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Three Essays on Dynamic Asset Pricing

by

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Three Essays on Dynamic Asset Pricing

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ABSTRACT

The purpose of this thesis is two-fold. The first purpose is to investigate efficiency, risk preferences, and equilibrium pricing, and thus contributing to the assessment of application of the dynamic asset pricing theory to derivatives. The second purpose is to propose an equilibrium model of derivative security prices when economic agents are heterogenous and the underlying asset is not marketable. This extends the dynamic asset pricing theory.

Chapter two is comprised of the first essay: "The Price of Risk for Bond Futures." The market price of risk implied by the futures price of bonds is obtained from market data assuming that the theoretical model used is correct. Exact formulas of bond futures prices under discrete, and continuous marking-to-market are derived using Vasicek's framework and Jamshidian's methodology. The market price of risk of the instantaneous interest rate and parameters of the interest rates process are estimated using Treasury bill futures prices from the Chicago Mercantile Exchange (CME), and Treasury bond futures prices from the Chicago Board of Trade (CBOT). The empirical procedure used for the estimation is the Generalized Method of Moments (GMM) of Hansen (1982). Since the market price of interest rate risk must reflect risk preferences, an issue raised in this essay is whether the estimated values of the market price of interest rate risk can be accounted for by assumptions about tastes and beliefs. The estimated market price of risk provides, along with the GMM tests, information about the reliability of the models of bond futures prices.

Chapter three is comprised of the second essay: "An Empirical Investigation of the Term Structure of Implied Volatilities in Currency Options". This chapter examines a fundamental hypothesis of dynamic asset pricing models; namely the no-arbitrage condition. If intertemporal arbitrage opportunities exist, expectations about prices are not formed rationally, and the market is inefficient. The
no-arbitrage condition is tested in this essay using the term structure of implied volatilities, and listed currency options from the Philadelphia Exchange.

The third and final essay entitled: "Futures Market Equilibrium with Heterogeneity," is presented in chapter four. This essay studies equilibrium in the futures market of a commodity in a single good economy, which is populated by heterogeneous producers and speculators. The commodity is traded in a spot market at the harvest time. The model illustrates the role of heterogeneity and non-tradeness in a futures market equilibrium.
RÉSUMÉ

L'objectif de cette thèse est de contribuer à l'examen et à l'extension de la théorie d'évaluation dynamique des actifs appliquée aux produits dérivés. Dans un premier temps, les préférences des investisseurs, l'efficience, et les prix d'équilibres dans les marchés de produits sont testés. Ensuite, un modèle de prix d'équilibre de contrats à terme lorsque les investisseurs sont hétérogènes est dérivé.

Le premier de la série de trois éssais composant la thèse est intitulé: "The price of Risk For Bond Futures." Cette recherche consiste à estimer les paramètres décrivant l'évolution des taux d'intérêt et la prime de risque unitaire à partir de modèles d'évaluation de prix à terme d'obligations. Les données utilisées dans l'étude empirique proviennent du Chicago Mercantile Exchange (CMM) et du Chicago Board of Trade (CBOT). La prime de risque unitaire implicite pourrait être indicateur consistence des préférences du modèle avec les données empiriques.

Le deuxième éssai de la thèse est intitulé: "An Empirical Investigation of the Term Structure of Implied Volatilities in Currency Options." Cette étude consiste à examiner une hypothèse fondamentale de la théorie d'évaluation dynamique des actifs: l'absence d'arbitrage. Condition nécessaire à l'efficience des marchés, l'absence d'arbitrage (intertemporel) est testée pour diverses options sur devises échangées à la bourse des valeurs de Philadelphie. L'étude qui exploite des modèles de structure à terme des volatilités implicites d'options, teste également l'adéquation des processus de taux de change retenus.

Le troisième et dernier éssai de la thèse est intitulé: "Futures Market Equilibrium with Heterogeneity." Ce travail de recherche consiste à déterminer l'équilibre en temps continu du marché à terme d'une denrée produite dans une économie à bien unique. Le marché au comptant de la denrée n'est ouvert qu'à la date de récolte qui coïncide avec l'échéance du contrat à terme. L'économie en question est composée de producteurs et spéculateurs hétérogènes. Le modèle permet de mettre en évidence le rôle de l'hétérogenité dans les demandes d'équilibre et de déterminer la dynamique des prix à terme sous des contraintes de transaction.
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CHAPTER 1: INTRODUCTION

During the past two decades, the dynamic asset pricing theory has grown as a distinctive part of financial economics. The traditional dynamic asset pricing models share a small set of basic assumptions: absence of arbitrage, single agent optimality, and equilibrium. The results obtained from these models are usually unified by two key concepts: state prices and martingales. This parsimonious presentation has contributed to the success of the theory. The conceptual and methodological development of the dynamic asset pricing theory is rooted in the decade spanning roughly from 1969 to 1979, called "the golden decade" by Duffie (1992). The advances occurred in both the discrete time and continuous time framework, with the latter considered as a useful approximation of the former.

Merton (1969) has given the initial impulse to the continuous time modelling with his explicit dynamic solution for optimal portfolio and consumption policies. These results set the stage for general equilibrium asset pricing models like those derived by Merton (1973) and later by Cox, Ingersoll and Ross (1985). In the discrete time framework, equilibrium asset pricing models have been mainly multiperiod extensions of the CAPM like that of LeRoy (1973), Rubinstein (1976a), and Lucas (1978), among others. However, the simplest multiperiod version of the CAPM is the continuous-time consumption-based CAPM derived by Breeden (1979).

The other major development in the dynamic asset pricing theory during that period is the contingent claims analysis. Contingent claims analysis is a technique for determining the price of derivative securities like futures contracts, options, loan guaranties, and mortgage backed securities. Although connected with the continuous-time portfolio models, the contingent claims analysis is primarily linked to the important breakthrough of Black and Scholes (1973) on the theory of option pricing. An influential discrete time simplification of the Black and Scholes model, the binomial option pricing model has been later presented by Cox, Ross and Rubinstein (1979). Building on the ideas of Black and Scholes (1973), Cox and Ross (1976) and Ross (1978), Harrison and Kreps (1979), and Harrison and Pliska (1981) have given to contingent claims analysis, and to the other aspects of dynamic asset pricing theory their current conceptual structure. Major innovations in dynamic asset pricing theory occurred during its first decade of existence. The present direction of the research in the field is the validation of existing models through empirical studies, and extension and reassessment of early models using alternative sets of assumptions.
The purpose of this thesis is two-fold. The first purpose is to investigate efficiency, risk preferences, and equilibrium pricing, and thus contributing to the assessment of application of the dynamic asset pricing theory to derivatives. The second purpose is to propose an equilibrium model of derivative security prices when economic agents are heterogeneous and the underlying asset is not marketable. This extends the dynamic asset pricing theory.

The remainder of this thesis is organized as follows. Chapter two is comprised of the first essay: "The Price of Risk for Bond Futures." The market price of risk implied by the futures price of bonds is obtained from market data assuming that the theoretical model used is correct. Exact formulas of bond futures prices under discrete, and continuous marking-to-market are derived using Vasicek's framework and Jamshidian's methodology. The market price of risk of the instantaneous interest rate and parameters of the interest rate process are estimated using Treasury bill futures prices from the Chicago Mercantile Exchange (CME), and Treasury bond futures prices from the Chicago Board of Trade (CBOT). The empirical procedure used for the estimation is the Generalized Method of Moments (GMM) of Hansen (1982). Since the market price of interest rate risk must reflect risk preferences, an issue raised in this essay is whether the estimated values of the market price of interest rate risk can be accounted for by assumptions about tastes and beliefs. The estimated market price of risk may provide, along with the GMM tests, information about the reliability of the models of bond futures prices.

Chapter three is comprised of the second essay: "An Empirical Investigation of the Term Structure of Implied Volatilities in Currency Options". This chapter examines a fundamental hypothesis of dynamic asset pricing models; namely the no-arbitrage condition. If intertemporal arbitrage opportunities exist, expectations about prices are not formed rationally, and the market is inefficient. The no-arbitrage condition is tested in this essay using the term structure of implied volatilities, and listed currency options from the Philadelphia Exchange.

The third and final essay entitled: "Futures Market Equilibrium with Heterogeneity," is presented in chapter four. This essay studies equilibrium in the futures market of a commodity in a single good economy, which is populated by heterogeneous producers, and speculators. The commodity is traded in a spot market at the harvest time. Each producer is endowed with a non-traded private technology. Both classes of agents trade in futures contracts written on the commodity, and in bonds.
CHAPTER 2: THE PRICE OF RISK FOR BOND FUTURES

2.1 Introduction

According to standard option theory, the value of a contingent claim depends on the dynamics of the underlying asset. For traded assets, a hedge portfolio consisting of the derivative asset and the underlying asset can be constructed so that the portfolio is without risk. If this portfolio earns other than the risk-free interest rate, arbitrage opportunities exist. A risk neutral valuation of the components of the portfolio is possible (i.e., the risk premia can be disregarded). Unfortunately, in the case of an interest rate derivative valuation, the state variable (interest rate) is not a traded asset. A hedge portfolio cannot be constructed so that risk is eliminated, and the valuation is preference free. The valuation of such a contingent claim requires the knowledge of the market price of interest rate risk.

When an interest rate contingent claim model is derived in a general equilibrium framework [such as Cox, Ingersoll, and Ross (CIR, 1985)], the market price of interest rate risk which is determined endogenously, depends on the characteristics of the economy. Alternatively, in partial equilibrium models of interest rate contingent claims, the underlying variable process and the market price of interest rate risk are exogenous. Such models usually assume a constant market price of interest rate risk [e.g., Vasicek (1977), Brennan and Schwartz (1977, 1979), Richard (1978), and Jamshidian (1989)]. Dothan (1978) provides a theoretical justification for a constant market price of interest rate risk, based on a continuous time CAPM with a logarithmic utility function for the consumer. CIR (1985), however, point out that partial equilibrium models of interest rate contingent claims can be internally inconsistent. First, no underlying equilibrium may be consistent with the assumption about the dynamics of the instantaneous interest rate. Second, arbitrarily selecting a functional form for the market price of interest rate risk may produce a model that violates the nonarbitrage condition. CIR (1981) show that the only version of the expectations theory of the term structure of interest rates that is obtained in a continuous time rational expectations equilibrium is the local expectations hypothesis. Therefore, the only valid equilibrium value for the market price of interest rate risk is zero.

Campbell (1985) argues that CIR's criticisms apply to versions of the pure expectations theory of the term structure, which states that the term premia are zero. However, much of the literature is concerned with a less restrictive theory, the expectations theory, which states that term premia are
constant through time.¹ He provides an example, based on a general equilibrium foundation for the Vasicek model, which shows that two versions of the expectations theory hold simultaneously in continuous time. Although the term premia are not strictly equal, the differences among them are second-order effects of bond yield variability. In the model of Vasicek, the market price of interest rate risk is a constant.²

As shown by Campbell using the Vasicek model, the market price of interest rate risk does not have to be zero for the partial equilibrium models to be consistent. Another consistency issue is whether the estimated market prices of interest rate risk yielded by these models can be accounted for by both the assumptions about tastes and beliefs, and the volatility of relevant economic data. For example, a very high estimated market price of interest rate risk might imply an unrealistic holding premium for the utility function assumed in the model, and the variance of interest rates.

The primary objective of this paper is to investigate the market price of interest rate risk for various models of bond futures prices. The market prices and parameters of the interest rate process are obtained from market data assuming that the model used is correct. Models of bond futures prices are derived in the Vasicek framework by using Jamshidian's methodology. The data used are Treasury bill futures prices from the Chicago Mercantile Exchange (CME), and Treasury bond futures prices from the Chicago Board of Trade (CBOT). The estimation procedure used is the Generalized Method of Moments (GMM) of Hansen (1982).

The contribution of this essay can be summarized as follows. First, models of continuous time bond futures prices under discrete marking-to-market are derived by using the fact that equispaced sampling of a continuous time AR-1 (the Ornstein-Uhlenbeck process) results in a discrete time AR-1. The formulas obtained are tractable and suitable for empirical investigation unlike a formula derived earlier by Chen (1992b). Second, meaningful estimates of the market price of interest rate risk are

¹Campbell defines two term premia as the primary objects of expectations theories. The first is the instantaneous holding premium, which is the expected difference at t between the instantaneous holding return on a bond which matures at T and the spot rate t. The second is the instantaneous forward premium, which is the expected difference at t between the forward rate and spot rate at t.

²A constant market price of risk need not imply constant term premia. Longstaff (1990) shows that the expectations hypothesis can imply time-varying term premia if the time frame for which the expectations hypothesis holds differs from the measurement period.
obtained and Treasury bond futures data are shown to be a valid alternative to the bond data used in previous studies to estimate such market prices.\textsuperscript{3} Since delivery options held by the short side of a Treasury bond futures contract do not appear to have an impact on the market price of interest rate risk, the very active Treasury bond futures market is a good source of data for empirical estimations. Estimates of the market price of interest rate risk obtained using models of Treasury bill futures prices data are far higher than those from other studies. Whether these estimates are too high to account for both a logarithmic utility function and the volatility of interest rate is an open empirical question. These results might be explained by market distortions. An alternative interpretation is that the models are misspecified. Estimations improve for long-lived futures contracts with long-term underlying bonds when a model with discrete marking-to-market is used.

The remainder of this essay is organized as follows. In section 2.2 the closed form solutions of futures prices of bonds are derived, and then used in the empirical investigation reported and discussed in section 2.3. Section 2.4 provides some concluding remarks.

2.2 The Model to Price Bond Futures

The continuous time futures price of a discount bond is derived below under the alternative assumptions of discrete and continuous marking-to-market. The new formula for a futures price under discrete marking-to-market is obtained directly from the continuous marking-to-market formula by taking advantage of the properties of the interest rate process rather than applying recursive methods as in previous studies.\textsuperscript{4} For the remainder of the paper, futures bonds are denoted by $U'(t,r,T,T')$, or $U$, where the the superscript $i = d$ (D), c (C), for discount bond futures under continuous (discrete) marking-to-market, and coupon bond futures under continuous (discrete) marking-to-market, respectively. The distinction between the subscript $t$ used for the futures price at time $t$, and the partial derivative of the futures price relative to time will always be clear from the context. The arguments of the futures price will be defined later.

\textsuperscript{3} See for example Brennan and Schwartz (1979), Dietrich-Campbell and Schwartz (1986), Longstaff (1989), and Duan (1992).

\textsuperscript{4} For discussions of the issues involved with continuous versus discrete timing to market, see Chen (1992b) and Flesaker (1993).
2.2.1 Futures prices under continuous marking-to-market

In this section, the Jamshidian (1989) methodology is followed to derive the futures price of a discount bond. While not new, these results are subsequently required to derive the discrete marking-to-market formula. Assume that the term structure of interest rates is completely determined by the instantaneous interest rate \( r(t) \) which follows a mean-reverting Gaussian process [as in Vasicek (1977)],

\[
    dr = a(r_0 - r)dt + \sigma dw,
\]

where \( \sigma, a, \) and \( r_0 \) are positive constants, and \( w(t) \) is a standard Wiener process. Prices of bonds and their derivative securities depend on \( r(t) \); that is, on their only state variable. The price of a security paying a rate \( h(r,t) \) continuously and yielding a terminal payoff \( g(r(T)) \) at time \( T \) is given as the solution of:

\[
    U_t = \frac{1}{2} \sigma^2 U_{rr} + a(\bar{r} - r)U_r - rU + h,
\]

\( \bar{r} = r_0 + \frac{\sigma}{a} \).

The market price of interest rate risk, given by \( \lambda \), is the expected instantaneous excess return above the riskless rate divided by the instantaneous standard deviation of return of the bond. According to Jamshidian (1989), the solution to (1)-(2) is:

\[
    U(r,t) = \mathbb{E}_{r,t}[\sum_{s} h(\bar{r}(s)) e^{-Y(t,s)}] + \int_{t}^{T} h(\bar{r}(s)) e^{-Y(t,s)} ds,
\]

\( \mathbb{E} \) denotes the expected value.

---

5The solution is based upon Friedman (1975), Theorem 5.3, chapter 6.
where \( \bar{r} \) is the "risk neutral interest rate process" as defined by:
\[
d\bar{r} = a(\bar{r} - \bar{r}) dt + \sigma dw, \quad \text{and} \quad Y(t,s) = \int \bar{r}(u) du.
\]

Suppose that \( U(r,t,T) \) is the futures price of a discount bond with a current price \( P(r,T,T') \), where \( T \) is the maturity date of the futures contract and \( T' \) is the maturity date of the discount bond. The futures price for delivery of a discount bond at time \( T \) is given as the solution of:
\[
U_t = \frac{1}{2} \sigma^2 U_{rr} + a(\bar{r} - r) U_r + h. \tag{4}
\]

Duffie and Stanton (1992) show that the resettlement price of a continuously resettled claim that is marked at the terminal date \( T \) to \( P(r(T),T,T') \) is:
\[
U(r,t,T) = E_{t,T}[P(r(T),T,T')]. \tag{5}
\]

Thus, equation (5) is the solution to equation (4). A formula of the bond futures price at any time \( t \) is obtained from (5) by observing as in Jamshidian (1989) that \( P(r(T),T,T') \), the bond price under the original process, is lognormal. Given the identical structure between \( r \) and \( \bar{r} \), \( P(\bar{r}(T),T,T') \), the

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\[6\] Arnold (1974, section 8.3) shows that \( E_{t,T}[\bar{r}(s)] = m(r,t,s) = e^{-a(s-t)}r + (1 - e^{-a(s-t)})\bar{r} \), and
\[
\text{var}_{r,T}[\bar{r}(s)] = \frac{\sigma^2 (1 - e^{-2a(s-t)})}{2a}.
\]

\[7\] A continuously resettled contingent claim is completely described by its maturity date, dividend rate and underlying process. A futures on a bond is a continuously resettled claim, with the bond price process as the underlying process, but with a zero dividend rate. A continuously resettled claim is continuously marked-to-market.
bond price under the risk neutralized process, is also lognormal. The term \( \log P(\bar{f}(T), T, T') \), where \( \log \) stands for the natural logarithmic function, is normal with conditional variance:

\[
\text{var}_{t,T'}[\log P(\bar{f}(T), T, T')] = \sigma^2 \frac{(1 - e^{-2a(T-t)}) (1 - e^{-a(T'-T)})^2}{2a^2}
\]

and conditional mean:

\[
\mathbb{E}_{t,T'}[\log P(\bar{f}(T), T, T')] = \frac{1}{2} k^2(t, T') - \frac{1}{2} k^2(t, T) - \eta(r, t, T') + \eta(r, t, T).
\]

According to a standard result in probability theory, if \( x = \log(y) \) is normal with mean \( \mu_x \) and variance \( \sigma_x^2 \), \( y \) is lognormal with mean \( \mu_y = \exp(\mu_x - 1/2\sigma_x^2) \). Hence, the futures price is:

\[
U^d(r, t, T, T') = \frac{P(r, t, T')}{P(r, t, T)} \exp \left[ -\frac{1}{2} \sigma^2 \frac{(1 - e^{-2a(T-t)}) (1 - e^{-a(T'-T)})^2}{2a^2} \right]. \tag{6a}
\]

The futures price of the discount bond is equal to its forward price multiplied by an exponential

\[
P(\bar{f}(T), T, T') = \exp \left[ -\frac{1}{2} k^2(T, T') - \eta(\bar{f}, T, T') \right]
\]

with

\[
\eta(r, t, T, T') = (T' - T) \bar{f} + (\bar{f}(T) - \bar{f}) (1 - e^{-a(T'-T)})
\]

and

\[
k^2 = \frac{\sigma^2}{2a^3} \left( 4e^{-a(T'-T)} - e^{-2a(T'-T)} + 2a(T' - T) - 3 \right).
\]
function which is the outcome of the continuous marking-to-market. The argument of this exponential function can be decomposed into the product of the covariance of \( Y(T,T') \) and \( \bar{f}(T') \), with a conditional variance of \( \bar{f}(T) \) and a negative constant. The futures price at time \( t \) becomes:

\[
U^d(r,t,T,T') = \frac{P[r,t,T']}{P[r,t,T]} \exp\left(- \frac{1}{\sigma^2} \text{cov}_{r,T} [\bar{f}(T'), \int_T^{T'} \bar{f}(u) \, du] \text{var}_{r,t}[\bar{f}(T)] \right) \tag{6b}
\]

This expression for the futures price is used in the next section to derive a discount bond futures price formula under discrete-time marking-to-market. However, before doing so, the formula for the futures price for the delivery of a coupon bond (i.e., delivery of a portfolio of discount bonds) is derived.

A coupon bond can be viewed as a portfolio of discount bonds consisting of \( a_i > 0 \) issues of each \( s_i \)-maturity discount bond. The value of the portfolio at time \( T \), where \( T < s_i \) for all \( i \), is:

\[
P_s = \sum a_i P(r(T),T,s_i) \tag{7}
\]

Thus, the futures price for the delivery of a portfolio of discount bonds at time \( T \) is:

\[
U^c = E_{r,T} \left[ \sum a_i P(\bar{f}(T),T,s_i) \right]
\]

\[
= \sum a_i E_{r,T} [P(\bar{f}(T),T,s_i)]
\]

\[
= \sum a_i U(r,t,T,s_i) \tag{8}
\]

In (8), the first equality is due to (5) and the definition of a portfolio, the second equality follows from the properties of the expectation operator, and the third equality is due to (5). The futures price for the delivery of a portfolio of discount bonds is equal to a portfolio of futures prices for the delivery of the individual discount bonds. While a similar formula is derived by Jamshidian (1989) for a portfolio of discount bonds, it has not yet been extended to the case of a futures price for the delivery of a portfolio of discount bonds.

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9 Chen (1992b) derives a similar result. It is easily shown that equation (6) herein is equivalent to Chen's equation (6). While Chen solves a differential equation, the general approach of Duffie and Stanton (1992) is used herein.
2.2.2 Futures price under discrete marking-to-market

In this section a closed-form continuous-time pricing formula is derived for a futures contract maturing at time T, which is written on a discount bond that matures at time T' and is discretely marked-to-market at an equispaced sampling interval L. Consider equations (6a) and (6b). From the previous section, it is known that the only part of the discount bond futures price formula which results from the continuous marking-to-market is:

\[ \exp\left(- \frac{1}{\sigma^2} \text{cov}_{r,T}[\tilde{f}(T'), \int_{T}^{T'} \tilde{f}(u) \, du] \text{var}_{r,T}[\tilde{f}(T)] \right). \]

The discrete marking-to-market equivalent of the above formula for an equispaced sampling interval L is:

\[ \exp\left(- \frac{\sigma^2}{2} \left( e^{-aLn} - e^{-aL(n' + n + 1)} \right) \right) \frac{1 - e^{-aL}}{1 - e^{-aL}}, \]

where \( n \) and \( n' \) are the number of equispaced sampling intervals L between \( t \) and \( T \), and \( T \) and \( T' \), respectively (see proof in the appendix). Therefore, the discount bond futures price at time \( t \) under discrete marking-to-market is:

\[ U^D(r,t,T,T') = \frac{P[r,t,T']}{P[r,t,T]} \exp\left(- \frac{\sigma^2}{2} \left( e^{-aLn} - e^{-aL(n' + n + 1)} \right) \right) \frac{1 - e^{-aL}}{1 - e^{-aL}}. \]  \( 8 \)

This formula is more tractable and compact than the alternative formula developed by Chen (1992b) for the same interest rate process by applying a backward dynamic programming methodology. The analytical differences between the outcomes for discrete and continuous marking-to-market are clearly shown by (8), unlike the case for Chen's formula. Furthermore, the representation of the futures price under discrete marking-to-market in (8) is similar to that of Flesaker (1993) for the interest rate model developed by Heath, Morton and Jarrow (1992). \(^{10}\)

\(^{10}\)Flesaker (1993) shows that a discount bond futures price under discrete marking-to-market is equal to the product of the forward price of the bond and an exponential term.
The coupon bond futures price formula under discrete marking-to-market, \( U^C(r,t,T,si) \), is derived as was the coupon bond futures price formula under continuous marking-to-market. This is a straightforward exercise left for the reader.

2.3 Empirical Study

In this section, the market price of interest rate risk and the parameters of the interest rate process are estimated, and the explanatory power of the developed models are assessed. The futures price model for the delivery of discount and coupon bonds are tested using data for the U.S. Treasury bill and bond futures contracts, respectively. The short side of a Treasury bond futures contract holds several delivery options (quality option, timing option, and wild card option) which theoretically should be accounted for when deriving a Treasury bond futures price. Not only are they very difficult to model simultaneously, but the resulting formulae are not suitable for the type of empirical analysis undertaken herein. Although recent empirical studies show that the delivery options are not significant, their effect on the market price of interest rate risk is tested later in this section.

2.3.1 Empirical Procedure

A description of the empirical procedure begins with the following definition of the futures pricing error:

\[
\epsilon_{tt} = U_t^i - U_t^m
\]

where \( U_t^i \) is the futures price of the bond at time \( t \) from the model, where the superscript \( i = d, D, c, C \) is defined as in section 2, and \( U_t^m \) is the futures price of the bond at time \( t \). The discrete time version of the short-term interest rate process is defined as in Chan, Karolyi, Longstaff and Sanders (1991):

\[11\text{Recent empirical evidence suggests that the value of delivery options has been exaggerated previously. For example, Hemler (1990) finds that the payoff obtained by switching from the bond cheapest to deliver three months prior to delivery to the one cheapest at time to delivery averages less than 30 basis points of par. Hegde (1988) reports the average value of a delivery option over the last quarter of the nearby futures contract to be less than .5% of the mean futures price. In contrast, Kane and Marcus (1986) find option values ranging from 1.39 to 4.6 percentage points of par. These values correspond to discounts of 1.9% to 6.2% from the equilibrium futures price that results in the absence of the delivery option.} \]
\[ e_{2t} = r_t - r_{t-1} - a (r_0 - r_{t-1}) \]  

(10)

\[ e_{3t} = e_{2t}^2 - \sigma^2, \]  

(11)

where \( a, r_0, \) and \( \sigma \) are the parameters of the continuous-time interest rate process defined earlier. The econometric approach adopted herein is to test (9)-(11) as a set of overidentifying restrictions on a system of moment equations using the Hansen (1982) Generalized Method of Moments (GMM). This technique is applied to test option pricing models by Rindell and Sandás (1991). The GMM estimators and their standard errors are consistent even if the disturbances \( e_u \) are conditionally heteroscedastic and serially correlated. The temporal aggregation problem, resulting from the estimation of a continuous-time process with discrete time data, is likely to affect the distribution of the disturbances in a minor fashion.\(^{12}\) Thus, the GMM approach is appropriate for this empirical study.\(^{13}\) A family of orthogonality conditions is constructed under this approach. Consider the vector of errors \( \mathbf{e} \), with elements \( e_{1t}, e_{2t}, \) and \( e_{3t} \) and define a vector of parameters \( \Theta \) with elements \( \sigma, a, \lambda \) and \( r_0 \). The vector of errors is a function of the parameters and is denoted as \( \mathbf{e} (\Theta) \). A trivial orthogonality condition is that the expected error vector, evaluated at some \( \Theta = \Theta_0 \), must be zero; that is:

\[ \mathbb{E}[\mathbf{e}(\Theta_0)] = 0. \]  

(12)

According to the unbiasedness hypothesis, the error vector should not be correlated with variables in the information set of the investor. Define \( \mathbf{z}_t \) as a \( g \)-dimensional vector of variables in the investors' information set at time \( t \). The unbiasedness hypothesis implies:

\[ \mathbb{E}[\mathbf{e}(\Theta_0) \otimes \mathbf{z}_t] = 0, \]  

(13)

\(^{12}\) Chan et al. (1991) acknowledge that the discretized process is only an approximation of the continuous time specification. The error introduced can be shown to be of second order if changes in the interest rate are measured over a short period of time, which is the case herein.

\(^{13}\) Chan et al. (1991)
where ⊗ denotes the Kronecker operator. The GMM procedure consists of replacing $E[e_t(\Theta)\otimes]$ with its sample counterpart, $g_T(\Theta)$, which for a sample of $T$ observations is:

$$g_T(\Theta) = \frac{1}{T} \sum_{t=1}^{T} e_t(\Theta) \otimes z_t.$$  \hfill (14)

The parameter estimates that minimize the following quadratic form are chosen:

$$J_T(\Theta) = e'(g'_T(\Theta) W g_T(\Theta))$$  \hfill (15)

where $W$ is a positive-definite symmetric weighting matrix. When the number of orthogonality conditions equals the number of parameters, a solution exists and $J_T = 0$. When the number of orthogonality conditions exceeds the number of parameters to be estimated, the model is said to have overidentifying restrictions, so that no $\Theta$ simultaneously solve all the moment conditions. Hansen (1982) shows that the minimized value of $J_T$ is chi-squared distributed under the null hypothesis that the model is true, with degrees of freedom equal to the number of overidentifying restrictions (that is, the number of orthogonality conditions less the number of parameters to be estimated). The chi-square statistic supplies a goodness-of-fit test for the model. A significance test of individual parameters is provided by the asymptotic covariance matrix for the GMM estimate of $\Theta$.

2.3.2 The market price of interest rate risk implied by Treasury bill data.

In this section, the market price of interest rate risk and the parameters of the short-term interest rate and discount bond futures price models are estimated. Data used are U.S. Treasury bill futures prices traded at the Chicago Mercantile Exchange (CME). Treasury bill futures contracts require delivery of a Treasury bill maturing three months from the first delivery day. The settlements occur for each delivery month during three successive delivery days. The first delivery day is the first day of the delivery month on which a three month Treasury bill is issued and an original one-year bill issue has three months remaining to maturity. The time period considered for the empirical estimations of the parameters implied by the Treasury bill futures prices runs from January 1983 to December 1989. The volume of trade of Treasury bill futures contracts depends on the time to maturity of the contract. While almost no trade occurs in the early period of the contract, the volume of contracts traded increases as the
maturity date approaches. The Treasury bill futures data are Wednesday settlement prices for contracts with various delivery months. The proxy of the short-term interest rate of Treasury bill futures prices is the one-week Treasury bill yield.

As outlined by Gallant (1987), the GMM estimators depend on the choice of instrumental variables. Since the optimal choice of instrument is still unresolved, the entire set or subsets of the following instrumental variables are used herein: a constant term of one; the one-period-lagged interest rate; the one-period-lagged trade volume; the one-period-lagged futures price; and the time to maturity.

The statistics and autocorrelation structure at lags of 1, 2, 3, 4 and 5 weeks of interest rates, bond futures price and volume samples are summarized in Table 2.1. The parameter estimates, the standard deviations of parameters, and the GMM minimized criterion ($\chi^2$) values for the system consisting of a model of a discount bond futures price under discrete marking-to-market and the short-term interest rate process are presented in Table 2.2. The $\chi^2$ test for goodness-of-fit suggests that the models cannot be rejected at the usual 95% confidence level. The parameter estimates, the standard deviations of the parameters and the GMM minimized criterion values for the model consisting of a discount bond futures price under continuous marking-to-market and the short-term interest rate process are presented in Table 2.3. The similarity of results obtained under the alternative marking-to-market assumptions in Tables 2.2 and 2.3 implies that the economic significance of marking interest rate futures contracts to the market on a daily basis is fairly trivial for futures contracts with maturity well below a year [as found earlier by Flesaker (1993)].

The estimates of the models provide a number of insights about the dynamics of short-term interest rates and the market price of interest rate risk. Since the parameter $a$ is significant, strong evidence exists for mean reversion in the short-term interest rate. The market price of interest rate risk $\lambda$ is inferred by the estimates of $\lambda \sigma$ and $\sigma^2$.\(^{14}\) $\lambda$ has the expected sign and is significantly different from zero for all the studied periods. The absolute values of the implied market price of interest rate risk are too high if compared with values from other studies reported in Table 2.7. While the lowest absolute value of the implied market price of interest rate risk herein is 2.27, the highest absolute value reported in other studies is 0.487. These unusual higher values obtained herein may be due to the pattern of trade

\(^{14}\) $\lambda \sigma$ and $a\sigma_0$ are estimated as individual parameters because the GMM procedure did not converge otherwise.
in the Treasury bill futures market. Trading in any particular futures contract is very low when first traded, increases over time, and is again low as the delivery date approaches. Thus, some price quotations might not be representative of the market value of the bond futures price due to market inefficiencies.\(^\text{15}\) \(\lambda\) is the only parameter which can capture specific aspects of the Treasury bill futures market, since the other variables are constrained by the two empirical equations of the short-term interest rate process. Thus, the implied market price of interest rate risk appears to be affected by market imperfections. An alternative interpretation of these results may be that the instantaneous holding premium implied by the market price of interest rate risk is too high to be consistent with the log utility function underlying the model, and the estimated volatility of the short-term interest rate.\(^\text{16}\) However, this claim should be investigated empirically before any reliable conclusion.

### 2.3.3 The market price of interest rate risk implied by Treasury bond data

In this section, the market price of interest rate risk and the parameters of the short-term interest rate and coupon bond futures price models are estimated. Data used are U.S. Treasury bond futures prices traded at the Chicago Board of Trade (CBOT). The Treasury bond futures price is determined from an instrument with a 8 percent coupon rate and yield, and 20 years time to maturity. A Treasury bond futures contract calls for the delivery of a Treasury bond at any day of the delivery month that, if callable, is not callable for at least 15 years from the first day of the delivery month or, if not callable, has a maturity of at least 15 years from the first day of the delivery month. The Treasury bond is treated as a portfolio of discount bonds, and delivery options are addressed later in this study. The market for Treasury bond futures is always very active. Weekly futures settlement prices are used from January 1989 to September 1991. The short-term interest rate used herein is the yield on three-month Treasury bills.

The parameter estimates, the standard deviations of the parameters and, the GMM minimized criterion (\(\chi^2\)) values for the system consisting of a model of a coupon bond futures price under discrete time marking-to-market and the short-term interest rate process are presented in Table 2.4. The \(\chi^2\) test

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\(^{15}\)Treasury bill futures prices may be affected (albeit temporarily) by other events. This is left to a future study.

\(^{16}\)The instantaneous holding premia implied by the values of the market price of interest rate risk reported in Table 2 are 2.1\%, 3.48\%, and 2.38\%, respectively.
for goodness-of-fit suggests that the models cannot be rejected at the usual 95% confidence level. The parameter estimates, the standard deviations of the parameters, and GMM minimized criterion values for the model of a coupon bond futures price under discrete marking-to-market and the short-term interest rate process are presented in Table 2.5. The $\chi^2$ test for goodness-of-fit suggests that these models cannot be rejected at the usual 95% confidence level. Unlike the Treasury bill futures, the values of the parameters for Treasury bond futures prices are different for the continuous marking-to-market model (Table 2.5) and discrete marking-to-market models (Table 2.4). Furthermore, the model with discrete marking-to-market fits the data better based on the $\chi^2$ test. This finding confirms the observation of Flesaker that the economic effects of daily marking-to-market are significant when a futures contract has a long time-to-maturity, and is written on a long-term bond. Over half of the futures price observations used herein are for contracts with time-to-maturity above one year, and the underlying instruments of the futures contracts are long-term bonds.

All of the estimated parameters in Table 2.4 are significant. The coefficients of mean reversion are low compared to those obtained for the period of 1983-1989. The difference may be explained by a temporal shift in the interest rate process or by the proxies used. The implied market price of risk is always low but significantly different from zero for the model under discrete marking-to-market. The $\lambda$ values are 0.101, 0.087 and 0.103, for the entire period and the two subperiods, respectively. The signs of the estimated values of $\lambda$ are consistent with the findings from other studies involving coupon bonds. Based on these results, the market price of interest rate risk is low but significantly different from zero. Moreover, it does not change much over time. While these results appear to be more reliable than those obtained from the Treasury bill futures, the effect of delivery options on the parameters needs to be checked before any conclusion is drawn.

2.3.4 Are delivery options significant?

While some empirical studies of the Treasury bond futures market conclude that delivery options

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17 The implied market price of interest risk has values 0.180, 0.093, and 0.045 when the model of futures price under continuous marking-to-market is used. The two first values are significant.

18 The holding premiums implied by the values of the market price of interest rate risk reported in Table 5 are -0.0013, -0.0010, and -0.0013.
held by the short side of the contract are of significant economic magnitude, most studies conclude otherwise. If such options have significant value, the performance of the model developed herein for coupon bond futures price would be suspect. Thus, a test of the effect of these options is necessary. To test for the effect of delivery options, variable $D$, is introduced into the model that equals unity for some futures contract assumed to be delivered at the maturity of the contract, and zero for all the remaining contracts assumed to be delivered at the beginning of the delivery period. Treasury bond futures contracts are usually settled at the very end of the delivery period. If some contracts were "wrongly" delivered at the beginning of the delivery period, then any difference in the pricing of the two sets of contracts is accounted for by the delivery options. The pricing discrepancy between the two sets of prices affects the market price of interest rate risk. Therefore, allowing for a dummy shift in the market price of interest risk, model (9)-(11) becomes:

$$
\varepsilon_{1t} = U_t^{c5} - U_t^m
$$

(9')

$$
\varepsilon_{2t} = r_t - r_{t-1} - a (r_0 - r_{t-1})
$$

(10')

$$
\varepsilon_{3t} = \varepsilon_{2t}^2 - \sigma^2,
$$

(11')

where,

$$
\eta (r,t,T,T') = (T' - T) \frac{r_0 a + \lambda \sigma + D_t \delta}{a} + (\bar{r} (T) - \frac{r_0 a + \lambda \sigma + D_t \delta}{a} ) (1 - e^{-z(T' - T)})
$$

for each discount bond futures price included in the coupon bond futures price. The results of the estimation of model (9)-(11) are summarized in Table 2.6. The $\delta$ estimates, which capture the effect of delivery options, are not significant at usual confidence levels. The market price of interest rate risk is low and significantly different from zero. Based on the best performing model for Treasury bond futures price, the local expectation hypothesis is rejected while the constant term premia hypothesis is not rejected.
2.4 Concluding remarks

Exact formulae of futures price for the delivery of discount bond and coupon bonds are derived under the alternative assumptions of discrete marking-to-market and continuous marking-to-market. Treasury bill futures prices and Treasury bond futures prices are used to assess the formulas, to determine the parameters of the interest rate process, and to estimate the market price of the short-term interest rate risk by using the GMM approach. The market price of interest rate risk is different from zero for each data set. This suggests that the local expectation hypothesis is not adequate for the partial equilibrium valuation of bonds and bond derivatives. The alternative hypothesis of constant holding and forward premia, and a constant market price of interest rate risk is not rejected for the best performing model applied to Treasury bond futures and three-month Treasury bill yields as interest rates. The market price of interest rate risk estimated with the Treasury bill futures data and one-week Treasury bill yields seems to be too high, which suggests the possibility of inefficiencies in the market. An alternative explanation is that the Vasicek model is not successful in describing the dynamics of short-term bonds and their derivative securities (i.e., futures contracts).

Excluding delivery options from the futures price formulas does not affect the parameters significantly. The strong effect of discrete marking-to-market on long-lived futures on long-term underlying bonds is supported herein.
Appendix to chapter 2: Derivation of the discount bond futures price formula under discrete marking-to-market

Consider the argument of the exponential function in equation (6a); that is:

\[-\frac{1}{\sigma^2} \text{cov}_{r,T}[\tilde{f}(T'), \int_{T}^{T'} \tilde{f}(u) du] \text{var}_{r,T}[\tilde{f}(T)].\]

This expression is obtained from the following mean-reverting gaussian process of the short-term interest rate:

\[dr = a(r_0 - r) dt + \sigma dw.\]

When the continuous-time interest rate process is at a uniform sampling interval (say, L), the resulting discrete-time process, \(R_p, p=0,1,2,\ldots\), follows a first-order linear normal autoregressive process of the form [Phadke and Wu (1974) and Vasicek (1977)]:

\[R_p = R_0 (1 - e^{-aL}) + e^{-aL} R_{p-1} + e^t.\]

Suppose that \(n\) equispaced samplings exist between \(t\) and \(T\), so that the contract is marked-to-market \(n\) times. The discrete-time equivalent of \(\text{var}_{r,T}[\tilde{f}(T)]\) is \(\sigma^2 (e^{-aLn})\). Since \(n\) equispaced samplings between \(t\) and \(T\) implies \(n'\) equispaced samplings between \(T\) and \(T'\),

\[\text{cov}_{r,T}[\tilde{f}(T'), \int_{T}^{T'} \tilde{f}(u) du]\]

becomes:

\[\sum_{k=1}^{n'} \text{cov}[R_{n'}, R_{n'-k}] = \frac{[1 - e^{-aLn'}]}{[1 - e^{-aL}]} \sigma^2,\]
[Nelson 1972, pp. 11 and 12]. By replacing the variance and covariance expressions with their discrete-time equivalents, the formula for bond futures prices under discrete marking-to-market is obtained.
Summary statistics are presented herein for the model variables and some instruments. The statistics include sample means, standard deviations (S.D.), minimum (Min), Maximum (Max), and autocorrelation of i order (si). Variables designated with 1 are used in the Treasury bill futures study, where the data are weekly from January 83 to December 89. Variables designated with 2 are used in the Treasury bond futures study, where the data are weekly from January 89 to September 91. The terms Futures, Interest and Volume designate futures prices, interest rates and the volume of trade, respectively.

### Table 2.1
Summary Statistics for the Model
Variables and Instruments

<table>
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<th>Variables</th>
<th>N</th>
<th>Mean</th>
<th>S.D.</th>
<th>Min</th>
<th>Max</th>
<th>s1</th>
<th>s2</th>
<th>s3</th>
<th>s4</th>
<th>s5</th>
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<td>.016</td>
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<tr>
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<tr>
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<td>000</td>
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Table 2.2

Model Estimates for Treasury Bill Futures

This table contains the estimates of the model for the discount bond futures price under discrete marking-to-market. The $U_t^D$ (U.S. Treasury bill futures prices) and short-term interest rates $r(t)$ (annualized one-week U.S. Treasury bill yields) are from January 83 to December 89. The parameters are estimated for the entire period and two equal length subperiods using the Generalized Method of Moments of Hansen (1982). Standard errors of the parameter estimates given in the parentheses. The $\chi^2$ statistics for tests of the overidentified restrictions imposed by the model are reported along with p-values in parentheses and the number of degrees of freedom (d.f.). The instruments used are $r_t$, the interest rate, and $V_t$, the volume of transactions. The model estimated is:

$$e_{1t} = U_t^D - U_t^m$$  \hspace{1cm} (9)

$$e_{2t} = r_t - r_{t-1} - \alpha (r_o - r_{t-1})$$  \hspace{1cm} (10)

$$e_{3t} = e_{2t}^2 - \sigma^2.$$  \hspace{1cm} (11)

The parameters $a$, $r_o$, $\lambda$ and $\sigma$ designate the mean reversion level, the long-term interest rate, the market price of interest rate risk, and the volatility of the short-term interest rate, respectively.

<table>
<thead>
<tr>
<th>Period</th>
<th>Sample Size</th>
<th>$\sigma^2$</th>
<th>$a$</th>
<th>$\lambda \sigma$</th>
<th>$\lambda$</th>
<th>$\alpha r_o$</th>
<th>$\chi^2$</th>
<th>d.f</th>
<th>(p-value)</th>
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<td>01/83-12/89</td>
<td>362</td>
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<td>0.4013**</td>
<td>-0.0245**</td>
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<td>(0.58)</td>
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</table>

* Significant at 0.05 level.
** Significant at 0.01 level.
Table 2.3
Model Estimates for Treasury Bill Futures

This table contains the estimates of the model for the discount bond futures price under continuous marking-to-market. The $U_t^d$ (U.S. Treasury bill futures prices) and the short-term interest rate $r(t)$ (annualized one-week U.S. Treasury bill yields) are from January 83 to December 89. The parameters are estimated for the entire period and two equal length subperiods using the Generalized Method of Moments of Hansen (1982). Standard errors of the parameter estimates given in the parentheses. The $\chi^2$ statistics for tests of the overidentified restrictions imposed by the model are reported along with the p-values in parentheses and the number of degrees of freedom (d.f.). The instruments used are $r_n$, the interest rate, and $V_n$, the volume of transactions. The model estimated is:

$$
\varepsilon_{1t} = U_t^d - U_t^m
$$

$$
\varepsilon_{2t} = r_t - r_{t-1} - a (r_0 - r_{t-1})
$$

$$
\varepsilon_{3t} = \varepsilon_{2t}^2 - \sigma^2.
$$

The parameters $a$, $r_0$, $\lambda$ and $\sigma$ designate the mean reversion level, the long-term interest rate, the market price of interest rate risk, and the volatility of the short-term interest rate, respectively.

<table>
<thead>
<tr>
<th>Period</th>
<th>Sample Size</th>
<th>$\sigma^2$</th>
<th>$a$</th>
<th>$\lambda \sigma$</th>
<th>$\lambda$</th>
<th>$ar_0$</th>
<th>$\chi^2$</th>
<th>d.f</th>
<th>(p-value)</th>
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* Significant at 0.05 level.
** Significant at 0.01 level.
Table 2.4

Model Estimates for Treasury Bond Futures

This table contains the estimates of the model for the coupon bond futures price under discrete marking-to-market. The $U_t^C$ (U.S. Treasury bond futures prices) and the short-term interest rate $r(t)$ (annualized three-month U.S. Treasury bill yields) are from January 89 to September 91. The parameters are estimated for the entire period and two equal length subperiods using the Generalized Method of Moments of Hansen (1982). Standard errors of the estimated parameters are given in the parentheses. The $\chi^2$ statistics for tests of overidentified restrictions imposed by the model are reported along with the p-values in parentheses and the number of degrees of freedom (d.f.). The instruments used are $r_t$, the interest rate, and $t$, the time-to-maturity of the futures contract. The model estimated is:

$$e_{1t} = U_t^C - U_t^m$$

(9)

$$e_{2t} = r_t - r_{t-1} - \alpha (r_0 - r_{t-1})$$

(10)

$$e_{3t} = e_{2t}^2 - \sigma^2.$$  

The parameters $\alpha$, $r_0$, $\lambda$, and $\sigma$ designate the mean reversion level, the long-term interest rate, the market price of interest rate risk, and the volatility of the short-term interest rate, respectively.

<table>
<thead>
<tr>
<th>Period</th>
<th>Sample Size</th>
<th>$\sigma^2$</th>
<th>$\alpha$</th>
<th>$\lambda\sigma$</th>
<th>$\lambda$</th>
<th>$\alpha r_0$</th>
<th>$\chi^2$ (p-value)</th>
<th>d.f</th>
</tr>
</thead>
<tbody>
<tr>
<td>01/89-09/19</td>
<td>140</td>
<td>0.0000870**</td>
<td>0.0463**</td>
<td>0.001683**</td>
<td>.180</td>
<td>0.003846**</td>
<td>6.68 (0.24)</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.00013)</td>
<td>(0.006)</td>
<td>(0.00040)</td>
<td></td>
<td>(0.00037)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>01/89-05/90</td>
<td>70</td>
<td>0.0001397**</td>
<td>0.0255**</td>
<td>0.000535**</td>
<td>.045</td>
<td>0.002629**</td>
<td>3.89 (0.56)</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.00023)</td>
<td>(0.008)</td>
<td>(0.00079)</td>
<td></td>
<td>(0.00070)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>05/90-09/91</td>
<td>70</td>
<td>0.0001534**</td>
<td>0.0306**</td>
<td>0.001161**</td>
<td>.093</td>
<td>0.00306**</td>
<td>3.94 (0.56)</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.00017)</td>
<td>(0.003)</td>
<td>(0.00032)</td>
<td></td>
<td>(0.00019)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* Significant at 0.05 level.
** Significant at 0.01 level.
Table 2.5

Model Estimates for Treasury Bond Futures

This table contains the estimates of the model for the coupon bond futures price under continuous marking-to-market. The \( U_t \) (U.S. Treasury bond futures prices) and the short-term interest rate \( r(t) \) (annualized three-month U.S. Treasury bill yields) are from January 89 to September 91. The parameters are estimated for the entire period and two equal length subperiods using the Generalized Method of Moments of Hansen (1982). Standard errors of the estimated parameters are given in the parentheses. The \( \chi^2 \) statistics for tests of the overidentified restrictions imposed by the model are reported along with the p-values in parentheses and the number of degrees of freedom (d.f.). The instruments used are \( r_t \), the interest rate, and \( t \), the time-to-maturity of the futures contract. The model estimated is:

\[
\begin{align*}
\epsilon_{1t} &= U_t - U_t^m \\
\epsilon_{2t} &= r_t - r_{t-1} - \alpha (r_0 - r_{t-1}) \\
\epsilon_{3t} &= \epsilon_{2t}^2 - \sigma^2.
\end{align*}
\]

The parameters \( \alpha \), \( r_0 \), \( \lambda \) and \( \sigma \) designate the mean reversion level, the long-term interest rate, the market price of interest rate risk, and the volatility of the short-term interest rate, respectively.

<table>
<thead>
<tr>
<th>Period</th>
<th>Sample Size</th>
<th>( \sigma^2 )</th>
<th>( \alpha )</th>
<th>( \lambda \sigma )</th>
<th>( \lambda )</th>
<th>( \alpha r_0 )</th>
<th>( \chi^2 ) (p-value)</th>
<th>d.f</th>
</tr>
</thead>
<tbody>
<tr>
<td>01/89-09/19</td>
<td>140</td>
<td>0.0001748** (0.00004)</td>
<td>0.0478** (0.008)</td>
<td>0.001336** (0.00053)</td>
<td>.101</td>
<td>0.003846** (0.00037)</td>
<td>5.92 (0.000004)</td>
<td>5</td>
</tr>
<tr>
<td>01/89-05/90</td>
<td>70</td>
<td>0.0001633** (0.00018)</td>
<td>0.0400** (0.009)</td>
<td>0.001122** (0.00009)</td>
<td>.087</td>
<td>0.002629** (0.00070)</td>
<td>3.42 (0.000018)</td>
<td>5</td>
</tr>
<tr>
<td>05/90-09/91</td>
<td>70</td>
<td>0.0001695** (0.00018)</td>
<td>0.0391** (0.004)</td>
<td>0.001341** (0.00048)</td>
<td>.103</td>
<td>0.00306** (0.00019)</td>
<td>3.33 (0.000019)</td>
<td>5</td>
</tr>
</tbody>
</table>

* Significant at 0.05 level.
** Significant at 0.01 level.
Table 2.6

Model Estimates for Treasury Bond Futures

This table contains the estimates of the model for the coupon bond futures price under discrete marking-to-market. The $U_t^c$ (U.S. Treasury bond futures prices) and the short-term interest rate $r(t)$ (annualized three-month U.S. Treasury bill yields) are from January 89 to September 91. The parameters are estimated for the whole period and two equal length subperiods using the Generalized Method of Moments of Hansen (1982). Standard errors of the parameters estimates are given in the parentheses. The $\chi^2$ statistics for tests of overidentified restrictions imposed by the model are reported along with the p-values in parentheses and the number of degrees of freedom (d.f.). The instruments used are, $r_t$ the interest rate and $t$, the time-to-maturity of the futures contract. The model estimated is:

\[ e_{1t} = U_t^c - U_t^m \]  
\[ e_{2t} = r_t - r_{t-1} - a (r_0 - r_{t-1}) \]  
\[ e_{3t} = e_{2t}^2 - \sigma^2. \]

The parameters $a$, $r_0$, $\lambda$ and $\sigma$ designate the mean reversion level, the long-term interest rate, the market price of interest rate risk, and the volatility of the short-term interest rate, respectively.

<table>
<thead>
<tr>
<th>Period</th>
<th>$\sigma^2$</th>
<th>$a$</th>
<th>$\lambda\sigma$</th>
<th>$a r_\sigma$</th>
<th>$\delta$</th>
<th>$\chi^2$</th>
<th>d.f</th>
<th>(p-value)</th>
</tr>
</thead>
<tbody>
<tr>
<td>01/89-09/91</td>
<td>0.0000977**</td>
<td>0.02070**</td>
<td>0.000680*</td>
<td>0.0019360</td>
<td>0.1887</td>
<td>4.39</td>
<td>4</td>
<td>(0.000003)</td>
</tr>
<tr>
<td>140</td>
<td>(0.000003)</td>
<td>(0.0080)</td>
<td>(0.00030)</td>
<td>(0.000680)</td>
<td>(0.126)</td>
<td>(0.35)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>01/89-05/90</td>
<td>0.0001626**</td>
<td>0.04570**</td>
<td>0.000494</td>
<td>0.0018480</td>
<td>0.3809</td>
<td>3.43</td>
<td>4</td>
<td>(0.000030)</td>
</tr>
<tr>
<td>70</td>
<td>(0.000030)</td>
<td>(0.0240)</td>
<td>(0.00114)</td>
<td>(0.001980)</td>
<td>(0.877)</td>
<td>(0.49)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>05/90-09/91</td>
<td>0.0001248**</td>
<td>0.02050**</td>
<td>0.000735*</td>
<td>0.0018480</td>
<td>0.1706</td>
<td>2.70</td>
<td>4</td>
<td>(0.000032)</td>
</tr>
<tr>
<td>70</td>
<td>(0.000032)</td>
<td>(0.0070)</td>
<td>(0.00020)</td>
<td>(0.000730)</td>
<td>(0.103)</td>
<td>(0.61)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* Significant at 0.05 level.
** Significant at 0.01 level.
Table 2.7

The Market Price of Risk Estimates for other Studies

The market price of the short-term interest rate risk $\lambda$ is reported herein. The standard errors are given in the parentheses if they were reported in these studies.

<table>
<thead>
<tr>
<th>Author</th>
<th>$\lambda$</th>
<th>Model of Interest rate Process</th>
<th>Procedure</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brennan and Schwartz (1979)</td>
<td>0.0335</td>
<td>General Stochastic Model of Short- and Long-term Interest Rates</td>
<td>Quadratic Interpolation</td>
</tr>
<tr>
<td>Dietrich-Campbell Schwartz (1979)</td>
<td>0.260</td>
<td>General Stochastic Model</td>
<td>Generalized Least Square</td>
</tr>
<tr>
<td>Longstaff (1989)</td>
<td>-0.487 (0.011)</td>
<td>Square Root Model</td>
<td>Generalized Method of Moments</td>
</tr>
<tr>
<td>Duan (1992)</td>
<td>-0.141 (0.380)</td>
<td>Double Square Root Model</td>
<td>Maximum Likelihood</td>
</tr>
<tr>
<td></td>
<td>0.360 (0.887)</td>
<td>Ornstein-Uhlenbeck Model</td>
<td></td>
</tr>
</tbody>
</table>
CHAPTER 3: AN EMPIRICAL INVESTIGATION OF THE TERM STRUCTURE OF IMPLIED VOLATILITIES IN CURRENCY OPTIONS

3.1 Introduction

The valuation of a derivative security depends on the dynamics of its underlying asset. For assets whose volatility changes over time, a constant volatility model like the Black-Scholes model is not always appropriate for option pricing. As a result, Hull and White (1987), Johnson and Shanno (1987), Scott (1987), Wiggins (1987), and Duan (1995) derive alternative models under the assumption of stochastic volatility. They find that stochastic volatility has a significant impact on option prices. If no risk premium exists for bearing volatility risk, the value of an at-the-money option is approximately equal to the Black-Scholes value, where volatility is equal to the average expected volatility of the underlying asset over the option's life.\(^{19}\) Hence, if the stochastic volatility model is well specified, the at-the-money implied volatility obtained using the market price and the Black-Scholes formula should be approximately equal to the average expected volatility over the time to maturity. The relationship between the average and implied volatilities for a given process can be exploited to derive the so-called "term structure" of implied volatilities. The term structure of implied volatilities is the relationship between the implied volatilities obtained by applying the inverse of the Black-Scholes formula to the theoretical values of two options on an asset following a specific process, where the options only differ in their maturities.

The term structure of implied volatilities can be used to construct a test of efficiency of the option market. As stated in Heynen, Kemna and Vorst (1994), if a specific asset return model describes the data\(^{19}\)This result follows because the Black-Scholes formula is nearly linear in volatility for at-the-money calls.
correctly, and the pricing model used to uncover the implied volatility is assumed to be accurate, then the term structure test checks whether the restrictions on implied volatilities are consistent with the hypothesis that investors' expectations about average volatility are formed rationally. Conversely, if the expectation hypothesis is admitted, then the term structure of implied volatilities can be used to test the dynamics of the underlying asset, and ultimately the option pricing formula obtained from those dynamics.

Stein (1989) is the first author to construct a test of efficiency of the option market with the term structure of implied volatilities. Based on a sample of Standard & Poor 100 options, Stein concludes that long-term volatilities overreact to changes in short-term volatilities which result in inefficiencies in the option market studied. Heynen, Kemna and Vorst (1994) perform ex-ante efficiency tests on the relationship between short-term and long-term implied volatilities. They reject the joint hypothesis of a correct specification of asset dynamics and ex-ante efficiency in the option market when the stock return volatility follows a mean reverting GARCH(1,1) process. They are unable to reject this joint hypothesis for the case of a mean reverting EGARCH(1,1) process. Campa and Chang (1993) test the expectation hypothesis on the term structure of implied volatilities with over-the-counter options. They detect overreactions in the long-rate to current changes in the short-rate resulting in option mispricing for many cases. Lamoureux and Lastrapes (1993) consider market overreactions to recent volatility shocks as an alternative interpretation of the results of tests conducted on stock options traded on the Chicago Board Options Exchange (CBOE). Diz and Finucane (1993) question the methodology of Stein. Based on an alternative set of tests, they find no indication of overreaction in the market for index options. Canina and Figlewski (1993) find implied volatility to be a poor forecast of subsequent realized volatility for Index options. Therefore, widespread use of implied volatility in other models as an ex-ante measure of perceived asset price risk is debatable. They conjecture that the rejection of the implied volatility as a
good proxy for the future volatility might be related to the difficulty of arbitrage trade for the index options studied.\textsuperscript{20} The conjecture is indirectly substantiated in a simulation study by Sheikh and Vora (1994) discussed later.

Since the evidence concerning overreaction in the market is mixed, more studies are required in various settings to better understand the reaction of investors to the arrival of new information in the option market. This study performs a joint test of the expectation hypothesis on the term structure of implied volatilities, and the dynamics of the exchange rate of the Japanese yen, the Swiss franc and the Deutschmark against the American Dollar. The implied volatilities of relative changes of exchange rates are estimated using the formula for currency options derived by Garman and Kohlhagen (1983).\textsuperscript{21} The derivation of the term structure of implied volatilities is based on Heynen, Kemna and Vorst (1994). The joint hypothesis of a correct model specification and ex-ante efficiency is not rejected for the Japanese yen and the Swiss franc, and this hypothesis is rejected for the Deutschmark. The rejection could be interpreted as evidence that inefficiencies caused by noise exist in the Deutschmark currency-option market during the studied period. However, other specifications for the Deutschmark exchange rate process need to be tested before a reliable conclusion on inefficiencies. The results reported herein can also be considered as an implicit test of the currency options equivalent of the Duan Garch-option model for stock options.

The conditional variance of the relative changes of the exchange rate is assumed to follow a GARCH(1,1) process which is specified in the next section. While researchers have recognized that asset returns exhibit both fat-tailed marginal distributions and volatility clustering since Fama (1965) and

\textsuperscript{20} Arbitrage involving currency options is relatively easy to do.

\textsuperscript{21} This model for currency options is equivalent to the Black and Scholes model for stock options.
Mandelbrot (1966), only recently have applied researchers explicitly modelled these features. These empirical observations are interpreted as evidence that volatilities of financial asset prices are stochastic and that the innovations in the process are persistent. One tool used to describe such changing variances is the autoregressive conditional heteroscedasticity (ARCH) class of models.\(^{22}\)

The failure of traditional time series models to capture stylized facts about short-run exchange rate movements has lead many authors to consider the ARCH class of models as alternatives. Hsieh (1989), McCurdy and Morgan (1988), Kugler and Lenz (1990), Pappell and Sayer (1990), Heynen and Kat (1993), and Kodres (1993) show that the simple symmetric linear GARCH(1,1) model may provide a good model of the second-order dynamics of most exchange rates series.\(^{23}\) Theodossiou (1994) uses an EGARCH model with a generalised error distribution to investigate the properties of five major Canadian exchange rates.

The remainder of this essay is organized as follows. In section 3.2, the restrictions on average expected volatilities are derived. In section 3.3, the use of implied volatility as a proxy for average expected volatilities is discussed. In section 3.4, the data are described. In section 3.5, the empirical results are reported; and in section 3.6, some concluding remarks are offered.

### 3.2 Restrictions on Average Expected Volatility

The restrictions on relative option prices that are implied by a GARCH(1,1) model of the

\(^{22}\)See Bollerslev et al. (1992) for an extensive review.

\(^{23}\)Kodres finds that a GARCH(1,1) specification is adequate to describe the exchange rate dynamics of the Deutschmark and the Swiss franc versus the American dollar. In the case of the British pound, the Canadian dollar, and the Japanese yen, a Garch(2,1) model fits the data best.
volatility of the underlying asset are developed in this section. These restrictions are derived in Heynen, Kemna and Vorst (1994) based on the approach developed by Stein (1989) for a continuous time mean-reverting AR1 process.

Consider a discrete time economy, and let $S_t$ be the asset price at time $t$. Its one period rate of return is assumed to be lognormally distributed under measure $P$ from a complete probability space $(\Omega, \mathcal{F}, P)$. That is,

$$\ln\frac{S_t}{S_{t-1}} = \mu_t + \epsilon_t$$  \hspace{1cm} (1)

Assume that the error $\epsilon_t$ evolves as a GARCH(1,1) process. Then:

$$\epsilon_t \mid \Phi_{t-1} \sim N(0, \sigma_t^2) \text{ under measure } P$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$  \hspace{1cm} (2)

where,

$\Phi_t$ is the information set containing all information up to and including time $t$,

$\sigma_t$ is the volatility of the relative changes of the exchange rate at time $t$, and

$\alpha_0, \alpha_1, \beta_1$ are time independent parameters, where $\alpha_1 + \beta_1 < 1$.

The average expected volatility for a period of time running from $t$ to $T$, $\sigma^2_{\alpha}(t,T)$, is defined as:

$$\sigma^2_{\alpha}(t,T) = \frac{1}{T} \sum_{k=1}^{T} E_t[\sigma^2_{\alpha,k}]$$  \hspace{1cm} (3)
where $E_r$ denotes the conditional expectation at time $t$. Heynen, Kemna and Vorst (1994) show that, when $\sigma_{t+k}^2$ is replaced by its value in equation (2), equation (3) becomes:

$$
\sigma_m^2(t,T) = \sigma_m^2 + (\sigma_{t+1}^2 - \sigma_m^2) \frac{1}{T} \frac{1 - \Psi}{1 - \Psi} \quad (4)
$$

where $\sigma_m^2 = \frac{\sigma_0}{1 - \alpha - \beta_1}$ and $\Psi = \alpha_1 + \beta_1$. Equation (4) relates the expected average volatility over the period $(t,T)$ to the short-term volatility, $\sigma_{t+1}^2$, and the parameters of the GARCH(1,1) process. Since the expected average appears to be mean-reverting, it tends towards a long-term mean volatility $\sigma_\infty^2$ for large values of $T$. For small values of $T$, the average expected volatility tends towards the short-term volatility $\sigma_{t+1}^2$. Although the short-term volatility cannot be observed, it is possible to derive a relationship between two average expected volatilities covering two different times-to-maturity, $T_1$ and $T_2$. Equating the short term variances from the two average expected volatilities yields:

$$
\sigma_m^2(t,T_1) = \sigma_m^2 + \frac{T_2}{T_1} \frac{\Psi_1 - 1}{\Psi_2 - 1} \left( \sigma_m^2(t,T_2) - \sigma_m^2 \right) \quad (5)
$$

Notice that the constant of proportionality depends only on the GARCH(1,1) parameters, and $T_1$ and $T_2$.

The true volatility of the exchange rate need not be known provided that a proxy exists. The existence of such a proxy is discussed in the next section.

3.3 The Implied Volatility as a Proxy for the Average Expected Volatility

24 The time-varying volatility may be modelled as a function of other variables. Lamoureux and Lastrapes (1993) include time- $t$-1 implied volatility in the equation describing time- $t$ volatility. Bessembinder and Seguin (1993) link volatility to expected and unexpected volume shocks. Mahieu and Schotman (1994) find that all bilateral exchange rates, expressed vis-a-vis a common numeraire currency, contain at least one common factor due to the numeraire effect.
The implied volatilities of stock prices and exchange rates are usually measured by inverting the Black-Scholes option pricing formula and the Garman-Kohlhagen currency-option pricing formula, respectively. The Garman-Kohlhagen formula is considered as an extension of the Black-Scholes formula to currency options.

The implied volatility obtained from the Black-Scholes formula is used as a proxy for the average volatility expected to prevail over the life of an option in several studies. In a world of stochastic volatility, the substitution of average expected volatilities by implied volatilities is justified only if the following conditions are satisfied: first, there is no risk for bearing volatility risk; and second, the option price is linear in volatility.\textsuperscript{25}

Presumably, the Black-Scholes formula which is linear in volatility is used to uncover implied volatilities because of its simplicity. Sheikh and Vora (1994) examine if the Black-Scholes implied volatilities are unbiased estimates of the true average volatility over the life of an option if the volatility changes stochastically, and what happens if the first condition above is not respected (i.e, the underlying asset volatility is correlated to consumption). They implement a careful simulation of option prices under alternative asset price processes and compute implied volatilities. They conclude that implied volatilities measure accurately the average volatility over the life of an option, and that calls that are deep-in-the-money or deep-out-of-money should not be used for computing implied volatilities. The results obtained by Sheikh and Vora confirm those of Heynen, Kemna and Vorst (1994), which are obtained for Duan's GARCH-option model of stock prices. A simulation study performed with an extension of Duan's GARCH-option model to currency option prices also may find that the implied volatility is a good proxy

\textsuperscript{25}See Stein (1989), note 3.
of the average expected volatility over the remaining life of the option.\(^{26}\)

If \(C_i\) is the price obtained from the GARCH-option model, the implied volatility is measured by inverting the Black-Scholes or the Garman and Kohlhagen (1983) formula. Assume that \(g_i\) is the Garman and Kohlhagen currency option formula. To uncover the volatility implied by the value \(C_i\), solve \(C_i = g_i(\sigma_i)\) so that \(\sigma_i = g_i^{-1}(C_i)\). For an at-the-money option with time to maturity \((t, T)\), \(\sigma_i(t, T) = \sigma_{av}(t, T)\), where \(\sigma_{av}(t, T)\) is the expected average volatility of the exchange rate return process as derived in equation (4). Equation (5) can thus be rewritten as:

\[
\sigma_i^2(t, T_1) = \sigma_m^2 + \frac{T_2}{T_1} \frac{\psi_{T_1}^{T_1-1}}{\psi_{T_2}^{T_2-1}} (\sigma_i^2(t, T_2) - \sigma_m^2),
\]

where \(T_1 > T_2\). Equation (6) holds when the theoretical option values \(C_i\) are used to recover the implied volatilities. If the exchange rate return follows the model described by equations (1) and (2), and a risk neutral valuation exists for the return process, then the market price of a currency option \(C^n_i\) should be equal to its theoretical value \(C_i\). When implied volatilities are uncovered with option market values such that \(\sigma_i = g_i^{-1}(C^n_i)\), equation (6) may be used to test the joint hypothesis of a correct option model

---

\(^{26}\)The Duan Garch-option model can be extended to currency options. Given that a risk neutral measure \(Q\) that satisfies a local risk neutral valuation relationship, LRNVR, exists [see Duan (1995) for necessary conditions for a LRNVR to hold], the GARCH-option model applied to currency options is as follows:

\[
\ln \frac{S_t}{S_{t-1}} = \ln(1 + r_f - r_d) - \frac{1}{2} \sigma_t^2 + \xi_t,
\]

where \(\xi_t \sim N(0, \sigma_t^2)\) and

\[
\sigma_t^2 = \alpha_0 + \alpha_1 (\xi_{t-1} - \lambda \sigma_{t-1}^2)^2 + \beta_1 \sigma_{t-1}^2.
\]

\[
C_i = e^{-r_i(T-t)} E^Q \{ \max(S_T - K, 0) | \phi_i \},
\]

where \(r_i = \ln(1 + r_d)\), and \(r_d\) is the domestic interest rate, and \(r_f\) is the foreign interest rate.
specification and the assumption that expectations are formed rationally.

3.4 The Data

Two sets of data are used in this study. The first set consists of the daily spot prices from the New York foreign exchange market on the U.S dollar versus the Japanese yen, the Swiss franc and the Deutschmark for the period from January 23, 1987 to January 21, 1988. This results in 250 observations for each exchange rate series.

The second set of data consists of the implied volatilities for call options on three currencies traded at the Philadelphia Exchange from January to June 1988: Japanese yen, Swiss franc, and Deutschmark. The implied volatilities for other major currencies are not used because trading in the currencies was thin during the period studied. Two series of prices are selected for each exchange rate; namely, one for short-term call options, and one for long-term call options. Short-term options have times-to-maturity of less than 40 days, while long-term options have times-to-maturity of more than 80 days. As in Xu and Taylor (1994), the following exclusion criteria are used to remove the uninformative option observations from the studied samples:

---

27Only currencies with enough option data are studied. For a call option with less than one year to maturity and domestic interest rates greater than foreign interest rates, the European boundary condition exceeds the American. No early exercise premium exists, and the American feature has no value [Adams and Wyatt (1987)]. Therefore, the Garman and Kohlhagen model for European call options is also suitable for American call options. This allows for more observations for currencies with lower interest rates than the American dollar during the sample period, such as the Japanese yen, the Deutschmark and the Swiss franc. There is not enough data to perform the tests for other major currencies (the British Pound, the French franc and the Italian lira) because American call options are excluded from the samples because interest rates related to these currencies are higher than the American interest rates during the period of study.

28The notation used herein is standard: S is the spot rate; X is the exercise price; T is the time to the expiry date as measured in years; \( r_d \) is the domestic interest rate; and \( r_f \) is the foreign interest rate.
i) Options with time to expiration of less than 15 calendar days;

ii) Options violating the European boundary conditions

\[ C < S e^{-rT} - X e^{-rT}; \]

iii) Options violating the American boundary conditions, \( C < S - X; \)

iv) Option premia less than or equal to .015 cents; and

v) Options that are 'far' in- or out-of-the-money; i.e., \( X < .975S \) or \( X > 1.025S. \)

Criterion i) eliminates options with very short times to maturity because implied volatilities are unreliable for such options. Criteria ii) and iii) eliminate options violating the boundary conditions for European and American options. Criterion iv) excludes options for which the discrete market prices are likely to distort the calculation of implied volatilities. Criterion v) is used to retain only at-the-money options, since the implied volatilities computed from such options seem to be the best proxies of expected average volatilities. Thus, the implied volatilities are obtained by applying the inverse of the Garman and Kohlhagen option model for currency options to the option prices, using only the closing prices of the nearest-the-money options.

3.5 Empirical Results

In this section, efficiency of the currency-option and the validity of the GARCH(1,1) market and the GARCH(1,1) process for the conditional variance of the relative changes of exchange rates are jointly tested by examining restrictions on implied volatilities. The estimated parameters of the GARCH(1,1) process are compared to those of alternative ARCH/GARCH models. The estimated parameters of the process are used to derive the term structure of implied volatilities, and to perform the tests of the joint
3.5.1 Maximum Likelihood Estimation of the GARCH Models

Different model assumptions for the volatility of the relative changes of various exchange rates are tested now. The spot exchange rates $S_t$ are converted to continuously compounded rates of return using:

$$y_t = \ln\left(\frac{S_t}{S_{t-1}}\right).$$

Some descriptive statistics for $y_t$ for the Japanese yen, the Swiss franc and the Deutschmark are reported in Table 3.1. All the series exhibit leptokurtic behaviour. The unconditional sample kurtosis of the Japanese yen of 1.62 (which is the highest among the three series) is below the normal value of three by more than four standard errors. The Wald statistic for the null hypothesis of normality is rejected for all three series.\(^{29}\) The autocorrelations and partial autocorrelations of the (squared) exchange rate return series are presented in Table 3.2. The Ljung-Box (1978) portmanteau test statistic for up to the tenth-order serial correlation in the residuals yields values that are not significant at the usual 5% level of confidence for all three currencies (see Ljung-Box 1, Table 3.1).\(^{30}\) In contrast, the squared residuals are not uncorrelated over time. The Ljung-Box test statistic for no serial correlation in the squares is significant for all three series (see, Ljung-Box 2, Table 3.1). The absence of dependence in the

---

\(^{29}\)Under the null hypothesis of normality, the Wald statistic is distributed as chi-squared with two degrees of freedom. The five percent critical value from the chi-squared table for two degrees of freedom is 5.99.

\(^{30}\)The Ljung-Box statistic for up to the tenth-order serial correlation follows a $\chi^2_{10}$ distribution.
conditional first moment and the dependence in the conditional second moment indicates that ARCH models are adequate choices for the representation of the volatility processes of the series studied.

To describe the volatility processes of the return series, the following three alternative models are considered: the GARCH(1,1), the GARCH(0,1), and the ARCH(1). The return series are modelled as

\[ y_t = \mu + \epsilon_t \quad (8) \]

\[ \epsilon_t \mid \phi_{t-1} \sim N(0, \sigma_t^2) \]

\[ \sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \alpha_2 \epsilon_{t-2}^2 + \beta_1 \sigma_{t-1}^2 \quad (9) \]

In (9), the ARCH(1) process is obtained if \( \beta_1 = 0 \) and \( \alpha_2 = 0 \); the GARCH(0,1) process is obtained if \( \alpha_1 = 0 \) and \( \alpha_2 = 0 \); the GARCH(1,1) process is obtained if \( \alpha_2 = 0 \); and the ARCH(2) process is obtained if \( \beta_1 = 0 \). Standard maximum likelihood techniques are used to estimate and test these ARCH/GARCH models. Assuming that the process is described by equations (8) and (9), the log-likelihood, \( \log L \), is given, apart from some initial conditions, by:

\[ \log L = -\frac{T}{2} \log (2\pi) - \frac{1}{2} \sum_{t=1}^{T} \left[ \log \sigma_t^2 + \frac{\epsilon_t^2}{\sigma_t^2} \right] \quad (10) \]

where all the terms are as defined earlier.

Maximum likelihood estimates of the parameters for the alternative models are presented in Tables 3.3, 3.4 and 3.6. The estimates are obtained using the Berndt, Hall, Hall and Hausman (1974)
algorithm using numerical derivatives. For the Japanese yen, the Ljung-Box test statistic for the standardized residuals and the standardized squared residuals from the estimated GARCH(1,1) and the ARCH(1) model do not indicate any first-order or second-order serial dependence. However, for the GARCH(0,1) model, some second-order serial dependence still remains. For the Swiss franc, the GARCH(1,1) and the GARCH(0,1) fit the data well. Both the GARCH(1,1) and the ARCH(1) models are adequate descriptions of the conditional volatility of the relative change of the Deutschmark exchange rate. However, likelihood ratio test results indicate that the optimal choice of model for the three exchange rates series is the GARCH(1,1) process.\textsuperscript{31}

The GARCH(1,1) model provides a good description of the second-order dynamics of the Japanese yen, the Swiss franc and the Deutschmark return series. However, it should be noted that the excess kurtosis in the raw data is not captured by the assumption of conditional normality as shown by the values of the Wald test statistic.\textsuperscript{32}

3.5.2 Tests of the Term Structure of Implied Volatilities

The parameters of the GARCH(1,1) process estimated in the previous section are used to specify the ex-ante relationship between the long- and short-term implied volatilities derived earlier. For each currency, the joint hypothesis of a correct model specification and the assumption that average volatility expectations are formed rationally is tested. The equation to be tested is:

\textsuperscript{31}If \(L_a\) and \(L_n\) are the maximum log likelihood values under the null and alternative hypotheses, respectively, then the test statistic \(-2[L_a - L_n]\) is asymptotically chi-squared distributed with degrees of freedom equal to the difference in the number of parameters under the null and the alternative.

\textsuperscript{32}Estimation of the different models under the assumption of a standardized t-distribution does not change the parameter estimations, and different tests do not change the results significantly. For further discussion of this issue, see McCurdy and Morgan (1987), Hsieh (1989), Baillie and Bollerslev (1989), and Bollerslev, Chou and Kroner (1992).
where $T_1 > T_2$. The parameter estimates for equation (11) for the different processes are presented in Table 3.6. The residual in equation (11) should be white noise; that is, $E(\epsilon_t) = 0$ and $E(\epsilon_t \epsilon_{t-j}) = 0$ for every $j$. The test is performed on 103 pairs of implied volatilities for the relative change of the Japanese yen exchange rate, 103 pairs of the relative change of the Swiss franc exchange rate, and 103 pairs of the relative change of the Deutschmark exchange rate. The test results for these three currencies are summarized in Table 3.7. Based on the t-values of the means of the residuals and the Ljung-Box test statistics, the assumption of white noise cannot be rejected for the Swiss franc and the Japanese yen. Thus, the joint hypothesis of a correct model specification and ex-ante efficiency of the option market cannot be rejected for the Swiss franc and the Japanese yen. This result is consistent with recent studies of the term structure using implied volatilities in the currency and the stock markets. For example, Heynen, Kemna and Vorst (1994) cannot reject the joint hypothesis of a correct model specification and ex-ante efficiency in the case of an EGARCH(1,1) conditional variance of stock returns. Similarly, based on a study of volatility expectations for four currencies, Xu and Taylor (1994) conclude that there is no evidence that currency option markets overreact.

Based on the t-values and the Ljung-Box statistics, the assumption of white noise is rejected for the GARCH(1,1) model for the conditional variance of the Deutschmark exchange rate return. The significant positive mean error suggests that the forecasts of long-term average volatility by investors in the case of the Deutschmark are larger than expected from the term structure relation. Such overreaction might be explained by the arrival of contradictory information into the market. In the first half of 1988, contrary to the economic fundamentals, the Deutschmark lost value against the American
dollar and other currencies.\textsuperscript{33} The contradictory information may be interpreted as a noisy signal about the long-term exchange rate between the American dollar and the Deutschmark. Noise may lead to "excess" long-term exchange rate volatility which is captured by the term structure of implied volatilities. However, other specifications for the Deutschmark exchange rate process need to be tested before reaching any reliable conclusion about inefficiencies during the period under review.

An alternative possibility is that the rejection is due to a mis specification of the exchange rate volatility dynamics. In practical terms, this suggests that the GARCH(1,1) estimate of the unconditional variance (i.e., the major determinant of the term structure relationship) is not correct. However, based on a sensitivity analysis summarized in Table 3.8, a change in the conditional volatility does not change the initial results (that is, a rejection of the term structure relationship for the Deutschmark).

3.6 Concluding remarks

In this essay, ex-ante tests of the relationship between short- and long-term implied volatilities are performed for the exchange rates of the Japanese yen, the Swiss franc and the Deutschmark versus the American dollar. The model selected for the conditional volatilities of the exchange rate relative change series for these currencies is the GARCH(1,1). The joint hypothesis of a correct model specification and ex-ante efficiency is not rejected for the Japanese yen and the Swiss franc, but is rejected for the Deutschmark. The rejection could be interpreted as evidence that inefficiencies caused by noise exist in the Deutschmark currency-option market during the studied period. An alternative possibility is that the rejection is due to a mis specification of the exchange rate volatility dynamics. While tests of the hypothesis for other major currencies (like the British pound, the French franc, the Italian lira

\textsuperscript{33}Bundesbank Annual Report 1988.
and the Canadian dollar) are of interest, the size of the available data samples does not allow for meaningful analyses.

The results reported herein contribute to both sides of the ongoing debate about whether option markets are efficient. Interesting avenues for future research are to extend tests of the term structure of implied volatilities to other specifications of the exchange rate dynamics and to determine the conditions under which inefficiencies arise, if any.
Table 3.1
Sample Statistics for the Japanese Yen, the Swiss Franc and the Deutschmark Exchange Rate Returns

The sample size for each currency is 250 observations. The period runs from January 1987 to January 1988, and excludes week-ends and holidays. Ljung-Box 1 and Ljung-Box 2 are the Ljung-Box test statistics for up to the tenth order serial correlation in the residuals and the squared residuals, respectively. Under the null hypothesis that the residuals are uncorrelated, these statistics are asymptotically distributed as chi-square, with 10 degrees of freedom. Under the assumption of normality, the asymptotic standard errors for kurtosis and skewness are respectively given by \((24/N)^{0.5}\) and \((6/N)^{0.5}\), where \(N\) denotes the numbers of observations. For \(N = 250\), these errors are given by 0.3098 and 0.1549, respectively.

<table>
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<th></th>
<th>Japanese yen</th>
<th>Swiss franc</th>
<th>Deutschmark</th>
</tr>
</thead>
<tbody>
<tr>
<td>Observations</td>
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<td>250</td>
<td>250</td>
</tr>
<tr>
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<td>-.00043762</td>
<td>-.0006192</td>
</tr>
<tr>
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<td>.005690</td>
<td>.0052232</td>
</tr>
<tr>
<td>Variance</td>
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<td>.0000272</td>
</tr>
<tr>
<td>Skewness</td>
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<td>-.05003</td>
<td>.15738</td>
</tr>
<tr>
<td>Kurtosis</td>
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<td>.89</td>
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<td>2.01</td>
<td>1.96</td>
</tr>
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<td>Wald Statistic</td>
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<td>53.06</td>
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<tr>
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<td>7.26</td>
<td>5.25</td>
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<tr>
<td>Ljung-Box 2</td>
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<td>34.30</td>
<td>33.30</td>
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Table 3.2

Autocorrelations for the first and second moments of the Japanese yen, the Swiss franc, and the Deutschmark exchange rate returns

Autocorrelations of the exchange rate return and squared exchange rate return series are calculated for lags up to ten days. The numbers in parentheses give the standard errors of autocorrelations.

<table>
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<tr>
<th>Lag</th>
<th>Japanese yen Return</th>
<th>Japanese yen Squared return</th>
<th>Swiss franc Return</th>
<th>Swiss franc Squared return</th>
<th>Deutschmark Return</th>
<th>Deutschmark Squared return</th>
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<td>.0060</td>
<td>.0259</td>
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<td>(.0631)</td>
<td>(.0631)</td>
<td>(.0631)</td>
<td>(.0631)</td>
<td>(.0631)</td>
</tr>
<tr>
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<td>.0134</td>
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Table 3.3

Japanese Yen

Maximum likelihood estimations of the ARCH(1), GARCH(0,1) and GARCH(1,1) models are presented herein. A full estimation of the ARCH(2) model was not achieved. The sample size for each estimation is 250 observations. The period runs from January 1987 to January 1988, and excludes week-ends and holidays. An asterisk (*) indicates significance at the 5 percent level using a t-test. Ljung-Box 1 and Ljung-Box 2 are the Ljung-Box test statistics for up to the tenth-order serial correlation in the residuals and the squared residuals, respectively. Under the null hypothesis that the residuals are uncorrelated, these statistics are asymptotically distributed as chi-square with 10 degrees of freedom.

<table>
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<th>Parameters</th>
<th>ARCH(1)</th>
<th>ARCH(2)</th>
<th>GARCH(0,1)</th>
<th>GARCH(1,1)</th>
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Table 3.4
Swiss Franc

Maximum likelihood estimations of the GARCH(0,1) and GARCH(1,1) models are presented herein. A full estimation of the ARCH(1) and the ARCH(2) models was not achieved. The sample size for each estimation is 250 observations. The period runs from January 1987 to January 1988, and excludes week-ends and holidays. An asterisk (*) indicates significance at the 5 percent level using a t-test. Ljung-Box 1 and Ljung-Box 2 are the Ljung-Box test statistics for up to the tenth-order serial correlation in the residuals and the squared residuals, respectively. Under the null hypothesis that the residuals are uncorrelated, these statistics are asymptotically distributed as chi-square with 10 degrees of freedom.

<table>
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<tr>
<th>Parameters</th>
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<td>n/a</td>
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TABLE 3.5

Deutschmark

Maximum likelihood estimations of the ARCH(1) and GARCH(1,1) models are presented herein. A full estimation of the ARCH(2) and GARCH(0,1) models was not achieved. The sample size for each estimation is 250 observations. The period runs from January 1987 to January 1988, and excludes week-ends and holidays. An asterisk (*) indicates significance at the 5 percent level using a t-test. Ljung-Box 1 and Ljung-Box 2 are the Ljung-Box test statistics for up to the tenth-order serial correlation in the residuals and the squared residuals, respectively. Under the null hypothesis that the residuals are uncorrelated, these statistics are asymptotically distributed as chi-square with 10 degrees of freedom.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>ARCH(1)</th>
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<th>GARCH(0,1)</th>
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<td>n/a</td>
<td>n/a</td>
<td>.0009350*</td>
</tr>
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<td>(.0004445)</td>
<td></td>
<td>n/a</td>
<td>(.0004424)</td>
</tr>
<tr>
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<td>n/a</td>
<td>.0000032</td>
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<td>(.0000046)</td>
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<td>n/a</td>
<td>(.0000019)</td>
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<tr>
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<td>n/a</td>
<td>n/a</td>
<td>.0794320*</td>
</tr>
<tr>
<td></td>
<td>(.0549260)</td>
<td></td>
<td>n/a</td>
<td>(.0329820)</td>
</tr>
<tr>
<td>α₂</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
</tr>
<tr>
<td>β₁</td>
<td>n/a</td>
<td></td>
<td></td>
<td>.8549210*</td>
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<tr>
<td></td>
<td>n/a</td>
<td></td>
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<td>(.0544180)</td>
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<td>n/a</td>
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<td>Ljung-Box 2</td>
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<td>n/a</td>
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<td>n/a</td>
<td>.17</td>
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<td>KURTOSIS</td>
<td>.46</td>
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<td>n/a</td>
<td>1.44</td>
</tr>
</tbody>
</table>
Table 3.6

Term structure of implied volatility test statistics on the Japanese yen, the Swiss franc and the Deutschmark

Some descriptive statistics of the residuals $\epsilon_i$ specified in the case for a GARCH(1,1) term structure are listed herein. Ljung-Box 1 is the Ljung-Box test statistics for up to the tenth-order serial correlation in the residuals. Under the null hypothesis that the residuals are uncorrelated, these statistics are asymptotically distributed as chi-square with 10 degrees of freedom.

<table>
<thead>
<tr>
<th>Statistics</th>
<th>Japanese yen</th>
<th>Swiss franc</th>
<th>Deutschmark</th>
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</thead>
<tbody>
<tr>
<td>Mean</td>
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<td>.008302</td>
<td>.0072131</td>
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<tr>
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<td>(.006970)</td>
<td>(.007908)</td>
<td>(.003343)</td>
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<tr>
<td>T-value</td>
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<td>1.04</td>
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<tr>
<td>Ljung-Box 1</td>
<td>2.58</td>
<td>2.40</td>
<td>70.80</td>
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</table>

Table 3.7

Parameters of the term structure relationships for the Japanese yen, the Swiss franc, and the Deutschmark

<table>
<thead>
<tr>
<th>Currency</th>
<th>$\sigma_m$</th>
<th>$\psi$</th>
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<tr>
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<td></td>
<td></td>
</tr>
<tr>
<td>Japanese yen</td>
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<td>(.033811)</td>
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<tr>
<td>Swiss franc</td>
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<td>.908354</td>
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<tr>
<td></td>
<td>(.074669)</td>
<td>(.074669)</td>
</tr>
<tr>
<td>Deutschmark</td>
<td>.111805</td>
<td>.934353</td>
</tr>
<tr>
<td></td>
<td>(.037824)</td>
<td>(.037824)</td>
</tr>
</tbody>
</table>

* $\sigma_m$ is the unconditional volatility, annualised (250 days)
** $\psi = \alpha_i + \beta_i$, is the sum of two parameters of the GARCH(1,1) model.
Table 3.8

Term Structure Test for Different Levels of Unconditional Volatility

<table>
<thead>
<tr>
<th>$\sigma_n$</th>
<th>Mean</th>
<th>T-value</th>
<th>Ljung-Box 1</th>
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<td>.116586</td>
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</tr>
<tr>
<td>.050000</td>
<td>.005294</td>
<td>1.84</td>
<td>63.3</td>
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<tr>
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<td></td>
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<td>.150000</td>
<td>.007189</td>
<td>2.18</td>
<td>64.5</td>
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<td>(.003287)</td>
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</table>
CHAPTER 4: FUTURES MARKET EQUILIBRIUM WITH HETEROGENEITY

4.1 Introduction

This paper studies hedging and equilibrium in the futures market for a commodity in a single good economy, which is populated by heterogeneous producers and speculators. The commodity is traded in a spot market at harvest time. Each producer is endowed with a non-traded private technology and trades in futures contracts in order to reduce her quantity and price risk. Speculators invest their initial wealths in bonds and take positions in futures contracts written on the commodity.

The model is motivated by the observation that a spot market is opened only at harvest time for some commodities. In such a setting, the producer faces both a price and a quantity risk. However, most of the optimal hedging literature deals only with price risk. Hirshleifer (1990, 1991) discusses the effects of both types of risk on the hedging decision but does not model them simultaneously. A simple way to represent the quantity and price risks of each producer is to model them as a private cash flow risk, where the cash position of a hedger is the present value of her terminal cash flows. The difficulty in solving the equilibrium under these conditions is that both the futures price and the cash position of a hedger are endogenous. In more traditional models, the cash position, which is the fixed quantity of the commodity held by the producer, is independent of the futures price.

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34 Fresh strawberries are an example of an agricultural product which has a spot market at harvest time, and which can be traded in futures market in the interim period by farmers, agro-industrial firms, consumers, and speculators.

35 See, for example, Anderson and Danthine (1980)
Although equilibrium futures price and portfolio policies are determined simultaneously, the determination herein proceeds in two stages for mathematical convenience. First, the demand of each agent is derived assuming that the parameters of the futures price are known. Then, individual demands are aggregated to obtain the equilibrium futures price at any point in time.

The optimal demand for a futures contract depends on tastes, on the resolution of uncertainty, and on the formulation of the hedging problem. Early contributions in the optimal hedging literature usually assumed that hedgers were monoperiodic expected utility maximizers (see for instance Stein (1961), Johnson (1960), Anderson and Danthine (1980) and Losq (1982)). A common result from most of these papers is that the optimal hedge ratio consists of a pure hedge component, and a mean variance component.

More recent models are derived in a continuous-time framework, whereby the hedger maximizes the expected utility of intertemporal consumption subject to a wealth-budget or cash-budget dynamic constraint. The cash-budget formulation has been used by Ho (1984) in a model in which a farmer, subject to both output and price risk during the production period, hedges a non-traded position. The optimal demand for futures contracts depends on the exogenous cash position (the output), and includes both a mean-variance efficiency component and a Merton-Breeden dynamic hedging component. The wealth-budget formulation, initiated by Stulz (1984), is used by Adler and Detemple (1988) to solve a problem similar to that of Ho. The resulting demand for futures contracts depends on the output, but also includes a mean-variance efficiency term, a dynamic hedging component, and a minimum-variance component. However, for myopic investors, both formulations yield tractable optimal demands, similar to those obtained in single period models. Therefore, investors are assumed to have logarithmic utility functions in the model developed herein.
The demand for futures contracts depends on the uncertainty structure of the spot and the futures market. A producer that hedges against price risk to protect a non-traded position has a demand dependent on the output. Without trade constraints, and in the absence of frictions, the demand for futures contracts is independent of the output [see Briys, Crouhy and Schlesinger (1990)]. In the model developed herein, a producer is concerned with both quantity and price risk, and thus hedges against the risk of individual cash flows.

The optimal demand for futures contracts by each investor is derived by assuming temporarily that the parameters of the futures price are known, and by applying the equivalent martingale method developed by Karatzas, Lehoczky and Shreve (1987). Diversity in cash flow risk leads to different hedging strategies. The demand for futures contracts by a producer depends on her wealth, the present value of her terminal production and the relative risk process of the futures contract. This structure of demand is motivated by a desire to diversify, and to insure the actual value of the terminal production. Producers are long or short in the futures contract if there is heterogeneity in cash flow risks. A similar result is obtained by Hirshleifer (1991) in a discrete-time model where heterogeneity in the resolution of uncertainty leads to different hedging strategies. In the model developed herein, the differential in cash flow risk can also be interpreted as a differential in the resolution of uncertainty.

Once the optimal demand for futures contracts is derived, the aggregation follows easily. A continuous-time equilibrium exists in the futures market when producers account for both quantity and price risk by hedging against their individual and terminal cash flow risks. Stochastic cash positions and equilibrium futures price are endogenous, unlike previous equilibrium models which ignore quantity risk. The parameters of the futures price depend on wealth and the relative risk process. The volatility of the futures price is stochastic and increases, on average, as the time to maturity decreases. This maturity
effect in the model is consistent with Samuelson (1965). The stochastic volatility obtained in the model is also consistent with the recent empirical evidence on the behaviour of daily futures prices.\footnote{See, for example, Milonas (1986) and Anderson (1985).} The positive mean of the instantaneous growth rate of futures prices indicates normal backwardation in the model. The relative risk process of futures prices is a weighted sum of the present values of all the terminal productions of hedgers.

Aggregation is made easier by assuming a logarithmic utility function which allows for the treatment of heterogeneity. The relative risk process of futures prices in the model of heterogeneous agents developed herein has a weighted-average structure. Detemple and Murthy (1994), who appear to be the first to deal with heterogeneity in an intertemporal production economy, obtain a similar result for the equilibrium interest rate.

The remainder of the chapter is organized as follows. In section 4.2, the economy is described. In section 4.3, optimal demands for futures contracts by each type of investor are derived. In section 4.4, equilibrium futures price is derived; and in section 4.5, some concluding remarks are offered.

4.2 The Economy

Consider a continuous-time production economy with a Brownian uncertainty structure that is populated with two classes of agents. The first class consists of producers indexed by $i = 1, 2, \ldots, n$, and the second class consists of speculators indexed by $k = 1, 2, \ldots, m$.

Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a finite time horizon $[0, T]$. On $(\Omega, \mathcal{F}, \mathbb{P})$, define a Brownian motion $z$ with values in $\mathbb{R}$. Also assume the following:
A1) All producers grow the same commodity, and each producer is endowed with a specific production technology with a stochastic constant return to scale.\(^{37}\) The value of the production in process \(S_t^i\) satisfies
\[
dS_t^i = S_t^i(\mu^i dt + \sigma^i dz_t),
\]
where the constant parameter \(\mu^i\) represents the expected instantaneous change in the rate of production growth, and \(\sigma^i\) is the constant local variance of the process. \(S_t^i\) is the production at harvest time, expressed as a cash flow in units of the numeraire (that is, the commodity). The production in process \(S_t^i\) is not tradeable, and \(S_0 > 0\).

A2) Producers and speculators have free and unlimited access to a financial market where a futures contract written on the specified commodity is traded. The settlement price of this contract, \(F_t\), satisfies
\[
dF_t = \alpha(.,t)dt + \delta(.,t)dz_t,
\]
The production growth rate and the futures contract have the same uncertainty structure, and \(\alpha\) and \(\delta\) are determined in equilibrium later.

A3) An instantaneously riskless bond is available and is held by both types of agents. The roll-over value of an investment \(I\) in the riskless rate solves
\[
dI_t = r I_t dt, \quad I_0 = 1,
\]
\(^{37}\)The aggregate production at maturity time \(T\) is homogenous. However, the differences in the stochastic constant returns to scale lead to differences in the individual quantities of production at maturity time \(T\).
where $r$ is deterministic.

**A4)** A producer $i$ chooses a trading strategy $\{N^i_t\}$ in the futures contract which is adapted to her information set so as to maximize the utility of her terminal consumption $C^i_T = S^i_T + X^i_T$. The terminal gain from trading in the futures contract $X_T$ can be written as:

$$X^i_T = \int_0^T e^{-rT_s} N^i_s dF_s,$$

where $X^i_T$ is the terminal value of a margin account. The position $N^i_t$ is credited (debited) with any gain (loss) caused by price changes. The margin account is also credited with interest at the continuously compounded interest rate $r$. Borrowing is possible at the same interest rate if the investor incurs losses that cause the value of the account to become negative. Transaction costs are ignored.

**A5)** A producer $i$ solves:

$$\max \{N^i_t\} E\left[e^{-\beta T} \log (C^i_T)\right]$$

subject to

$$s.t. \ E^* [e^{0 \cdot C^i_T}] = x^i + E^* [e^{0 \cdot S^i_T}],$$

where $x^i$ is the initial cash-position of producer $i$, and $\beta$ is a subjective discount rate which is identical across agents. Equations (1) and (2) are the static equivalents of the maximization of terminal wealth under the wealth-budget dynamic constraint [see Karatzas, Lehoczky, and Shreve (1987)]. $E^*$ is the expectation under a probability measure $Q$ which is equivalent to $P$. 
A6) A speculator trades in the futures contract. He chooses a strategy \( \{N_t^k\} \) so as to maximize his terminal wealth \( X_T^k \), which satisfies:

\[
X_T^k = \int_0^T e^{r^k s} N_s^k dF_s
\]

A7) The utility function of speculator \( k \) is logarithmic. Thus the speculator solves:

\[
\max_{\{N_t^k\}} E[e^{-\beta T \log(W_T^k)}] \tag{3}
\]

subject to

\[
E[e^{-r^k W_T^k}] \leq x^k \tag{4}
\]

where \( x^k \) is the initial wealth of speculator \( k \), \( \beta \) is the subjective discount rate used by producers, and \( W_T^k \) is the terminal wealth. Equations (3) and (4) are also equivalent to the static problem of terminal wealth maximization under a dynamic wealth-budget constraint. \( E^* \) is the expectation under a probability measure \( Q \) which is equivalent to \( P \).

4.3 Optimal Demands for the Futures Contract

In this section, the optimization problems for producers and speculators are solved when a futures contract is available and the interest rate is exogenous. Before deriving the demands for investment, the present value of terminal production \( S_T^k \) are determined.

Lemma 1
Assume that a bounded equilibrium relative risk process \( \langle \theta_t = \frac{a_t}{\delta_t} \rangle \) exists for the futures price at time \( t \).

The present value of the terminal production \( S_t^i \) is:

\[
V_t^i = E^*[e^{-\int_{t}^{\tau} \rho dv} | S_t^i | \mathcal{F}_t],
\]

where \( E^* \) is the expectation under the equivalent martingale measure \( Q \).

Proof: see the appendix.

The futures contract is a continuously traded asset in an economy with only one source of uncertainty. Therefore, the market is complete, and the present value, \( V_t^i \), of each terminal production, \( S_t^i \), is unique. The volatility of \( V_t^i \) is of interest here because of its relevance to the hedging decision of the producer. As we do not yet know the exact form of the relative risk process, the volatility of the present value of terminal production is:
\[ a_t^i V_t^i = \sigma^i V_t^i - E \left[ e^{-\int_t^T \theta_{-i} d\bar{z} - \int_t^T S_t D_i \theta_i d\bar{z} - \int_t^T |\mathcal{F}_t|} \right. \]

where \( D_i \) is the Malliavin derivative operator at time \( t \), and \( \bar{z} \) is the Brownian motion under the equivalent martingale measure \( Q \).\(^{38}\) The second expression on the right-hand side (RHS) of equation (5) is justified by the uncertainty about the future evolution of the relative risk process at time \( t \). If the relative risk process is deterministic, the Malliavin derivatives are equal to zero.

The allocation problem of the individual agent given that the relative risk process of futures price is known is now solved. The demands for the futures contract by producers and by speculators are given in proposition 1 and proposition 2, respectively, below.

**Proposition 1**

Consider the problem described by equations (1) and (2). The optimal demand for the futures contract by producer \( i \) at time \( t \) is:

\[ N_t^i = \frac{W_t^i \theta_t^i}{\delta_t} - \frac{V_t^i \sigma_t^i}{\delta_t}, \]

where \( W_t^i = X_t^i + V_t^i \) (wealth) is the absolute risk tolerance of investor \( i \)'s utility function, and \( V_t^i \) is the...

\(^{38}\)For a detailed introduction to Malliavin calculus, see Ocone (1988). For previous financial applications of Malliavin calculus, see Ocone and Karatzas (1991) for the optimal portfolio problem, and Detemple and Zapatero (1991) for the asset pricing problem.
present value of the cash flows generated by the position in the commodity at time T.\textsuperscript{39}

For the proof see the appendix.

The first and second terms on the RHS of equation (6) are the demand for mean variance efficiency and the hedge ratio, respectively. The present value of the non-traded position in the commodity is insured. The producer uses futures contracts to offset variations in the present value $V_t$ of the cash flow $S_t$ received from her non-traded position at maturity. The diversity in initial endowments and technologies implies that some producers might be short in the futures contract at any time $t$, whereas others are long in the futures contract. The net position of a producer in futures contracts is determined by the difference between the volatility of her wealth and the volatility of the present value of her terminal production. Producers with highly (un)certainty terminal cash flows smooth their terminal consumptions by buying (selling) futures contracts. The fact that some producers are long when others are short in the futures contract is a consequence of heterogeneity. Hirshleifer (1991) obtains a similar result for a discrete time model in which heterogeneity concerns the resolution of uncertainty for producers. In contrast, heterogeneity is part of the equilibrium in the present model, and the hedging position of the producer is determined by the relative volatility of her production process. In the Hirshleifer model, arrival of information is independent of the equilibrium futures price, and the assumption of additively separable preferences, complete markets, and a non-random endowment of the numeraire are required to obtain a martingale property for the futures price. This martingale property of the futures price is instrumental in deriving the demand for futures contracts. No restriction on the futures or the spot price of the commodity are needed to solve for the equilibrium in the next section of the paper.

\textsuperscript{39}See Detemple and Murthy (1994).
Before doing so, the optimal demand for investment by speculators is derived. The difference between a speculator and a producer herein is only in the position in the non-traded technology.

**Proposition 2**

For the problem described by equations (3) and (4) the optimal demand for futures contracts by a speculator \( k \) is:

\[
N_k^t = W_k^t \frac{\theta_t}{\delta_t},
\]

where \( W_k^t \) (wealth) is the absolute risk tolerance of the speculator's utility function. Demands for futures contracts by speculators depend on the mean and the variance of the instantaneous changes in the futures price.

### 4.4 Equilibrium with Heterogenous Agents

The equilibrium value, \( \Theta_t \), of the relative risk process of the futures price at any time \( t \) is derived in this section. The market clearing solution is straightforward if producers hedge only against the price risk. In the model developed herein, producers hedge against fluctuations of present values of their respective terminal cash flows, which depend on the equilibrium value of the relative risk process for futures price. This equilibrium value depends on the present values of the terminal cash flows of all producers. Thus, the present values of the terminal cash flows and the relative risk process are endogenous to the model developed herein.
Proposition 3

For the economy described in section 2, a unique equilibrium relative risk process of the futures price exists at each time which satisfies:

\[
\theta_t = \sum_{i} b_i^t \sigma_{it},
\]

where:

\[
b_i^t = \frac{V_i^t}{W_t} - \sum_{i=1}^{n} V_i^t + \sum_{i=1}^{n} X_i^t + \sum_{k=1}^{m} W_k^t,
\]

\[
dW_t^i = [W_t^i(r_t + \theta_i^2) - V_t^i\sigma_i^t \theta_i^t]dt + W_t^i \theta_i d\tilde{z}_t,
\]

\[
dW_t^k = W_t^k(r_t + \theta_k^2)dt + W_t^k \theta_k d\tilde{z}_t,
\]

\[
dW_G_t = W_G_t r_t dt + W_G_t \theta_t d\tilde{z}_t,
\]

\[
\theta_t = \sum_{i} b_i^t \sigma_{it},
\]

Proof: see the appendix.

Equation (8) characterizes the relative risk process in equilibrium, and is obtained by observing that a zero net supply of the futures contract exists in equilibrium. The equilibrium value of the relative risk process depends on the ratios b_i's, and the volatilities \sigma_i's which are also determined endogenously. This weighted-average characterization follows from a structural property of linear technologies with logarithmic utility [for another example, see Detemple and Murthy (1994)].

Proposition 4

The equilibrium futures price process for the delivery of aggregate production at time T is described by:
\[ dF_t = W_{Gr} \theta_t^2 dt + W_{Gr} \theta_t dz_t. \]  

(10)

Proof: see the appendix.

Based on proposition 4, the futures price is driven by wealth, rather than by spot prices as in other models. The premium paid by the short side to the long side of the contract diminishes with increasing aggregate risk tolerance (that is, increasing aggregate wealth). Price changes per unit of time are stochastic, and have conditional variances that increase with time, as the present values of terminal production increase.\(^40\) This behaviour is consistent with the Samuelson hypothesis, which states that changes per unit of time in the volatility of futures price increase as the time to maturity decreases. This

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\(^40\)An alternative expression for the instantaneous volatility of the futures price change at time \( t \) is:

\[ \sum_{i} \nu_{i,t} \sigma_{\nu_{i,t}} = \sum_{i} \nu_{i,t} \sigma_{i} - \sum_{i} E \left[ \int_{t}^{T} \sigma_{i}^2 \, dt \right] \]

The instantaneous conditional volatility at time \( t \) of the futures price change at time \( t+j \), \( dF_{t+j} \), is denoted by:

\[ E \left[ \sum_{i} \nu_{i,t} \sigma_{\nu_{i,t}} \mid \mathcal{F}_t \right]. \]

This conditional volatility increases with the time index \( j \).
futures price process is consistent also with the normal backwardation hypothesis.41

The futures price dynamics described by proposition 4 also appear to be consistent with the empirical evidence. Although the behaviour of futures prices is still an open question, several studies have uncovered a maturity effect in the volatility of futures price series. Anderson (1985) reports a maturity in the daily prices for commodities. Milonas (1986) concludes that a strong maturity effect exists in 10 out of 11 futures price series examined (including agriculturals, financials and metals).

4.5 Concluding Remarks

Hedging and equilibrium are studied herein for a setting in which: (1) heterogeneous producers hedge to reduce their respective terminal cash flow risks, which emanate from non-traded production technologies, by trading in futures contracts written on the commodity they grow, and (2) speculators invest their initial wealth in bonds and trade in futures contracts. The distinctive feature of the model is that cash flow (and not price) risk drives trades in futures contracts. Consequently, the futures price

4I The non-tradeness of production technologies also contributes to the increase of futures price volatility. To see this, define $f_t$ as the futures price for the delivery of one unit of the weighted-average cash flows at time $T$. The futures price at time $t$ is obtained by solving the stochastic differential equation

$$df_t = \frac{dF}{\Sigma V^i} = \frac{W_{dt}}{\Sigma V^i} \rho dt + \frac{\rho}{\Sigma V^i} \left( \frac{\Sigma V^i}{\Sigma V^i} \right) dz_t - \int_{t}^{T} \left[ D \theta_s dz_s | \mathcal{F}_t \right] \right) dz_t.$$

The first part of the volatility of $df_t$ is stationary, the second part of the volatility tends to zero as $t$ tends to $T$. The volatility of $df_t$ is negatively correlated to time to maturity as the volatility of $dF_t$, which means that the present values of terminal cash flows are not the only factors contributing to the increase of volatility. The increasing volatility property is also explained by the fact that a non-traded position held by a producer cannot be changed before time $T$. So, at any time $t$, the time remaining before the next trade in the production technology is equal to the time to maturity of the futures contract. As time to maturity decreases, so does the time until the next trade in the production technology, increasing the resolution rate of production uncertainty. This is consistent with Anderson and Danthine (1983), who establish an explicit link between time pattern of the volatility of futures prices and that of the resolution of production uncertainty.
and the cash position are endogenous.

This theoretical framework yields some interesting insights. First, the net position in futures contracts depends on the relative volatility of the producer cash flows. Producers with cash flow uncertainty, which are resolved less rapidly than the aggregate, will tend to go short in order to smooth their terminal consumption, or alternatively, will hold futures contracts. This result differs from that obtained in traditional models in which producers sell futures contracts to hedge their non-traded positions against spot price fluctuations. This result is consistent with practice where the distinction between hedgers and speculators is ambiguous. Second, changes in the equilibrium futures price depend on wealth and the relative risk process. The futures price dynamics resulting from this model are consistent with empirical evidence for commodity futures price, where trading is absent. Specifically, the model predicts an inverse relation between volatility of futures price increments and maturity (the Samuelson hypothesis).

An interesting avenue for future research would be to study a similar model in an incomplete market setting. This might be able to account better for the ARCH behaviour reported in studies of futures prices.42

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Appendix to chapter 4

Proof of lemma 1

Assume that an equilibrium bounded relative risk process $\Theta_t$ exists. The terminal production $S^i_T$ obtained by a producer $i$ from her non-traded technology can be attained by using bond and futures contracts. For the admissible strategy $N^i_t$:

$$dV^i_t = N^i_t dF_t + (V^i_t - N^i_t) r_t dt, \quad V^i_0.$$  \hfill (11)

$$V^i_T = V^i_0 + \int_0^T N^i_s \delta_s ds + N^i_s \delta_s d\bar{z}_s + r_s V^i_s ds.$$  \hfill (12)

Therefore the time $t$ value is:

$$E^*[e^{-\int_0^T r_s ds} S^i_T] = V^i_0. \hfill (13)$$
Proof of proposition 1

Consider the optimal consumption of a producer $i$ at time $T$, $C^*_i$. Since the futures price has the same uncertainty structure as the non-traded technology, it can be used to obtain the terminal consumption. It is known that:

$$C^*_i = Y^i_T + S^i_T \quad (14)$$

Define $\eta_T = e^{(\theta T - \frac{1}{2} \sigma_T^2)}$ and $\xi_T = e^{\theta T} \eta_T$ where $\theta_T = \frac{\alpha_T}{\delta_T}$ for $\tau$ in $[0,T]$ denotes the relative risk process of the traded technology correlated to the non-traded position of producer $i$. According to Karatzas, Lehoczky and Shreve (1987), the optimal terminal consumption can be written as $C^*_i = I(y^i \xi_T)$, where $I(.)$ is a strictly decreasing function. From equation (14),

$$Y^i_t = E^*[e^{-\int^T_t (\int^T_t r_v dv)} | \mathcal{F}_t] e^{\int^T_t (\int^T_t r_v dv)}$$

where $E^*$ is the expectation under the measure $Q$. Notice that $z_t$ is a $P$-Brownian motion and $\xi_t = z_t + \Theta t$ is a $Q$-Brownian motion. From equation (15),

$$Y^i_t = E^*[e^{-\int^T_t (\int^T_t r_v dv)} I(y^i \xi_T) - S^i_T | \mathcal{F}_t] e^{\int^T_t (\int^T_t r_v dv)}$$

The first part on the right-hand side (RHS) of (16) is the present value at of the optimal wealth $W^i_t$ time $t$. The second part of the RHS of equation (16) is the present value, $V^i_t$, of the final production $S^i_T$ at time $t$. This value is derived in lemma 1. The volatility of $V^i_t$ is $\sigma^i_n V^i_t$. To derive the volatility of $W^i_t$ for an investor with a logarithmic utility function, first note that:
where $y^*$ is the Lagrange multiplier. Thus, the volatility of $W_i^*$ is:

$$-(y^* b_i e^{-\theta T})^{-1} \eta_i^{-2} (-\eta_i \theta_i) dz_i + \eta_i^{-1} (b_i e^{-\theta T})^{-1} \rho_i dz_i,$$

or $W_i^* \Theta_i$ the volatility of $y^*$ is $\rho_i = 0$.

The optimal demand for futures contract by producer $i$ is obtained by equating the instantaneous volatility of the RHS of equation (17) with the volatility of the RHS of equation (16) to get:

$$N_i^i = W_i^i \Theta_i - V_i^i \sigma_{i}.$$

The optimal demand for the futures contract by a producer $i$ is:

$$N_i^{ii} = \frac{\theta_i}{\delta_t} W_i^i - \frac{V_i^i \sigma_{i}}{\delta_t}.$$

Proof of proposition 2

The demand for investment by a speculator is obtained by setting $V_i$ to zero in equation (18).

Proof of proposition 3

The relative risk process of the futures price is obtained by setting the aggregate demand for futures contracts by producers and speculators to zero, because the futures contracts are in zero net supply. The existence and the uniqueness of this process is shown by first assuming that the conditions of proposition
are respected. Then, the aggregate production at time \( t \) can be written as a function of the relative risk process \( \Theta_t \) as:

\[
A_t(\Theta) = \sum V_i^t = E^*[e^{-\int_0^t \theta_s ds} A_t | \mathcal{F}_t],
\]

(19)

where:

\[
A_T = \sum S_i^T
\]

(20)

The instantaneous variation of aggregate production is:

\[
\sum_i V_i^t r_i^t dt + \sum_i V_i^t \sigma_i^t d\tilde{z}_i.
\]

(21)

Using the equilibrium condition \( \theta_t = \sum b_i^t \sigma_i^t \) and setting \( \sum_i X_i^t + \sum_k W_k^t = l_t \), the following process is obtained for \( \tilde{A}_t \) (the value of \( A_t \) when using equilibrium condition), with:

\[
d\tilde{A}_t = \tilde{A}_t r_t dt + (\tilde{A}_t + l_t) \theta_t d\tilde{z}_t
\]

(22)

which is equivalent to:

\[
d\tilde{A}_t = [\tilde{A}_t (r_t + \theta_t^2) + l_t \theta_t^2] dt + (\tilde{A}_t + l_t) \theta_t d\tilde{z}_t
\]

(23)

**Remark:** Since \( l_t \) is independent of \( \Theta_t \) and the \( V_i^t \)'s, and a zero net supply of the futures contracts exists in equilibrium, then:

\[
d l_t = l_t r_t dt + \sum_i N_i^t (\alpha_i dt + \delta_i dz_i),
\]

\[
d l_0 = \sum x_i.
\]
depends only on the interest rate and the initial cash position of agents, both of which are exogenous.

The system of equations (19)-(23) is consistent if and only if:

$$\tilde{A}_T = A_T$$  \hspace{1cm} (24)

Thus, at each point in time $t$, $\tilde{A}_t$ and $\Theta_t$ should be such that $\tilde{A}_T = A_T$. The above problem can be viewed as a design problem in which $\tilde{A}_t$ and $\Theta_t$ are chosen at each time $t$ such that the terminal condition is met. This is the equivalent to solving a backward stochastic differential equation of the type studied by Pardoux and Peng (1990); namely:

$$x(t) + \int_0^1 f(s, x(s), y(s)) \, ds + \int_0^1 g(s, x(s), y(s)) \, dW_s = X. \hspace{1cm} (25)$$

The conditions for the existence of a unique solution to equation (25) are:

a) $f(.,0,0)$ and $g(.,0,0)$ belong to $M^2(0,1,R)$, the set of integrable and progressively measurable functions relative to the information set.

b) $c > 0$ exists such that

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| + |g(t,x_1,y_1) - g(t,x_2,y_2)| \leq c (|x_1 - x_2| + |y_1 - y_2|)$$

for all $x_1, x_2 \in R$, and $y_1, y_2 \in R$, $(t,w)$ a.e.

c) An $a$ exists such that:

$$|g(t,x,y_1) - g(t,x,y_2)| \leq a |y_1 - y_2|$$

for all $x \in R$, and $y_1, y_2 \in R$, $(t,w)$ a.e.

In the case of $\tilde{A}_t$:

$$f(t, x(t), y(t)) = \tilde{A}_t (r_t + \Theta_t^2) + l_t(\Theta_t^2), \quad \Theta_t = \sum_i \beta_i \sigma_i;$$
\[ g(t, x(t), y(t)) = (\ddot{A}_t + l(t))\theta_t. \]

Condition a) obviously holds for the differential equation of \( \ddot{A}_t \). Condition b) is the Lipschitz condition. Since \( \Theta_t \) is bounded, the Lipschitz condition holds for the differential equation of \( \ddot{A}_t \). Condition c) also holds for the differential equation of \( \ddot{A}_t \). This is apparent after setting \( a = \min \{ l_t / t \in [0, T] \} \), and knowing that \( l_t \) is always positive. When these three conditions are respected, a unique couple \((\ddot{A}_t, \Theta_t)\) solving the differential equation (21) with the terminal condition (24) exists at each time \( t \). The solution to the system (21)-(24) is equivalent to an optimal allocation made by a central planner.

That the solution is compatible with equilibrium in the initial decentralised economy is now shown. If a unique couple \((\ddot{A}_t, \Theta_t)\) solution exists to the system of (21)-(24) at each time \( t \), then:

\[
\ddot{A}_t^* = E^*[e^{-\int_{t}^{T} r_s dv} A_T | \mathcal{F}_t]
\]

\[
\ddot{A}_t = \sum_{i} E^*[e^{-\int_{t}^{T} r_s dv} S_t^i | \mathcal{F}_t]
\]

The volatility of \( \ddot{A}_t^* \) is then:

\[
\sum_{i} \nu_t^i \sigma_{\nu_t}^i
\]

However, from equation (22) the volatility of \( \ddot{A}_t^* \) is:

\[
(\ddot{A}_t^* + l_t)\theta_t^*.
\]

Equating the two volatilities leads to the equilibrium condition of the decentralised economy:
Proof of proposition 4

The futures price, \( F_t \), for the delivery of aggregate production is obtained by setting the instantaneous volatility of \( dF_t \) at time \( t \), \( \delta_t \), to \( \sum_i \nu_i \sigma^i_{vt} \), and then, using the relation \( \alpha_t / \delta_t = \Theta_t \) to determine the optimal \( \alpha_t \).
CHAPTER 5: MAJOR FINDINGS, IMPLICATIONS
AND DIRECTIONS FOR FUTURE RESEARCH

This thesis used a dynamic asset pricing framework to investigate the efficiency of currency option markets, risk preferences of investors implied by a partial equilibrium model of bond futures prices, and to study equilibrium in futures market with heterogeneity. The major findings of the three essays are summarized next.

In the first essay, models of continuous time bond futures prices under discrete marking-to-market are derived by using the fact that equispaced sampling of a continuous time AR-1 (the Ornstein-Uhlenbeck process) results in a discrete time AR-1. The formulas obtained are tractable and suitable for empirical investigation unlike a formula derived earlier by Chen (1992b). Second, meaningful estimates of the market price of interest rate risk are obtained, and Treasury bond futures data are shown to be a valid alternative to the bond data used in previous studies to estimate such market prices. Since delivery options held by the short side of a Treasury bond futures contract do not appear to have an impact on the market price of interest rate risk, the very active Treasury bond futures market is a good source of data for empirical estimations. Since estimates of the market price of interest rate risk obtained using models of Treasury bill futures prices data are too high to account for both a logarithmic utility function, and the volatility of interest rates, these models models are rejected. These results might be explained by market distortions. An alternative interpretation is that the models are misspecified. Estimations improve for long-lived futures contracts with long-term underlying bonds when a model with discrete marking-to-

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market is used.

In the second essay, ex-ante tests of the relationship between short- and long-term implied volatilities are performed for the exchange rates of the Japanese yen, the Swiss franc and the Deutschmark versus the American dollar. The model selected for the conditional volatilities of the exchange rate relative return series for these currencies is the GARCH(1,1). The joint hypothesis of a correct model specification and ex-ante efficiency is not rejected for the Japanese yen and the Swiss franc, and is rejected for the Deutschmark. This rejection is interpreted as evidence that inefficiencies caused by noise exist in the Deutschmark currency-option market during the studied period. The results reported herein contribute to both sides of the ongoing debate about whether option markets are efficient.

In the third essay, hedging and equilibrium are studied herein for a setting in which: (1) heterogeneous producers hedge to reduce their respective terminal cash flow risks, which emanate from non-traded production technologies, by trading in futures contracts written on the commodity they grow, and (2) speculators invest their initial wealth in bonds and trade in futures contracts. The distinctive feature of the model is that cash flow (and not price) risk drives trades in futures contracts. Consequently, the futures price and the cash position are endogenous. This theoretical framework yields some interesting insights. First, the net position in futures contracts depends on the relative volatility of a producer's cash flows. Producers with cash flows uncertainty which are resolved less rapidly than the aggregate, will tend to go short in order to smooth their terminal consumption, or alternatively will hold futures contracts. This result differs from that obtained in traditional models in which producers sell futures contracts to hedge their non-traded positions against spot price fluctuations. This result is consistent with what is observed in practice where the distinction between hedgers and speculators is ambiguous. Second, changes in the equilibrium futures price depend on wealth and the relative risk
process. The futures price dynamics resulting from this model are consistent with empirical evidence for commodity futures price, where trading is absent. Specifically, the model predicts an inverse relation between volatility of futures price increments and maturity (the Samuelson hypothesis).

Several directions for future research emerge from this thesis. First, the research methodology used herein for the Japanese yen, the Swiss franc and the Deutschmark should be applied to other major currencies, and other periods, or in relation to market events, so that the conditions under which inefficiencies arise are determined. Second, further investigations of the market price of interest rate risk taking into account confounding events, and the possibility of structural breaks might yield richer results. It would be interesting to test Treasury bill futures price models with data sampled exclusively when the number of transactions is high, and the risk of inefficiencies low. Third, a model of equilibrium in an incomplete futures market with heterogeneity might explain the GARCH behaviour of some futures prices.
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