7	-	0.2104	0	0.6109
0.8	-	0.2548	0	0.7463
0.9	-	0.3082	0	0.8786

Table 1.2: Estimated probability of systemic event in the infinite population limit for the cases (from left to right) with (i) no major bank, (ii) total default probability ($\bar{p}_{SE}$) with a major bank, (iii) conditional default probability ($\bar{p}_{SE|MS}$) with a non-defaulting major bank, and (iv) conditional default probability ($\bar{p}_{SE|MD}$) with a defaulting major bank

Table 1.1) in the presence of a major bank is higher than the case without a major bank in the interbank market.

(b) Probability of Systemic Event

We use a linear regression[3] to estimate the total and conditional probabilities of the systemic event. The results are shown in Table 1.2.

We first investigate the probability of systemic event in the scenario without a major bank. In this case the market state is equal to the mean field of log-monetary reserves ($\bar{x}_t$). In the numerical experiments all the banks in the system start borrowing and lending with zero log-monetary reserves at the beginning of the time period. This setting leads to a special case where the mean-field does not change and stays at zero during the whole period. Therefore, the market state will never reach the default threshold. This could be seen from the mean-filed equation (1.23). Therefore the estimated systemic risk is equal

[3]See the "Regression Model 2" in Section 1.8.

to zero as shown in the first column of Table 1.2.

For the scenarios with a major bank shown in the second column of Table 1.2, the total probability $\bar{p}_{SE}$ of the systemic event is zero when the relative size of the major bank is given by $G = 0.1$. However, for the larger relative sizes $\bar{p}_{SE}$ becomes positive and increases with the size G. Now if we examine the conditional systemic risk $\bar{p}_{SE|MS}$ given a non-defaulting major bank in the third column of Table 1.2, we observe that it is always equal to zero no matter the relative size G. Therefore a successful major bank makes the system stable and hence the systemic risk zero in the infinite-population limit. It is worth mentioning that this result is slightly different for the finite-population case with only 10 small banks. For more details we refer the reader to Section 1.7.2. We now examine the systemic risk $\bar{p}_{SE|MD}$ for the case with a defaulting major bank in the last column of Table 1.2. We observe that the presense of a defaulting major bank increases $\bar{p}_{SE|MD}$. Moreover, the larger the relative size G of the major bank, the higher is $\bar{p}_{SE|MD}$.

Hence, although with the presence of a successful major bank the systemic risk is zero, if it goes bankrupt the likelihood of a financial catastrophe is much higher.

(c) Trajectories of Banks

We plot the simulated trajectories of log-monetary reserves for 10 representative small banks and the major agent, and the market state in Fig. 1.1. As can be seen, when the relative size G of the major bank increases, the trajectories of the minor banks, the major bank and the market state evolve more closely to each other. Moreover, the failure of the major bank drags down the whole system quickly and makes it a disaster. This is while for a smaller size $G = 0.1$ of the major bank, despite its default the market state remains above the default threshold. These plots confirm the fact that the failure of a major bank with a larger relative size G has a larger detrimental effect as it significantly increases the default probability of a representative minor bank and the risk of a financial disaster.

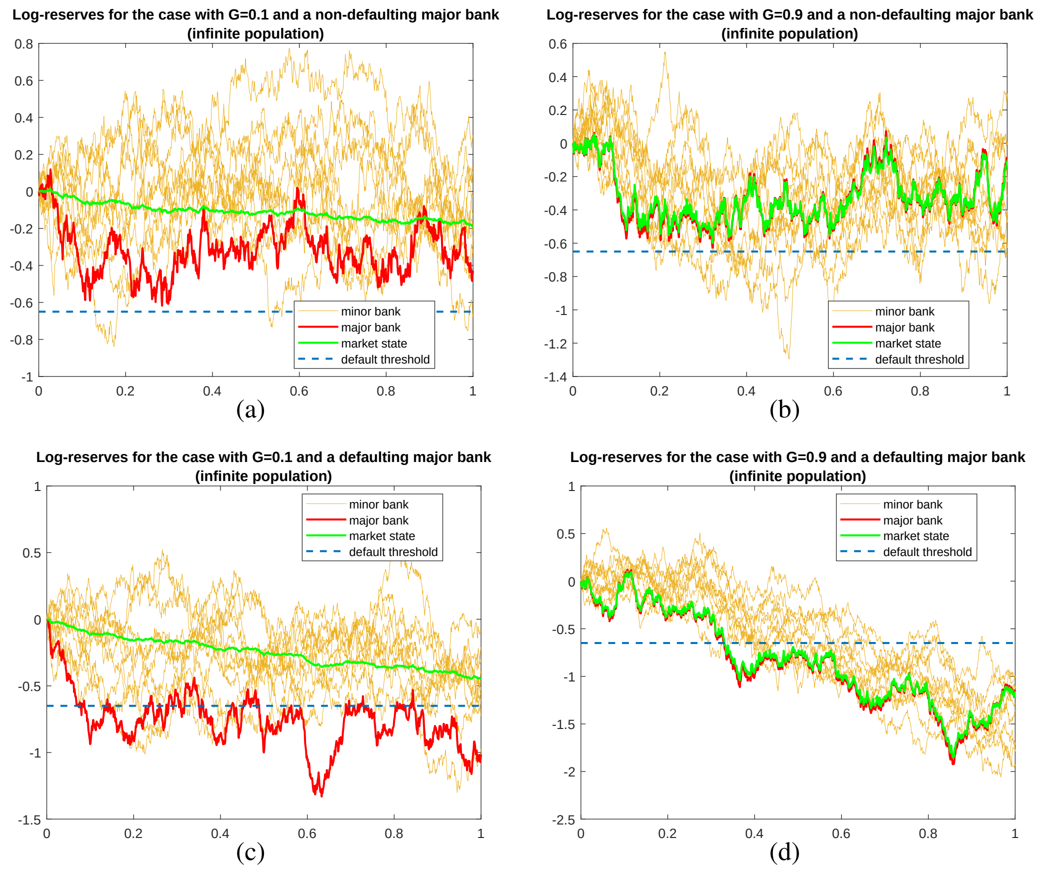


Figure 1.1: Simulated trajectories for 10 representative minor banks, the major bank and the market state in the infinite-population limit for the cases with ($a = 5$): (a) $G = 0.1$ and a non-defaulting major bank, (b) $G = 0.9$ and a non-defaulting major bank, (c) $G = 0.1$ and a defaulting major bank, and (d) $G = 0.9$ and a defaulting major bank. In all cases.

1.7.2.2 Impact of Mean-Reversion Rate

In this section, we assume that the major bank and the mean field (mass) of minor banks are of the same size ($F = G = 0.5$). We then examine the impact of mean-reversion rate in interbank transactions on the individual default probability and systemic risk. We note that the mean reversion rate a of a representative minor bank is considered to be different from that a_0 of the major bank due to their different characteristics and position in the market. However, they are related through the clearing condition (1.6b) which for our case gives rise to $a_0 = 0.5a$. From the financial perspective, a higher mean-reversion rate translates to a higher frequency in lending and borrowing activities. Hence the major

a	No Major Bank	With a Major Bank		
		Total	Conditional	
			Non-defaulting Major	Defaulting Major
		$\bar{p}_i$	$\bar{p}_{i\|MS}$	$\bar{p}_{i\|MD}$
1	0.4030	0.4379	0.3147	0.6086
2	0.3668	0.4199	0.2859	0.6247
3	0.3251	0.3932	0.2511	0.6392
4	0.2806	0.3794	0.2260	0.6570
5	0.2362	0.3580	0.1928	0.6579
6	0.1941	0.3428	0.1766	0.6716
7	0.1561	0.3340	0.1547	0.7007
8	0.1230	0.3148	0.1353	0.6959
9	0.0952	0.3159	0.1226	0.7131
10	0.0725	0.2966	0.1112	0.7190

Table 1.3: Estimated default probability of a representative minor bank in the infinite population limit for the cases (from left to right) with (i) no major bank, (ii) total default probability ($\bar{p}_i$) with a major bank, (iii) conditional default probability ($\bar{p}_{i|MS}$) with a non-defaulting major bank, and (iv) conditional default probability ($\bar{p}_{i|MD}$) with a defaulting major bank

bank trades in a lower frequency than a representative minor bank given the same distance from their respective tracking signal, respectively, $\bar{x}_t$ and $0.5(x_t^0 + \bar{x}_t)$. This could be due to some market frictions and conditions explained in Section 1.3.3. To investigate the impact of the mean reversion rates on the system we change the value of a in the set $\{1, 2, \ldots, 10\}$ and summarize the results in the following sections.

(a) Default Probability of a Representative Minor Bank

In this part, we investigate the effect of increasing the mean-reversion rate a (or equivalently a_0) on the default probability of a representative minor bank. We use a linear regression model, the details of which is provided in Section 1.8. The estimated probabilities are shown in Table 1.3.

From the first column of Table 1.3, the default probability of a minor bank decreases as the mean-reversion rate a increases in a market without a major bank. Hence when a minor bank trades with a higher frequency, it is less likely to end up in the default region. A similar impact is observed in an interbank market where a major bank is present (see the second column of Table 1.3). As the mean-reversion rate increases, the total default probability of a minor bank decreases. However, we observe that, in comparison with the case without a major bank, (i) the amount of decrease is not as large, and (ii) the default probability for each value of a is higher. These observations can be justified by examining the conditional default probabilities. According to the third columns of Table 1.3, a successful major bank could improve the status of a minor bank to some extent. However, if the major bank gets in trouble, the default probability of a minor bank increases with a. This is because the major bank's log-reserve is reflected in the market state, which a minor bank seeks to follow as closely as it can. Therefore when a minor bank increases its frequency of trading (mean-reversion rate), it tends to converge more quickly to the market state brought down by a defaulting major bank. Hence, this negative externality offsets the benefit the minor bank obtains from having a major bank in the market. Therefore, the total default probability is higher than that of a market without a major bank for each value of a, and it decreases with a with a smaller slope.

(b) Probability of Systemic Risk

We follow a similar approach and use a regression model detailed in Section 1.8 to investigate the impact of mean reversion rate on systemic risk. The results are summarized in Table 1.4.

The zero values for the systemic risk in a market without a major bank in the first column of Table 1.4 can be explained in the same manner as in Section 1.7.1. For the case where there exists a major bank, the total probability of systemic event slightly increases with the mean-reversion rate a. We also observe that the conditional systemic risk given a non-defaulting major bank in the market is zero. However when the major bank goes bankrupt, the conditional systemic risk increases significantly.

(c) Trajectories of Banks

We plot the simulated trajectories for the log-monetary reserves of 10 representative minor banks, the major bank, and the market state for the cases with $a = 1$ and $a = 10$ in Fig. 1.2.

a	No Major Bank	With a Major Bank		
		Total	Conditional	
			Non-defaulting Major	Defaulting Major
		$\bar{p}_{SE}$	$\bar{p}_{SE\|MS}$	$\bar{p}_{SE\|MD}$
1	0	0.1158	0	0.2762
2	0	0.1056	0	0.2671
3	0	0.1094	0	0.2989
4	0	0.1082	0	0.3041
5	0	0.1202	0	0.3384
6	0	0.1126	0	0.3353
7	0	0.1230	0	0.3745
8	0	0.1198	0	0.3741
9	0	0.1300	0	0.3971
10	0	0.1286	0	0.4157

Table 1.4: Estimated probability of systemic event in the infinite population limit for the cases (from left to right) with (i) no major bank, (ii) total default probability ($\bar{p}_{SE}$) with a major bank, (iii) conditional default probability ($\bar{p}_{SE|MS}$) with a non-defaulting major bank, and (iv) conditional default probability ($\bar{p}_{SE|MD}$) with a defaulting major bank

Our results are consistent with those in (Fouque and Sun, 2013). When the mean-reversion rate 'a' increases from 1 to 10, there is a larger flocking effect such that the trajectories of minor banks evolve much more closely to each other. Moreover, from panel (c) and (d) of Fig. 1.2, we observe that a higher mean-reversion rate could delay the default of the major bank. However, when the major bank goes bankrupt it drags down the market state and hence the minor banks more quickly.

1.7.2.3 Mean-Reversion Rate Across Time

In the interbank market modelled in Section 1.4, a representative minor bank and the major bank implement their best-response strategy (derived in Section 1.5) to minimize the costs they incur by transacting in the market. After substituting the optimal strategies of the major bank (1.19) and a representative minor bank (1.52) in their respective dynamics of the log-monetary reserves (respectively, (1.1) and (1.4)), we can view how transactions with the central bank affect the log-monetary reserves. We find that the optimal strategies increase the mean-reversion rate by adding a time-varying component

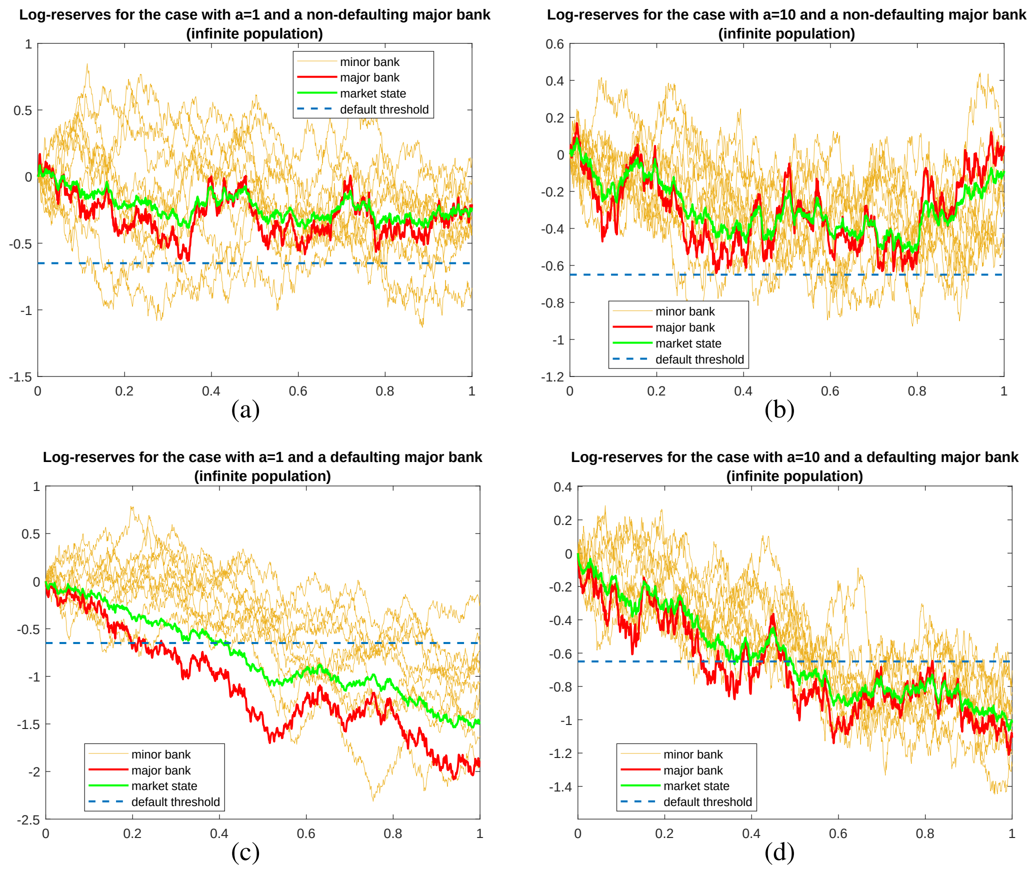
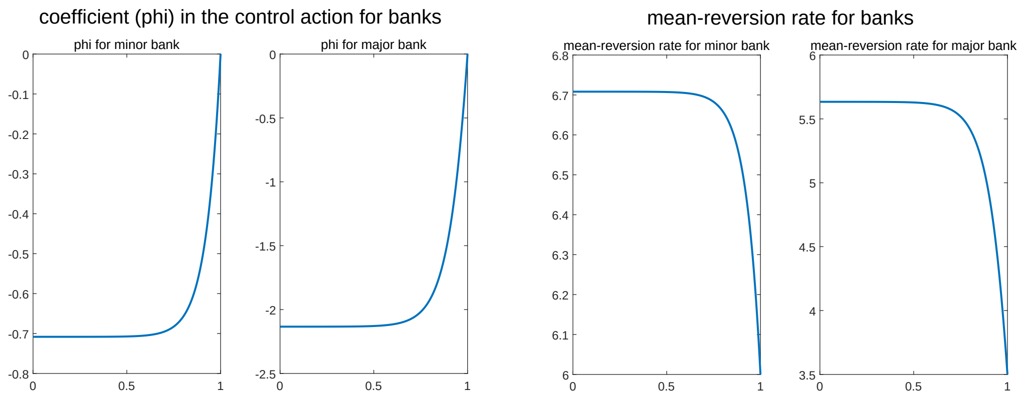


Figure 1.2: Simulated Trajectories for 10 representative minor banks, the major bank and the market state in the infinite-population limit for the cases with $(G = 0.5, F = 0.5)$: (a) $a = 1$ and a non-defaulting major bank, (b) $a = 10$ and a non-defaulting major bank, (c) $a = 1$ and a defaulting major bank, and (d) $a = 10$ and a defaulting major bank

$\left(q_0 - \phi_t^0\right)$ and $\left(q - \phi_t\right)$, respectively, for the major bank and a representative minor bank. The evolution of ϕ_t^0 and ϕ_t over time is depicted in panel (a) of Fig. 1.3. Moreover the evolution of the total mean reversion rates $\left(a + q - \phi_t\right)$ and $\left(a_0 + q_0 - \phi_t^0\right)$ is depicted in panel (b) of Fig. 1.3. We observe that the presence of a central bank provides the market participants with extra liquidity and increases the frequency of their transaction activities (note that $q - \phi_t > 0$ and $q_0 - \phi_t^0 > 0$). In our model, banks only trade during a fixed time period $[0, T]$ and they are not concerned about what happens after T. Banks start borrowing and lending activities with a higher mean-reversion rate since they can not forecast too far away. However, as time elapses, they are approaching the end of the trading horizon



(a) ϕ_t *and* ϕ_t^0 (b) $\left(a+q-\phi_t\right)$ *and* $\left(a_0+q_0-\phi_t^0\right)$

Figure 1.3: Simulation Results (from left to right): (a) solution of the ODE system, and (b) the mean-reversion level of different banks after adding the optimal control ($a=5, a_0=2.5, F=0.5, G=0.5, q=q_0=1$)

and the uncertainty in the market decreases. Hence they reduce their trading frequency naturally.

1.7.2 Finite Population

In this section we model the financial system with a finite number of minor banks described by (1.1)-(1.5) in Section 1.3. The strategies taken by banks in the finite population are the mean field strategies obtained in the infinite-population limit, where the mean field $\bar{x}_t$ is replaced by the empirical average $x_t^{(N)}$ of the log-monetary reserves of minor banks and the finite-population log-monetary reserves are used. We consider a setup where there are 10 minor banks and one major bank in the financial system. In the mean field game methodology, the equilibrium of a large population game is approximated by the Nash equilibrium of the limiting game when the number of agents goes to infinity. To show the quality of this approximation for our setup,we plot the mass effect of minor banks and the market states in both finite and infinite population cases Fig. 1.4. We observe that the trajectories of the mass effect $(x_t^{(N)}, \bar{x}_t)$ and those of the market state $(F\bar{x}_t+Gx_t^0, Fx_t^{(N)}+Gx_t^0)$ evolve closely. Hence the behaviour of the system in the infinite

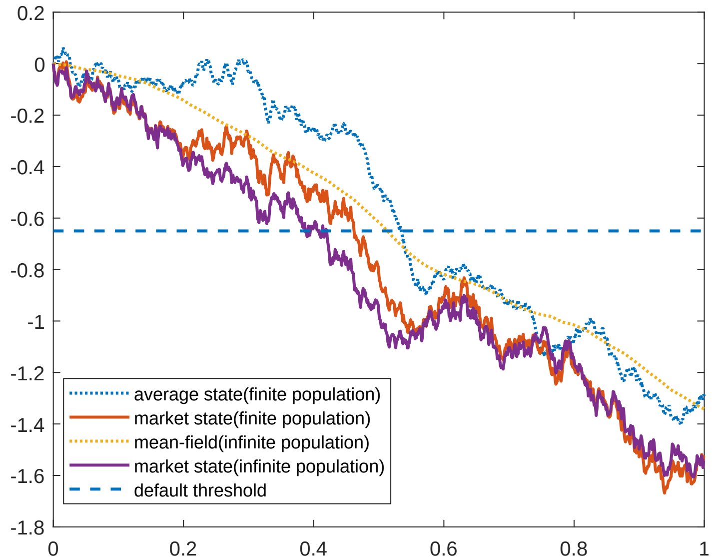


Figure 1.4: Convergence of the average state and the market state in the finite population to corresponding quantities in the infinite population

population is a good approximation to that in the finite population even when the number of minor banks in the finite population is relatively small.

In the remainder of this section we perform 50000 simulations for various settings to estimate the default probabilities introduced in Section 1.6 using Monte Carlo simulations. In particular, we change the relative size G and the mean reversion rate a_0 of the major bank and use regression analysis to estimate $p_i, p_{i|MD}, p_{i|MS}, p_{SE}, p_{SE|MD}, p_{SE|MS}$ respectively by $\bar{p}_i, \bar{p}_{i|MD}, \bar{p}_{i|MS}, \bar{p}_{SE}, \bar{p}_{SE|MD}, \bar{p}_{SE|MS}$.

1.7.3.1 Impact of Relative Size of Major Bank

In this section, we simulate the system with 10 minor banks and run the same regression as we did in Section 1.7.1. We examine how changing the relative size G of the man in the set $\{0.1, 0.2, \ldots, 0.9\}$ affects total and conditional default probabilities $p_i, p_{i|MD}, p_{i|MD}, i \in \{1, 2, \ldots, 10\}$, and systemic risks $\bar{p}_{SE}, \bar{p}_{SE|MD}, \bar{p}_{SE|MS}$.

We use linear regression[4] to examine the variation of the default probability of a representative minor bank-$i, i \in \{1, 2, \ldots, 10\}$. The obtained results are summarized in Ta-

[4] See the "Regression Model 1" in Section 1.8

G	No Major Bank	With a Major Bank		
		Total	Conditional	
			Non-defaulting Major	Defaulting Major
		$\bar{p}_i$	$\bar{p}_{i\|MS}$	$\bar{p}_{i\|MD}$
0	0.3405	-	-	-
0.1	-	0.3367	0.2248	0.5112
0.2	-	0.3422	0.2084	0.5617
0.3	-	0.3548	0.1980	0.6157
0.4	-	0.3645	0.1920	0.6615
0.5	-	0.3785	0.1919	0.7020
0.6	-	0.3873	0.1915	0.7295
0.7	-	0.3985	0.1939	0.7557
0.8	-	0.4033	0.1945	0.7762
0.9	-	0.4098	0.1984	0.7923

Table 1.5: Estimated default probability of a representative minor bank in finite population for the cases (from left to right) with (i) no major bank, (ii) total default probability ($\bar{p}_i$) with a major bank, (iii) conditional default probability ($\bar{p}_{i|MS}$) with a non-defaulting major bank, and (iv) conditional default probability ($\bar{p}_{i|MD}$) with a defaulting major bank

ble 1.5. It shows a similar trend in the changes in the estimated default probability of a representative minor bank compared with the simulated results in the infinite population. A successful major bank could improve the status of the minor bank in the system. However the failure of the major bank may drag down the minor bank and hence increase its default probability. These two effects lead to a higher total default probability for a representative minor agent.

Next, we use a linear regression[5] to estimate the total and conditional probabilities of the systemic event. The results are shown in Table 1.6. The systemic risk generally show similar trends in the finite population (Table 1.6) and the infinite population (Table 1.2) cases. However, despite the infinite-population case we observe nonzero systemic risks for the cases where (i) there is no major bank, and (ii) there is a non-defaulting major agent. This is due to the finite number of minor banks. Moreover, we observe that the conditional systemic risk $\bar{p}_{SE|MS}$ given a non-defaulting major bank decreases with the relative size G of the major bank. Moreover, the total systemic risk increases with G.

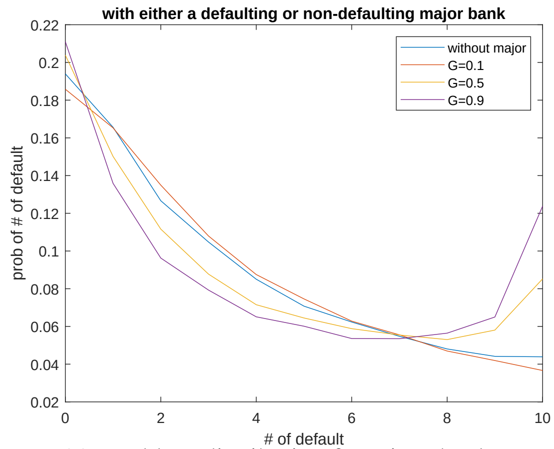
[5]See the "Regression Model 2" in Section 1.8.

G	No Major Bank	With a Major Bank		
		Total	Conditional	
			Non-defaulting Major	Defaulting Major
		$\bar{p}_{SE}$	$\bar{p}_{SE\|MS}$	$\bar{p}_{SE\|MD}$
0	0.0376	-	-	-
0.1	-	0.0288	0.0018	0.0709
0.2	-	0.0445	0.0004	0.1169
0.3	-	0.0726	0.0001	0.1931
0.4	-	0.1100	0	0.2994
0.5	-	0.1521	0	0.4155
0.6	-	0.1929	0	0.5299
0.7	-	0.2364	0	0.6494
0.8	-	0.2744	0	0.7644
0.9	-	0.3150	0	0.8850

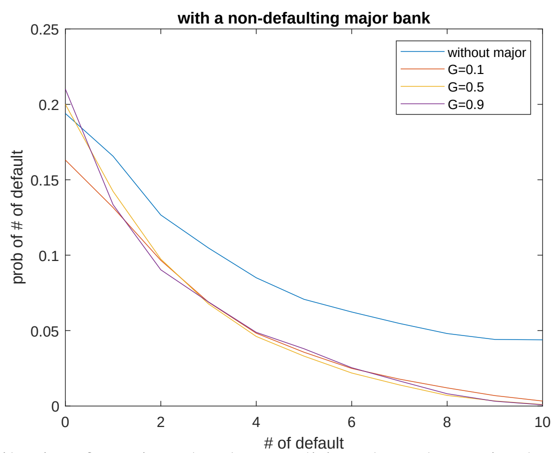
Table 1.6: Estimated probability of systemic event in finite population for the cases (from left to right) with (i) no major bank, (ii) total default probability ($\bar{p}_{SE}$) with a major bank, (iii) conditional default probability ($\bar{p}_{SE|MS}$) with a non-defaulting major bank, and (iv) conditional default probability ($\bar{p}_{SE|MD}$) with a defaulting major bank

This is because a defaulting major bank may significantly increase the systemic risk.

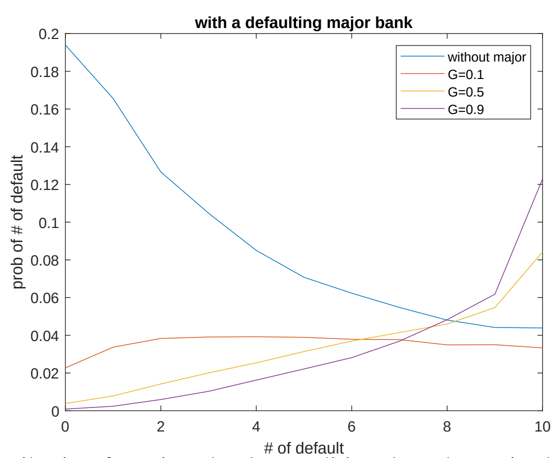
Finally we plot the loss distribution in Fig. 1.5. From panel (a), we notice the tail gets fatter as we increase the relative size of the major bank. This means that the probability of extreme events (either a large number of minors go to default together or no minor bank end up in bankruptcy increases. Panel (b) of Fig. 1.5 shows that having a non-defaulting major bank may improve the stability of the system compared with the case where there is no major bank. The loss distribution in the former is much lower than in the latter. As the relative size of the major agent increases the loss distribution in panel (b) remains almost the same. The only difference lies in the left tail, i.e. the probability of the extreme event where no bank ends up in default increases with G. From panel (c) of Fig. 1.5, when there is a defaulting major bank in the market, the right tail of the loss distribution becomes fatter as the relative size G of the major agent increases. Hence the probability of the extreme event where almost all agents wind up in default increases.



(a) total loss distribution for minor banks



(b) loss distribution for minor banks conditional on the major bank not default



(c) loss distribution for minor banks conditional on the major bank default

Figure 1.5: Loss distribution ($a = 5$): (a) total loss distribution for minor banks, (b) loss distribution for minor banks conditional on the major bank not default, and (c)loss distribution for minor banks conditional on the major bank default

a	No Major Bank	With a Major Bank		
		Total	Conditional	
			Non-defaulting Major	Defaulting Major
		$\bar{p}_i$	$\bar{p}_{i\|MS}$	$\bar{p}_{i\|MD}$
1	0.4341	0.4397	0.2999	0.6361
2	0.4135	0.4246	0.2692	0.6544
3	0.3869	0.4090	0.2419	0.6700
4	0.3666	0.3939	0.2146	0.6869
5	0.3435	0.3770	0.1920	0.6999
6	0.3213	0.3661	0.1732	0.7117
7	0.3002	0.3524	0.1551	0.7234
8	0.2831	0.3438	0.1393	0.7336
9	0.2629	0.3315	0.1259	0.7441
10	0.2482	0.3249	0.1162	0.7507

Table 1.7: Estimated default probability of a representative minor bank in finite population for the cases (from left to right) with (i) no major bank, (ii) total default probability ($\bar{p}_i$) with a major bank, (iii) conditional default probability ($\bar{p}_{i|MS}$) with a non-defaulting major bank, and (iv) conditional default probability ($\bar{p}_{i|MD}$) with a defaulting major bank

1.7.3.2 Impact of Mean-reversion Rate

In this section, we investigate the effect of increasing mean-reversion rate a (or equivalently a_0) on the default probability of a representative minor bank and the systemtic risk.

From Table 1.7, the variations in the estimated default probabilities of a representative minor bank show a similar trend in the finite population and in the infinite population cases. A non-defaulting major bank reduces the default probability of a representative minor bank in the system. However a defaulting major bank may increases this probability. These two effects lead to a higher total default probability of the representative minor bank compared with the case in the absence of a major bank. However, this probability decreases with the mean-reversion rate a.

Now we use a regression model[6] to examine the systemic risk in the finite population (Table 1.8). Despite the infinite population case, the estimated probability of the systemic event is nonzero in the market without a major bank in the finite-population case. We

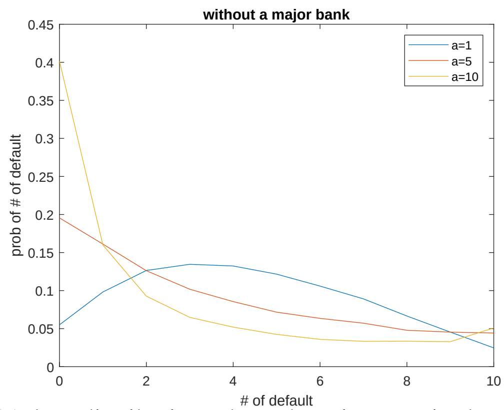
[6]See the "Regression Model 4" in Section 1.8.

a	No Major Bank	With a Major Bank		
		Total	Conditional	
			Non-defaulting Major	Defaulting Major
		$\bar{p}_{SE}$	$\bar{p}_{SE\|MS}$	$\bar{p}_{SE\|MD}$
1	0.0378	0.1470	0	0.3534
2	0.0377	0.1448	0	0.3590
3	0.0377	0.1480	0	0.3792
4	0.0373	0.1499	0	0.3950
5	0.0377	0.1500	0	0.4120
6	0.0363	0.1543	0	0.4307
7	0.0365	0.1541	0	0.4437
8	0.0368	0.1570	0	0.4561
9	0.0362	0.1577	0	0.4741
10	0.0356	0.1616	0	0.4914

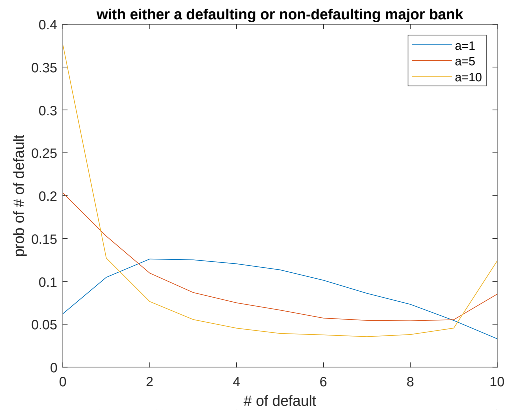
Table 1.8: Estimated probability of systemic event in finite population for the cases (from left to right) with (i) no major bank, (ii) total default probability ($\bar{p}_{SE}$) with a major bank, (iii) conditional default probability ($\bar{p}_{SE|MS}$) with a non-defaulting major bank, and (iv) conditional default probability ($\bar{p}_{SE|MD}$) with a defaulting major bank

emphasize that in this case the empirical average $x_t^{(N)}$ of log-monetary reserves are used in the market model instead of the mean field. We observe that the systemic risk slightly decreases with the mean-reversion rate a in the case without a major bank. However, when a major bank is present the total systemic risk $\bar{p}_{SE}$ increases with a. We can obtain a better understanding of this result by examining the conditional systemic risks. We observe that indeed when there is a non-defaulting major bank in the market, the systemic risk $\bar{p}_{SE|MS}$ is down to zero, which means that the system is slightly more stable. This is while in the case of a defaulting major agent, the systemic risk $\bar{p}_{SE|MD}$ increases significantly with a.

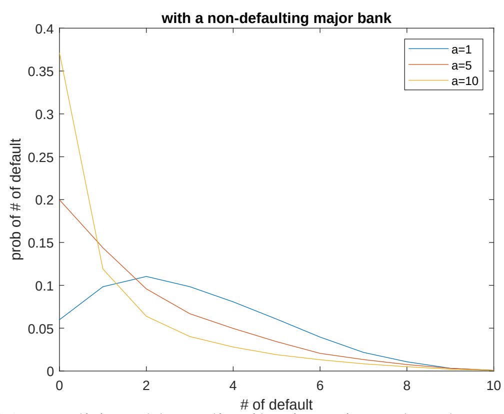
Next we plot the loss distribution of minor banks in Fig. 1.6. By comparing panels (a) and (b), we conclude that they have a very similar shape. As the mean-reversion rate increases, the tails become fatter . The distinction mainly lies in the right tail which represents the scenario where all minor banks simulated end up in default. Panel (b) shows a fatter right tail than panel (a). This confirms the results in Table 1.8, i.e. the model with a major bank is more likely to lead to a financial crisis. By inspecting panels (c) and (d) of Fig. 1.6, we get a clearer view of the role played by the major bank. First



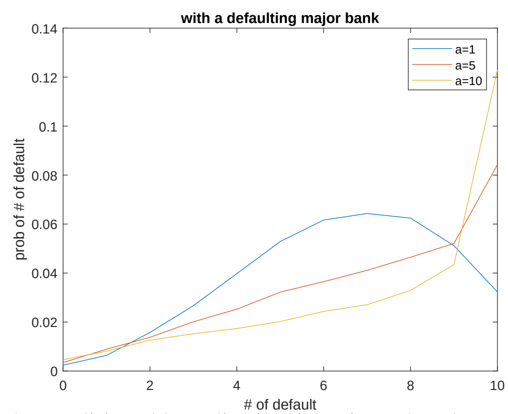
(a) loss distribution where there is no major bank



(b) total loss distribution where there is a major bank



(c) conditional loss distribution given that the major bank defaults



(d) conditional loss distribution given that the major bank defaults

Figure 1.6: Loss distribution ($G = 0.5, F = 0.5$): (a) loss distribution for minor banks without major bank, (b) total loss distribution for minor banks with major bank, (c) loss distribution for minor banks conditional on the major bank not default, and (d)loss distribution for minor banks conditional on the major bank default

of all, the left tails in panels (a) and (c) are almost the same. However, the fatter right tale in (a) indicates a higher systemic risk. Now if we compare panels (c) and (d), the former has a fatter left tail and the latter a fatter right tail and, indicating a low systemic risk in the case of non-defaulting major bank and a high systemic risk in the case of a defaulting major bank.

1.8 Concluding Remarks

The purpose of the current study is to investigate the impact of the presence of a major bank on the default probability of a representative minor bank and the systemic risk. Based on the numerical results, we have come to the conclusion that having a major bank in the market may have two opposite effects on the system depending on whether it defaults or not. In the case where the major bank does not default, a representative minor agent is less likely to end up in default. Moreover, the systemic risk decreases. This positive impact increases with the relative size and the mean-reversion rate of the major agent (equivalently the mean reversion rate of the minor agent). However, in an interbank market where the major bank defaults, a representative minor bank has a higher default risk. Moreover, the systemic risk increases significantly. Indeed, the failure of a large bank is highly likely to bring the whole system down. This negative impact becomes larger with the relative size and the mean-reversion rate of the major bank. Our results show that the negative impact of a major bank prevails its positive impact. We observe that the total default probability of a representative minor bank and the systemic risk are higher in a market where there exists a major bank compared to the case where there is no major bank. Obviously, the higher the relative size of the major agent and the mean reversion rate in the market, the higher is the systemic risk.

These findings have significant implications for the understanding how a major bank may affect the interbank market. Specifically, they could be used to develop policies and regulations that improve the stability of such markets. In particular, it is not healthy for the economy to have a very large bank due to its extreme negative externality. Policymakers may set regulations to prevent banks from becoming too large or impose higher capital requirement and stricter regulations on large banks to ensure that they would not end up in default.

With regard to the research methods, some limitations need to be acknowledged. The current study is limited by the fact that the model is relatively simple. Hence, it cannot capture all the characteristics of the interbank activities. Further experimental investigations are needed to estimate the impact of other parameters on the whole system. The model could be advanced by taking more factors into consideration, for example, considering multiple group of minor banks with different characteristics and risk sensitivity. Moreover, it would be interesting to study systemic risk and the behaviour a major bank which might be bailed out by the government once in crisis. Moreover, a more comprehensive sensitivity analysis can be performed to investigate the impact of other model parameters rather than the relative size of the major bank and the mean reversion rate of banks.

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Appendix A

The choice of Default Threshold

In this section, we would like to explain how we select an appropriate default threshold. It is crucial to choose it carefully. A proper value is that we might have sufficient sample observations to make some kind of explanation and conclusion.

First of all, it is important to note that the default threshold is chosen in an environment where there is no major bank, since we want to eliminate the impact of the major bank on the minors so that we can see an unconditional distribution of small banks' log-currency reserves at the end of the period that we are considering. We examine the stability of the different quantiles of the log-monetary reserves using the Monte Carlo simultaion to calculate the quantiles of the distribution for the minor banks' log-monetary reserves at the end of the period. It is shown in Table 1 that the first ten quantiles of the distribution for the log-monetary reserves for the minor bank. For example, only around 400 minor banks (%1 out of 40000 minor banks) in each simulation will end up in the region where their log-monetary reserves will be less than -0.49. We could conclude from the Table 1 that the quantiles are stable across different simulations.

Then, by changing the parameter '*a*' from 1 to 10, we select the default threshold for the subsequent simulations. We show the simulation results in Table 2. We think the -0.65 which is the 1% quantile when we set the parameter '*a*' equal to 5 is a good choice. It ensures that the default probability is not too far from what reality shows when the parameter '*a*' is relatively small. And also we could have some default observations as

%	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
1	-0.49	-0.49	-0.49	-0.49	-0.49	-0.49	-0.49	-0.49	-0.49	-0.49
2	-0.44	-0.44	-0.44	-0.43	-0.43	-0.44	-0.44	-0.43	-0.43	-0.44
3	-0.40	-0.40	-0.40	-0.40	-0.40	-0.40	-0.40	-0.40	-0.40	-0.40
4	-0.37	-0.37	-0.37	-0.37	-0.37	-0.37	-0.37	-0.37	-0.37	-0.37
5	-0.35	-0.35	-0.35	-0.35	-0.35	-0.35	-0.35	-0.35	-0.35	-0.35
6	-0.33	-0.33	-0.33	-0.33	-0.33	-0.33	-0.33	-0.33	-0.33	-0.33
7	-0.31	-0.31	-0.31	-0.31	-0.31	-0.31	-0.31	-0.31	-0.31	-0.31
8	-0.30	-0.30	-0.30	-0.30	-0.30	-0.30	-0.30	-0.30	-0.30	-0.30
9	-0.28	-0.28	-0.28	-0.28	-0.28	-0.28	-0.28	-0.28	-0.28	-0.28
10	-0.27	-0.27	-0.27	-0.27	-0.27	-0.27	-0.27	-0.27	-0.27	-0.27

Table 1: First ten quantiles for the distribution of the log-monetary reserves across different values taken by parameter 'a' ($a = 1, F = 0.5, G = 0.5$)

%	$a=1$	$a=2$	$a=3$	$a=4$	$a=5$	$a=6$	$a=7$	$a=8$	$a=9$	$a=10$
1	-0.98	-0.87	-0.78	-0.71	-0.65	-0.61	-0.57	-0.54	-0.52	-0.49
2	-0.87	-0.76	-0.68	-0.63	-0.58	-0.54	-0.51	-0.48	-0.46	-0.43
3	-0.79	-0.70	-0.63	-0.57	-0.53	-0.49	-0.46	-0.44	-0.42	-0.40
4	-0.74	-0.65	-0.58	-0.53	-0.49	-0.46	-0.43	-0.41	-0.39	-0.37
5	-0.69	-0.61	-0.55	-0.50	-0.46	-0.43	-0.41	-0.38	-0.36	-0.35
6	-0.66	-0.58	-0.52	-0.47	-0.44	-0.41	-0.38	-0.36	-0.34	-0.33
7	-0.62	-0.55	-0.49	-0.45	-0.41	-0.39	-0.36	-0.34	-0.33	-0.31
8	-0.59	-0.52	-0.47	-0.43	-0.39	-0.37	-0.35	-0.33	-0.31	-0.30
9	-0.57	-0.50	-0.45	-0.41	-0.38	-0.35	-0.33	-0.31	-0.30	-0.28
10	-0.54	-0.48	-0.43	-0.39	-0.36	-0.34	-0.32	-0.30	-0.28	-0.27

Table 2: Different quantiles for the distribution of the log-monetary reserves across 10 simulations ($F = 0.5, G = 0.5$)

we increase the parameter 'a'.

Regression Model

In this section, we provide the details about the regression models used in the analysis of numerical part.

(i) Regression Model 1

$$PD(Minor\ banks) = \alpha_0 \cdot \mathbb{I}_{G=0} + \mathbb{I}_{major\ exist}\Big(\sum_{i=0.1}^{0.9} \beta_i \cdot \mathbb{I}_{G=i} + \mathbb{I}_{major\ default}\sum_{i=0.1}^{0.9} \gamma_i \cdot \mathbb{I}_{G=i}\Big) \tag{88}$$

- $PD(Minor\ banks)$ is the dependent variable representing the simulated default probability of minor banks.

- $\mathbb{I}_{G=i}$ represents the dummy variables dividing the simulated results into 10 different groups. It takes 1 when the data comes from the simulation in which $G = i$ and 0 otherwise. We need to notice that the dummy $\mathbb{I}_{G=0}$ represents the observations from the model without a major bank.

- The estimated slope coefficient α_0 is the difference between the average default probability of minor banks in the group where there exist a non-defaulting major bank and that in another group without a major bank. We could show the average default probability of minor banks in the system with a successful major bank as .

$$\alpha_0 = PD(Minor\ banks\ default \mid without\ a\ major\ bank) \tag{89}$$

- The estimated slope coefficient β_i is the difference between the average default probability of minor banks in the group where there exist a non-defaulting major bank and that in another group without a major bank. We could show the average default probability of minor banks in the system with a successful major bank as Eq. (91).

$$\beta_i = PD(Minor\ banks\ default \mid G = i\ and\ major\ bank\ not\ default) - PD(Minor\ banks\ default \mid without\ a\ major\ bank) \tag{90}$$

$$PD(Minor\ banks\ default \mid G = i\ and\ major\ bank\ not\ default) = \alpha_0 + \beta_i \tag{91}$$

- The estimated slope coefficient γ_i is the difference between the average default probability of minor banks in the system with a defaulting major bank and that in another system with a successful major bank. And the average default probability of minor banks in the system with a failed major bank could be expressed as Eq. (93).

$$\begin{aligned} \gamma_i = PD(Minor\ banks\ default \mid G = i\ and\ major\ bank\ default) - \\ PD(Minor\ banks\ default \mid G = i\ and\ major\ bank\ not\ default) \end{aligned} \tag{92}$$

$$PD(Minor\ banks\ default \mid G = i\ and\ major\ bank\ default) = \alpha_0 + \beta_i + \gamma_i \tag{93}$$

(ii) Regression Model 2

$$PD(Systemic\ event) = \alpha_0 \cdot \mathbb{I}_{G=0} + \mathbb{I}_{major\ exist}\left(\sum_{i=0.1}^{0.9} \beta_i \cdot \mathbb{I}_{G=i} + \mathbb{I}_{major\ default}\sum_{i=0.1}^{0.9} \gamma_i \cdot \mathbb{I}_{G=i}\right) \tag{94}$$

- $PD(Systemic\ event)$ is the probability of the market state $\left(F\bar{x}_t + Gx_t^0\right)$ ending in the default region and it is the dependent variable in this regression.

- $\mathbb{I}_{G=i}$ represents the dummy variables dividing the simulated results into 10 different groups. It takes 1 when the data comes from the simulation in which $G = i$ and 0 otherwise. We need to notice that the dummy $\mathbb{I}_{G=0}$ represents the observations from the model without a major bank.

- The estimated slope coefficient α_0 is the average probability of systemic event in the scenario where there is no major bank.

$$\alpha_0 = PD(systemic\ event \mid without\ a\ major\ bank) \tag{95}$$

- The estimated slope coefficient β_i is the difference between the average probability of systemic event for the system having a successful major bank and that for another system without a major bank. We could express the average probability of the systemic event as Eq. (97).

$$\beta_i = PD(systemic\ event \mid G = i\ and\ major\ bank\ not\ default) - PD(systemic\ event \mid without\ a\ major\ bank) \quad (96)$$

$$PD(systemic\ event \mid G = i\ and\ major\ bank\ not\ default) = \alpha_0 + \beta_i \quad (97)$$

- The estimated slope coefficient γ_i is the difference between the average probability of the systemic event in the environment having a failed major bank and that in another system with a successful major bank. And we could express the average probability of the systemic event in the group with a defaulting major bank as Eq. (99).

$$\gamma_i = PD(systemic\ event \mid G = i\ and\ major\ bank\ default) - PD(systemic\ event \mid G = i\ and\ major\ bank\ not\ default) \quad (98)$$

$$PD(systemic\ event \mid G = i\ and\ major\ bank\ default) = \alpha_0 + \beta_i + \gamma_i \quad (99)$$

(iii) Regression Model 3

$$PD(Minor\ banks) = \sum_{i=1}^{10} \alpha_i \cdot \mathbb{I}_{a=i} + \mathbb{I}_{major\ exist}\left(\sum_{i=1}^{10} \beta_i \cdot \mathbb{I}_{a=i} + \mathbb{I}_{major\ default}\sum_{i=1}^{10} \gamma_i \cdot \mathbb{I}_{a=i}\right) \quad (100)$$

- $PD(Minor\ banks)$ is the dependent variable representing the simulated default probability of minor banks.

- $\mathbb{I}_{a=i}$ represents the dummy variables dividing the simulated results into 10 different groups. It takes 1 when the data comes from the simulation in which $a = i$ and 0 otherwise.

- The estimated slope coefficient α_0 is the average default probability of minor banks coming from the model without major banks.

$$\alpha_i = PD(Minor\ banks\ default \mid a = i\ and\ without\ a\ major\ bank) \tag{101}$$

- The estimated slope coefficient β_i is the difference between the average default probability of minor banks in the system with a successful major bank and that in another system without a major bank. The average default probability of minor banks in the system having a non-defaulting major bank is Eq. (103).

$$\begin{aligned}\beta_i = PD(Minor\ banks\ default \mid a = i\ and\ major\ bank\ not\ default) - \\ PD(Minor\ banks\ default \mid a = i\ and\ without\ a\ major\ bank)\end{aligned} \tag{102}$$

$$PD(Minor\ banks\ default \mid a = i\ and\ major\ bank\ not\ default) = \alpha_i + \beta_i \tag{103}$$

- The estimated slope coefficient γ_i is the difference between the average default probability of minor banks in the environment having a failed major bank and that in the system with a successful major bank. The average default probability of minor banks in the system having a defaulting major bank could be expressed as Eq. (105).

$$\begin{aligned}\gamma_i = PD(Minor\ banks\ default \mid a = i\ and\ major\ bank\ default) - \\ PD(Minor\ banks\ default \mid a = i\ and\ major\ bank\ not\ default)\end{aligned} \tag{104}$$

$$PD(Minor\ banks\ default \mid a = i\ and\ major\ bank\ default) = \alpha_i + \beta_i + \gamma_i \tag{105}$$

(iv) Regression Model 4

$$PD(Systemic\ event) = \sum_{i=1}^{10} \alpha_i \cdot \mathbb{I}_{a=i} + \mathbb{I}_{major\ exist}\left(\sum_{i=1}^{10} \beta_i \cdot \mathbb{I}_{a=i} + \mathbb{I}_{major\ default}\sum_{i=1}^{10} \gamma_i \cdot \mathbb{I}_{a=i}\right) \tag{106}$$

- $PD(Systemic\ event)$ is the dependent variable representing the simulated probability of systemic event.

- $\mathbb{I}_{a=i}$ represents the dummy variables dividing the simulated results into 10 different groups. It takes 1 when the data comes from the simulation in which $a = i$ and 0 otherwise.

- The estimated slope coefficient α_0 is the average probability of systemic event coming from the model without major banks.

$$\alpha_i = PD(Systemic\ event \mid a = i\ and\ without\ a\ major\ bank) \tag{107}$$

- The estimated slope coefficient β_i is the difference between the average probability of systemic event in the system with a successful major bank and that in another system without a major bank. The average probability of systemic event in the system having a non-defaulting major bank is Eq. (109).

$$\beta_i = PD(Systemic\ event \mid a = i\ and\ major\ bank\ not\ default) - PD(Systemic\ event \mid a = i\ and\ without\ a\ major\ bank) \tag{108}$$

$$PD(Systemic\ event \mid a = i\ and\ major\ bank\ not\ default) = \alpha_i + \beta_i \tag{109}$$

- The estimated slope coefficient γ_i is the difference between the average systemic risk in the environment having a failed major bank and that in the system with a successful major bank. The average systemic risk in the system having a defaulting major bank could be expressed as Eq. (111).

$$\gamma_i = PD(Systemic\ event \mid a = i\ and\ major\ bank\ default) - PD(Systemic\ event \mid a = i\ and\ major\ bank\ not\ default) \tag{110}$$

$$PD(Systemic\ event \mid a = i\ and\ major\ bank\ default) = \alpha_i + \beta_i + \gamma_i \tag{111}$$