Sampled-data Nash equilibria in differential games with impulse controls

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Abstract

We study a class of deterministic two-player nonzero-sum differential games where one player uses piecewise-continuous controls to affect the continuously evolving state while the other player uses impulse controls at certain discrete instants of time to shift the state from one level to another. The state measurements are made at some given instants of time, and players determine their strategies using the last measured state value. We provide necessary and sufficient conditions for the existence of sampled-data Nash equilibrium for a general class of differential games with impulse controls. We specialize our results to a scalar linear-quadratic differential game, and show that the equilibrium impulse timing can be obtained by determining a fixed point of a Riccati like system of differential equations with jumps coupled with a system of non-linear equality constraints. By reformulating the problem as a constrained non-linear optimization problem, we compute the equilibrium timing, and level of impulses. We find that the equilibrium piecewise continuous control and impulse control are linear functions of the last measured state value. Using a numerical example, we illustrate our results.

1 Introduction

Recently, there has been renewed interest in the study of differential games with impulse controls, where the state is controlled by two players, with at least one being able to affect the continuously evolving state variable at certain discrete instants of time only [1, 20]. In such games, the number and timing of interventions besides their level are also decision variables. Settings where such dynamic interactions arise include option pricing [18], pollution regulation [20], exchange rate interventions [1], cybersecurity [25], and related problems [2]. Two recent papers that have studied impulse control games are [9] and [14]. In [9], the authors provide an extension of the two-player stochastic impulse game to an N-player game ($N > 2$) and also study its corresponding mean-field game while in [14], a controller-stopper game is studied where one player uses impulse controls while the other player can stop the game at any time. A solution concept for these games is the Nash equilibrium, where the strategies of the players use the information that is available to them at the time when they make their decisions [7]. Nash equilibrium in differential games with impulse controls have been obtained before under two different information structures, namely, open-loop and perfect-state feedback information structures. In the open-loop information structure, the equilibrium controls of the

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players are obtained assuming that players have access to only the initial state, whereas with the perfect-state feedback information structure, players make their decisions using the state measurements at each instant of time in the game. One limitation of using open-loop strategies is that they are not strongly time-consistent [7, 8], whereas the feedback equilibrium strategies require state measurements to be made at each instant of time in the game. In many real-world problems, such as, economic data collected from the surveys, position of players in pursuit-evasion games, and measuring the quality of goods, acquiring continuous state measurements is costly. As a result, state information is available to the players at some (discrete) sampling instants only, and the players determine their sampled-data controls [26, 5], using the past (time-sampled) state measurements. It appears that Nash equilibrium in differential games with impulse controls and sampling has not been studied before in the literature.

In [26], the authors have introduced a deterministic two-player nonzero-sum differential game where state measurement is made at discrete instants of time, and both players use piecewise-continuous strategies. The sampled-data controls of the players are assumed to be functions of the last measured state value, and players implement open-loop controls between the sampling instants. The authors showed that the equilibrium of linear-quadratic differential games can be obtained by solving a system of Riccati equations coupled with a system of differential equations that determine the terminal conditions on the Riccati equations. Reference [3] studies a stochastic linear-quadratic differential game where players have access to the sampled-data state information as well as the sampling times. A time-variant zero-sum linear-quadratic differential game was studied in [5] where it was shown that the minimax sampled-data controller can be obtained by solving a generalized Riccati-differential equation. Reference [6] has provided a characterization of the minimax controller of a switching system with sampled state information. In contrast to the aforementioned research that deals with piecewise continuous controls, [16] has derived the Nash equilibrium of a class of stochastic linear-quadratic differential games assuming that the admissible strategies are constant between the state measurements.

In this paper, we consider a general class of deterministic two-player nonzero-sum differential games where the two players are endowed with different kinds of controls (discrete and piecewise-continuous). In particular, Player 1 uses piecewise continuous controls to affect the continuous evolution of the state whereas Player 2 uses impulse controls to shift the state value instantaneously from one level to another at the impulse instants that are endogenously determined by Player 2 in addition to the number of impulse instants. The more general case with both players using continuous and impulse controls can be easily studied using our model. However, for the application of our work to problems involving regulation and cybersecurity, we restrict our focus here to the canonical game model with one player using piecewise-continuous controls and the other player using impulse controls.

The objectives of this research are two-fold: Our first goal is to provide necessary and sufficient conditions for the existence of Nash equilibrium. Our second objective is to specialize our results to scalar linear-quadratic differential games (LQDGs) which are widely used in economics, engineering and management science domains (see [7, 22, 8]) as they allow for the possibility of modeling real-world problems involving non-linear returns to scale.

Our contributions can be summarized as follows:

(i) The paper provides, for the first time, necessary and sufficient conditions for the existence of Nash equilibrium in a differential game with impulse controls, where the players' strategies are functions of the state values measured at certain discrete time instants; see Theorem 1.

(ii) For the case of LQDGs with exogenously given impulse instants, Theorem 2 provides a system of Riccati like equations with jumps, which characterizes the sampled-data Nash equilibrium.

(iii) For LQDGs with a given number of impulses in each sampling interval, Theorem 3 shows that the equilibrium timing of impulses can be obtained as a solution of a system of Riccati equations (with jumps) provided that the impulse instants satisfy a system of non-linear equality constraints.

The rest of the paper is organized as follows: In Section 2, we introduce the canonical two-player differential game model. Section 3 provides necessary and sufficient conditions for the existence of sampled-data Nash equilibrium for the canonical model. In Section 4, we specialize the results to a scalar linear-quadratic differential game. We illustrate the theoretical results using a numerical example in Section 5. Finally, Section 6 provides concluding remarks.

2 Model

In this paper, we consider a deterministic two-player differential game of finite duration $T < \infty$ where both players can affect a continuously evolving state variable $x(t) \in \mathbb{R}^p$ to maximize their individual payoffs. However, the two players are equipped with different types of controls. Player 1 can continuously influence the dynamics of the state variable using her piecewise continuous controls $u(t) \in \Omega_u$, while Player 2 is able to intervene and cause jumps in the state variable at certain discrete instants of time τ_i $(i = 1, 2, \cdots, k)$. We assume that Ω_u is a bounded and convex subset of $\mathbb{R}^{m_1}$. When Player 2 does not intervene in the game, the state variable is continuous and its dynamics are controlled entirely by Player 1 so that the state variable evolves as follows:

$$\dot{x}(t) = f(x(t), u(t)),\; x(0^-) = x_0,\; \text{for}\, t \neq \{\tau_1, \tau_2, ..., \tau_k\}, \tag{1}$$

where $f : \mathbb{R}^p \times \Omega_u \to \mathbb{R}^p$, the initial value of the state variable is given by $x_0 \in \mathbb{R}^p$ (a known parameter), $x(\tau_i^-) = \lim_{t\uparrow\tau_i} x(t)$, $x(\tau_i^+) = \lim_{t\downarrow\tau_i} x(t)$, and 0^- denotes the time instant just before 0. The state variable is assumed to be left-continuous at points of discontinuity. At an impulse instant τ_i, Player 2 intervenes in the game to shift the state from $x(\tau_i^-)$ to $x(\tau_i^+)$ by using an impulse of size $v_i \in \Omega_v$, that is,

$$x(\tau_i^+) - x(\tau_i^-) = g(x(\tau_i^-), v_i),\; i \in \{1, 2, \cdots, k\}, \tag{2}$$

where $g : \mathbb{R}^p \times \Omega_v \to \mathbb{R}^p$. We assume that Ω_v is a bounded and convex subset of $\mathbb{R}^p$. The number of impulses $k \in \mathbb{N}$ (the set of natural numbers), and timing of impulses τ_i are decision variables of Player 2 in addition to the levels of impulses. The impulse controls are denoted by $\tilde{v} = \{(\tau_i, v_i),\; i = 1, 2, \cdots, k\}$.
In this differential game, Player 1 attempts to maximize the following objective:

$$J_1(x_0, u(\cdot), \tilde{v}) = S_1(x(T^+)) + \int_0^T F_1(x(t), u(t))dt - \sum_{i=1}^{k} G_1(x(\tau_i^-), v_i), \tag{3}$$

and Player 2 uses the impulse controls (τ_i, v_i) to maximize the objective

$$J_2(x_0, u(\cdot), \tilde{v}) = S_2(x(T^+)) + \int_0^T F_2(x(t), u(t))dt - \sum_{i=1}^{k} G_2(x(\tau_i^-), v_i), \tag{4}$$

where $F_1,\, F_2 : \mathbb{R}^p \times \Omega_u \to \mathbb{R}$, $G_1,\, G_2 : \mathbb{R}^p \times \Omega_v \to \mathbb{R}$, and $S_1,\, S_2 : \mathbb{R}^p \to \mathbb{R}$. For Player 1, F_1 denotes the running payoff, G_1 denotes the intervention cost at the impulse instants, and S_1 is the terminal payoff. For Player 2, the running payoff is given by F_2, while G_2 represents the intervention cost at the impulse instants, and S_2 denotes the terminal payoff.

In a differential game, the Nash equilibrium depends on the state information that the players use to determine their strategies (see [7], [22]). We assume that the state measurement is made at certain discrete instants of time $t_n, n \in \{1, 2, \cdots, N\}$, with the corresponding state values denoted by $x_1, x_2, \cdots, x_N$ such that $0 = t_1 < t_2 < \cdots < t_{N-1} < t_N = T$. The sampled-data controls of Player 1 are given by

$$u(t) = \gamma(t; x(t_n)) \in \Omega_u,\; \text{for}\, t_n \leq t < t_{n+1},\; n \in \mathcal{N}' = \{1, 2, \cdots, N-1\},\; \gamma \in \Gamma,$$

where $\gamma : [t_n, t_{n+1}] \times \mathbb{R}^p \to \Omega_u$ is a sampled-data state feedback controller of Player 1 and the strategy set of Player 1 is denoted by Γ. Similarly, the impulse levels of Player 2 are given by

$$v_{i,n} = \delta(\tau_{i,n}; x(t_n)) \in \Omega_v, \text{ for } t_n \le \tau_{i,n} < t_{n+1},\ n \in \mathcal{N}',\ \delta \in \Delta,$$

where $\tau_{i,n}$ denotes the timing of impulse in the sampling interval (t_n, t_{n+1}), $\delta : [t_n, t_{n+1}] \times \mathbb{R}^p \to \Omega_u$ is a sampled-data state feedback controller for Player 2 and Δ denotes the strategy set of Player 2.

The objective functions of the players over the sub-interval $[t_n, T]$, initialized at the sampling instant t_n with the corresponding state $x(t_n) = x_n$ are given by

$$\begin{aligned} J_1(x_n, \gamma_{[t_n,T]}, \delta_{[t_n,T]}) = S_1(x(T^+)) + \sum_{j=n}^{N-1} \int_{t_j}^{t_{j+1}} F_1(x(t), \gamma(t; x(t_j)))dt \\ - \sum_{j=n}^{N-1} \sum_{i=1}^{k_j} \mathbb{1}_{\tau_{i,j} \ge t_n} G_1(x(\tau_{i,j}^-), \delta(\tau_{i,j}; x(t_j))), \end{aligned} \tag{5a}$$

$$\begin{aligned} J_2(x_n, \gamma_{[t_n,T]}, \delta_{[t_n,T]}) = S_2(x(T^+)) + \sum_{j=n}^{N-1} \int_{t_j}^{t_{j+1}} F_2(x(t), \gamma(t; x(t_j)))dt \\ - \sum_{j=n}^{N-1} \sum_{i=1}^{k_j} \mathbb{1}_{\tau_{i,j} \ge t_n} G_2(x(\tau_{i,j}^-), \delta(\tau_{i,j}; x(t_j))), \end{aligned} \tag{5b}$$

where the strategies $\gamma_{[t_n,T]}$ and $\delta_{[t_n,T]}$ are restrictions of γ and δ to the interval $[t_n, T]$, and $\Gamma_{[t_n,T]}$ and $\Delta_{[t_n,T]}$ denote the corresponding admissible strategy sets of Player 1 and Player 2, respectively. The state dynamics are given by

$$\dot{x}(t) = f(x(t), \gamma(t; x_n)),\ x(t_n^-) = x_n, \text{ for } t_n \le t < t_{n+1},\ t \ne \tau_{i,n},\ n \in \mathcal{N}', \tag{5c}$$

$$x(\tau_{i,n}^+) - x(\tau_{i,n}^-) = g(x(\tau_{i,n}^-), \delta(\tau_{i,n}; x_n)), \text{ for } i \in \mathcal{I}^n = \{1, 2, \cdots, k_n\}, \tag{5d}$$

where k_n denotes the equilibrium number of impulses in the sampling interval $[t_n, t_{n+1}]$.[1] From (5a)-(5b), it is clear that each player can influence the payoff of their opponent directly through their controls, and indirectly by changing the state variable.

Remark 1. *The above canonical differential game model (5a-5d) can be used to study problems in cyber-security and pollution regulation where the state can be the security level of a system for a software firm or the level of pollution, with the running payoff of one player, say Player 1, decreasing with state and that of the other player, increasing with state. Player 1 continuously invests in reducing the state except at the impulse instants wherein Player 2 intervenes in the game to instantaneously shift the state to a higher value. Consequently, Player 1 incurs a state-dependent cost at the impulse instant.*

Clearly, the admissible controls in the aforementioned real-world applications satisfy the following definition:

Definition 1. $(\tau_{i,n}, v_{i,n})$, $i \in \mathcal{I}^n$, $n \in \mathcal{N}'$, *is an admissible impulse control of Player 2 if the impulse instants satisfy the following increasing monotone sequence property:*

$$t_n < \tau_{1,n} < \tau_{2,n} < \cdots < \tau_{k_n,n} < t_{n+1}, \tag{6}$$

where $k_n < \infty$, $v_{i,n} \ne 0$.

[1] In the paper, the necessary and sufficient conditions are obtained for any finite equilibrium number of impulses. However, in the case of linear-quadratic games, we fix the number of impulses in each sampling interval (see Section 4).

In the above definition, we have assumed that the impulses cannot occur at the sampling instants as Player 2 can either measure the state or use her controls, but cannot do both at the same time (see [23] and the references therein).

In this paper, we seek to determine the sampled-data Nash equilibrium of the differential game (5a-5d), which is defined as follows:

Definition 2. *The strategy profile* (γ^*, δ^*) *is a sampled-data Nash equilibrium of the differential game (5a-5d), if the restrictions of* γ^* *and* δ^**, denoted by* $\gamma^*_{[t_n,T]}$ *and* $\delta^*_{[t_n,T]}$*, to any subgame that starts at the sampling time* t_n *with state measurement* x_n *satisfy the following inequalities:*

$$J_1(x_n, \gamma^*_{[t_n,T]}, \delta^*_{[t_n,T]}) \geq J_1(x_n, \gamma_{[t_n,T]}, \delta^*_{[t_n,T]}),\ \forall \gamma_{[t_n,T]} \in \Gamma_{[t_n,T]}, \tag{7a}$$

$$J_2(x_n, \gamma^*_{[t_n,T]}, \delta^*_{[t_n,T]}) \geq J_2(x_n, \gamma^*_{[t_n,T]}, \delta_{[t_n,T]}),\ \forall \delta_{[t_n,T]} \in \Delta_{[t_n,T]}. \tag{7b}$$

Remark 2. *The sampled-data Nash equilibrium strategies of the differential game (5a-5d) for* $t \in [0, T]$ *when restricted to* $[t_n, T]$ *are also the Nash equilibrium strategies of the subgame that starts at* t_n*. As a result, the sampled-data Nash equilibrium strategies are strongly time-consistent [4] if the perturbation of state can occur only at the sampling instants* t_n*,* $n = \{1, 2, \cdots, N\}$*. At all other time instants, that is,* $t \neq t_n$*, the sampled-data Nash equilibrium strategies are weakly time-consistent [4].*

Remark 3. *When sampling is done at the initial and final time only, then the sampled-data Nash equilibrium coincides with the open-loop Nash equilibrium of a differential game. It is shown in [26] that the sampled-data equilibrium controls approach the closed-loop controls as the number of sampling intervals increases.*

3 Necessary and Sufficient Conditions

In this section, we derive a set of necessary and sufficient conditions for the existence of sampled-data Nash equilibrium in differential games with impulse controls.

The approach to determine the sampled-data Nash equilibrium can be summarized as follows. Suppose the sampling instants are given by $t_1, t_2, \cdots, t_N$. For $t \in [t_n, t_{n+1})$, players use open-loop strategies $\gamma^*(t; x_n)$ and $\delta^*(t; x_n)$, which are functions of the last measured state value x_n, that is, for any given initial state x_n, Player 1 determines the open-loop controls in the sampling interval and Player 2 determines the equilibrium number, timing, and levels of impulses. The payoff of each player at $(t_n, x(t_n))$ is a salvage value for the open-loop game between t_{n-1} and t_n. Therefore, starting from the last sampling interval $[t_{N-1}, T)$ with salvage values S_1 and S_2, we can recursively obtain the equilibrium strategies for all the sampling intervals $[t_n, t_{n+1})$, $n \in \mathcal{N}'$.

First, we define the Hamiltonians of the two players that will be used in the necessary conditions for the existence of sampled-data Nash equilibrium. The continuous Hamiltonians of Player 1 and Player 2 are given, respectively, by

$$H_1(x(t), u(t), \lambda_1(t)) = F_1(x(t), u(t)) + \lambda_1(t)^T f(x(t), u(t)), \tag{8a}$$

$$H_2(x(t), u(t), \lambda_2(t)) = F_2(x(t), u(t)) + \lambda_2(t)^T f(x(t), u(t)), \tag{8b}$$

where $\lambda_1(.)$ and $\lambda_2(.)$ denote the co-states of Player 1 and Player 2, respectively. The impulse Hamiltonian of Player 2 is given by

$$H_2^I(x(t), v, \lambda_2(t)) = -G_2(x(t), v) + \lambda_2(t)^T g(x(t), v). \tag{9}$$

Given the strategies, γ and δ, the value-to-go functions of Player 1 and Player 2 at the sampling instants $t_{n+1},\ n \in \mathcal{N}'$ are given, respectively, by

$$\begin{aligned} V_1(t_{n+1}, x_{n+1}) = S_1(x(T)) &+ \sum_{j=n+1}^{N-1} \int_{t_j}^{t_{j+1}} F_1(x(t), \gamma(t; x_j))dt \\ &- \sum_{j=n+1}^{N-1} \sum_{i=1}^{k_j} \mathbb{1}_{\tau_{i,j} \geq t_{n+1}} G_1(x(\tau_{i,j}^-), \delta(\tau_{i,j}; x_j)), \end{aligned} \tag{10a}$$

$$\begin{aligned} V_2(t_{n+1}, x_{n+1}) = S_2(x(T)) &+ \sum_{j=n+1}^{N-1} \int_{t_j}^{t_{j+1}} F_2(x(t), \gamma(t; x_j))dt \\ &- \sum_{j=n+1}^{N-1} \sum_{i=1}^{k_j} \mathbb{1}_{\tau_{i,j} \geq t_{n+1}} G_2(x(\tau_{i,j}^-), \delta(\tau_{i,j}; x_j)), \end{aligned} \tag{10b}$$

with $V_1(T, x(T)) = S_1(x(T))$, and $V_2(T, x(T)) = S_2(x(T))$. We denote the equilibrium payoffs of Player 1 and Player 2 at t_{n+1} by $V_1^*(t_{n+1}, x_{n+1})$ and $V_2^*(t_{n+1}, x_{n+1})$, respectively which are obtained by substituting the equilibrium strategies γ^* and δ^* in (10a) and (10b).

To derive a set of necessary conditions for the existence of a Nash equilibrium, we make the following assumptions:

Assumption 1. *(a) The function $f : \mathbb{R}^p \times \Omega_u \to \mathbb{R}^p$ is Lipschitz continuous in x uniformly in u.*

(b) Between the sampling instants, the functions F_1, F_2, G_1, G_2 are continuous, and have continuous partial derivatives with respect to their arguments.

(c) For all strategies γ and δ of Player 1 and Player 2, respectively, the value-to-go functions V_1 and V_2 are continuous, and have continuous partial derivatives with respect to the state at the sampling instants.[2]

The following theorem provides a set of necessary conditions for the existence of sampled-data Nash equilibrium of the differential game (5a-5d).

Theorem 1. *Suppose the sampling instants are given by $t_1, t_2, \cdots, t_N$ with $0 = t_1 < t_2 < \cdots < t_N = T$, and Assumption 1 holds. Let (γ^*, δ^*) be the sampled-data Nash equilibrium of the differential game described by (5a-5d). Then, there exist piecewise continuous and piecewise differentiable functions $\lambda_1(.)$ and $\lambda_2(.)$ with $\lambda_1(t) \in \mathbb{R}^n$ and $\lambda_2(t) \in \mathbb{R}^n$ such that the following conditions hold for $t \in [t_n, t_{n+1}), n \in \mathcal{N}'$: The equilibrium control of Player 1 satisfies*

$$u^*(t) = \arg\max_{u \in \Omega_u} H_1(x^*(t), u, \lambda_1(t)), \forall t \notin \mathcal{T}^n = \{\tau_{1,n}^*, \tau_{1,n}^*, \cdots, \tau_{k_n^*,n}^*\}. \tag{11a}$$

At the impulse instant $\tau_{i,n}^*$, $i \in \mathcal{I}^n$, the equilibrium control of Player 2 satisfies

$$v_{i,n}^* = \arg\max_{v \in \Omega_v} H_2^I(x^*(\tau_{i,n}^{*-}), v, \lambda_2(\tau_{i,n}^{*+})). \tag{11b}$$

The equilibrium strategies of Player 1 and Player 2 are given, respectively, by $\gamma^*(t; x_n) = u^*(t), \forall t \in [t_n, t_{n+1})$, $t \notin \mathcal{T}^n$ and $\delta^*(\tau_{i,n}^*; x_n) = v_{i,n}^*$, $\forall i \in \mathcal{I}^n$.

[2] In Section 4, we show directly that the conditions of Assumption 1 are satisfied for the linear-quadratic differential game studied there.

The maximized Hamiltonian and impulse Hamiltonian functions are given, respectively, by

$$H_1^*(x^*(t),\lambda_1(t)) = H_1(x^*(t),u^*(t),\lambda_1(t)),\ \forall t \notin \mathcal{T}^n, \tag{11c}$$

$$H_2^{I^*}(x^*(\tau_{i,n}^{*-}),\lambda_2(\tau_{i,n}^{*+})) = H_2^I(x^*(\tau_{i,n}^{*-}),v_{i,n}^*,\lambda_2(\tau_{i,n}^{*+})),\ i \in \mathcal{I}^n, \tag{11d}$$

the equilibrium state and co-states satisfy for $t \notin \mathcal{T}^n$,

$$\dot{x}^*(t) = f(x^*(t),u^*(t)),\ x^*(t_n) = x_n, \tag{11e}$$

$$\dot{\lambda}_1(t) = -H_{1x}^*(x^*(t),\lambda_1(t)),\ \lambda_1(t_{n+1}) = \frac{\partial V_1^*(t_{n+1},x^*(t_{n+1}))}{\partial x},\quad V_1^*(T,x^*(T)) = S_1(x^*(T)), \tag{11f}$$

$$\dot{\lambda}_2(t) = -H_{2x}^*(x^*(t),u^*(t),\lambda_2(t)),\ \lambda_2(t_{n+1}) = \frac{\partial V_2^*(t_{n+1},x^*(t_{n+1}))}{\partial x},\quad V_2^*(T,x^*(T)) = S_2(x^*(T)), \tag{11g}$$

the jumps in the state and co-state variables satisfy for $i \in \mathcal{I}^n$

$$x^*(\tau_{i,n}^{*+}) = x^*(\tau_{i,n}^{*-}) + g(x^*(\tau_{i,n}^{*-}),v_{i,n}^*), \tag{11h}$$

$$\lambda_1(\tau_{i,n}^{*-}) = (I + (g_x(x^*(\tau_{i,n}^{*-}),v_{i,n}^*))^T)\lambda_1(\tau_{i,n}^{*+}) - G_{1x}(x^*(\tau_{i,n}^{*-}),v_{i,n}^*), \tag{11i}$$

$$\lambda_2(\tau_{i,n}^{*-}) = \lambda_2(\tau_{i,n}^{*+}) + H_{2x}^{I*}(x^*(\tau_{i,n}^{*-}),\lambda_2(\tau_{i,n}^{*+})), \tag{11j}$$

and the following Hamiltonian continuity condition holds:

$$H_2(x^*(\tau_{i,n}^{*+}),u^*(\tau_{i,,n}^{*+}),\lambda_2(\tau_{i,n}^{*+})) = H_2(x^*(\tau_{i,n}^{*-}),u^*(\tau_{i,n}^{*-}),\lambda_2(\tau_{i,n}^{*-})). \tag{11k}$$

Proof For $t \in [t_n, t_{n+1})$, Player 1 and Player 2 play their open-loop Nash equilibrium strategies, $\gamma^*(t;x_n)$ and $\delta^*(\tau_{i,n};x_n)$, that depend on the last measured state value x_n. The salvage values of the two players at t_{n+1} are given by (10a) and (10b).

Given the equilibrium strategy $\delta^*(\tau_{i,n}^*, x_n)$ of Player 2 in the sampling interval $[t_n, t_{n+1})$, Player 1 solves a non-standard optimal control problem given in (7a) due to jumps in the state and the additional cost at the impulse instant. Suppose Assumption 1 holds. Then, the optimality conditions for Player 1 are given in (11a), (11e), (11f), (11h), (11i) (see [21], [25]), with co-state at t_{n+1} given by the gradient of the equilibrium payoff of Player 1 at t_{n+1}. Next, for Player 1's open-loop equilibrium strategy, $\gamma^*(t;x_n)$ in $[t_n, t_{n+1})$, Player 2 solves the impulse optimal control problem (7b). The necessary conditions for the existence of the impulse controls follow from [10], [11], [15], and are given by (11b), (11e), (11h), (11g), (11j), (11k), where the co-state at t_{n+1} is given by the gradient of the equilibrium payoff of Player 2 at t_{n+1}. ■

The necessary conditions yield candidates for the sampled-data Nash equilibrium. In each sampling interval, the players use open-loop Nash equilibrium strategies, and the game is solved using backward translation starting from the last sampling interval. Consequently, if the sufficient conditions (to be given next) for the open-loop Nash equilibrium are satisfied in each sampling interval, then the candidate solutions identified by using the necessary conditions are indeed the sampled-data Nash equilibrium strategies.

A set of sufficient conditions for the existence of sampled-data Nash equilibrium for the differential game described by (5a-5d) is given as follows:

Proposition 1 (Theorem 3, [25]). *Let Assumption 1 hold. Suppose that in each sampling interval $[t_n, t_{n+1})$, $n \in \mathcal{N}'$, the initial state is x_n, and there exist feasible solutions $(\gamma^*(t;x_n), \delta^*(\tau_{i,n}^*;x_n))$ with corresponding state* *trajectory $x^*(.)$, and co-state trajectories $\lambda_1(.)$ and $\lambda_2(.)$, such that the conditions given in Theorem 1 are satisfied. Also, if in each sampling interval, the maximized Hamiltonian $H_1^*(x(t), \lambda_1(t))$ of Player 1 is concave in $x(t)$ for all $\lambda_1(t)$, the Hamiltonian $H_2(x(t), u^*(t), \lambda_2(t))$ of Player 2 is concave in $x(t)$, the value-to-go functions for Player 1 and Player 2 given by (10a) and (10b) are concave in $x(t_{n+1})$, $\lambda_1^T g(x(t), v) - G_1(x(t), v)$ is concave in $x(t)$, and the impulse Hamiltonian $H_2^I(x(t), v, \lambda_2(t))$ of Player 2 is jointly concave in $(x(t), v)$, then (γ^*, δ^*), obtained by concatenating the (open-loop) strategies $(\gamma^*(t; x_n), \delta^*(\tau_{i,n}^*; x_n))$ for $t \in [t_n, t_{n+1})$, are indeed the sampled-data Nash equilibrium strategies of the differential game described by (5a-5d).*

4 A Scalar Linear-quadratic Differential Game

In this section, we specialize the results in Theorem 1 to a one-dimensional linear-quadratic differential game with impulse controls, where state measurements are made at the sampling instants t_n, $n \in \mathcal{N} = \{1, 2, \cdots, N\}$ such that $0 = t_1 < t_2 < \cdots < t_N = T$.

We study the following scalar linear-quadratic differential game with impulse controls (referred to as iLQDG from here on):

$$\begin{aligned} J_1(x_0, u(\cdot), \tilde{v}) = & \frac{1}{2} f_1 x(T)^2 + s_1 x(T) \\ & + \frac{1}{2} \sum_{n=1}^{N-1} \int_{t_n}^{t_{n+1}} \left(h_1 x(t)^2 + 2 w_1 x(t) + c_u u(t)^2 \right) dt \\ & + \sum_{n=1}^{N-1} \sum_{i=1}^{k_n} \frac{z_1}{2} x(\tau_{i,n}^-)^2 + d_1 x(\tau_{i,n}^-), \end{aligned} \tag{12a}$$

$$\begin{aligned} J_2(x_0, u(\cdot), \tilde{v}) = & \frac{1}{2} f_2 x(T)^2 + s_2 x(T) \\ & + \sum_{n=1}^{N-1} \int_{t_n}^{t_{n+1}} \left(\frac{1}{2} h_2 x(t)^2 + w_2 x(t) \right) dt + \sum_{n=1}^{N-1} \sum_{i=1}^{k_n} \frac{1}{2} c_v v_{i,n}^2, \end{aligned} \tag{12b}$$

$$\begin{aligned} & \dot{x}(t) = a x(t) + b u(t), \ \forall t \notin \mathcal{T}^n, \ n \in \mathcal{N}', \ x(0) = x_0, \\ & x(\tau_{i,n}^+) = x(\tau_{i,n}^-) + g v_{i,n}, \ \forall i \in \mathcal{I}^n = \{1, 2, \cdots, k_n\}, \ n \in \mathcal{N}', \end{aligned} \tag{12c}$$

where $b \neq 0$, $g \neq 0$, c_u, $c_v < 0$, z_1, $d_1 < 0$, f_j, $h_j < 0$, w_j, $s_j > 0$, $j \in \{1, 2\}$, and the state at the sampling instants $t_1, t_2, \cdots, t_N$ is denoted by $x_1, x_2, \cdots, x_N$.

As indicated earlier, our objective here is to apply the general results obtained in Section 3 to a specialized scalar linear-quadratic differential game. Therefore, in the iLQDG (12), we have taken a specific form of the objective function of Player 2 to be able to obtain semi-analytic solutions for time and state pairs at which an equilibrium impulse occurs.[3] This is analogous to the stopping set condition that is obtained with the feedback information structure in stochastic impulse games with threshold type impulse controls [1]. We make the following assumptions on the equilibrium controls of the players:

Assumption 2. *In each sampling interval, Player 1's strategy space $\Gamma_{[t_n, t_{n+1})}$ is the set of locally square-integrable functions, that is,*

$$\Gamma_{[t_n, t_{n+1})} := \left\{ u(t) \in \mathbb{R}, \ t \in [t_n, t_{n+1}) \ \middle| \ \int_{t_n}^{t_{n+1}} u(t)^2 dt < \infty \right\}, \tag{13}$$

[3] [25] obtained the Hamiltonian continuity condition for a general class of linear-quadratic differential games under the open-loop information structure.

and Player 2's controls satisfy Definition 1.

Assumption 3. *The controls $u(t)$ of Player 1 and equilibrium impulse levels v_i of Player 2 lie in the interior of the control sets Ω_u and Ω_v.*

4.1 Necessary Conditions

Before considering the case where the number, timing, and levels of impulses are determined by Player 2, we consider the differential game (12) with exogenously given impulse instants.

Theorem 2. *Let $t_1, t_2, \cdots, t_N$ denote the sampling instants, and suppose that Assumptions 2 and 3 hold. Let the equilibrium impulse instants be given by $\tau^*_{i,n}, \forall i \in \mathcal{I}^n = \{1, 2, \cdots, k_n\}, n \in \mathcal{N}' = \{1, 2, \cdots, N-1\}$. Then γ^* and δ^* given in (15a)-(15b) are the equilibrium strategies of Player 1 and Player 2, respectively, assuming that the following Riccati equation (14a) has a solution with no finite escape time [4] in the entire sampling interval $[t_n, t_n + 1]$, $n \in \mathcal{N}'$, and α_j, β_j, p_j, q_j, r_j for $j \in \{1, 2\}$ satisfy the system of equations (14b)-(14t) below.*

for $t \notin \mathcal{T}^n$:

$$\dot{\alpha}_{1,n}(t) = -2\alpha_{1,n}(t)a + \frac{b^2}{c_u}\alpha_{1,n}(t)^2 - h_1, \alpha_{1,N}(T) = f_1, \tag{14a}$$

$$\dot{\beta}_{1,n}(t) = \beta_{1,n}(t)\left(\frac{b^2}{c_u}\alpha_{1,n}(t) - a\right) - w_1, \forall t \notin \mathcal{T}^n, \beta_{1,N}(T) = s_1, \tag{14b}$$

$$\dot{\alpha}_{2,n}(t) = -2\alpha_{2,n}(t)a + \frac{b^2}{c_u}\alpha_{2,n}(t)\alpha_{1,n}(t) - h_2,\ \alpha_{2,N}(T) = f_2, \tag{14c}$$

$$\dot{\beta}_{2,n}(t) = -\beta_{2,n}(t)a + \frac{b^2}{c_u}\beta_{1,n}(t)\alpha_{2,n}(t) - w_2,\ \beta_{2,N}(T) = s_2, \tag{14d}$$

$$\alpha_{j,n}(t_{n+1}) = p_{j,n+1}(t_{n+1}),\ \beta_{j,n}(t_{n+1}) = q_{j,n+1}(t_{n+1}),\ j = \{1, 2\}, \tag{14e}$$

$$\dot{p}_{1,n}(t) = -h_1 - 2(a - \frac{b^2}{c_u}\alpha_{1,n}(t))p_{1,n}(t) - \frac{b^2}{c_u}\alpha_{1,n}(t)^2, \tag{14f}$$

$$\dot{q}_{1,n}(t) = \frac{b^2\beta_{1,n}(t)}{c_u}(p_{1,n}(t) - \alpha_{1,n}(t)) - q_{1,n}(t)(a - \frac{b^2}{c_u}\alpha_{1,n}(t)) - w_1, \tag{14g}$$

$$p_{j,n}(t_{n+1}) = p_{j,n+1}(t_{n+1}),\ q_{j,n}(t_{n+1}) = q_{j,n+1}(t_{n+1}),\ j \in \{1, 2\}, \tag{14h}$$

$$\dot{p}_{2,n}(t) = -h_2 - 2(a - \frac{b^2}{c_u}\alpha_{1,n}(t))p_{2,n}(t), \tag{14i}$$

$$\dot{q}_{2,n}(t) = p_{2,n}(t)\frac{b^2}{c_u}\beta_{1,n}(t) - (a - \frac{b^2}{c_u}\alpha_{1,n}(t))q_{2,n}(t) - w_2, \tag{14j}$$

$$p_{j,N}(T) = f_j,\ q_{j,N}(T) = s_j,\ j = \{1, 2\} \tag{14k}$$

for $i \in \mathcal{I}^n$:

$$\mu(\tau^{*+}_{i,n}) = \frac{c_v}{c_v + g^2\alpha_{2,n}(\tau^{*+}_{i,n})}, \tag{14l}$$

$$p_{1,n}(\tau^{*-}_{i,n}) = p_{1,n}(\tau^{*+}_{i,n})\mu(\tau^{*+}_{i,n})^2 + z_1, \tag{14m}$$

[4] If the solution $y(t)$ of a non-linear ordinary differential equation $\dot{y} = f(y, t), y(0) = y_0$ becomes unbounded as $t \to t_e$ where $t_e < \infty$, then t_e is called the finite escape time [24, 19]. In [19], it is shown that the Riccati differential equation $\dot{y}(t) = sy(t)^2 + 2ay(t) + h$, $y(T) = q_T$ has a solution for every $T > 0$ if $d = a^2 - hs \geq 0$ and $q_T > \frac{a+\sqrt{d}}{s}$.

$$q_{1,n}(\tau_{i,n}^{*-}) = \mu(\tau_{i,n}^{*+}) \left(-\mu(\tau_{i,n}^{*+})p_{1,n}(\tau_{i,n}^{*+})\beta_{2,n}(\tau_{i,n}^{*+}) + q_{1,n}(\tau_{i,n}^{*+})\right) + d_1, \tag{14n}$$

$$\alpha_{1,n}(\tau_{i,n}^{*-}) = \alpha_{1,n}(\tau_{i,n}^{*+})\mu(\tau_{i,n}^{*+}) + z_1, \tag{14o}$$

$$\beta_{1,n}(\tau_{i,n}^{*-}) = \beta_{1,n}(\tau_{i,n}^{*+}) - \frac{g^2\alpha_{1,n}(\tau_{i,n}^{*+})\beta_{2,n}(\tau_{i,n}^{*+})}{c_v + g^2\alpha_{2,n}(\tau_{i,n}^{*+})} + d_1, \tag{14p}$$

$$\alpha_{2,n}(\tau_{i,n}^{*-}) = \mu(\tau_{i,n}^{*+})\alpha_{2,n}(\tau_{i,n}^{*+}), \tag{14q}$$

$$\beta_{2,n}(\tau_{i,n}^{*-}) = \mu(\tau_{i,n}^{*+})\beta_{2,n}(\tau_{i,n}^{*+}), \tag{14r}$$

$$p_{2,n}(\tau_{i,n}^{*-}) = \frac{\mu(\tau_{i,n}^{*+})^2}{c_v}\left(c_v p_{2,n}(\tau_{i,n}^{*+}) + g^2\alpha_{2,n}(\tau_{i,n}^{*+})\right), \tag{14s}$$

$$q_{2,n}(\tau_{i,n}^{*-}) = \mu(\tau_{i,n}^{*+})q_{2,n}(\tau_{i,n}^{*+}) + \frac{\beta_{2,n}(\tau_{i,n}^{*+})c_v(-p_2(\tau_{i,n}^{*+}) + g^2\alpha_{2,n}(\tau_{i,n}^{*+}))}{(c_v + g^2\alpha_{2,n}(\tau_{i,n}^{*+}))^2}. \tag{14t}$$

The equilibrium strategies of Player 1 and Player 2, as dictated by the necessary conditions, are given by

$$\gamma^*(t;x_n) = -\frac{b}{c_u}\Big(\alpha_{1,n}(t)\big(\phi(t,\tau_{i,n}^{*+})(\phi(\tau_{i,n}^{*-},t_n)\Big(1 - \frac{g^2}{c_v}\alpha_{2,n}(\tau_{i,n}^{*-})\mathbb{1}_{t>\tau_{i,n}^*}\Big)x_n - \mathbb{1}_{t>\tau_{i,n}^*}\phi(\tau_{i,n}^{*-},t_n)\frac{g^2}{c_v}\beta_{2,n}(\tau_{i,n}^{*-}) + \varphi(\tau_{i,n}^{*-},t_n)) + \varphi(t,\tau_{i,n}^{*+})\big) + \beta_{1,n}(t)\Big), \tag{15a}$$

$$\delta^*(\tau_{i,n}^{*-};x_n) = -\frac{g}{c_v}\Big(\alpha_{2,n}(\tau_{i,n}^{*-})\left(\phi(\tau_{i,n}^{*-},t_n)x_n + \varphi(\tau_{i,n}^{*-},t_n)\right) + \beta_{2,n}(\tau_{i,n}^{*-})\Big), \tag{15b}$$

where $t \in [t_n, t_{n+1})$, $\tau_{0,n}^* := t_n$, $\tau_{k_n^*+1,n}^* := t_{n+1}$, *and* $\forall i \in \{0\} \cup \mathcal{I}^n$,

$$\dot{\phi}(t,\tau_{i,n}^*) = \left(a - \frac{b^2}{c_u}\alpha_{1,n}(t)\right)\phi(t,\tau_{i,n}^*), \forall t \in (\tau_{i,n}^*, \tau_{i+1,n}^*), \phi(\tau_{i,n}^*,\tau_{i,n}^*) = 1, \tag{16a}$$

$$\varphi(t,\tau_{i,n}^{*-}) = -\int_{\tau_{i,n}^{*-}}^{t}\phi(h,\tau_{i,n}^{*-})\frac{b^2}{c_u}\beta_{1,n}(h)\,dh, \forall t \in (\tau_{i,n}^*, \tau_{i+1,n}^*), \tag{16b}$$

$$\phi(\tau_{i+1,n}^{*-},t_n) = \phi(\tau_{i+1,n}^{*-},\tau_{i,n}^{*+})\phi(\tau_{i,n}^{*-},t_n)\mu(\tau_{i,n}^{*-}), \tag{16c}$$

$$\varphi(\tau_{i+1,n}^{*-},t_n) = \phi(\tau_{i+1,n}^{*-},\tau_{i,n}^{*+})\varphi(\tau_{i,n}^{*-},t_n)\mu(\tau_{i,n}^{*-}) \tag{16d}$$

$$-\frac{g^2}{c_v}\beta_{2,n}(\tau_{i,n}^{*-})\phi(\tau_{i+1,n}^{*-},\tau_{i,n}^{*+}) + \varphi(\tau_{i+1,n}^{*-},\tau_{i,n}^{*+}). \tag{16e}$$

Proof Given the equilibrium control of Player 2, we obtain necessary conditions for iLQDG using (11a), (11e), (11f), (11i). The Hamiltonian of Player 1 is given by

$$H_1(x(t),u(t),\lambda_1(t)) := \frac{1}{2}h_1x(t)^2 + w_1x(t) + \frac{1}{2}c_uu(t)^2 + \lambda_1(t)(ax(t) + bu(t)),$$

where $\lambda_1(t)$ is the co-state of Player 1. Using (11a) and Assumption 3 on interior solutions, the first-order condition yields

$$H_{1u}(x^*(t),u^*(t),\lambda_1(t)) = 0 \Rightarrow u^*(t) = -\frac{b}{c_u}\lambda_1(t). \tag{17}$$

From (11e) and (11f), the equilibrium state and co-state trajectory at the non-impulse instants evolve as follows:

$$\dot{x}^*(t) = ax^*(t) - \frac{b^2}{c_u}\lambda_1(t),\ x^*(t_{n+1}) = x_{n+1}, \tag{18}$$

$$\dot{\lambda}_1(t) = -a\lambda_1(t) - h_1 x^*(t) - w_1,\ \lambda_1(t_{n+1}) = \frac{\partial V_1^*(t_{n+1}, x^*(t_{n+1}))}{\partial x}. \tag{19}$$

From (11i), the jump in the co-state at the impulse instants is given by

$$\lambda_1(\tau_{i,n}^{*-}) = \lambda_1(\tau_{i,n}^{*+}) + z_1 x^*(\tau_{i,n}^{*-}) + d_1. \tag{20}$$

The Hamiltonian, and the impulse Hamiltonian of Player 2 are given by

$$H_2(x(t), u(t), \lambda_2(t)) := \frac{1}{2}h_2 x(t)^2 + w_2 x(t) + \lambda_2(t)(ax(t) + bu(t)),$$
$$H_2^I(v_i, \lambda_2(\tau_i^+)) := \frac{1}{2}c_v v_i^2 + \lambda_2(\tau_{i,n}^+) g v_i,$$

where $\lambda_2(t)$ is the co-state of Player 2. From (11g), we obtain the dynamics of the co-state of Player 2 at the non-impulse instants as follows:

$$\dot{\lambda}_2(t) = -a\lambda_2(t) - h_2 x(t) - w_2,\ \forall t \in (t_n, t_{n+1}),\ n \in \mathcal{N}',$$
$$\lambda_2(t_{n+1}) = \frac{\partial V_2^*(t_{n+1}, x_{n+1}^*)}{\partial x}. \tag{21}$$

The co-state is equal to the gradient of the value function of Player 2 at the sampling instants because of our assumption that there are no impulses at the sampling instants.
Player 2's objective is quadratic in state, and thus we can guess the form of corresponding co-state to be linear in state, that is,

$$\lambda_2(t) = \alpha_{2,n}(t)x^*(t) + \beta_{2,n}(t),\ \forall t \in [t_n, t_{n+1}),\ n \in \mathcal{N}'. \tag{22}$$

Taking the derivative of (22) with respect to time and using the derivatives of state and co-state from (18) and (21), we arrive at

$$-a(\alpha_{2,n}(t)x^*(t) + \beta_{2,n}(t)) - h_2 x^*(t) - w_2 = \dot{\alpha}_{2,n}(t)x^*(t)$$
$$+ \alpha_{2,n}(t)\left(ax^*(t) - \frac{b^2}{c_u}(\alpha_{1,n}(t)x^*(t) + \beta_{1,n}(t))\right) + \dot{\beta}_{2,n}(t).$$

Upon comparing the coefficients, we obtain (14c) and (14d). From (21), we obtain,

$$\alpha_{2,n}(t_{n+1})x^*(t_{n+1}) + \beta_{2,n}(t_{n+1}) = \frac{\partial V_2^*(t_{n+1}, x_{n+1}^*)}{\partial x}.$$

Using the necessary condition (11b) and Assumption 3 on interior impulse levels, the first-order condition yields

$$H_{1v_i}(v_{i,n}^*, \lambda_2(\tau_{i,n}^*)) = 0 \Rightarrow v_{i,n}^* = -\frac{g}{c_v}\lambda_2(\tau_{i,n}^{*+}),\ \forall i \in \mathcal{I}^n, n \in \mathcal{N}'. \tag{23}$$

Since $v_{i,n}^*$ are the equilibrium impulse levels, it follows from (11h) that the jump in the state is given by

$$x^*(\tau_{i,n}^{*+}) = x^*(\tau_{i,n}^{*-}) - \frac{g^2}{c_v}\lambda_2(\tau_{i,n}^{*+}),\ \forall i \in \mathcal{I}^n, n \in \mathcal{N}', \tag{24}$$

and from (11j), we have that the co-state of Player 2 is continuous, that is

$$\lambda_2(\tau_{i,n}^{*-}) = \lambda_2(\tau_{i,n}^{*+}),\ \forall i \in \mathcal{I}^n, n \in \mathcal{N}. \tag{25}$$

We substitute (22) in (25) to obtain

$$\alpha_{2,n}(\tau_{i,n}^{*-})x^*(\tau_{i,n}^{*-}) + \beta_{2,n}(\tau_{i,n}^{*-}) = \alpha_{2,n}(\tau_{i,n}^{*+})x^*(\tau_{i,n}^{*+}) + \beta_{2,n}(\tau_{i,n}^{*+}). \tag{26}$$

Next, we substitute (24) in the above equation to obtain

$$\begin{aligned}\alpha_{2,n}(\tau_{i,n}^{*-})x^*(\tau_{i,n}^{*-}) + \beta_{2,n}(\tau_{i,n}^{*-}) &= \beta_{2,n}(\tau_{i,n}^{*+}) \\ &+ \alpha_{2,n}(\tau_{i,n}^{*+})\left(x(\tau_{i,n}^{*-}) - \frac{g^2}{c_v}(\alpha_{2,n}(\tau_{i,n}^{*-})x(\tau_i^{*-}) + \beta_{2,n}(\tau_{i,n}^{*-}))\right).\end{aligned}$$

The above equation holds for all x, thus leading to (14q) and (14r).
Given that the objective of Player 1 is quadratic in state, we can guess the form of corresponding co-state also to be linear in the state, so that

$$\lambda_1(t) = \alpha_{1,n}(t)x^*(t) + \beta_{1,n}(t),\ \forall t \in [t_n, t_{n+1}),\ n \in \mathcal{N}'. \tag{27}$$

We substitute (27) in (20) to obtain the following relation at the impulse instants:

$$\begin{aligned}&\alpha_{1,n}(\tau_{i,n}^{*-})x^*(\tau_{i,n}^{*-}) + \beta_{1,n}(\tau_{i,n}^{*-}) \\ &\quad= \alpha_{1,n}(\tau_{i,n}^{*+})x^*(\tau_{i,n}^{*+}) + \beta_{1,n}(\tau_{i,n}^{*+}) + z_1x^*(\tau_{i,n}^{*-}) + d_1 \\ &\quad= \alpha_{1,n}(\tau_{i,n}^{*+})(x^*(\tau_{i,n}^{*-}) - \frac{g^2}{c_v}(\alpha_{2,n}(\tau_{i,n}^{*-})x(\tau_{i,n}^{*-}) + \beta_{2,n}(\tau_{i,n}^{*-})) \\ &\qquad\qquad + z_1x^*(\tau_{i,n}^{*-}) + d_1 + \beta_{1,n}(\tau_{i,n}^{*+}).\end{aligned}$$

Upon comparing the coefficients, we obtain (14o) and (14p).
Taking the derivative of (27) with respect to time and using the derivatives of state and co-state from (18) and (19), we obtain

$$\begin{aligned}&-a(\alpha_{1,n}(t)x^*(t) + \beta_{1,n}(t)) - h_1x^*(t) - w_1 \\ &\quad= \dot\alpha_{1,n}(t)x^*(t) + \alpha_{1,n}(t)\left(ax^*(t) - \frac{b^2}{c_u}(\alpha_{1,n}(t)x^*(t) + \beta_{1,n}(t))\right) + \dot\beta_{1,n}(t).\end{aligned}$$

Upon comparing the coefficients, we obtain (14a) and (14b), where $\alpha_{1,n}(t_{n+1})x(t_{n+1}) + \beta_{1,n}(t_{n+1}) = \frac{\partial V_1^*(t_{n+1},x^*{}_{n+1})}{\partial x}$. The value-to-go for Player 1 is given by

$$\begin{aligned}V_1(t_n,x_n) = \sum_{i=0}^{k_n^*}\Bigg(\frac{1}{2}\left(\int_{\tau_{i,n}^{*+}}^{\tau_{i+1,n}^{*-}}\left(h_1x(t)^2 + 2w_1x(t) + c_uu(t)^2\right)dt\right) \\ + \frac{1}{2}z_1x(\tau_{i,n}^{*-})^2 + d_1x(\tau_{i,n}^{*-})\Bigg) + V_1(t_{n+1},x_{n+1}),\end{aligned} \tag{28}$$

where $\tau_{k_n^*+1}^* := t_{n+1}$. Next, we know that for all x,

$$\begin{aligned}&\int_{\tau_{i,n}^{*+}}^{\tau_{i+1,n}^{*-}}\left(\frac{1}{2}\dot p_{1,n}(t)x(t)^2 + p_{1,n}(t)x(t)\dot x(t) + \dot q_{1,n}(t)x(t) + q_{1,n}(t)\dot x(t) + \dot r_{1,n}(t)\right)dt \\ &\quad - \frac{1}{2}p_{1,n}(t)x(t)^2\Big|_{\tau_{i,n}^{*+}}^{\tau_{i+1,n}^{*-}} - q_{1,n}(t)x(t)\Big|_{\tau_{i,n}^{*+}}^{\tau_{i+1,n}^{*-}} - r_{1,n}(t)\Big|_{\tau_{i,n}^{*+}}^{\tau_{i+1,n}^{*-}} = 0,\ i \in \mathcal{I}^n,\ n \in \mathcal{N}'.\end{aligned}$$

Substituting $\dot{x}(t) = ax(t)+bu(t)$ in the above equation, adding it to (28), and using $u^*(t) = -\frac{b}{c_u}(\alpha_{1,n}(t)x^*(t)+\beta_{1,n}(t))$, we obtain the equilibrium value-to-go for Player 1. Since the equilibrium control maximizes the value-to-go for Player 1, (14f)-(14g), (14m)-(14n) and the following conditions hold for all $n \in \mathcal{N}'$:

$$
\begin{aligned}
\dot{r}_{1,n}(t) &= \frac{b^2}{c_u}\left(q_{1,n}(t)\beta_{1,n}(t) - \beta_{1,n}(t)^2\right), \forall t \notin \mathcal{T}^n, r_{1,n}(t_{n+1}) = r_{1,n+1}(t_{n+1}),\\
r_{1,n}(\tau^{*-}_{i+1,n}) &= r_{1,n}(\tau^{*+}_{i+1,n}) - \left(\frac{g^2\beta_2(\tau^{*+}_{i+1,n})}{c_v + g^2\alpha_{2,n}(\tau^{*+}_{i+1,n})}\right)\Bigg(q_{1,n}(\tau^{*+}_{i+1,n})\\
&\qquad - \frac{g^2 c_v \beta_2(\tau^{*+}_{i+1,n})p_{1,n}(\tau^{*+}_{i+1,n})}{2(c_v + g^2\alpha_{2,n}(\tau^{*+}_{i+1,n}))}\Bigg),
\end{aligned}
$$

where p_1, q_1 and r_1 are continuous at the sampling instants because there are no impulses at the sampling instants (see Definition 1). Therefore, the equilibrium value-to-go is given by

$$
V_1^*(t_n, x_n) = \frac{1}{2}p_{1,n}(t_n)x_n^2 + q_{1,n}(t_n)x_n + r_{1,n}(t_n),\ \forall n \in \mathcal{N}'. \tag{29}
$$

The value-to-go for Player 2 is given by

$$
V_2(t_n, x_n) = \sum_{i=0}^{k_n}\left(\int_{\tau^+_{i,n}}^{\tau^-_{i+1,n}}\left(w_2 x(t) + \frac{1}{2}h_2 x(t)^2\right)dt + \frac{1}{2}c_v v^2_{i+1,n}\right) + V_2(t_{n+1}, x_{n+1}). \tag{30}
$$

For all x, we have

$$
\begin{aligned}
&\int_{\tau^+_{i,n}}^{\tau^-_{i+1,n}}\left(\frac{1}{2}\dot{p}_{2,n}(t)x(t)^2 + p_{2,n}x(t)\dot{x}(t)+\dot{q}_{2,n}(t)x(t) + q_{2,n}(t)\dot{x}(t) + \dot{r}_{2,n}(t)\right)dt\\
&-\frac{1}{2}p_{2,n}(t)x(t)^2\Big|_{\tau^+_{i,n}}^{\tau^-_{i+1,n}} - q_{2,n}(t)x(t)\Big|_{\tau^+_{i,n}}^{\tau^-_{i+1,n}} - r_{2,n}(t)\Big|_{\tau^+_{i,n}}^{\tau^-_{i+1,n}} = 0,\ \forall i \in \mathcal{I}^n,\ n \in \mathcal{N}'.
\end{aligned}
$$

Substituting $\dot{x}(t) = ax(t)+bu^*(t)$ in the above equation, adding it to (30) and using the equilibrium controls $(\tau^*_{i,n}, v^*_{i,n})$, $i \in \mathcal{I}^n$, $n \in \mathcal{N}'$ yields the equilibrium value-to-go. Taking $u^*(t)$ as given, the equilibrium control of Player 2 maximizes the value-to-go for Player 2 for all x, so that (14i)-(14j), (14s)-(14t), and the following relations hold:

$$
\begin{aligned}
\dot{r}_{2,n}(t) &= q_{2,n}(t)\frac{b^2}{c_u}\beta_{1,n}(t),\ \forall t \notin \mathcal{T}^n, r_{2,n}(t_{n+1}) = r_{2,n+1}(t_{n+1}),\ r_{2,N}(T) = 0,\\
r_{2,n}(\tau^{*-}_{i+1,n}) &= r_{2,n}(\tau^{*+}_{i+1,n}) + \frac{g^2\beta_{2,n}(\tau^{*+}_{i+1,n})^2}{2(c_v + g^2\alpha_{2,n}(\tau^{*+}_{i+1,n}))^2}\left(p_{2,n}(\tau^{*+}_{i+1,n})g^2 + c_v\right)\\
&\qquad - \frac{g^2 q_{2,n}(\tau^{*+}_{i+1,n})\beta_{2,n}(\tau^{*+}_{i+1,n})}{c_v + g^2\alpha_{2,n}(\tau^{*+}_{i+1,n})},\ \forall i \in \mathcal{I}^n,
\end{aligned}
$$

and the value-to-go is given by

$$
V_2^*(t_n, x_n) = \frac{1}{2}p_{2,n}(t_n)x_n^2 + q_{2,n}(t_n)x_n + r_{2,n}(t_n),\ \forall n \in \mathcal{N}'. \tag{31}
$$

Since the co-state is equal to the gradient of value function at the sampling instants (see (21)), we have

$$
\alpha_{2,n}(t_{n+1})x_{n+1} + \beta_{2,n}(t_{n+1}) = p_{2,n+1}(t_{n+1})x_{n+1} + q_{2,n+1}(t_{n+1}),\ \forall n \in \mathcal{N}'.
$$

Since the above relation holds for all x_n, we obtain $\alpha_{2,n}(t_{n+1}) = p_{2,n+1}(t_{n+1})$ and $\beta_{2,n}(t_{n+1}) = q_{2,n+1}(t_{n+1})$. Using (27) in (18), we obtain

$$\dot{x}^*(t) = \left(a - \frac{b^2}{c_u}\alpha_{1,n}(t)\right) x^*(t) - \frac{b^2}{c_u}\beta_{1,n}(t) \tag{32}$$

$$\Rightarrow x^*(\tau_1^{*-}) = \phi(\tau_{1,n}^{*-}, t_n)x_n + \varphi(\tau_{1,n}^{*-}, t_n), \tag{33}$$

where (16a)–(16e) hold and $\tau_{k_n^*+1,n} := t_{n+1}$. Define

$$x^*(\tau_{i,n}^{*-}) = \phi(\tau_{i,n}^{*-}, t_n)x_n + \varphi(\tau_{i,n}^{*-}, t_n),\ \forall i = \mathcal{I}^n, \tag{34a}$$

$$x^*(\tau_{i+1,n}^{*-}) = \phi(\tau_{i+1,n}^{*-}, t_n)x_n + \varphi(\tau_{i+1,n}^{*-}, t_n),\ \forall i = \mathcal{I}^n\backslash\{k_n\}. \tag{34b}$$

From (32), we obtain

$$\begin{aligned}
x^*(\tau_{i+1,n}^{*-}) &= \phi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+})x^*(\tau_{i,n}^{*+}) + \varphi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+}) \\
&= \phi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+})\big(x^*(\tau_{i,n}^{*-}) - \frac{g^2}{c_v}\left(\alpha_{2,n}(\tau_{i,n}^{*-})x^*(\tau_{i,n}^{*-}) + \beta_{2,n}(\tau_{i,n}^{*-})\right)\big) + \varphi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+}) \\
&= \phi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+})\phi(\tau_{i,n}^{*-}, t_n)\left(1 - \frac{g^2}{c_v}\alpha_{2,n}(\tau_{i,n}^{*-})\right)x_n + \varphi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+}) \\
&\quad -\frac{g^2}{c_v}\beta_{2,n}(\tau_{i,n}^{*-})\phi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+}) + \phi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+})\varphi(\tau_{i,n}^{*-}, t_n)\left(1 - \frac{g^2}{c_v}\alpha_{2,n}(\tau_{i,n}^{*-})\right).
\end{aligned}$$

Upon comparing with (34b), we obtain

$$\begin{aligned}
\phi(\tau_{i+1,n}^{*-}, t_n) &= \phi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+})\phi(\tau_{i,n}^{*-}, t_n)\left(1 - \frac{g^2}{c_v}\alpha_{2,n}(\tau_{i,n}^{*-})\right), \\
\varphi(\tau_{i+1,n}^{*-}, t_n) &= \phi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+})\varphi(\tau_{i,n}^{*-}, t_n)\left(1 - \frac{g^2}{c_v}\alpha_{2,n}(\tau_{i,n}^{*-})\right) \\
&\quad -\frac{g^2}{c_v}\beta_{2,n}(\tau_{i,n}^{*-})\phi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+}) + \varphi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+}).
\end{aligned}$$

The equilibrium state evolves according to the following equation:

$$\begin{aligned}
x(t) = \phi(t, \tau_{i,n}^{*})(\phi(\tau_{i,n}^{*-}, t_n)&\left(1 - \frac{g^2}{c_v}\alpha_{2,n}(\tau_{i,n}^{*-})\mathbb{1}_{t>\tau_{i,n}^*}\right)x_n \\
&-\mathbb{1}_{t>\tau_{i,n}^*}\phi(\tau_{i,n}^{*-}, t_n)\frac{g^2}{c_v}\beta_{2,n}(\tau_{i,n}^{*-}) + \varphi(\tau_{i,n}^{*-}, t_n)) + \varphi(t, \tau_{i,n}^{*+})) \\
&\forall t \in (\tau_{i,n}^*, \tau_{i+1,n}^*), i \in \mathcal{I}^n \cup \{0\}, n \in \mathcal{N}',
\end{aligned} \tag{35}$$

where $\tau_{0,n}^* := 0$. Then, from (17) and (23), the equilibrium strategies of Player 1 and Player 2 are given by (15a) and (15b), respectively. ■

Remark 4. *Even when the timing of impulses is given, the system of Riccati equations* (14a) *for Player 1 defined in each sampling interval* $[t_n, t_{n+1}]$ *differs from the corresponding system obtained in classical differential games [26] because* $\alpha_1(\cdot)$ *also jumps due to the interventions by Player 2 in addition to the update in* $\alpha_1(\cdot)$ *at the sampling instants.*

The above theorem characterizes the equilibrium with exogenously given impulse instants. If the number and timing of impulses are determined by Player 2, the impulse instants must satisfy the Hamiltonian continuity condition (11k) in addition to (14a)–(14t).

Theorem 3. *Suppose that $t_1, t_2, \cdots, t_N$ are the sampling instants, and Assumptions 2 and 3 hold. Then $\tau^*_{i,n}$, $i \in \mathcal{I}^n$, $n \in \mathcal{N}'$ are the equilibrium impulse instants if*

$$
\begin{aligned}
x(\tau^*_{i,n}) &= \phi(\tau^{*-}_{i,n}, t_n)x_n + \varphi(\tau^{*-}_{i,n}, t_n) \\
&= \frac{c_u g^2(g^2 h_2 - 2c_v a)\beta_2(\tau^{*-}_{i,n}) + 2c_v(c_v b^2 d_1 - w_2 g^2 c_u)}{2(c_v c_u h_2 g^2 - b^2 c_v^2 z_1) - c_u g^2(g^2 h_2 - 2c_v a)\alpha_2(\tau^{*-}_{i,n})},
\end{aligned} \tag{36a}
$$

where ϕ and φ satisfy (16a)-(16e), and the Riccati equation (14a) has no finite escape time in the entire sampling interval $[t_n, t_{n+1}]$, $n \in \mathcal{N}'$.

Proof From the continuity condition (11k) on the Hamiltonian and using (17), (20), (27), (24), and (25), we obtain

$$
\begin{aligned}
c_u g^2(g^2 h_2 - 2c_v a)\lambda_2(\tau^{*-}_{i,n})^2 + 2c_v(c_v b^2 d_1 - w_2 g^2 c_u)\lambda_2(\tau^{*-}_{i,n}) \\
+ 2\lambda_2(\tau^{*-}_{i,n})(b^2 c_v^2 z_1 - c_v c_u h_2 g^2)x(\tau^{*-}_{i,n}) = 0.
\end{aligned}
$$

$\lambda_2(\tau^{*-}_{i,n}) = 0$ implies that the equilibrium impulse level is zero. From Definition 1, $v^*_{i,n}$ cannot be equal to zero if $\tau^*_{i,n}$ is an admissible impulse instant. Thus, an impulse occurs if

$$
x(\tau^{*-}_{i,n}) = \frac{c_u g^2(g^2 h_2 - 2c_v a)\beta_2(\tau^{*-}_{i,n}) + 2c_v(c_v b^2 d_1 - w_2 g^2 c_u)}{2(c_v c_u h_2 g^2 - b^2 c_v^2 z_1) - c_u g^2(g^2 h_2 - 2c_v a)\alpha_2(\tau^{*-}_{i,n})}.
$$

On substituting (34a) in the above equation, we arrive at (36a). ■

Remark 5. *An impulse occurs at equilibrium whenever the state trajectory intersects the time varying function $\xi(t)$, given by*

$$
\xi(t^-) = \frac{c_u g^2(g^2 h_2 - 2c_v a)\beta_2(t^-) + 2c_v(c_v b^2 d_1 - w_2 g^2 c_u)}{2(c_v c_u h_2 g^2 - b^2 c_v^2 z_1) - c_u g^2(g^2 h_2 - 2c_v a)\alpha_2(t^-)}.
$$

4.2 Non-linear Optimization

Let $\tau_{1,n}, \tau_{2,n}, \cdots, \tau_{k_n,n}$ denote the admissible impulse instants for a given number of impulses, k_n, in each sampling interval $[t_n, t_{n+1})$, $n \in \mathcal{N}'$. From Definition 1, we have

$$
\tau_{1,n} < \tau_{2,n} < \cdots < \tau_{k_n,n}.
$$

The above strict ordering can be represented as

$$
D_n \boldsymbol{\tau}_n < \mathbf{0}, \tag{37}
$$

where

$$
D_n := \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix}_{(k_n-1)\times k_n}, \quad \boldsymbol{\tau}_n := \begin{bmatrix} \tau_{1,n} \\ \vdots \\ \tau_{k_n,n} \end{bmatrix}, \forall n \in \mathcal{N}'.
$$

At the equilibrium impulse instants, the Hamiltonian continuity condition (36a) holds for the iLQDG formulated by (12). The equilibrium impulse instants are obtained by finding the fixed-point solution of the Riccati like system of equations (14a)–(14t) and the system of non-linear equality constraints (36a). Alternatively, this problem can be viewed as the following constrained non-linear optimization problem:

$$\underset{\{\boldsymbol{\tau}_n\}_{n\in\mathcal{N}'}}{\text{argmin}} \quad \sum_{n=1}^{N-1}\sum_{i=1}^{k_n}\Big(x(\tau_{i,n}^-)-\xi(\tau_{i,n}^-)\Big)^2 \tag{38a}$$

$$\text{subject to} \quad \mathbf{1}.(t_n+s)\leq \boldsymbol{\tau}_n \leq \mathbf{1}.(t_{n+1}-s) \qquad \forall n\in\mathcal{N}' \tag{38b}$$

$$D_n\boldsymbol{\tau}_n \leq -\mathbf{1}.s \qquad \forall n\in\mathcal{N}', \tag{38c}$$

where $s>0$ is a slack variable, and

$$\xi(\tau_{i,n}^-)=\frac{c_u g^2(g^2h_2-2c_v a)\beta_2(\tau_{i,n}^-)+2c_v(c_v b^2 d_1-w_2 g^2 c_u)}{2(c_v c_u h_2 g^2-b^2c_v^2 z_1)-c_u g^2(g^2 h_2-2c_v a)\alpha_2(\tau_{i,n}^-)}. \tag{38d}$$

The above problem can be solved using interior point algorithms [13] or sequential quadratic programming methods [12].

Remark 6. *If there are no cross terms between the state and control variables and the players' objectives and state dynamics are linear in the state, then the resulting class of linear-in-the-state differential games with impulse controls can be shown to be degenerate, that is, Player 2's impulse optimal control problem is decoupled from Player 1's optimization problem. For non-degenerate linear-in-the-state games, the non-linear optimization procedure in Section 4.2 can be used to numerically compute the equilibrium.*

5 A Numerical Example

In this section, we illustrate the theory developed in the previous two sections using a numerical example.

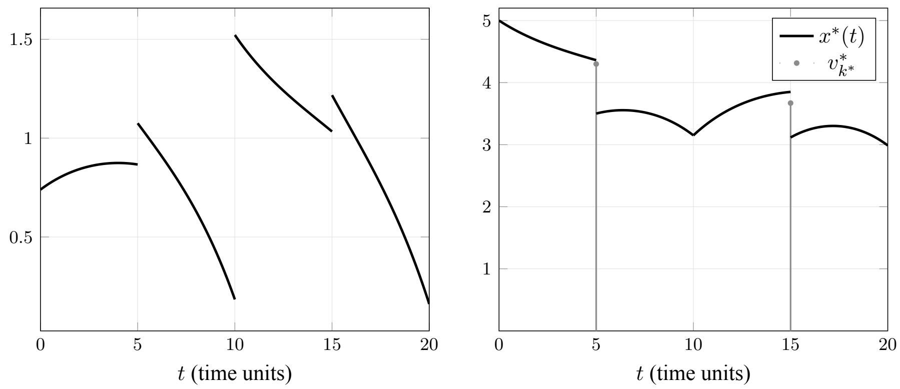
Consider a dynamic game with scalar linear dynamics, and with time horizon $T=20$. Player 1 uses piecewise continuous sampled-data state feedback controls while Player 2 uses impulse controls. The state measurements are made at three instants of time, that is, $t_1=0$, $t_2=10$, $t_3=20$. Player 1 and Player 2 maximize their respective objective functions

$$\begin{aligned}J_1(x_0,u(\cdot),\tilde{v}) &= x(20)(10-x(20))-\sum_{n=1}^{2}\int_{t_n}^{t_{n+1}}\left(x(t)^2-10x(t)+2.5u(t)^2\right)dt\\ &\quad -\sum_{n=1}^{2}0.25x(\tau_n^-)^2\\ J_2(x_0,u(\cdot),\tilde{v}) &= -2x(20)^2-\sum_{n=1}^{2}\int_{t_n}^{t_{n+1}}x(t)^2dt-\sum_{n=1}^{2}v_n^2,\end{aligned}$$

and the state dynamics are given by

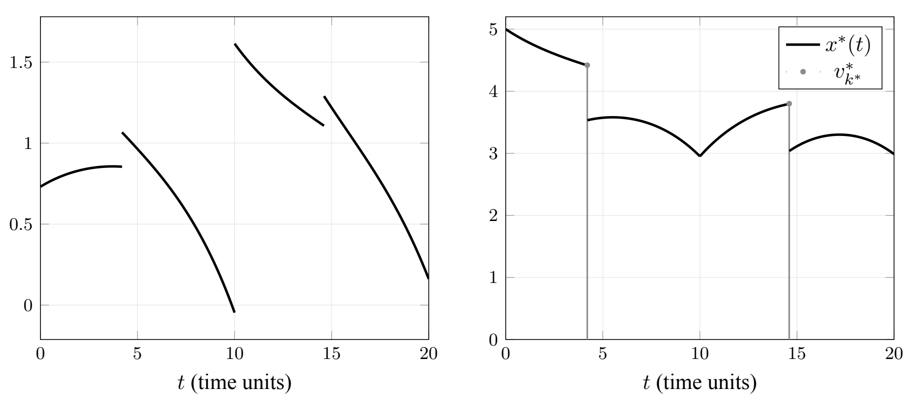
$$\begin{aligned}\dot{x}(t)&=-0.1x(t)+0.4u(t),\ t\notin\{\tau_1,\tau_2\},\ x(0)=5,\\ x(\tau_i^+)&=x(\tau_i^-)-0.2v_i,\ i\in\{1,2\}.\end{aligned}$$

First, we analyze the case where the impulses are periodic, that is, $\tau_1=5$ and $\tau_2=15$. The equilibrium control of Player 1, given in Figure 1(a), jumps at the impulse instants because of the jump in her co-state caused by the impulse control of Player 2. The state trajectory, and the equilibrium impulse levels of Player 2 are shown in Figure 1(b). At equilibrium, Player 1 incurs a loss of 518.51, while Player 2 incurs a loss of 96.54.



(a) $u^*(t)$ **(b)** $x^*(t)$ and v^*

Figure 1: Equilibrium controls, and state trajectory with periodic impulses.



(a) $u^*(t)$ **(b)** $x^*(t)$, $v^*_{k^*}$ and $\xi(t)$

Figure 2: Equilibrium candidate controls, and state and co-state trajectories.

Table 1: Equilibrium payoff of Player 1 and Player 2 for varying numbers of impulses in each sampling interval.

Different cases	# Impulses	Impulse instants	Payoffs	
	(k_1, k_2)	τ^*	J_1	J_2
Periodic impulses	$(1, 1)$	$\{5, 15\}$	-518.51	-96.54
	$(1, 0)$	5	-525.67	-158.06
	$(0, 1)$	15	-52.47	-224.67
Endogenous impulses	$(1, 1)$	$\{4.20, 14.60\}$	-553.20	-73.38
	$(1, 0)$	7.09	-338.74	-226.59
	$(0, 1)$	14.56	-114.33	-212.30
No impulses	$(0, 0)$	–	126.94	--396.18

k_1 and k_2 denote the number of impulses in the sampling intervals $(0, 10)$ and $(10, 20)$, respectively.

Next, we determine the equilibrium when the impulse instants in each sampling interval are determined by Player 2, and there is only one impulse in each sampling interval. The impulse timing is characterized by the Hamiltonian continuity condition (11k).

The (candidate) equilibrium impulses occur at $\tau_1^* = 4.20$ and $\tau_2^* = 14.60$, and at equilibrium, the losses of Player 1 and Player 2 are given by 553.20 and 73.38, respectively when Player 2 applies one impulse in each sampling interval. The piecewise continuous equilibrium control of Player 1 is shown in Figure 2(a) and equilibrium impulse levels of Player 2 are shown in Figure 2(b). Since the sufficient conditions in Proposition 1 hold for Player 2, the candidates indeed constitute Nash equilibrium. In Table 1, we consider periodic impulses, endogenous impulse timings, as well as no impulses in the game. It can also be seen in Table 1 that the equilibrium payoff of Player 2 is the highest if she decides on the number and timing of the impulses, applying them at $\tau_1^* = 4.20$ and $\tau_2^* = 14.60$.

6 Conclusions

In this paper, we have derived necessary and sufficient conditions for the existence of sampled-data Nash equilibrium in a general class of two-player nonzero-sum differential games with impulse controls, where only one of the players controls the impulses (their number, timing, and magnitudes). For a special case of differential games with scalar linear state dynamics and general quadratic objective functions for the players, we have shown that the sampled-data Nash equilibrium can be obtained by determining the fixed point of a system of Riccati like equations with jumps coupled with non-linear equality constraints. We have further shown for the same class of differential games that the piecewise continuous equilibrium control of Player 1 and equilibrium impulse control of Player 2 are linear in the most recently measured state value, and provide a numerical procedure to determine the equilibrium strategies.

For the future, it would be interesting to apply our results to case studies in pollution regulation, exchange rate interventions, and cybersecurity. One extension of our work would be to differential games where both players use continuous as well as impulse controls. Another extension would be to differential games with more than two players, and particularly in the high population regime.

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References

[1] Aïd, R., Basei, M., Callegaro, G., Campi, L., Vargiolu, T.: Nonzero-sum stochastic differential games with impulse controls: A verification theorem with applications. Mathematics of Operations Research **45**(1), 205 – 232 (2020)

[2] Aïd, R., Campi, L., Li, L., Ludkovski, M.: An impulse-regime switching game model of vertical competition. Dynamic Games and Applications (2021). DOI 10.1007/s13235-021-00381-4. ePub ahead of print, March 22

[3] Başar, T.: On the existence and uniqueness of closed-loop sampled-data Nash controls in linear-quadratic stochastic differential games. In: K. Iracki, K. Malanowski, S. Walukiewicz (eds.) Optimization Techniques, pp. 193 – 203. Springer Berlin Heidelberg, Berlin, Heidelberg (1980)

[4] Başar, T.: Time consistency and robustness of equilibria in noncooperative dynamic games. In: F.V. der Ploeg, A. de Zeeuw (eds.) Dynamic Policy Games in Economics, pp. 9 – 54. Springer International Publishing, North-Holland (1989)

[5] Başar, T.: Optimum $H\infty$ designs under sampled state measurements. Systems & Control Letters **16**(6), 399 – 409 (1991)

[6] Başar, T.: Minimax control of switching systems under sampling. Systems & Control Letters **25**(5), 315 – 325 (1995)

[7] Başar, T., Olsder, G.: Dynamic Noncooperative Game Theory, 2nd edition. Society for Industrial and Applied Mathematics, Philadelphia, PA (1999)

[8] Başar, T., Haurie, A., Zaccour, G.: Nonzero-Sum Differential Games. In: T. Başar, G. Zaccour (eds.) Handbook of Dynamic Game Theory, pp. 61 – 110. Springer International Publishing, Cham (2018)

[9] Basei, M., Cao, H., Guo, X.: Nonzero-sum stochastic games and mean-field games with impulse controls. Mathematics of Operations Research (2021). (Forthcoming)

[10] Blaquière, A.: Differential games with piece-wise continuous trajectories. In: P. Hagedorn, H.W. Knobloch, G.J. Olsder (eds.) Differential Games and Applications, pp. 34 – 69. Springer, Berlin (1977)

[11] Blaquière, A.: Necessary and sufficient conditions for optimal strategies in impulsive control and application. In: M. Aoki, A. Morzzolla (eds.) New Trends in Dynamic System Theory and Economics, pp. 183 – 213. Academic Press, New York (1977)

[12] Büskens, C., Maurer, H.: Sqp-methods for solving optimal control problems with control and state constraints: adjoint variables, sensitivity analysis and real-time control. Journal of Computational and Applied Mathematics **120**(1), 85 – 108 (2000)

[13] Byrd, R.H., Hribar, M.E., Nocedal, J.: An interior point algorithm for large-scale nonlinear programming. SIAM Journal on Optimization **9**(4), 877 – 900 (1999)

[14] Campi, L., De Santis, D.: Nonzero-sum stochastic differential games between an impulse controller and a stopper. Journal of Optimization Theory and Applications **186**(2), 688–724 (2020)

[15] Chahim, M., Hartl, R.F., Kort, P.M.: A tutorial on the deterministic Impulse Control Maximum Principle: Necessary and sufficient optimality conditions. European Journal of Operational Research **219**(1), 18 – 26 (2012)

[16] Drăgan, V., Ivanov, I.G., Popa, I.L.: Stochastic linear quadratic differential games in a state feedback setting with sampled measurements. Systems & Control Letters **134**, 104,563 (2019)

[17] Dockner, E.J., Jorgensen, S., Long, N.V., Sorger, G.: Differential Games in Economics and Management Science. Cambridge University Press (2000)

[18] El Farouq, N., Barles, G., Bernhard, P.: Deterministic minimax impulse control. Applied Mathematics and Optimization **61**(3), 353 – 378 (2010)

[19] Engwerda, J.C.: LQ Dynamic Optimization and Differential Games. Wiley, New York (2005)

[20] Ferrari, G., Koch, T.: On a strategic model of pollution control. Annals of Operations Research **275**(2), 297 – 319 (2019)

[21] Geering, H.: Continuous-Time Optimal Control Theory for Cost Functionals Including Discrete State Penalty Terms. IEEE Transactions on Automatic Control **21**(6), 866 – 869 (1976)

[22] Haurie, A., Krawczyk, J.B., Zaccour, G.: Games and Dynamic Games. World Scientific, Singapore (2012)

[23] Imer, O.C. and Başar, T. To measure or to control: optimal control with scheduled measurements and controls. In *2006 American Control Conference*, pages 1003–1008. 2006.

[24] Jacobson, D.H.: Extensions of Linear-Quadratic Control, Optimization and Matrix Theory. Academic Press, London-New York (1977)

[25] Sadana, U., Reddy, P.V., Zaccour, G.: Nash equilibria in non-zero sum differential games with impulse control. European Journal of Operational Research. **295**(2), 792 – 805 (2021)

[26] Simaan, M., Cruz Jr., J.: Sampled-data Nash controls in non-zero-sum differential games. International Journal of Control **17**(6), 1201 – 1209 (1973)