Lot-Sizing Models with Simultaneous Backlogging and Lost Sales

by

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Abstract

To contribute to the growing trend of adapting lot-sizing models to represent realistic situations in an effort to propose more practical solutions, this thesis studies a multi-item capacitated lot-sizing problem that simultaneously considers both backlog and lost sales. We first review the Classic Lot-sizing Problem (CLSP) formulation along with its extensions with backlog and lost sales. For better compatibility, we propose the Facility Location Reformulation (FLR) to formulate our problem and highlight the reasons why. We develop a formulation using FLR that simultaneously considers backlog and lost sales. We also propose several extensions to our formulation, first by introducing two different lost sales approaches: *fixed proportion lost sales* and *variable proportion lost sales*. The formulations are also tested using different assumptions for backlog: *unlimited backlog*, *restricted backlog*, and *multiple customer types*. We extend the traditional backlog concept to include a “multiple customer types” backlog assumption in which customers have different willingness to wait. We summarize the results and make comparisons between the different formulations on their performance and structure of their solution. We then evaluate the impact that certain parameters have on the performance of the formulations by conducting sensitivity analyses on the following parameters: capacity, customer’s willingness to wait, lost sales cost, and length of backlog restriction.

Table of Contents

List of Figures

List of Tables

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1. Introduction

Lot-sizing models are Mixed Integer Programming (MIP) models used to determine the optimal timing and level of production. In their most basic form, lot-sizing models consider the trade-off between setup costs and holding costs, while satisfying deterministic and dynamic demand. Deterministic and dynamic demand describes a situation in which demand is known up front but can vary over time. Oftentimes in the lot-sizing literature, it is assumed that demand must be completely satisfied on time. However in practice, this demand assumption often does not hold true and companies face situations where demand cannot be satisfied on time. In this thesis, we study lot-sizing models in which demand possibly cannot be satisfied on time. We describe this inability to satisfy demand on time as a *stock-out*. Once faced with a stock-out, there are different options that can be taken, two of which are backlogging the unsatisfied demand and incurring lost sales.

Problem Description

When faced with unsatisfied demand due to limited capacity or other reasons, it may be possible to backlog this demand. The term backlogging is used to describe the situation in which unsatisfied demand in a specific period is satisfied by production in later periods. Over the years, researchers have studied models that consider backlogging. However, it is usually assumed that backlogged demand is eventually all satisfied during a later period within the planning horizon, at a given cost. Once again, this is often not the case in practice. Industries are becoming more saturated and competitive and costumers are willing to shop elsewhere if their demand cannot be immediately satisfied. When producers are faced with stock-outs, demand can sometimes be lost. Within the lot-sizing literature, both backlog and lost sales have been studied separately.

In Zangwill (1966), an extension to the standard Mixed Integer Programming (MIP) formulation of the basic lot-sizing problem was developed to include the possibility of planned backlog. Other researchers have then developed other formulations with backlog (Lambrecht and Vander Eecken 1978a, Pochet and Wolsey 1988). This research subsequently lead to studies on different problems considering backlog. Service-level constraints, which limit the use of backlog, have been studied by Gade and Küçükyavuz (2013) and Gruson *et al.* (2018). The concept of lost sales in lot-sizing problems has also been studied. Sandbothe and Thompson (1990) were some of the first to explore this idea. Their formulation forbids the use of backlog or any alternative means of satisfying demand in case of a stock-out. When demand is not met on time, lost sales, along with a penalty cost, are incurred for every unit of demand lost. As an alternative to allowing backlog and lost sales, researchers have also included the option of outsourcing in lot-sizing problems. This scenario involves portions of the production being completed by a third-party, rather than in-house. Inspired by the popular use of outsourcing in practice, Zhang (2015) studied a capacitated lot-sizing problem with constant production capacity and unlimited outsourcing.

In this thesis, we study a capacitated multi-item lot-sizing problem, which simultaneously considers the possibility of both backlog and lost sales. The combination of these two concepts has not yet been pursued in the literature. Lot-sizing models are increasingly being adapted to represent realistic situations in an effort to propose more practical solutions. We strive to continue that trend by proposing new lot-sizing problems, where both backlogging and lost sales can be options as a means of dealing with a stock-out. In this problem, once faced with a stock-out, units of demand will either be backlogged or become lost sales. When considering these two concepts simultaneously, one must first determine how the two interact with each other.

In combination with backlog, there are two possibilities of lost sales that are studied: a fixed-proportion and a variable-proportion version. Before defining the two versions, it is important to note that in our study, we treat each unit of demand as a separate individual customer. Customers with multiple units of demand are a more complex problem that can be an interesting extension for future research. In both the fixed and variable version, when a stock-out occurs, a minimum fixed percentage of the unsatisfied demand is lost. This fixed percentage represents the customers who are not willing to wait in case of a stock-out. For the fixed-proportion lost sales, the remaining unsatisfied demand must be backlogged and satisfied in later periods. As for the variable-proportion lost sales, the company decides whether the remaining unsatisfied demand will be lost sales or become backlog that must be satisfied in later periods.

Since stock-outs have a negative impact on customer loyalty, backlog service-level constraints have been modeled in an effort to limit the loss of customer goodwill (Gade and Küçükyavuz, 2013). Others have used penalty costs to punish demand that is not met on-time either with a lost sales penalty cost (Sandbothe and Thompson, 1990), or a backlog penalty cost (Pochet and Wolsey 1988, Gruson *et al.* 2018). Our model uses both types of costs to represent the negative impact of stock-outs. The backlog penalty cost is used to represent the costs associated with backlogging an item (administration cost, schedule changing cost, extra transportation cost, loss of goodwill, etc.), while the lost sales penalty cost will be used to represent the lost profit and the loss of customer goodwill.

Research Question and Contributions

Our research is inspired by highly competitive and short seasonal industries like that of fast fashion. Companies within these industries have similar product offerings, and subsequently some customers are more willing to shop elsewhere if stock-outs occur while others are willing to wait.

Although these types of industries inspired this thesis, there are many other situations that are relevant. As long as customers have substitutes available, there can be lost sales. Even in special cases, there may be lost sales without any substitutes available. For instance there is a loss of interest or need in a product. The important idea to note is that our thesis applies to situations where, in case of a stock-out, some customers are willing to wait (leading to backlog) and others are not willing to wait (leading to lost sales). We study a production planning problem of a manufacturer that faces deterministic demand under the assumption that every unit of demand is represented by a unique customer. The goal is to develop various mathematical optimization models for a manufacturer that will represent different relationships between backlog and lost sales.

Lot-sizing research has veered towards problems that portray realistic business scenarios. The possibility of stock-outs is a concept that is common throughout multiple industries. Solutions for stock-outs, such as backlog, outsourcing, and lost sales have all been studied. In a realistic production setting, multiple options and consequences are in play. Considering both backlog and lost sales simultaneously could provide insight for a more realistic production planning problem. In addition, this will allow us to analyze the trade-off between backlog and lost sales. This thesis aims to contribute to the movement of providing more practical solutions by answering the following research question: How can one formulate and solve a multi-item capacitated lot-sizing problem that considers backlog and lost sales simultaneously?

The contribution of this thesis is sixfold. (1) We propose new lot-sizing problems, which incorporate backlog and lost sales simultaneously. (2) We extend the backlog concept to model customers that have a different willingness to wait. (3) We develop a formulation for each of the proposed problems. (4) We conduct computational tests to evaluate the performance of the new formulations versus similar existing formulations. (5) We analyze the structure of the solutions, and the trade-off between backlog and lost sales. (6) We conduct sensitivity analyses to determine the impact of changing some key parameters.

Methodology

In order to conduct the analysis, CPLEX 12.6.3.0 will be used to solve test instances for different formulations which are modelled using the OPL coding environment. We conduct our computational experiments on two datasets developed by Trigeiro *et al.* (1989). The datasets are adapted to fit our new formulations prior to the tests. The new formulations are tested, along with similar existing formulations, for comparison purposes. The formulations are then tested under different assumptions such as backlog restrictions and multiple customer types. Lastly, we change certain parameters in the dataset to perform sensitivity analyses.

The thesis is organized as follows. In Section 2, we review the related existing literature on our topic. Section 3 will summarize the existing formulations for the Classical Lot-sizing Problem (CLSP) and Facility Location Reformulation (FLR) that incorporate either backlog or lost sales. Next in Section 4, we develop two new reformulations that include both backlog and lost sales simultaneously and describe potential alterations to the formulations that can be made to incorporate different assumptions. In Section 5, we compare the computational results of multiple formulations, different assumption, and parameter settings, and analyze the results. In Section 6, we conduct a sensitivity analysis on multiple key parameters. Lastly in Section 7, we provide a conclusion to the thesis.

2. Literature Review

In this section, we first provide a brief overview of the literature on the topic of lot-sizing, specifically when demand is dynamic. Then, we look more closely at lot-sizing models that consider backlog. We follow by examining the use of lost sales or similar concepts in lot-sizing models. Next, we review alternative reformulations that have been developed to solve comparable lot-sizing problems. Finally, we summarize the main areas of research that we focus on and highlight the papers that motivate our thesis.

2.1 Lot-sizing

Lot-sizing models, in general, aim to determine the optimal timing (when) and level of production (how much). The model's goal is to minimize the total cost while adhering to all of the constraints and satisfying the demand. Lot-sizing models can be classified using different angles in terms of their timescale, demand behaviour, and the time horizon (Jans and Degraeve, 2008). The classical Economic Order Quantity (EOQ) model developed by Harris (1913) is seen as the base form of lot-sizing models. The EOQ model assumes a continuous timescale, no production capacity constraints, and a deterministic and constant demand rate with only one type of product. One of the contributions of the EOQ model is that it highlights the trade-off between setup and holding costs. Many extensions of this model have been explored. A thorough review on these EOQ extensions is provided by Silver *et al.* (1998). One such extension, named the Economic Lot Scheduling Problem (ELSP), considers multiple products and a constant production rate (Elmaghraby 1978, Zipkin 1991). In contrast to the EOQ model, which is simpler to solve, the ELSP is NP-hard (Elmaghraby 1978, Zipkin 1991). The ELSP model represents a more realistic problem, wherein companies produce and deliver multiple products.

Even though the EOQ model has been extended in many different ways, it often has too many unrealistic assumptions to provide a practical production plan. Researchers have extended this model to include more realistic conditions by relaxing some of the previous assumptions. One of the EOQ model's biggest assumption is that demand is constant. Since this is rarely the case, researchers have moved their focus to dynamic demand, which permits the demand to vary over a time horizon.

2.2 Dynamic Demand

Lot-sizing research has moved towards using dynamic demand to obtain more realistic production plans over a time horizon. Dynamic demand assumes that the demand for items can vary over time. The goal of these models is to satisfy aggregate demand through production during each period (Robinson, 2009). Wagner and Whitin (1958) were the first to introduce the classical dynamic lot-sizing model where dynamic demand must be satisfied in each given period. The original model has since been extended in different directions. Robinson *et al.* (2009) state that there are four predominant types of classical dynamic demand lot-sizing problems that are studied, as indicated in Table 1.

	Capacitated	Uncapacitated
Single-item	S-CLSP	S-ULSP
Multi-item	M-CLSP	M-ULSP

Table 1. Types of Lot-sizing Problems

They are categorized according to two factors: (1) single or multiple products, (2) capacitated or uncapacitated. A review of different models for lot-sizing problems can be found in

Jans and Degraeve (2008). In this thesis, we focus on multi-item capacitated lot-sizing problems (M-CLSP).

2.3 Backlog

When faced with stock-out, backlogging demand can be a possible option. This means that demand in a specific period can be satisfied by production in later periods. Zangwill (1966, 1969) adapted the original uncapacitated lot-sizing model (ULSP) of Wagner and Whitin (1958) to allow backlogging. He proposed an algorithm with a concave holding and backlogging cost structure along with a limited number of periods that a demand could be backlogged for. For the capacitated lot-sizing model with backlog, Swoveland (1975) provided a model with piecewise concave cost functions for production, holding, and backlogging.

Pochet and Wolsey (2006) present a basic MIP formulation for the ULSP. The formulation is based on three types of variables: production, inventory, and production setup. The authors also provide an extension to the formulation that includes backlog. All variables, including backlog, have an associated cost and the objective is to minimize their sum.

Researchers have also experimented with service-level constraints with backlog. Gade and Küçükyavuz (2013) study a multi-item dynamic lot-sizing problem with backlog and a service-level constraints. For instance, one of the constraints limits the number of periods that demand can be backlogged for. They propose the service-level constraint as an alternative to having a backlog cost in the objective function, and then develop a shortest path algorithm to solve the resulting formulation. Another service-level constraint that is often considered is a constraint that ensures that a certain percentage of demand is met on time in each period (Gruson *et al.*, 2018). Gruson *et al.* (2018) extend on Gade and Küçükyavuz (2013), and they propose various service-level constraints. The authors use the facility location reformulation to help distinguish between backorders and backlog. By doing so, they were able to study different service level constraints at an individual level, item-by-item, period-by-period, and globally. Their results concluded that different constraints can lead to very different solutions.

Backlog has also been combined with the concept of time-windows (Absi *et al.* 2011, Lee *et al.* 2001, Huang 2008, Brahimi *et al.* 2006). Absi *et al.* (2011) study a single uncapacitated lot-sizing problem (USLP) with production time-windows, early production and backlog. Within this problem, demand can be met with on-time production at a unit cost, early production at a penalty cost, or backlog at a cost for backorders.

2.4 Lost Sales

Lost sales occur when demand is not met on time and the customer no longer wants the product, either because the product is not needed anymore, or because he decides to go to a competitor to purchase a similar item. Companies may incur lost sales when faced with stock-out and backlog is not permitted. When lost sales occur, the firm may avoid paying the unit production cost, but faces a loss in revenue and customer goodwill. There is hence a cost associated to each unit of lost sales.

Sandbothe and Thompson (1990) were some of the first to explore the lot-sizing problem with the concept of lost sales. They examined two versions of a capacitated problem with the possibility of lost sales where production capacity either remains constant over a time horizon, or fluctuates. In this formulation backlog is forbidden. When the producers are not able to meet demand on time, lost sales are incurred and they suffer a penalty cost for each lost order.

Aksen *et al.* (2003) examined an uncapacitated version of the single-item deterministic lot-sizing problem, where demand cannot be backlogged but does not have to be satisfied. If demand is not satisfied, the cost of a lost sale is equal to the lost revenue. This paper highlights the trade-off between production cost and sales revenue. They first present a profit-maximization model, and then reformulate it as a cost-minimization model. Aksen *et al.* (2003) do so by including the unmet demand decision variable as a component of the total cost function.

Typically in production planning, the ability to meet customer demand is limited by production capacity. Liu *et al.* (2007) explore an alternate realistic scenario, where demand is constrained by inventory capacity instead of production capacity. The paper studies a single-item capacitated lot-sizing problem (CLSP) with time varying inventory restrictions and lost sales. They associate a penalty cost to each unit of lost sales. The authors assume, in this case, that the lost sales cost is greater than the production cost. Once inventory reaches its capacity for a given period, the unsatisfied demand is absorbed by the lost sale variable. Liu *et al.* (2007) do not consider the addition of backlog into their model.

Absi *et al.* (2011) consider a single-item uncapacitated lot-sizing problem with production time windows, and early production. They address a single-level problem with competing interests between customers and suppliers. This type of problem simulates an environment with strict time restrictions and tight capacity levels. The model does allow items to be produced before the given time window. Companies may opt for this option at a given early production penalty cost. The proposed model uses production time windows, therefore if the demand cannot be produced within a given time interval, there is a stock-out. Absi *et al.* (2011) propose two formulations that deal with this stock-out in a different way. The first formulation includes early production costs and lost sales, in which a cost is incurred for producing before the time window, and a lost sales cost for not satisfying demand before the deadline. The second formulation was previously mentioned in Section 2.3 and involves the consideration of early production with backlog. In this case, there is an extra cost for backlog, however there is no potential for lost sales, nor is there any penalty for customer goodwill. The authors do not consider a problem with backlog and lost sales together. The research illustrates the trade-off between satisfying customers by adhering to due dates, and pleasing suppliers by respecting release dates (Absi *et al.*, 2011).

The research paper by Absi *et al.* (2011) is the inspiration for this thesis. The authors mentions the idea of merging all three variables (early production, backlog, and lost sales), but indicates that these are not generally considered simultaneously. We do believe, however, that considering backlog and lost sales simultaneously in an environment without time windows can be insightful. By removing the concept of production time windows, early production is no longer applicable. Items that are produced in periods before they are to satisfy demand, simply become inventory, and are charged a holding cost in each period until they are consumed. We extend this work by proposing reformulations that include both backlog and lost sales simultaneously.

2.5 Facility Location Reformulation

The classic MIP formulation for the multi-item capacitated (CLSP) and uncapacitated lot-sizing problem (ULSP) has its limitations when backlog is considered (Gruson *et al.*, 2018). The CLSP formulation is able to identify backlog but not backorders. Backorders represent unsatisfied demand on an individual level while backlog represents the sum of multiple backorders which have not been satisfied yet. For the formulation we aim to develop, we require the information of backorders. The distinction between backlog and backorders will be further discussed in Section 3.4.

Faced with this issue, we must turn to a different formulation that is capable of handling this type of problem. Krarup and Bilde (1977) proposed the facility location reformulation, which solves the ULSP as a linear program (LP). Researchers have used this reformulation, or extensions of it, to include factors such as backlog, service levels, and multi-processed items (Gade and Küçükyavuz 2013, Gruson *et al.* 2018, Denizel *et al.* 2008). The reformulation is shown to be able to provide better LP relaxations and improve computing times (Denizel *et al.* 2008, Nemhauser and Wolsey 1988). Typically, the facility location problem consists of finding the optimal facilities so that demand is satisfied at minimum cost (Daskin 1995). Krarup and Bilde (1997) reformulate the model specifically to address lot-sizing problems.

Continuing with Krarup and Bilde (1977), the reformulation uses a production variable that is able to track separately in which period demand is produced and satisfied. This information facilitates the calculation of the total production cost, along with the associated penalty costs. Gade and Küçükyavuz (2013) use a similar reformulation to solve lot-sizing problems with service level constraints. Gruson *et al.* (2018) extend this reformulation by including backlog and backorder costs.

2.6 Target Area of Research

Inspired by the research of Absi *et al.* (2011), we aim to develop a lot-sizing model that considers both backlog and lost sales. We study a relevant scenario used by Gade and Küçükyavuz (2013), which examines both the capacitated and uncapacitated multi-item lot-sizing problem. The facility location reformulation proposed by Krarup and Bilde (1977), will be used to solve the problem. We develop extensions to our new model that consider different backlog assumptions and different types of lost sales. Upon optimization of our models using CPLEX, we analyze the results and the relationship between backlog and lost sales. In addition, we examine the results between our model and its various extensions. Table 2 summarizes which of our focused topics are covered by the reviewed research. Note that in Absi *et al.* (2011), both backlog and lost sales are considered, but not simultaneously.

	Capacitated	Multi-item	Backlog	Lost Sales	FLR
Absi *et al.* (2011)			X	X	
Gade and Küçükyavuz (2013)		X	X		X
Gruson *et al.* (2018)	X	X	X		X
Krarup and Bilde (1977)					X
Liu *et al.* (2007)	X			X	
Pochet and Wolsey (1988)	X	X	X		
Sandbothe and Thompson (1990)	X			X	

Table 2. Summary of Related Research

3. Overview of Various Existing Formulations

In this section, we describe the transition of the standard capacitated multi-item lot-sizing model (CLSP) towards a reformulation that incorporates both backlog and lost sales simultaneously. In Section 3.1, we review the classic MIP formulation presented by Pochet and Wolsey (2006). We then explore two adapted versions of the model which take into account backlog (Section 3.2) and lost sales (Section 3.3) separately. Due to the MIP formulation's limitations, we turn to different formulations that are capable to handle our problem better. Section 3.4 briefly reviews the concepts behind the classic facility location problem. In Section 3.5 we propose the facility location reformulation developed by Krarup and Bilde (1977) as a possible solution to our problem. We present the facility location reformulation with backlog (Section 3.6) and lost sales (Section 3.7) separately. In Section 3.8, we develop our reformulation which consider both backlog and lost sales simultaneously. This section proposes two models with different types of lost sales: a fixed and a variable version. In the following sections, extensions to our two developed formulations are proposed. In Section 3.9, we rewrite the formulation to consider a scenario where there is no restrictions on the number of periods demand can be backlogged for. In Section 3.10, we propose a formulation that considers multiple customer types instead of the common single customer type approach.

3.1 Basic Formulation: Classic Multi-Item Capacitated Lot-Sizing Problem (CLSP)

The premise of the basic lot-sizing problem is to determine the optimal time and batch sizes that must be produced to satisfy a dynamic demand in a discrete and finite time horizon. Pochet and Wolsey (2006) present the classic MIP formulation which uses three important sets of variables: inventory, production, and setup variables. General notations for the capacitated multi-item lot-sizing formulations we will be using are as follows:

Parameters:

T Set of time periods {1, …, *m*}

N Set of items {1, …, *n*}

SC_{it} The setup cost of production for item *i* during period *t*, $\forall i \in \mathrm{N}, \forall t \in \mathrm{T}$

VC_{it} The unit production cost for item *i* during period *t*, $\forall i \in \mathrm{N}, \forall t \in \mathrm{T}$

HC_{it} The unit holding cost for item *i* during period *t*, $\forall i \in \mathrm{N}, \forall t \in \mathrm{T}$

Cap_t The production capacity during period *t*, $\forall t \in \mathrm{T}$

UT_{it} The unit production time for item *i* during period *t*, $\forall i \in \mathrm{N}, \forall t \in \mathrm{T}$

ST_{it} The setup time for item *i* during period *t*, $\forall i \in \mathrm{N}, \forall t \in \mathrm{T}$

d_{it} Demand of item *i* for period *t*, $\forall i \in \mathrm{N}, \forall t \in \mathrm{T}$

I_{i0} The amount of inventory of item *i* at the beginning of period 1, $\forall i \in \mathrm{N}$

M_{it} $= Min\{\, (\sum_{l=t}^{m} d_{il}), (\frac{Cap_t - ST_{it}}{UT_{it}})\}$

Decision Variables:

X_{it} The amount of item *i* produced in period *t*, $\forall i \in \mathrm{N}, \forall t \in \mathrm{T}$

I_{it} The amount of inventory for item *i* at the end of period *t*, $\forall i \in \mathrm{N}, \forall t \in \mathrm{T}$

Y_{it} Set up variable for item *i* in period *t*, takes on the value of 1 if production takes place and becomes 0 otherwise, $\forall i \in \mathrm{N}, \forall t \in \mathrm{T}$

CLSP

$$\text{Min} \quad \sum_{t\in T}\sum_{i\in N}(VC_{it}X_{it} + HC_{it}I_{it} + SC_{it}Y_{it}) \qquad (1)$$

$$S.T. \quad I_{i,t-1} + X_{it} = d_{it} + I_{it} \qquad \forall i \in \mathrm{N}, \forall t \in \mathrm{T} \ (2)$$

$$X_{it} \leq M_{it}Y_{it} \qquad \forall i \in \mathrm{N}, \forall t \in \mathrm{T} \ (3)$$

$\sum_{i \in N}(UT_{it}X_{it} + ST_{it}Y_{it}) \leq Cap_t$ $\forall t \in$ T (4)

$I_{i0} = 0$ $\forall i \in$ N (5)

$Y_{it} \in \{0,1\}$ $\forall i \in$ N, $\forall t \in$ T (6)

$X_{it}, I_{it} \geq 0$ $\forall i \in$ N, $\forall t \in$ T (7)

We first consider the capacitated multi-item lot-sizing model, as formulated by Trigeiro *et al.* (1989), as an appropriate representation of a production planning problem for a manufacturer that faces a deterministic demand. The objective function (1) is to minimize the sum of the unit production cost, inventory cost, and setup cost. This model attempts to find the optimal solution while adhering to a number of restrictions. Constraints (2) are the balancing constraints that ensure that all demand is satisfied by either production or inventory. Figure 1 provides a graphical illustration of the structure of the inventory balance constraints for an uncapacitated single-item lot-sizing model.

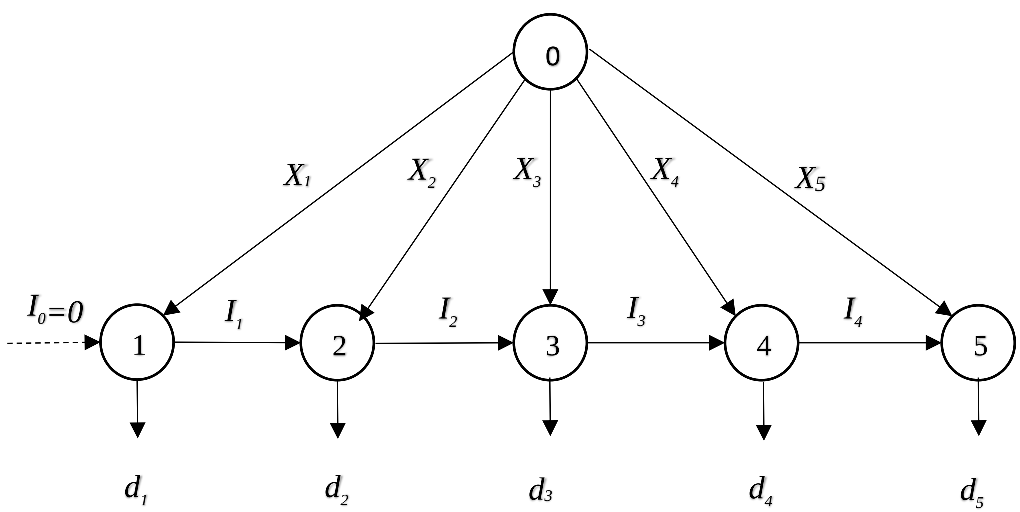

Figure 1. Graphical Representation of the Inventory Balance Constraints for the Single-Item ULSP

In this example the time horizon consists of five periods and there is no starting inventory. For every period, we must satisfy a given demand represented by d_t. The nodes represent each period labelled from 1 to 5, while the arcs represent the flow of products. Each period node must obtain products from incoming arcs, either production (X_t) or inventory (I_t), in order to satisfy the demand (outgoing arc) in that period.

The setup constraints (3) force a setup for production during each period if production is strictly positive. Constraints (4) are the capacity restrictions in terms of units of time consumed by both regular production and production setups. Constraints (5) set the initial inventory to zero. Constraints (6) and (7) are binary and non-negativity constraints respectively.

3.2 CLSP with Backlog

This classic formulation can be further extended to consider the concept of backlog (Pochet and Wolsey 1988, Zangwill 1969). In this scenario, we allow demand to be satisfied at a later period at a penalty. Let B_{it} equal the number of items i that are backlogged at the end of period t ($\forall i \in$ N, $\forall t \in$ T), and let BC_{it} be the unit cost incurred for backlogging one item of i at the end of period t. Furthermore, we define M'_{it} as follows: $M'_{it} = Min\{\,(\sum_{l=1}^{m} d_{il}), (\frac{Cap_t - ST_{it}}{UT_{it}})\}$. With the newly added decision variable and cost parameter, the formulation is adapted as follows

CLSP-B

$$\text{Min} \quad \sum_{t\in T}\sum_{i\in N}(VC_{it}X_{it} + HC_{it}I_{it} + SC_{it}Y_{it} + BC_{it}B_{it}) \qquad (8)$$

$$S.T. \quad (4), (5), (6), and$$

$$I_{i,t-1} + X_{it} + B_{it} = d_{it} + I_{it} + B_{i,t-1} \qquad \forall i \in \text{N}, \forall t \in \text{T} \ (9)$$

$$X_{it} \leq M'_{it}Y_{it} \qquad \forall i \in \text{N}, \forall t \in \text{T} \ (10)$$

$$X_{it}, I_{it}, B_{it} \geq 0, B_{im} = 0 \qquad \forall i \in \mathrm{N}, \forall t \in \mathrm{T} \quad (11)$$

The objective function (8) minimizes the total cost which now includes backlog cost. The backlog variable is added to constraints (9) which allows that demand can be satisfied at a later period. In the setup forcing constraints (10), the M'_{it} represents a large number that is now equal to the following:

Constraints (11) have replaced constraints (7) as the updated non-negativity constraints. In addition, these constraints ensures that there cannot be backlog remaining at the end of the time horizon. In both models at this point, we assume no lost sales. By forcing $B_{im} = 0$, we assure that all demand must be satisfied by production within the time horizon. The important question of this problem is when each item will be produced. Depending on when each item is produced, different costs will be incurred.

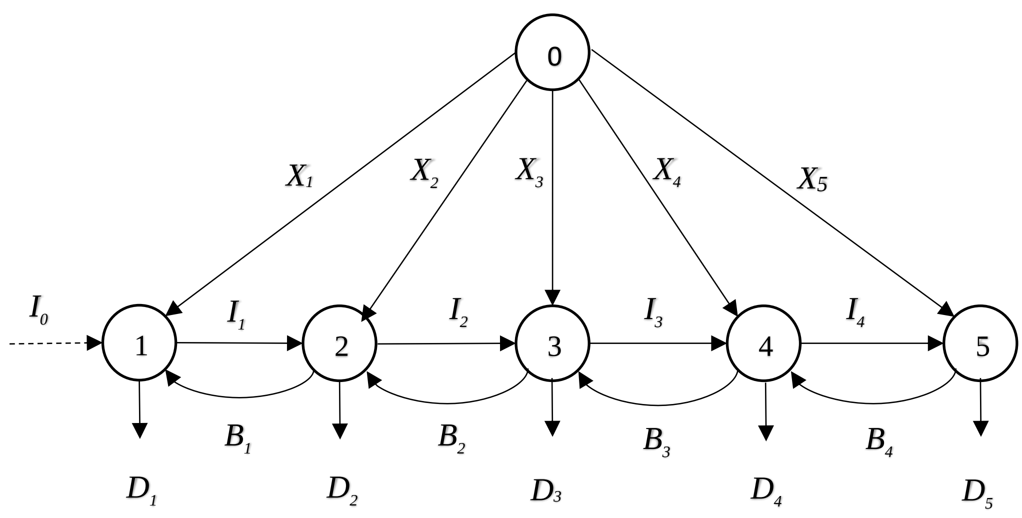

Figure 2. Graphical Representation of the Inventory Balance Constraints for the Single-Item ULSP with Backlog

Figure 2 provides a visual for the added backlog variable to the original structure of the inventory balance constraints for a single-item ULSP with backlog. In this formulation, there is more flexibility for satisfying demand due to the addition of another incoming arc. Each period t now has three methods to satisfy demand: on-time production (X_t), inventory (I_{t-1}), and backlog (B_t). For every method used there is an associated cost to it. Throughout our thesis we denote t as the period in which demand is satisfied and denote k as the period in which the item is produced. Demand that is produced in the same period ($k = t$) will not have any additional cost other than the variable production cost (VC_{it}). For instance, every unit of D_3 that is satisfied by X_3 will only incur the variable production cost. As for products that were produced before the demand period, they will have an additional inventory cost (HC_{it}) for each period equal to ($t - k$) periods. Therefore, if D_3 is satisfied by X_1, a holding cost will be incurred for both periods it was held in inventory (I_1 and $I_2 \geq 0$). Products that were produced later than the demand period, will have an additional backlog cost (BC_{it}) for ($k - t$) periods. Similarly to inventory, if D_3 is satisfied by X_5, a backlog cost will be incurred for both periods it was backlogged (B_3 and $B_4 \geq 0$).

For the context of our research, we assume that $HC_{it} \leq BC_{it}$. Although the CLSP formulation is a correct formulation to solve the lot-sizing problems, it has limitations that prevents it from being effective when backlog is added to the formulation. We discuss these limitations in greater depth in Section 3.4.

3.3 CLSP with Lost Sales

Oftentimes in CLSP models we assume that all demand will eventually be satisfied. However, customers may chose not to wait in case of a stock-out. They may opt to purchase the product from a competitor, purchase a substitute product, or decide not to purchase it at all. To incorporate this concept of lost sales, we can construct a reformulation of the CLSP with lost sales (Sandbothe and Thompson 1990, Absi *et al.* 2011, Aksen *et al.* 2003). When potential demand is lost because it cannot be met on time, the company incurs lost sales. Lost sales is a variable commonly found in profit maximization formulations. However, Aksen *et al.* (2003) explain how lost sales can also be used in a cost minimization formulation.

Aksen *et al.* (2003) study a single-item lot-sizing problem where the firm has to decide whether or not to satisfy demand in a period. In this problem, it is assumed that demand cannot be backlogged and all unsatisfied demand becomes lost sales. Aksen *et al.* (2003) defines p_t as the unit revenue (unit-selling price) in period *t*, $\forall t \in$ T. They also define L_t as the number of lost sales in period *t.* Typically when dealing with lost sales, the model is a profit maximization formulation and the objective function is written as follows:

$$(Max)\ Profit\ (\pi) = \sum_{t=1}^{T} p_t(d_t - L_t) - \sum_{t=1}^{T}(SC_tY_t + VC_tX_t + HC_tI_t)$$

The components of the objective function consist of the total revenue minus the relevant costs (setup, production, and holding). In some cases, it can happen that companies facing low demand with low gross marginal profit in a certain period, may find it more profitable to lose that demand (Aksen *et al.*, 2003). Total revenue over the time horizon is not constant and therefore the total profit cannot be maximized just by minimizing the total cost. To account for this, the paper makes a fundamental assumption that the gross marginal profit ($p_t - VC_t$) is non-negative in each period *t*, in other words, the unit selling price is greater than or equal to the unit production cost ($p_t \geq VC_t$). The authors then rearrange the objective function as follows:

$$(Max)\ Profit\ (\pi) = \sum_{t=1}^{T} p_t d_t - \sum_{t=1}^{T}(SC_tY_t + VC_tX_t + HC_tI_t + p_tL_t)$$

The first term of the objective function is now a data-dependent constant that can be removed. By removing the constant, the formulation can easily be turned into a minimization problem. We can rewrite the objective function as follows:

$$(Min) - Profit\ (\pi) = \sum_{t=1}^{T} (SC_t Y_t + VC_t X_t + HC_t I_t + p_t L_t) - const.$$

This derivation and further details can be found in Aksen *et al.* (2003). With this new formulation, the unit revenue parameter (p_t) acts as a lost sales cost. In practice, the cost associated to a lost sale can be even larger than p_t when the cost of the loss of customer goodwill is included.

The assumption of a non-negative profit margin proposed by Aksen *et al.* (2003) and Liu *et al.* (2007), is used in this thesis as is reflects the realistic production planning problem we are trying to emulate.

For our formulation with lost sales, we consider a scenario where there is no longer the possibility of backlog. If the demand cannot be met on time, lost sales are incurred. Figure 3 shows the new network where backlog is prohibited and lost sales occur. Note the new incoming arcs (L_t). The lost sales arcs differ from the backlog arcs discussed in the previous section. Lost sales provides an alternative means to balancing the demand. Using the lost sales alternative rather than matching the demand with production, however comes at a high cost.

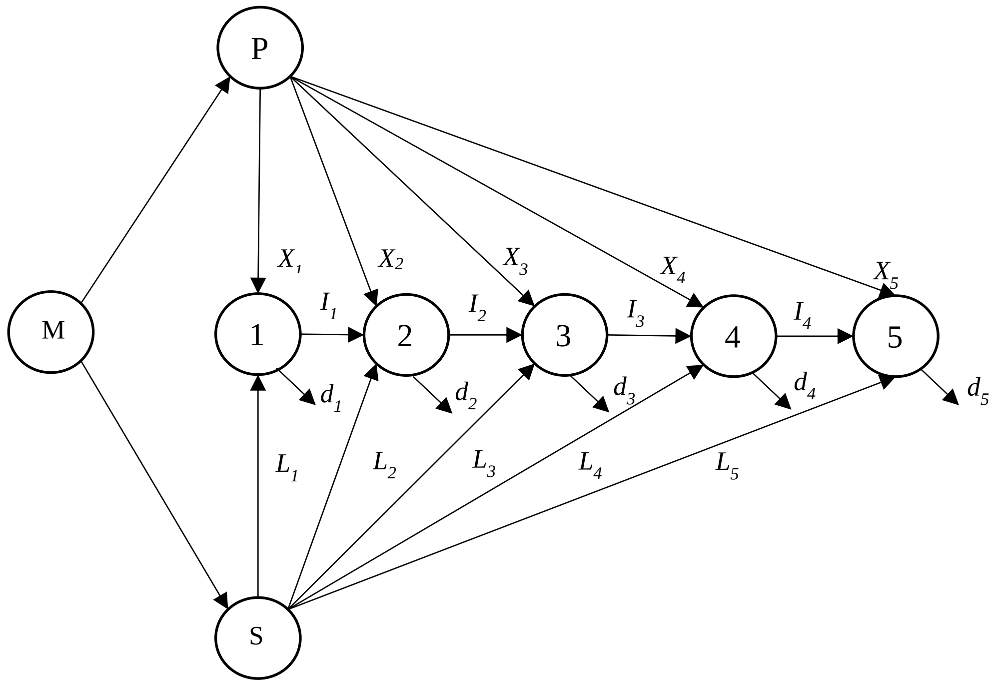

Figure 3. Graphical Representation of the Inventory Balance Constraints for the Single-Item ULSP with Lost Sales (*Adapted from* Sandbothe and Thompson, 1990)

To incorporate lost sales in the CLSP formulation, we let L_{it} be the amount of lost sales for item *i* in period *t* ($\forall i \in$ N, $\forall t \in$ T), and let LC_{it} equal the unit cost incurred for the lost sale of item *i* in period *t* ($\forall i \in$ N, $\forall t \in$ T). LC_{it} is at least equal to the unit selling price as explained in Aksen *et al.* (2003), but can be larger if the cost related to the loss of customer goodwill is included. The new formulation proposed by Aksen *et al.* (2003) is written as follow:

CLSP-L

$$\text{Min} \quad \sum_{t \in T} \sum_{i \in N} (VC_{it}X_{it} + HC_{it}I_{it} + SC_{it}Y_{it} + LC_{it}L_{it}) \qquad (12)$$

$$S.T. \quad (4), (5), (6), and$$

$$I_{i,t-1} + X_{it} + L_{it} = d_{it} + I_{it} \qquad \forall i \in \text{N}, \forall t \in \text{T} \; (13)$$

$$X_{it} \leq M_{it}Y_{it} \qquad \forall i \in \text{N}, \forall t \in \text{T} \; (14)$$

$$X_{it}, I_{it}, L_{it} \geq 0 \qquad \forall i \in \mathrm{N}, \forall t \in \mathrm{T} \quad (15)$$

In the CLSP with lost sales formulation, the lost sales cost replaces the cost for backlog in the objective function (12). For the purpose of clarity, we refer to index k as the period in which item *i* is produced, and let index *t* indicate the period in which demand is to be satisfied. In the balancing constraints (13), the demand (d_{it}) in period *t* can now be satisfied by on-time production ($X_{ik}, k = t$), early production ($X_{ik}, k < t$) or lost sales (L_{it}). The lost sales variable can be interpreted as a ghost production variable. In the first two models, total demand must be satisfied before the end of the time horizon. Production was also the only method to meet demand. Now with the possibility of lost sales, production is no longer forced to satisfy all demand. In case of lost sales, the variable production cost (VC_{it}) may not be charged, only the cost related to lost sales (LC_{it}) will be. Total demand over the time horizon can now be satisfied by either production or lost sales. Constraints (14) enforce the production setup and returns to its original form from the basic CLSP formulation. Constraints (15) are the non-negativity constraints.

3.4 Facility Location Problem

While the MIP formulations presented above are capable of providing optimal solutions for these basic cases, Gruson *et al.* (2018) explain how they have a major disadvantage which prevents us from using them any further. The classic formulation is capable of identifying backlog, however not *backorders*. This can cause issues when we develop our formulation with backlog and lost sales, because we need to be able to measure stock-out levels. When no lost sales are present, the backorder level in period *t* refers to the quantity of unsatisfied demand in the given period, whereas backlog in period *t* represents the number of backorders from the beginning of the time horizon up to period *t* that have still not been met at the end of period *t* (Gruson *et al.* 2018, Gade and Küçükyavuz 2013). The backlog variable is the sum of the remaining (i.e. yet satisfied) backorders which could represent multiple combinations of backorders. For instance, if B_{i1}=1 and B_{i2}=2, this could represent two different situations with respect to backorders. There could be one backorder accumulated in both periods that have not yet been satisfied or one backorder in period 1 that was satisfied in period 2 and two new backorders accumulated in period 2. The classical formulation is unable to distinguish between these two cases. This is explained in more detail in Gruson *et al.* (2018).

In order to combine both backlog and lost sales simultaneously, we need to know when the stock-out occurred and how many units were backordered. This will enable the penalty cost to be properly accounted for. To further extend our problem while allowing us to keep track of when a product was backordered and for how long it has been backlogged for, we use the facility location reformulation of the lot-sizing problem proposed by Krarup and Bilde (1977). In addition, it will give us the ability to apply restrictions on the number of periods an item can be backlogged for. Furthermore, the facility location reformulation typically provides a better linear programming relaxation, and hence improves the computational time (Krarup and Bilde, 1977).

Before we review the Krarup and Bilde (1977) reformulation, we first review the formulation for the facility location problem (FLP) that it was based on. The FLP was originally developed to select the best placement of *v* facilities with unrestricted size in order to minimize the total cost for satisfying given demand at *w* customer locations.

Daskin (2008) describes a facility location model that aims to minimize the sum of the fixed and variable costs while satisfying all demand zones. The variable cost (c_{ij}) is the total cost for satisfying all customers in zone *j* from facility *i*, while the fixed cost (f_i) is the cost for opening facility *i*. This formulation has two sets of decision variables, (x_{ij}) which is the fraction of the customer's demand in zone *j* that is satisfied by facility i. The second is a binary variable (y_i) that is equal to 1 if facility *i* is open and 0 otherwise. In this formulation, we let V be the set of facilities $\{1,\ldots, v\}$, and let W represent the set of customer locations $\{1,\ldots, w\}$. The formulation looks as follows:

FLP

$$\text{Min} \quad \sum_{i\in V}\sum_{j\in W} c_{ij}x_{ij} + \sum_{i\in V} f_i y_i \qquad (16)$$

$$S.T. \quad \textstyle\sum_{i\in V} x_{ij} = 1 \qquad \forall j \in \mathrm{W} \ (17)$$

$$x_{ij} \le y_i \qquad \forall i \in \mathrm{V}, \forall j \in \mathrm{W} \ (18)$$

$$y_{ij} \in \{0,1\} \qquad \forall i \in \mathrm{V}, \forall j \in \mathrm{W} \ (19)$$

$$x_{ij} \ge 0 \qquad \forall i \in \mathrm{V}, \forall j \in \mathrm{W} \ (20)$$

The objective function (16) minimizes the total variable and fixed cost. Constraints (17) ensure that demand for each customer zone is satisfied, while constraints (18) only allow demand to be satisfied by opened facilities.

3.5 Facility Location Reformulation

Krarup and Bilde (1977) proposed the facility location reformulation of the lot-sizing problem. This reformulation gives us the ability to distinguish between backlog and backorders. For a backorder in a given period, we are now able to place limits on the number of periods an item can be backlogged for. This formulation provides the flexibility and structure that allows us to formulate a problem with both lost sales and backlog simultaneously.

Facility Location Reformulation without Backlog or Lost Sales

In the facility location reformulation, a new variable Z_{ikt} denotes the quantity of product *i* produced in period *k* to satisfy demand in period *t* ($\forall i \in$ N, $\forall t \in$ T, $\forall k \in$ T). In the basic formulation where we assume no backlog, the following relationships hold:

$$X_{it} = \sum\nolimits_{k=t}^{m} Z_{itk}$$

$$d_{it} = \sum\nolimits_{k=1}^{t} Z_{ikt}$$

The first equality states that production in period *t* can only be used to satisfy demand in *t* or in a later period. While the second equality states that all demand must be satisfied by early or on-time production during the time horizon.

The facility location reformulation of the capacitated lot-sizing problem is written as follows:

FLR

$$\text{Min} \quad \sum_{i \in N} \sum_{t \in T} \sum_{k=1}^{t} VC_{ik} Z_{ikt} + \sum_{i \in N} \sum_{t \in T} SC_{it} \, Y_{it} + \sum_{i \in N} \sum_{t \in T} \sum_{k=1}^{t} \left(\sum_{l=k}^{t-1} HC_{il} \right) Z_{ikt} \tag{21}$$

$$S.T. \quad \sum\nolimits_{k=1}^{t} Z_{ikt} = d_{it} \qquad \forall i \in \text{N}, \forall t \in \text{T} \ (22)$$

$$Z_{ikt} \leq d_{it} Y_{ik} \qquad \forall i \in \text{N}, \forall t \in \text{T}, \forall k \in \text{T} \mid k \leq t \ (23)$$

$$\sum\nolimits_{i \in N} \sum\nolimits_{l=t}^{m} UT_{it} \, Z_{itl} + \sum\nolimits_{i \in N} ST_{it} Y_{it} \leq Cap_t \qquad \forall t \in \text{T} \ (24)$$

$$Y_{it} \in \{0,1\} \qquad \forall i \in \text{N}, \forall t \in \text{T} \ (25)$$

$$Z_{ikt} \geq 0 \qquad \forall i \in \text{N}, \forall t \in \text{T}, \forall k \in \text{T} \quad (26)$$

In the facility location reformulation, the objective function and constraints have similar interpretations as in the standard CLSP model. The objective function (21) minimizes the sum of the unit production cost, setup cost and inventory cost. Constraints (22) ensure that all demand must be satisfied. Constraints (23) are the setup constraints, while constraints (24) are the capacity constraints. Constraints (25) and (26) are the binary and non-negative constraints respectively.

The Z_{ikt} variable contains the information of which item is produced, when it was produced, and when it satisfied demand. With this variable, it is now possible to differentiate between backlog and backorders (Gruson *et al.*, 2018). At the same time, this variable eliminates the need for the production (X_{it}), inventory (I_{it}), and backlog (B_{it}) variables that were needed in the CLSP formulation. Instead, these variables can be identified by the configuration of the *k*, *t* indices. When *k* = *t*, Z_{ikt} refers to the number of items *i* produced and satisfied in period *t*. When $k < t$, Z_{ikt} equals the number of products made in period *k* that have been held in inventory for $(t - k)$ periods, until its demand was satisfied in period *t*. For example, Z_{i13} indicates the number of units of item *i* that were produced in period 1 and consumed in period 3. From this, we can gather that these units were held in inventory for two periods and the associated holding cost can be calculated.

3.6 Facility Location Reformulation with Backlog

If backlog is considered, demand for period *t* can now be satisfied by production in any period *k*. This leads to the following relationships:

$$X_{it} = \sum_{k=1}^{m} Z_{itk}$$

$$d_{it} = \sum_{k=1}^{m} Z_{ikt}$$

When $k > t$, Z_{ikt} equals the number of products made in period *k* that were backordered in period *t* and have been backlogged for $(k-t)$ periods. A backlog cost BC_{it} is added to the objective function for each period an item is backlogged for. With backlog considered the resulting formulation is as follows:

FLR-B

$$\text{Min} \quad \sum_{i\in N}\sum_{t\in T}\sum_{k=1}^{t} VC_{ik}Z_{ikt} + \sum_{i\in N}\sum_{t\in T} SC_{it}\,Y_{it} + \sum_{i\in N}\sum_{t=1}^{m}\sum_{k=1}^{t}\left(\sum_{l=k}^{t-1} HC_{il}\right)Z_{ikt}$$
$$+\sum_{i\in N}\sum_{t=1}^{m-1}\sum_{k=t+1}^{m}\left(\sum_{l=t}^{k-1} BC_{il}\right)Z_{ikt} \qquad (27)$$

$S.T.$ $(25), (26), and$

$$\textstyle\sum_{k=1}^{m} Z_{ikt} = d_{it} \qquad \forall i\in \text{N}, \forall t\in \text{T} \ (28)$$

$$Z_{ikt} \leq d_{it}Y_{ik} \qquad \forall i\in \text{N}, \forall t\in \text{T}, \forall k\in \text{T} \ (29)$$

$$\textstyle\sum_{i\in N}\sum_{l=1}^{m} UT_{it}\, Z_{itl} + \sum_{i\in N} ST_{it}Y_{it} \leq Cap_t \qquad \forall t\in \text{T} \ (30)$$

The objective function (27) now includes the sum of the backlog costs. Constraints (22), (23), and (24) are changed into constraints (28), (29), and (30) to include the possibility of satisfying a demand from any period. Since in constraints (28) all demand must be satisfied within the time horizon, there is no backlog possible at the end of the horizon and therefore makes the previous $B_{im} = 0$ constraints obsolete.

As the cost function for each item depends on *when* they were produced, we can simplify the cost coefficient as follows:

$$C_{ikt} = \begin{cases} \left(VC_{ik} + \sum_{l=k}^{t-1} HC_{il}\right) & if\ k < t \\ VC_{ik} & if\ k = t \\ \left(VC_{ik} + \sum_{l=t}^{k-1} BC_{il}\right) & if\ k > t \end{cases}$$

With the new cost coefficient the objective function can now be written as follows:

$$\text{Min} \quad \sum_{i \in N} \sum_{t \in T} \left(SC_{it} Y_{it} + \sum_{k=1}^{m} C_{ikt}\, Z_{ikt} \right) \tag{31}$$

Moving forward in this thesis, we build upon the above formulation to simplify the notations.

Up to this point we have considered that backlog has no restrictions, and therefore demand can be satisfied during any period in the time horizon. We can add a restriction on the number of periods an item can be backlogged for. Let *r* be the maximum number of periods an item can be backlogged for. We define the set of R as R={1,…, r}. The stocked-out items that become backlogged will now be restricted to *r* periods before they must be satisfied. The resulting model would be rewritten as follows:

FLR-BR

Min (31)

$$S.T. \quad (25), (26), and$$

$$\sum_{k=1}^{\min\{m,t+r\}} Z_{ikt} = d_{it} \qquad \forall i \in \mathrm{N}, \forall t \in \mathrm{T} \quad (32)$$

$$Z_{ikt} \leq d_{it} Y_{ik} \qquad \forall i \in \mathrm{N}, \forall t \in \mathrm{T}, \forall k \in \mathrm{T} \mid 1 \leq k \leq min\{m, t+r\} \quad (33)$$

$$\sum_{i \in N} \sum_{l=\max\{t-r,1\}}^{m} UT_{it} Z_{itl} + \sum_{i \in N} ST_{it} Y_{it} \leq Cap_t \qquad \forall t \in \mathrm{T} \quad (34)$$

Constraints (32) are the balancing constraints which now restrict backlog to *r* periods. Constraints (33) are the altered setup constraints, while constraints (34) enforce the capacity restrictions.

The restricted backlog assumptions will be used when introducing our later models in section 3.8 and 3.9 which consider both backlog and lost sales simultaneously. However, an explanation on how to formulate the new problems with unlimited backlog will be provided as well.

3.7 Facility Location Reformulation with Lost Sales

In order to develop the facility location reformulation that considers lost sales, we return to a situation where backlog is prohibited and all demand that is not satisfied on time becomes lost sales. The resulting formulation is written as follows:

FLR-LS

$$\text{Min} \quad \sum_{i \in N} \sum_{t \in T} \left(SC_{it} Y_{it} + LC_{it} L_{it} + \sum_{k=1}^{t} C_{ikt} \, Z_{ikt} \right) \tag{35}$$

$S.T.$ $(23), (24), (25), and$

$$\textstyle\sum_{k=1}^{t} Z_{ikt} + L_{it} = d_{it} \qquad \forall i \in \text{N}, \forall t \in \text{T} \tag{36}$$

$$Z_{ikt}, L_{it} \geq 0 \qquad \forall i \in \text{N}, \forall t \in \text{T}, \forall k \in \text{T} \tag{37}$$

The objective function (35) aims to minimize the sum of all the costs (production, holding, and setup) which now also include the cost for lost sales. It is important to note that since backlog is not allowed ($k \leq t$), C_{ikt} consists of only two cost possibilities: on-time production ($k = t$), and a product that has been held in inventory ($k < t$). The new balancing constraints (36) no longer restrict all demand to be satisfied by production. They now allow the possibility of demand being lost if not satisfied on time. Constraints (37) are the new non-negativity constraints which include the lost sales decision variable. Formulation FLR-LS shows that adding lost sales without the possibility of backlog can be easily formulated. Once you consider both options simultaneously the formulation becomes more complex, and several variants are possible depending on the assumption with respect to the relationship between lost sales and backlog.

4. Facility Location Reformulation with Simultaneous Backlog and Lost Sales

The goal of this thesis is to develop a formulation that can solve a multi-item lot-sizing problem that considers both backlog and lost sales. In addition, we aim to test its performance and efficiency versus similar formulations for comparative analysis. We formulate several variants for this new problem with different assumptions. Table 3 summarizes all the formulations that will be examined for our thesis. Note that the specifics of all listed formulations in Table 3 either have already been reviewed or will be explained in the later sections.

	No Lost Sales	Fixed Lost Sales	Variable Lost Sales
No Backlog	FLR	FLR-LS	N.A.
Unlimited Backlog	FLR-B	FLR-B-FL	FLR-B-VL
Restricted Backlog	FLR-BR	FLR-BR-FL	FLR-BR-VL
Backlog with Multiple Customer Types	FLR-BM	FLR-BM-FL	FLR-BM-VL

Table 3. Formulation Categories for Multi-item CLSP

The basic lot-sizing model without backlog or lost sales is modeled as a facility location reformulation (FLR).

As shown in Table 3, with respect to the lost sales, there are three variants of multi-item capacitated lot-sizing formulations that we examine:

1. No lost sales,
2. Fixed lost sales,
3. Variable lost sales.

Both fixed and variable lost sales describe a scenario where in case a stock-out occurs, a fixed percentage of the unsatisfied demand automatically becomes lost sales. In a model with variable lost sales, the fixed percentage of unsatisfied demand is however a minimum and additional lost sales can be incurred if it is better for the company's interest. The concepts of fixed versus variable sales will be further explained in Sections 4.1 and 4.2.

There are also four different assumptions considered for backlog:

1. Backlog is forbidden,
2. The number of periods an item can be backlogged for is unlimited,
3. The number of periods an item can be backlogged for is restricted,
4. Backlog restrictions are dependent on multiple customer types.

The case of multiple customer types describes a scenario where there are multiple customer types each with a different willingness to wait for stocked-out demand. This assumption of backlog with multiple customer types is new in the literature and extends the concept of the traditional single customer type. The concept of multiple customer types is further explained in Section 4.3.

All formulations listed in Table 3 are similar in structure. In fact, all formulations can be derived from the FLR-BM-FL formulation by either removing constraints, altering constraints, or changing a parameter. A detailed explanation on how to reverse engineer the FLR-BM-FL formulation into the others will be provided in Section 4.4. Note that it is not possible to have a formulation without backlog and with variable lost sales. The formulations that only consider one of the two concepts (backlog and lost sales) such as FLR-LS, FLR-B, and FLR-BR, have already been reviewed in our previous sections. In our next sections we propose new formulations that consider both backlog and lost sales simultaneously. Furthermore, we explore these new formulation using different assumptions on backlog.

4.1 Facility Location Reformulation with Simultaneous Backlog and Fixed Lost Sales

When considering lost sales with the possibility of backlog, there are two different types that can be explored: *fixed lost sales* and *variable lost sales*. In the case of fixed lost sales, we assume that when a stock-out occurs in period *t*, a fixed percentage of the unsatisfied demand will result in lost sales while the rest remains as backlog until that demand is satisfied. The interpretation of fixed lost sales is valid in case a company estimates that α percent of the customers who are faced with a stock-out are willing to wait for their item, while (1 – α) is not willing to wait, and their demand will be lost if there is a stock-out.

We first define S_{it} as the number of items *i* stocked-out in period *t.* A stock-out for a specific item *i* and period *t* is the amount of demand d_{it} that is not satisfied on time and is calculated as follows: $S_{it} = (\sum_{k=t+1}^{\min\{m,t+r\}} Z_{ikt} + L_{it})$. In addition, we let α be the percentage of stock-out for a specific period that will remain as backlog, and let (1-α) be the remainder of the stock-out that becomes lost sales. The backlogged stock-outs are, however, restricted to *r* periods of backlog. The resulting formulation with restricted backlog and fixed lost sales is as follows:

FLR-BR-FL

$$\text{Min} \quad \sum_{i=1}^{N}\sum_{t=1}^{m}\left(SC_{it}Y_{it} + LC_{it}L_{it} + \sum_{k=1}^{\min\{m,t+r\}} C_{ikt}\,Z_{ikt}\right) \tag{38}$$

$$S.T. \quad (25), (33), (34), (37), and$$

$$\sum_{k=1}^{\min\{m,t+r\}} Z_{ikt} + L_{it} = d_{it} \qquad \forall i \in \text{N}, \forall t \in \text{T} \tag{39}$$

$$L_{it} = (1-\alpha)(\sum_{k=t+1}^{\min\{m,t+r\}} Z_{ikt} + L_{it}) \qquad \forall i \in \text{N}, \forall t \in \text{T} \tag{40}$$

The objective function (38) minimizes the sum of production cost, holding cost, setup cost, backlog cost, and cost of lost sales. Constraints (39) ensures that demand is either satisfied by production or becomes lost sales. Constraints (40) sets $(1-\alpha)$ percentage of the stocked-out demand in period *t* as lost sales. If we look at this constraint more closely, $\sum_{k=t+1}^{\min\{m,t+r\}} Z_{ikt}$ represents backorders for demand in period *t* according to the definition in Gruson *et al.* (2018). When we take $(\sum_{k=t+1}^{\min\{m,t+r\}} Z_{ikt} + L_{it})$ altogether, it represent the total stocked-out demand in period *t*, consisting of the amount backordered and the lost sales.

The case where there are no restrictions on backlog can also be modeled. This type of formulation has the objective function and constraints written as follows:

FLR-B-FL

$$\text{Min} \quad \sum_{i=1}^{N}\sum_{t=1}^{m}\left(SC_{it}Y_{it} + LC_{it}L_{it} + \sum_{k=1}^{m} C_{ikt}\, Z_{ikt}\right) \tag{41}$$

$S.T.$ $(25), (30), (37), and$

$$\sum_{k=1}^{m} Z_{ikt} + L_{it} = d_{it} \qquad \forall i \in \mathrm{N}, \forall t \in \mathrm{T} \tag{42}$$

$$Z_{ikt} \le d_{it} Y_{ik} \qquad \forall i \in \mathrm{N}, \forall t \in \mathrm{T}, \forall k \in \mathrm{T} \tag{43}$$

$$L_{it} = (1-\alpha)(\sum_{k=t+1}^{m} Z_{ikt} + L_{it}) \qquad \forall i \in \mathrm{N}, \forall t \in \mathrm{T} \tag{44}$$

The objective function (41) is the same as in FLR-BR-FL, but without the backlog restriction of *r* periods. Constraints (42) are the demand balancing constraints that enable demand to be satisfied by any period in the time horizon. Constraints (43) are the setup constraints.

Constraints (44) are the lost sales constraints which force a fixed percentage of the stock-outs to become lost sales.

Note that as an alternative, model FLR-B-FL can be obtained from model FLR-BR-FL by setting $r = m$-1.

To help clarify the fixed lost sales models, we review a small example with two items (n = 2) and a planning horizon of four periods (m = 4). For the purpose of this example, we use the FLR-B-FL formulation which does not impose any backlog restrictions. In Tables 4 and 5, we lay out the parameters. In this example, the costs parameters may change depending on the item, but remain the same during each period. An instance where costs change depending on the period could also be solved by our formulation. As a final note, we set α = 50%.

Parameters	**Item 1 (i=1)**	**Item 2 (i=2)**
Variable Cost (VC_{it})	0	0
Holding Cost (HC_{it})	1	1
Setup Cost (SC_{it})	25	50
Lost Sales Cost (LC_{it})	12	12
Backlog Cost (BC_{it})	3	3
Unit Production Time (UT_{it})	1	3
Setup Time (ST_{it})	2	5

Table 4. List of Values for Cost and Time Parameters

Parameters	**Items (i)**	**Period 1 (t=1)**	**Period 2 (t=2)**	**Period 3 (t=3)**	**Period 4 (t=4)**
Demand (d_{it})	i=1	10	5	10	10
	i=2	2	2	6	3
Capacity (Cap_t)		20	30	25	35
Percentage of fixed lost sales (1-α)		50%			

Table 5. List of Values for Demand and Capacity

Once we solve this dataset using CPLEX 12.6.3.0, we obtain the optimal objective function of 219. We can break down our solution by examining the values of our decision variables. Table 6 summarizes the solution of item 1, while Table 7 reviews the solution of item 2.

Z_{ikt} Variables	**t=1**	**t=2**	**t=3**	**t=4**	**Total Cost**
Y_{1t}	1	1	1	0	25+25+25=75
Z_{11t}	10	0	0	0	0
Z_{12t}	0	15	0	0	0
Z_{13t}	0	0	10	10	10
Z_{14t}	0	0	0	0	0
L_{1t}	0	0	0	0	0
S_{1t}	0	0	0	0	
d_{1t}	10	15	10	10	
Objective Function					85

Table 6. Solution for the Decision Variables for Item 1 using FLR-B-FL

Z_{ikt} Variables	t=1	t=2	t=3	t=4	Total Cost
Y_{2t}	0	0	0	1	50
Z_{21t}	0	0	0	0	0
Z_{22t}	0	0	0	0	0
Z_{23t}	0	0	0	0	0
Z_{24t}	1	1	3	3	9+6+9=24
L_{2t}	1	1	3	0	60
S_{2t}	2	2	6	0	
d_{2t}	2	2	6	3	
Objective Function					134

Table 7. Solution for the Decision Variables for Item 2 using FLR-B-FL

In this example, we have item 1 which has a low setup cost and item 2 which has a higher setup cost. The low setup cost encourages more frequent production in order to avoid extra costs for inventory, backlog, and lost sales. Notice that item 1 never incurs lost sales. It is optimal for item 1 to be produced on-time in almost every period. The sole reason for the demand of period 4 being produced in period 3 is that the capacity is needed in period 4 for item 2. With regards to item 2, it is more economical to produce all the demand over the horizon in one period because of its large setup cost. Production would rather receive the additional penalties for backlog and lost sales then pay for the high setup cost. The lost sales and backlog penalties are also less detrimental for item 2 because the demand in units is relatively low. If we refer to Table 7, all on-time production is located in the cell where $k = t$. These cells result in no costs other than the initial cost to set up production for that given period. All early production (where products were held in inventory for one or more periods) are shown in the cells where $k < t$. For the cells where $k > t$, they indicate the backlogged demand that was produced the period k.

Backlogged demand incurs a backlog cost for every period that demand is not satisfied for. Let's take Z_{24t} in Table 7 as an example. The calculations for the backlog cost related to the production in period 4 are conducted as follows:

$$((BC_{23} + BC_{22} + BC_{21})Z_{241}) + ((BC_{23} + BC_{22})Z_{242}) + ((BC_{23})Z_{243})$$

$$\big((9) * 1\big) + \big((6) * 1\big) + \big((3) * 3\big) = 24$$

In the FLR-B-FL model, having backlog indicates that there was a stock-out. For each period t that a stock-out occurs, a percentage of the unmet demand in period (S_{it}) is lost. In this case (1-α) is 50 percent. The complete demand of 2 units in period 1 is stocked-out, which means one unit of d_{21}becomes lost sales ($L_{21} = 1$), and 1 unit is backlogged ($Z_{241} = 1$) as indicated in Table 7. For all periods in which stock-outs occur, the stock-out equals the number of lost sales plus the total number of backordered items produced in later periods. Using the stock-out example for item 2 in period 1, the equation is as follows:

$$S_{21} = L_{21} + Z_{221} + Z_{231} + Z_{241}$$

$$= 1 + 0 + 0 + 1 = 2$$

To summarize, this example demonstrates how the FLR-B-FL model functions. The Z variables are used to indicate when demand is satisfied by production. If ever the demand in period t cannot be satisfied on time, (1 - α) percent of the stocked-out items in period t is lost while the rest must be satisfied from production in later periods. If we place a time restriction on backlogged items as we do in the FLR-BR-FL formulation, then the backlogged item must be satisfied in the next r periods. For instance, if we set r = 2 in our previous example, our solution from Tables 6 and 7 is no longer valid. Variable Z_{241} has a value of one indicating that one unit of item 2 is produced in period 4 to satisfy demand in period 1. Since this one unit is backlogged for three periods, the solution is not valid and a new one must be generated. To illustrate the effect that backlog restrictions can have, we set *r* = 2 and solve the instances used in our previous example using the FLR-BR-FL formulation. The solution is summarized in Table 8 and 9.

Z_{ikt} Variables	**t=1**	**t=2**	**t=3**	**t=4**	**Total Cost**
Y_{1t}	1	1	0	1	25+25+25=75
Z_{11t}	10	0	0	0	0
Z_{12t}	0	15	10	0	10
Z_{13t}	0	0	0	0	0
Z_{14t}	0	0	0	10	0
L_{1t}	0	0	0	0	0
S_{1t}	0	0	0	0	
d_{1t}	10	15	10	10	
Objective Function					85

Table 8. Solution for the Decision Variables for Item 1 using FLR-BR-FL

Z_{ikt} Variables	**t=1**	**t=2**	**t=3**	**t=4**	**Total Cost**
Y_{2t}	0	0	1	1	50+50=100
Z_{21t}	0	0	0	0	0
Z_{22t}	0	0	0	0	0
Z_{23t}	1	0	5.667	0	6
Z_{24t}	0	1	0.167	3	6+0.5=6.5
L_{2t}	1	1	0.167	0	12+12+2=26
S_{2t}	2	2	0.333	0	
d_{2t}	2	2	6	3	
Objective Function					138.5

Table 9. Solution for the Decision Variables for Item 2 using FLR-BR-FL

The new objective function is 223.5, which is 4.5 more than the objective function for FLR-B-FL. Although the objective function increased by only a small amount, the solution is very different. The production plan for item 1 only has one change: demand for period 3 was produced in period 2 in order to leave available capacity in period 3 for item 2. As for item 2, the production plan added a second production setup to satisfy the backlog restriction. In period 3, the firm is forced to produce the backlogged demand from period 1. The rest of the capacity was spent satisfying demand for period 3. In period 4, the remaining backlogged demand and the demand for period 4 were produced. The production plan for item 2 using FLR-BR-FL had an additional setup. However, it incurred less lost sales than FLR-B-FL. From this example we notice the impact that a backlog restriction can have on the optimal solution. FLR-B-FL is a more flexible formulation than FLR-BR-FL, and therefore may provide better solutions. The example also illustrates a limitation of the model. In the lot-sizing literature, the production quantity is typically modeled as a continuous variable (see Pochet and Wolsey, 2006). As a consequence, the inventory, backlog, and lost sales variables are also continuous variables, and can hence take fractional values. This is a typical assumption in the standard lot-sizing models (Absi *et al*. 2011, Pochet and Wolsey 1988, Gade and Küçükyavuz 2013).

4.2 Facility Location Reformulation with Simultaneous Backlog and Variable Lost Sales

The second type of lost sales is *variable* lost sales. When a stock-out occurs for the demand in period *t*, a *minimum* percentage of the unsatisfied demand will result in lost sales while the rest remains as backlog until that demand is satisfied. The difference of the variable lost sales compared to the fixed lost sales is that the percentage of the stocked-out demand that will result in lost sales is a minimum and therefore the firm has the option to incur more lost sales if this provides a better solution. It sounds contradictory, however it may be optimal to incur more lost sales in times of high demand and strict capacity levels. Additional lost sales for some products can be beneficial if there are products with different characteristics in terms of backlog cost, lost sales cost, capacity usage, etc. The ability to take on additional lost sales also ensures the existence of a feasible solution. The interpretation of variable lost sales is valid in the same case as fixed lost sales, but with the addition that the company can choose that in case of a stock-out, the demand of some customers that are willing to wait will instead not be satisfied.

The resulting formulation for variable lost sales is the same as the fixed lost sales formulation (FLR-BR-FL), except for a minor adjustment to the lost sales constraints (40). The FLR-BR-VL formulation is written as follows:

FLR-BR-VL

Min (38)

$S.T.\quad (25), (33), (34), (37), (39)\ and$

$$L_{it} \geq (1-\alpha)(\sum_{k=t+1}^{\min\{m,t+r\}} Z_{ikt} + L_{it}) \qquad \forall i \in \mathrm{N}, \forall t \in \mathrm{T} \quad (45)$$

Constraints (45) is no longer an equality and instead is an inequality which allows the possibility to have more than the minimum lost sales required for every stock-out.

This variable lost sales formulation can also be written under the assumption that there is no restriction on backlog. The FLR-B-VL formulation would be constructed as follows:

FLR-B-VL

Min (41)

$S.T.\quad (25), (30), (37), (42), (43), and$

$$L_{it} \geq (1-\alpha)(\sum_{k=t+1}^{m} Z_{ikt} + L_{it}) \qquad \forall i \in \mathrm{N}, \forall t \in \mathrm{T} \quad (46)$$

Constraints (46) serve the same purpose as constraints (45), but do not consider backlog restrictions.

In essence, the fixed and variable lost sales formulations are similar with a minor adjustment to the lost sales constraints. However this adjustment has an impact on the formulation's ability to provide feasible solutions. For instance, with fixed lost sales, some instances might be infeasible if the capacity is not sufficient, whereas with variable lost sales, the company always has the possibility to have all the demand as lost sale, and hence the problem can never be infeasible.

4.3 Facility Location Reformulation with Multiple Customer Types

Up to this point, we have covered three types of reformulations with respect to the lost sales: FLR without lost sales, FLR with fixed lost sales, and FLR with variable lost sales. In addition, we have reviewed these formulations under different assumptions on backlog: no backlog, unlimited backlog, and restricted backlog. In all cases, we assume that customers share the same behaviour when faced with backlog. Specifically, it was assumed that customers that are backlogged all have the same maximum number of periods that they are willing to wait, which can be either restricted or unrestricted. In this section, we develop a formulation that assumes multiple customer types with different behaviours when faced with backlog. The customer behaviour is characterized by the maximum number of periods a customer is willing to wait.

In the case of multiple customer types, customers are assumed to have different tolerances for the maximum length of backlog. To incorporate this into our model, we redefine R as the set of possible maximum backlogged periods a customer is willing to wait $\{1, \ldots, r\}$. In addition, let B_q define the percentage of customers willing to wait a maximum of q periods, $\forall q \in$ R. We enforce a condition which sets $\sum_{q \in R} B_q = \alpha$. This condition ensures that the total percentage of customers willing to backlog are represented by one of the customer types. The percentage of customers not willing to wait for any periods of backlog and becoming lost sales when a stock-out occurs can be defined as $(1 - \sum_{q \in R} B_q)$.

With the addition of multiple customer types, the definition of the stock-out variable S_{it} which represents the number of item i stocked-out in period t remains the same as before: $S_{it} = (\sum_{k=t+1}^{\min\{m,t+r\}} Z_{ikt} + L_{it})$. In this model with multiple customer types, once a stock-out occurs, the unsatisfied demand will either be lost or be backlogged up to a maximum of r periods. Every customer type has a maximum number of periods they are willing to wait for backlog. For instance if a stock-out has occurred and $B_1 = 0.5$, then 50 percent of the unsatisfied customers are willing to wait a maximum of one period of backlog. If these customers' demand are not satisfied within one period of backlog, the firm will incur lost sales. If a stock-out has occurred with $B_1 = 0.5$ and $B_2 = 0.25$, then the 25 percent of customers who are willing to wait at most 2 periods along with the 50 percent of customers willing to wait one period may all be satisfied after one period of backlog.

To illustrate the multiple customer type assumption, we consider a problem with backlog and *fixed* lost sales. The FLR-BM-FL formulation has the same objective function (38) found in the FLR-BR-FL formulation, while its constraints are written as follows:

FLR-BM-FL

Min (38)

$$S.T. \quad (25), (33), (34), (37), (39), (40), and$$

$$\sum_{q=l}^{\min\{r,m-t\}} Z_{i,t+q,t} \leq \sum_{q=l}^{\min\{r,m-t\}} B_q * S_{it}$$

$$\forall i \in \mathrm{N}, \forall t \in \mathrm{T}, \forall l \mid 1 \leq l \leq \min(r, m\text{-}t) \quad (47)$$

Constraints (47) ensure that the items that are backlogged for at least *l* periods are restricted by the number of customers that are willing to wait for *l* periods or more. The customers willing to wait for *l* periods consist of all the customers willing to wait for a maximum of *l* periods or longer. To help clarify, we break down the set of constraints (47) using an example with one item, three customer types (i.e. with a maximum willingness to wait for 1, 2, and 3 periods) and four periods in the horizon. The constraints for a stock-out for item 1 related to the demand for period 1 are listed below:

$$L_{11} = S_{11} * ((1 - (B_1 + B_2 + B_3)) \tag{48}$$

$$L_{11} + Z_{121} + Z_{131} + Z_{141} = S_{11} \tag{49}$$

$$Z_{121} + Z_{131} + Z_{141} \leq (B_1 + B_2 + B_3) * S_{11} \qquad l = 1 \tag{50}$$

$$Z_{131} + Z_{141} \leq (B_2 + B_3) * S_{11} \qquad l = 2 \tag{51}$$

$$Z_{141} \leq B_3 * S_{11} \qquad l = 3 \tag{52}$$

Constraint (48) sets the number of lost sales equal to the percentage of customers not willing to wait for backlogged demand in case of a stock-out. Constraint (49) ensures that the stocked-out demand is either satisfied by production in the following three periods or becomes lost sales. Constraint (50) restricts the total backlogged demand to the cumulative number of all customers willing to wait at least one period. Constraint (51) restricts demand backlogged for 2 or 3 periods to the number of customers willing to wait at least two periods. Constraint (52) limits the demand backlogged for 3 periods to the number of customers willing to wait for a maximum of 3 periods.

To explain the FLR-BM-FL formulation as a whole, we return to the example in section 3.8. Our problem still has two items (n=2) and four periods (m=4), however we restrict the number of periods an item can be backlogged to two periods (r=2). The α variable is no longer relevant in this model. For the models with multiple customer types, we assign percentages (B_q) to each of the possible backlogged periods to denote what percentage of the customers are willing to wait maximum q periods. Let $B_1 = 0.30$ and $B_2 = 0.20$, while the remaining stocked-out items will be lost immediately. There is a maximum of 50 percent of customers willing to wait for one period, and a maximum of 20 percent are willing to wait two periods. This also signifies that for every stock-out, 50 percent of the stocked-out demand will be lost. Using the same parameters as in Table 4 and 5, our optimal solution for model FLR-BM-FL is as follows:

Z_{ikt} Variables	**t=1**	**t=2**	**t=3**	**t=4**	**Total Cost**
Y_{1t}	1	1	0	1	25+25+25=75
Z_{11t}	10	8	0	0	8
Z_{12t}	0	7	10	0	10
Z_{13t}	0	0	0	0	0
Z_{14t}	0	0	0	10	0
L_{1t}	0	0	0	0	0
S_{1t}	0	0	0	0	
d_{1t}	10	15	10	10	
Objective Function					93

Table 10. Solution for the Decision Variables for Item 1 using FLR-BM-FL

Z_{ikt} Variables	**t=1**	**t=2**	**t=3**	**t=4**	**Total Cost**
Y_{2t}	0	1	1	1	50+50+50=150
Z_{21t}	0	0	0	0	0
Z_{22t}	0.6	1.4	0	0	1.8
Z_{23t}	0.4	0.267	6	0	3.2
Z_{24t}	0	0.033	0	3	0.2
L_{2t}	1	0.3	0	0	15.6
S_{2t}	2	0.6	0	0	
d_{2t}	2	2	6	3	
Objective Function					170.8

Table 11. Solution for the Decision Variables for Item 2 using FLR-BM-FL

The optimal objective function of this example is 263.8, which is 40.3 higher than for FLR-BR-FL and 44.8 higher than for FLR-B-FL. The solution for item 1 using the FLR-BM-FL formulation exhibits a similar solution to the previously used FLR-BR-FL formulation. The minor changes are caused by the required capacity needed for item 2. The optimal solution for item 2 uses three production setups in periods 2, 3, and 4.

The effects of the multiple customer types are apparent in period 1 for item 2. In period 1, there is a stock-out of 2 units for item 2. In this situation the constraints for stocked-out demand in period 1 of item 2 consists of the following;

$$L_{21} = S_{21} * ((1 - (B_1 + B_2)) \tag{53}$$

$$L_{21} + Z_{221} + Z_{231} = S_{21} \tag{54}$$

$$Z_{221} + Z_{231} \leq (B_1 + B_2) * S_{21} \tag{55}$$

$$Z_{231} \leq B_2 * S_{21} \tag{56}$$

If we input all the values for this example, constraint (53) determines that for any period where there are stock-outs, fifty percent of them are lost immediately. Constraint (54) ensures that all stocked-out items either become backlog and are satisfied in a later period or become lost sales. As for the backlogged demand, constraint (55) shows that a maximum of fifty percent of the stocked-out items in period 1 can be satisfied in period 2 or 3. Constraint (56) ensures that a maximum of twenty percent of the stocked-out items in period 1 can be satisfied in period 3.

Using this example, we review how the optimal solution respects the multiple customer constraints. For item 2 a stock-out of 2 items occurs in period 1 ($S_{21} = 2$). By inserting the values into constraint (53) we obtain $L_{21} = 1$, which matches that of the optimal solution. The optimal solution in Tables 8 and 9 has $Z_{221} = 0.6$ and $Z_{231} = 0.4$. When the values are added to constraint (54), the right hand and left hand side of the equation are equal to each other. Constraints (55) and (56) are also respected when the value are inserted in the inequalities. For item 2 in period 2, a second stock-out occurs for 0.6 units. By using constraint 48, we can validate that $L_{22} = 0.3$. Upon inspection, the other three constraints are satisfied.

To summarize, this example highlights the similarities and differences between the formulations with multiple customer types and a single customer type, while illustrating the trade-offs that occur between lost sales and backlog.

A variable version of the formulation can be constructed by changing the *lost sales* constraints into an inequality (≥). To simplify the formulation, the *lost sales* constraints (46) can be removed altogether because it is implicitly stated in the *multiple customer type* constraints (47). The formulation includes objective function (38) and the following constraints;

FLR-BM-VL

Min (38)

$$S.T. \quad (25), (33), (34), (37), (39), (45), and (47)$$

The multiple customer types constraints (47) with *l* = 1 state that the total backlogged items for demand in period *t* is restricted by $\sum_{q \in R} \beta_q * S_{it}$. Since we assume that $\sum_{q \in R} \beta_q = \alpha$, the lost sales constraints (46) are redundant and therefore we remove them. The lost sales constraints (46) are there to set a minimum number of stocked-out items as lost sales. However the minimum lost sales is already enforced in the multiple customer types constraints as $\sum_{q \in R} \beta_q$ defines the maximum percentage of customers willing to wait for backlogged demand in case of a stock-out. Constraint (46) can be included in the formulation, but this will have no impact on the optimal solution.

4.4 Relationships between Formulations

Amongst the formulations we cover, there are some a priori relationships between the optimal objection function values that can be identified. Under the same backlog assumption, a relationship exist between formulations with different types of lost sales. Given the difference in the *lost sales constraints,* the variable lost sales formulations will always have an optimal objective function value less than or equal to the value of the fixed lost sales version of the formulations. It is important to clarify that these relationships hold amongst formulations that share the same assumptions on backlog. We define V_f as the optimal objective function for formulation *f*. The equation below illustrates this relationship if we assume that backlog is restricted to *r* periods.

$$V_{FLR-BR-VL} \leq V_{FLR-BR-FL}$$

A relationship can also be found between formulations with different backlog assumptions if we consider formulations with the same type of lost sales. Given that formulations with unrestricted backlog provide more flexibility, the optimal objection function value will be less than or equal to formulations with restricted backlog. Furthermore, due to the relationship between the α and β parameters (i.e. $\alpha = \sum_{q \in R} \beta_q$), formulations with multiple customer types will have an optimal objective function value larger than or equal to those of formulations with restricted backlog assuming they have both the same maximum restriction. The equation below illustrates this relationship if we assume that the formulations all consider fixed lost sales:

$$V_{FLR-B-FL} \leq V_{FLR-BR-FL} \leq V_{FLR-BM-FL}$$

4.5 Summary of Facility Location Reformulations

Using the facility location reformulation, we were able to successfully develop formulations that consider both backlog and lost sales under different backlog and lost sales assumptions. If we take a closer look, all formulations reviewed in Sections 3.5 through 3.10 have similar structures. Using the FLR-BM-FL model as the base formulation, we are able to derive all the other formulations by making a few alterations. To help with the reengineering of the formulation, we first revisit the FLR-BM-FL formulation below:

FLR-BM-FL

$$\text{Min} \quad \sum_{i=1}^{N}\sum_{t=1}^{m}\left(SC_{it}Y_{it} + LC_{it}L_{it} + \sum_{k=1}^{\min\{m,t+r\}} C_{ikt}\, Z_{ikt}\right) \tag{57}$$

$$S.T. \quad \sum_{k=1}^{\min\{m,t+r\}} Z_{ikt} + L_{it} = d_{it} \qquad \forall i \in \text{N}, \forall t \in \text{T} \tag{58}$$

$$Z_{ikt} \le d_{it}Y_{ik} \qquad \forall i \in \text{N}, \forall t \in \text{T}, \forall k \in \text{T} \mid 1 \le k \le \min\{m, t+r\} \tag{59}$$

$$\sum_{i\in N}\sum_{l=t-r}^{m} UT_{it}\, Z_{itl} + \sum_{i\in N} ST_{it}Y_{it} \le Cap_t \qquad \forall t \in \text{T} \tag{60}$$

$$L_{it} = (1-\alpha)\left(\sum_{k=t+1}^{\min\{m,t+r\}} Z_{ikt} + L_{it}\right) \qquad \forall i \in \text{N}, \forall t \in \text{T} \tag{61}$$

$$\sum_{q=l}^{\min\{r,m-t\}} Z_{i,t+q,t} \le \sum_{q=l}^{\min\{r,m-t\}} B_q * S_{it}$$

$$\forall i \in \text{N}, \forall t \in \text{T}, \forall l \mid 1 \le l \le \min(r, m\text{-}t) \tag{62}$$

$$Z_{ikt}, L_{it} \ge 0 \qquad \forall i \in \text{N}, \forall t \in \text{T}, \forall k \in \text{T} \tag{63}$$

$$Y_{it} \in \{0,1\} \qquad \forall i \in \text{N}, \forall t \in \text{T} \tag{64}$$

The *objective function* (57) along with the *demand balancing constraints* (58), the *setup constraints* (59), and the *capacity constraints* (60) are integral components to the basic structure of the facility location reformulation. These constraints exist in all the formulation, although possibly in a slightly different form. The *lost sales constraints* (61) exist in formulations with both backlog and lost sales and serves as a restriction on the number of stocked-out items that can be backlogged. The α parameter represents the percentage of stocked-out items that can be backlogged. By setting the parameter at different values we are able to create various other formulations. The *multiple customer type constraints* (62) exist in formulations that consider multiple customer types with different levels of patience when faced with backlog. The B_q parameter represents the percentage of customers that are willing to wait a maximum number of q periods for backlogged items. By setting the parameter at different values we are able to create various other formulations. Lastly, there are *non-negativity* (63) and *binary constraints* (64) which are essential to the formulations.

We are able to derive all formulations from the FLR-BM-FL formulation by either removing sets of constraints, restricting the parameters, or deleting variables. Table 12 shows the alterations necessary to derive each formulation.

	No Lost Sales	Fixed Lost Sales	Variable Lost Sales
No Backlog	**FLR** Set L_{it} = 0 $Z_{ikt} = 0$, if $k > t$ *Consequence:* No (61), (62)	**FLR-LS** Set α = 0% in (61) $Z_{ikt} = 0$, if $k > t$ *Consequence:* No (62)	N.A.
Unlimited Backlog	**FLR-B** $r = m$-1 $B_q = \alpha\ for\ q = r$ $B_q = 0\ for\ q < r$ $\alpha = \sum_{q \in R} B_q = 100\%$ *Consequence:* No L_{it} in (57) and(58) No (61), (62)	**FLR-B-FL** $r = m$-1 $B_q = \alpha\ for\ q = r$ $B_q = 0\ for\ q < r$ *Consequence:* No (62)	**FLR-B-VL** $L_{it} \geq (1\text{-}\alpha)^* S_{it}$ in (61) $r = m$-1 $B_q = \alpha\ for\ q = r$ $B_q = 0\ for\ q < r$ *Consequence* No (62)
Restricted Backlog	**FLR-BR** $B_q = \alpha\ for\ q = r$ $B_q = 0\ for\ q < r$ $\alpha = \sum_{q \in R} B_q = 100\%$ *Consequences:* No L_{it} in (57) and (58) No (61), (62)	**FLR-BR-FL** $B_q = \alpha\ for\ q = r$ $B_q = 0\ for\ q < r$	**FLR-BR-VL** $L_{it} \geq (1\text{-}\alpha)^* S_{it}$ in (61) $B_q = \alpha\ for\ q = r$ $B_q = 0\ for\ q < r$
Multiple Customer Types	**FLR-BM** $\alpha = \sum_{q \in R} B_q = 100\%$ *Consequences:* No L_{it} in (57) and (58) No (61)	**FLR-BM-FL** Base Formulation $\alpha = \sum_{q \in R} B_q \leq 100\%$	**FLR-BM-VL** $L_{it} \geq (1\text{-}\alpha)^* S_{it}$ in (61)

Table 12. Required Adjustments Needed to Derive Each Formulation from FLR-BM-FL

Starting with the bottom row in Table 12, FLR-BM-VL can be derived by simply changing the *lost sales constraints* in FLR-BM-FL into an inequality (≥), allowing for more lost sales to be incurred if necessary. Note that the original equality constraint (61) can be considered to be equal to the union of a "≤" and "≥" constraint. Changing this to a "≥" constraint can be done by removing the "≤" constraint. The FLR-BM formulation can be created setting the sum of the B_q parameters to 100 percent which results in the removal of the *lost sales constraint* (61) and the lost sales variable in constraints (57) and (58). This can be said for all formulations in Table 12 that do not allow lost sales.

In the "Restricted Backlog" row, the formulations share a common backlog restriction of *r* periods that is enforced for all stocked-out items. To formulate the restricted backlog case, starting from the multiple customer type case, we must set $B_q = \alpha$ for $q = r$, and set $B_q = 0$ for $q < r$. So there is only one class of customers and this ensures that customers faced with backlog all have the same patience, and are willing to wait for a maximum of *r* periods.

The formulations under the "Unlimited Backlog" assumption has two distinct differences compared to the restricted backlog formulations. They no longer have a backlog restriction and therefore $r = m\text{-}1$ where *m* is the last period in the time horizon. Because of the lifted restriction to backlog, we can also remove constraints (62) to formulate FLR-B-FL and FLR-B-VL.

To derive the formulations without backlog, we must first remove the possibility of backlog by setting $Z_{ikt} = 0$ when $k > t$. Additionally, to create the basic FLR formulation we must remove the L_{it} decision variable from the constraints (58), along with constraints (61) and (62) completely. We achieve this by setting all lost sales variables to zero. For the FLR-LS formulation, we remove constraints (62) and set α = 0% for constraints (61). The formulation where there is no backlog and variable lost sales does not exist. Variable lost sales exist to allow for more backlog to be incurred if necessary in case of a stock-out. However if there is no backlog, lost sales is the only option rendering the "variable lost sales concept" obsolete.

We have proposed new formulations that consider both backlog and lost sales simultaneously. This chapter reviewed at length these new formulations along comparable versions. In the following chapter, we explain the methods used to test the formulations, summarize the computational results and provide an analysis of these results.

5. Computational Results

Using the facility location reformulation, we have developed several variants of models that simultaneously consider both backlog and lost sales. Historically, studies have considered one of the variables using either a classical lot-sizing model or a reformulation. However both variables have not been considered together. To test the speed and efficiency of the new models, we run them using several data instances. In addition, we run the existing reformulations that were summarized in Sections 3.5 to 3.7 and compare the results.

5.1 Data Set

With the absence of an existing data set with backlog and lost sales parameters, we opt to adapt the instances of Trigeiro *et al.* (1989), which were originally intended for a standard capacitated lot-sizing model. These datasets have also been used in other research (Jans and Degraeve 2007, Sural *et al.* 2009, Fiorotto and de Araujo 2014, Fiorotto *et al.* 2015). There are two data sets we use to test our models: the F-set and the G-set. The F-set includes 70 instances each with 6 items (n=6) and a time horizon of 15 periods (m=15). The G-set includes a total of 71 instances, 46 of which follow the same format as the F-set with 6 items and 15 periods. The remaining 25 instances are broken up into 5 different structures with 5 instances each. The breakdown is shown in Table 13.

Instances	Items (*n*)	Periods (*m*)
1-50	6	15
51-55	12	15
56-60	24	15
61-65	6	30
66-70	12	30
71-75	24	30

Table 13. G-dataset Structure

The original dataset was constructed with the CLSP formulation in mind, which we reviewed in Section 3.1. Table 14 summarizes the original parameters developed by Trigeiro *et al.* (1989).

Parameters	**Notation**	**Range**	**Type of Variance**
Production setup cost	SC_i	200-1000	Product-Variant
Holding cost (per unit per period)	HC_i	1-5	Product-Variant
Setup time (time per period)	ST_i	10-50	Product-Variant
Unit Production time (time per unit)	UT	1	Static
Demand (per item per period)	d_{it}	Average of 100 per period	-

Table 14. Characteristics of the Original Parameters from Trigeiro *et al.* (1989)

The F and G datasets use nominal values throughout, and change one characteristic at a time. For the costs, the variable cost is set to zero while production and holding cost are randomly assigned within their respective ranges indicated in Table 14. The unit production time remained at 1 and the setup time varies between 10 to 50 units of time per period. The demand per item per period were randomly distributed with a mean of 100 units of demand. It is important to note that while the ranges listed in Table 14 were usually respected in the original datasets by Trigeiro *et al.* (1989), the datasets veered from these ranges on occasion. More information can be found in Trigeiro et al. (1989).

The F and G datasets use a static capacity level for all their instances, meaning that the capacity level for each period remains the same throughout the time horizon. Since in the original datasets, there are no backlog or lost sales, all demand can be satisfied on time. In order to create instances that have stock-outs, we chose to decrease the original capacity down to 92.5 percent of its original level. By doing so we can better expose the trade-off between early production, on-time production, late production, or no production under a tight capacity limit. The resulting capacity levels used are as follows:

Dataset (n, m)	Values
F (6,15)	[673, 984]
G (6,15)	[612, 673, 703, 722, 898]
G (12,15)	1347
G (24,15)	2694
G (6,30)	697
G (12,30)	1395
G (24,30)	2790

Table 15. Adapted Capacity Levels Used in F and G-datasets (Trigeiro *et al.* 1989)

In the F-dataset, there are 50 instances where the adapted capacity is set at 673, and 20 where it is at 984. As for the G-dataset, there are five different adapted capacity settings used for problems with 6 items and 15 periods. The problems of larger size, each have a set capacity level for their five instances. More details on the original dataset can be found in Trigeiro *et al*. (1989).

For the purpose of our thesis, we extend the dataset to include new backlog and lost sales parameters. The new parameters include the following ones: backlog cost (BC_{it}), lost sales cost (LC_{it}), a fixed percentage for stocked-out items remaining as backlog (α), the limit to the number of periods an item can be backlogged for (r), and a percentage of customers that are willing to wait a maximum number of q periods (β_q). While adding to the existing data set, we strived to keep the values consistent throughout the instances in order keep the initial structure of the data set. When setting the backlog and lost sales costs, we ensured that the cost of lost sales is always greater than if the item was backlogged and satisfied in a later period. We therefore set the rule that $LC_{it} > BC_{it} * r$. Furthermore note that $BC_{it} > HC_{it}$. The added parameters to the existing dataset are listed in Table 16.

	Parameters	Values
Backlog cost (per unit per period)	BC	[6,7]
Lost Sales cost (per unit)	LC	[25,30]
Backlog restriction (in periods)	r	4
Fixed percentage for stocked-out items remaining as backlog	α	0.75
Percentage of customers that are willing to wait a maximum number of *r* periods for $r \in \{1, 2, 3, 4\}$	β_r	[0.25, 0.20, 0.15, 0.15]

Table 16. Characteristics of the New Parameters

5.2 Models

In an effort to evaluate the efficiency and performance of our new models, the reformulations will be tested and compared to similar existing models. The FLR-B and FLR-LS models will be tested for the purpose of base case comparison. This will offer insights on how the performance and solution of the reformulation is affected when both concepts our considered together compared to individually. In addition, we test both our fixed and variable lost sales formulations under different backlog assumptions. The formulations we include in our study are listed in Table 17.

	No Lost Sales	Fixed Lost Sales	Variable Lost Sales
No Backlog	X	FLR-LS	X
Unlimited Backlog	FLR-B	FLR-B-FL	FLR-B-VL
Restricted Backlog	FLR-BR	FLR-BR-FL	FLR-BR-VL
Backlog with Multiple Customer Types	FLR-BM	FLR-BM-FL	FLR-BM-VL

Table 17. Breakdown of the Different Formulations that will be Evaluated

Unlimited Backlog

In the formulations with unlimited backlog, the backlog can be satisfied during any period in the time horizon. To evaluate this assumption we test FLR-B, FLR-B-FL, and FLR-B-VL.

Restricted Backlog

Formulations under the assumption of restricted backlog have a limit for the number of periods an item can be backlogged for (r). During the testing, we have set r=4. To determine if backlog restrictions have any impact, we will test and compare FLR-BR, FLR-BR-FL, and FLR-BR-VL.

Multiple Customer Types

In section 3.8, we introduce the concept of multiple customer types as an alternative to the commonly used single customer type assumption. We test three adapted formulations FLR-BM, FLR-BM-FL and FLR-BM-VL to analyze how multiple customer types affect the performance of the formulation. The single customer type consisted of a universal willingness to wait for backlogged items. The customer was either not willing to wait for their backlogged demand or willing to wait any number of periods (r being the maximum number of periods). In the multi-customer versions of the formulation, we add more layers to the customer's behaviour when faced with backlog. For our multi-customer formulations, we introduce four customer types with the following characteristics:

- 25 percent of all customers are willing to wait a maximum of one period
- 15 percent of all customers are willing to wait a maximum of two periods
- 10 percent of all customers are willing to wait a maximum of three periods
- 10 percent of all customers are willing to wait a maximum of four periods

In an effort to compare the formulations with one customer type versus multiple types, we set the parameters as follows:

$$\sum_{q \in R} B_q = \alpha$$

The above equality ensures that the minimum percentage of stock-out that becomes lost sales is the same.

Uncapacitated Formulations

For more insight we develop and test uncapacitated versions of all of our models. This will allow us to compare the basic structure of our formulations.

Exclusions

The classic FLR model was considered for testing and analysis as a base example of the reformulation without backlog or lost sales, but was ultimately excluded from our study. The FLR model would have mirrored the study done by Trigeiro *et al.* (1989). However, due to our reduction in capacity, many of the instances are infeasible. For this reason, we exclude this model from our analysis. We did however test the FLR formulation using the original instances to verify the consistency with the results obtained by Trigeiro *et al.* (1989).

Formulation Overview

To summarize, we have accumulated a number of different models that will be run and analyzed. They are listed in Table 18.

	No Lost Sales	Fixed Lost Sales	Variable Lost Sales
No Backlog	U-FLR	FLR-LS	X
Unlimited Backlog	FLR-B U-FLR-B	FLR-B-FL U-FLR-B-FL	FLR-B-VL U-FLR-B-VL
Restricted Backlog	FLR-BR U-FLR-BR	FLR-BR-FL U-FLR-BR-FL	FLR-BR-VL U-FLR-BR-VL
Backlog with Multiple Customers	FLR-BM U-FLR-BM	FLR-BM-FL U-FLR-BM-FL	FLR-BM-VL U-FLR-BM-VL

Table 18. List of Formulations used in our Analysis

5.3 Results

The formulations were coded in IBM Optimization Studio using OPL and solved with CPLEX 12.6.3.0. The optimization tolerance was set to zero using a single thread. A time limit of 1800 seconds was installed, any computations that went beyond that were stopped. We allocated 512.0 MB of working memory for the program to solve problems with 6 items and 15 periods. Once we got to the larger problems, the program was unable to process the problems without larger working memory. For all problems with more than 6 items and 15 periods, 5,000 MB of working memory were allocated. The computer used to run the programs had 8.00 GB installed RAM and a 1.70-2.40GHZ processor.

When summarizing and analyzing our results, we break it down into two sections. The first is a comparison of the performances between the different models. The performance measures include: the computing time, the optimal solution, and the LP time and gap. The second is a comparison of the structure of the solutions between different models. Our goal for this chapter is to highlight the differences in performance and structure of the solutions between different formulations.

5.3.1 Performance Comparison

F-Dataset

Table 19 summarizes our initial test results for all ten capacitated models using the F-dataset at a 92.5 percent capacity level. The column CPU Time indicates the average time used to solve an instance in seconds with a limit of 1800 seconds. The IP column indicates the average objective function value of the best found solution, while Gap signifies the average percentage difference between the best found solution and the best lower bound at the end of the optimization process. If this gap is equal to zero, then the solver has found the optimal solution. The LP column indicates the solution of the problem when binary variables (production setup) are relaxed as continuous variables, while LP Time indicates the average time it takes to solve the LP. The LP is calculated separately from the IP. Lastly, the LP Gap displays the average percentage difference between the best found solution and the LP relaxation value. All values in Table 19 are the average values of 70 separate instances of the same size.

Backlog	Lost Sales	Formulations	CPU Time (sec)	IP	Gap (%)	LP	LP Time (sec)	LP Gap (%)
No	Fixed	FLR-LS	168.41	43,974.15	0.03	42,479.13	0.02	3.393
Unlimited	No	FLR-B	30.24	41,460.68	0.00	40,654.04	0.02	1.944
	Fixed	FLR-B-FL	82.29	42,975.89	0.00	41,832.22	0.01	2.701
	Variable	FLR-B-VL	114.15	42,974.98	0.00	41,832.22	0.03	2.699
Restricted	No	FLR-BR	29.27	41,460.68	0.00	40,654.04	0.02	1.944
	Fixed	FLR-BR-FL	80.10	42,975.89	0.00	41,832.22	0.02	2.701
	Variable	FLR-BR-VL	78.79	42,974.98	0.00	41,832.22	0.02	2.699
Multiple	No	FLR-BM	80.04	41,892.85	0.00	40,890.19	0.02	2.351
	Fixed	FLR-BM-FL	172.41	43,054.38	0.03	41,843.81	0.02	2.848
	Variable	FLR-BM-VL	143.48	43,035.49	0.02	41,843.74	0.03	2.812

Table 19. Computational Results and Solutions for F-dataset at 92.5 percent Capacity Level

IP Computing Time

The results show that computing time is impacted by the type of formulation. Under the same backlog assumption, formulations that consider both backlog and lost sales have larger average computing times than those that only consider backlog. Our results also indicate that computing times for formulations with multiple customers are longer than for formulations that only consider one customer type. These results may be caused by the added complexity which the combination of backlog and lost sales and the multiple customer types bring to these formulations.

Formulations that do not allow lost sales (FLR-B, FLR-BR, and FLR-BM) are solved a lot quicker than those which do. FLR-LS has the longest average computing time apart from FLR-BM-FL. When we allow the possibility of backlog along with lost sales, it generally leads to a decrease in computing time.

Although there are some discrepancies in the average CPU Time between formulations with fixed and variable lost sales, there is no evidence that either one takes longer than the other.

IP Objective Function Value

If we compare the objective function values of the best found solutions in Table 19 for formulations under the same backlog assumption, we first notice that the average value of the best found objective function is very similar for both the fixed and variable lost sales versions. With the lost sales cost being so high, there is little benefit in incurring more lost sales than necessary. The difference between fixed and variable lost sales is the largest for the case with multiple customer types.

The average objective function of the optimal solution does not change when removing the backlog restriction. By looking at Table 19, we can see FLR-BR, FLR-BR-FL, and the FLR-BR-VL all have the same optimal solutions as the unrestricted backlog version of their formulations. This could be the result of a lenient restriction on backlog (i.e. the maximum backlog period). By reducing the value of *r*, we create a larger restriction on backlog which could produce a difference in the average objective function value of the optimal solution between formulations with unrestricted and restricted backlog. The average value of the optimal objective function does however increase slightly when the formulation includes the assumption of multiple customer types instead of a singular customer type. The multiple customer type formulations are less flexible and may be forced to incur more lost sales or additional production setups during periods of high demand. All formulations that consider both backlog and lost sales have similar average optimal solutions. The formulations with only one of the two concepts, however, have different solutions. The objective value of the optimal solution for FLR-BR decreased by approximately 3.0 percent, while FLR-LS increased by approximately 1.7 percent compared to formulations that consider both backlog and lost sales.

From the results in Table 19, we notice that formulations that only consider backlog provide the lowest optimal value of the objective function. In situations where demand for a period is low, it may be not be worth it to setup production. Instead, they can satisfy the demand during the next production period and pay a smaller backlog penalty cost instead of the more expensive production setup. To highlight this strategy (i.e. use backlog to reduce costs), we performed a separate study in which we compare the results of the U-FLR and U-FLR-BR formulations (i.e. the uncapacitated case). We refer to the uncapacitated versions to highlight the possible gains in using backlog versus a standard model without backlog. Table 20 demonstrates that backlog in fact is an effective tool to reduce total cost. The table compares the average optimal solution between the U-FLR and the U-FLR-BR formulations for both datasets and all problem sizes. The first row includes all 70 instances of the F-dataset, while the second includes 50 instances. The last five rows each have a sample size of five. Backlog's usefulness is hence not restricted to cases with capacity constraints, it can also be used as an effective strategy to cut costs for the uncapacitated case.

Problem Size	Avg. IP	
	U-FLR	U-FLR-BR
F(6, 15)	40,639.50	39,879.83
G(6, 15)	35,995.59	35,774.70
G(12, 15)	67,452.20	67,053.00
G(24, 15)	143,287.00	142,122.40
G(6, 30)	66,532.80	66,253.20
G(12, 30)	144,165.80	143,330.00
G(24, 30)	272,379.40	270,488.20

Table 20. Comparison of the Solution between U-FLR and U-FLR-B

As a final note for this section, all formulations have a low average optimality gap. When the gap is 0.00% it indicates that the optimal solution was found within the time limit. Only three formulations did not have an average gap of 0.00%, being FLR-LS, FLR-BM-FL, FLR-BM-FL. These formulations each had one instance that was not solved to optimality within the time limit. The instances of the F-dataset are of a manageable size with only 6 items and 15 periods considered. When the problem size becomes larger, it can take longer to find the optimal solution and in some cases it might not be possible to solve the problem within the time limit.

LP Solution

All formulations exhibited very low LP computing times, all less than three hundredths of a second. The LP gap, which represents the average percentage difference between the best found objective function value of the optimal solution and the value of the LP relaxation, differed between formulations. The LP gap was highest for FLR-LS and lowest for FLR-B and FLR-BR. The LP gap is seen to be slightly larger for formulations that assume multiple customer types compared to those that consider one customer type.

G-Dataset

We use the results of the G dataset to investigate how efficiently the formulations deal with larger sized problems. Table 21 summarizes the results of each formulation for the different sized problems.

The column with the header of “# of Time Limits”, indicates the number of times the program was not able to find the optimal solution within the time limit of 1,800 seconds. Each formulation has six rows each showing the average results for the different sized problems. In the

G-dataset, there are 50 instances with 6 items and 15 periods, and 5 instances for each of the other problem sizes. The capacity level is at 92.5 percent of the original dataset for all instances.

Backlog	Lost sales	Formulation	Size (n, m)	CPU Time (sec)	IP	Gap (%)	LP	LP Time (sec)	LP Gap (%)	# of Time Limits
No	Fixed	FLR-LS	(6, 15)	179.02	41,333.46	0.000	39,502.70	0.02	4.257	0
			(12, 15)	1,308.29	73,032.00	0.279	71,748.50	0.03	1.778	3
			(24, 15)	1,726.33	148,152.80	0.171	147,347.11	0.06	0.532	3
			(6, 30)	1,425.00	69,665.20	0.792	67,529.96	0.05	3.127	4
			(12, 30)	1,800.00	148,503.00	0.938	146,431.32	0.09	1.341	5
			(24, 30)	1,800.00	276,153.00	0.268	275,144.76	0.24	0.378	5
Unlimited	No	FLR-B	(6, 15)	46.57	38,711.91	0.000	37,630.57	0.02	2.740	0
			(12, 15)	999.89	69,415.00	0.024	68,775.29	0.04	0.924	1
			(24, 15)	1,387.11	144,080.40	0.064	143,696.20	0.11	0.267	3
			(6, 30)	1,492.77	68,713.40	0.392	67,137.78	0.07	2.351	2
			(12, 30)	1,800.00	146,221.00	0.485	145,090.94	0.25	0.764	5
			(24, 30)	1,657.78	272,943.00	0.183	272,306.62	0.52	0.245	4
	Fixed	FLR-B-FL	(6, 15)	57.73	40,295.27	0.000	38,847.71	0.01	3.507	0
			(12, 15)	1,664.53	71,530.28	0.307	70,511.47	0.03	1.443	3
			(24, 15)	1,800.02	146,879.37	0.161	146,259.07	0.07	0.421	4
			(6, 30)	1,612.72	69,565.89	0.757	67,522.53	0.05	2.999	3
			(12, 30)	1,800.02	148,220.25	0.789	67,522.53	0.13	1.190	5
			(24, 30)	1,800.02	275,901.90	0.257	275,045.26	0.35	0.324	5
	Variable	FLR-B-VL	(6, 15)	41.14	40,283.38	0.000	38,841.09	0.02	3.495	0
			(12, 15)	1,484.39	71,526.76	0.312	70,511.47	0.03	1.438	0
			(24, 15)	1,800.15	146,872.98	0.162	146,259.07	0.06	0.418	2
			(6, 30)	1,528.80	69,605.80	0.827	67,522.53	0.04	3.056	1
			(12, 30)	1,800.05	148,191.96	0.766	67,522.53	0.10	1.175	5
			(24, 30)	1,800.02	275,951.50	0.274	275,045.26	0.21	0.343	3
Restricted	No	FLR-BR	(6, 15)	54.31	38,711.91	0.000	37,630.57	0.02	2.740	0
			(12, 15)	906.56	69,415.00	0.014	68,775.29	0.03	0.924	1
			(24, 15)	1,407.94	144,080.40	0.057	143,696.20	0.08	0.267	3
			(6, 30)	1,311.46	68,700.00	0.348	67,137.78	0.05	2.331	2
			(12, 30)	1,800.00	146,217.80	0.462	145,090.94	0.20	0.761	5
			(24, 30)	1,657.19	272,937.40	0.173	272,306.62	0.41	0.242	4
	Fixed	FLR-BR-FL	(6, 15)	82.67	40,295.27	0.000	38,847.71	0.02	3.507	0
			(12, 15)	1,486.80	71,530.28	0.307	70,511.47	0.03	1.443	3
			(24, 15)	1,779.79	146,879.37	0.161	146,259.07	0.06	0.421	4
			(6, 30)	1,606.68	69,565.89	0.757	67,522.53	0.04	2.999	3
			(12, 30)	1,800.06	148,220.25	0.789	67,522.53	0.04	1.190	5
			(24, 30)	1,800.16	275,901.90	0.257	275,045.26	0.84	0.324	5
	Variable	FLR-BR-VL	(6, 15)	62.79	40,283.38	0.000	38,841.09	0.01	3.495	0
			(12, 15)	1,520.67	71,526.76	0.312	70,511.47	0.04	1.438	0
			(24, 15)	1,779.89	146,872.98	0.162	146,259.07	0.10	0.418	2
			(6, 30)	1,395.68	69,605.80	0.827	67,522.53	0.04	3.056	1
			(12, 30)	1,800.03	148,191.96	0.766	67,522.53	0.04	1.175	5
			(24, 30)	1,800.05	275,951.50	0.274	275,045.26	0.43	0.343	3
Multiple	No	FLR-BM	(6, 15)*	256.61	39,537.15	0.661	37,868.34	0.02	3.990	4
			(12, 15)	1,456.15	69,733.40	0.192	69,017.98	0.07	1.025	4
			(24, 15)	1,800.03	144,569.40	0.142	144,072.02	0.25	0.343	5
			(6, 30)	1,800.03	68,849.40	0.705	67,161.30	0.12	2.517	5
			(12, 30)	1,800.03	146,561.80	0.600	145,261.78	0.44	0.873	5
			(24, 30)	1,800.07	273,343.80	0.234	272,580.49	0.99	0.291	5
	Fixed	FLR-BM-FL	(6, 15)	193.66	40,526.24	0.000	38,887.93	0.03	3.912	1
			(12, 15)	1,512.20	71,583.67	0.339	70,520.89	0.04	1.502	3
			(24, 15)	1,800.02	146,908.41	0.181	146,265.12	0.08	0.436	5
			(6, 30)	1,519.95	69,580.79	0.870	67,522.53	0.05	3.021	3
			(12, 30)	1,800.05	148,416.86	0.915	146,408.05	0.17	1.309	5
			(24, 30)	1,800.03	275,922.35	0.261	275,045.26	0.32	0.331	5
	Variable	FLR-BM-VL	(6, 15)	83.25	40,459.06	0.000	37,781.17	0.02	3.360	0
			(12, 15)	935.16	71,602.46	0.412	69,351.57	0.06	1.067	1
			(24, 15)	804.62	146,924.71	0.197	145,278.30	0.06	0.240	2
			(6, 30)	630.24	69,580.05	0.878	67,139.87	0.05	2.063	1
			(12, 30)	1,800.02	148,306.68	0.862	145,690.62	0.13	0.847	5
			(24, 30)	1,280.68	275,879.33	0.251	273,657.73	0.32	0.197	3

Table 21. Computational Results and Solutions for G-dataset with Different Sized Problems at 92.5 percent Capacity Level

*Two instances were infeasible

For Table 21, we remark that there were two (6, 15) instances that were infeasible using FLR-BM. Infeasible solutions are possible when using this formulation because we assume that there is no backlog allowed at the end of the time horizon. This formulation does not allow for lost sales and assumes that $r = 4$. It also assumes that 50 percent of customers are only willing to wait one period of backlog. With all of this considered, FLR-BM is an inflexible formulation that can run into some infeasibility issues.

IP Computing Time

From our results, we can deduce that on average, larger problems take longer to solve, which is no surprise. For all formulations the computing time increased as the problem sizes increased. The computing times however do not follow a linear trend. Although we can conclude that in general, larger problems will take longer than smaller problems, there are instance where that is not the case. For example, if we look at the (12, 30) problems versus the (24, 30) problems, we observe that for all formulations, the (12, 30) instances have a larger computing time than the (24, 30) instances. To evaluate this further we examine the number of instances that reach the time limit before finding the optimal solution.

The number of times the program reaches the time limit before reaching optimality also increases as the problem size becomes larger. However as Figure 4 indicates, our results do not show a consistent correlation between the size of the problem and the number of instances that were not solved to optimality within the time limit of 30 minutes.

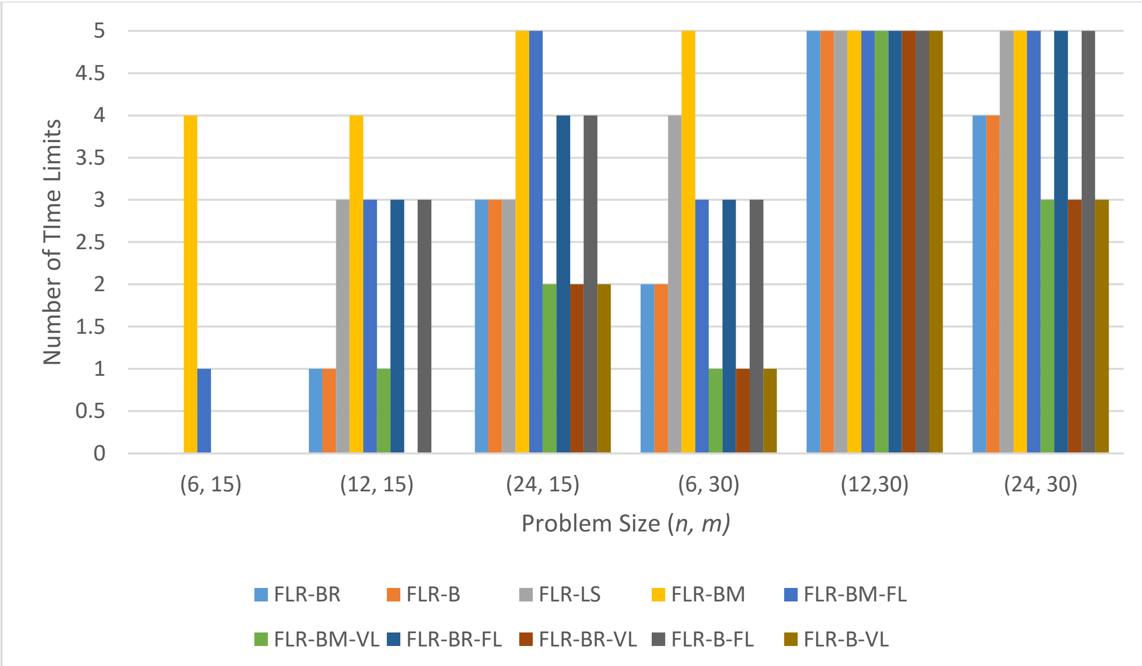

Figure 4. Comparison of Number of Instances in which the Time Limit was Reached for the G-dataset

In all cases, the G(12, 30) problems took longer on average than the G(24, 30) problems. The five G(12,30) instances reached the time limit for all formulations that we tested. The same cannot be said for the G(24, 30) problems, where the instances reached the time limit three to five times depending on the formulation. The reasons could be linked to the small sample size (n=5) used for the larger problems: the five G(12, 30) instances could have all been difficult to solve. Another reason could be that the adjustments in the capacity parameter made the G(12, 30) instances tighter than the G(24,30) instances.

IP Objective Function Value

From our results, we can evidently see that larger size problems produce a higher optimal objective function than smaller problems due to the added costs associated to the new items and periods. In Table 21, we notice that the formulations that performed best for the (6, 15) problems continue to perform best with larger problem sizes. FLR-B and FLR-BR have the lowest best found objective function. When only backlog is considered, there is more flexibility and less penalty costs. Under the assumption that the maximum cost for backlogging an item must be lower than the cost of lost sales ($LC_{it} > BC_{it} * r$), the better option is to backlog when possible. Realistically, when demand is stocked-out, customers may look for other options rather than wait. The models that consider multiple options simultaneously (backlog and lost sales), gave comparative solutions that were higher than the respective formulations with only backlog but lower than the respective formulations with only lost sales.

LP Solution

Both the G-dataset and F-dataset have a similar LP Gap of two to four percent for problems of 6 items and 15 periods. As the problems become larger, our results for the G-dataset show that the LP Gap decreases accordingly. Table 22 shows the average LP gap according to the size of the problems for all models. The LP gap appears to be more affected when the number of items are increased compared to the number of periods. These results are in line with the results of Trigeiro et al. (1989) who studied the capacitated lot-sizing problem without backlog or lost sales.

LP Gap (%)	Number of periods (*m*)	
Number of items (*n*)	15	30
6	3.500	2.752
12	1.298	1.062
24	0.376	0.302

Table 22. Average LP Gap Results for the Different Sized Problems of the G-dataset

5.3.2 Comparison of the Structure of the Solution

In the previous section, we have reviewed the optimal or best found objective function value and computation time of all models for both datasets. For greater insight, this section analyzes the structure of the solution between models. Table 23 and Table 24 display the weight each cost has within the total cost. The first column "Avg. Setup Cost" indicates the percentage of the total cost that is made up of the cost to setup production. The following three rows do the same for holding, backlog, and lost sales costs in that order. The "# of no LS" column, shows the number of instances that do not include lost sales in the optimal solution. The last column does the same for instances in which backlog is not included. The cells that have an "N/A", do not consider those variables in the formulation and therefore do not indicate any value. In Table 24, we use the average results of the entire G-dataset to make the results easier to compare.

Backlog	Lost Sales	Formulation	Avg. % Setup	Avg. % Holding	Avg. % Backlog	Avg. % Lost Sales
No	Fixed	FLR-LS	65.08	31.82	N/A	3.10
Unlimited	No	FLR-B	61.19	28.47	10.34	N/A
	Fixed	FLR-B-FL	63.87	31.34	2.01	2.78
	Variable	FLR-B-VL	63.87	31.38	1.99	2.76
Restricted	No	FLR-BR	61.19	28.47	10.34	N/A
	Fixed	FLR-BR-FL	63.87	31.34	2.01	2.78
	Variable	FLR-BR-VL	63.87	31.38	1.99	2.76
Multiple	No	FLR-BM	61.82	31.30	6.88	N/A
	Fixed	FLR-BM-FL	63.42	32.11	1.86	2.61
	Variable	FLR-BM-VL	63.61	31.92	1.82	2.65

Table 23. Cost Solution Structure for F-dataset at 92.5 percent Capacity Level (70 instances)

Backlog	Lost Sales	Formulation	Avg. % Setup	Avg. % Holding	Avg. % Backlog	Avg. % Lost Sales
No	Fixed	FLR-LS	62.55	32.30	N/A	5.14
Unlimited	No	FLR-B	59.15	30.21	10.64	N/A
	Fixed	FLR-B-FL	61.72	32.49	2.42	3.37
	Variable	FLR-B-VL	61.81	32.37	2.17	3.66
Restricted	No	FLR-BR	59.11	30.25	10.64	N/A
	Fixed	FLR-BR-FL	61.72	32.49	2.42	3.37
	Variable	FLR-BR-VL	61.81	32.37	2.17	3.66
Multiple	No	FLR-BM	60.25	31.33	8.29	N/A
	Fixed	FLR-BM-FL	61.63	32.36	2.55	3.46
	Variable	FLR-BM-VL	61.71	32.31	2.41	3.57

Table 24. Cost Solution Structure for G-dataset at 92.5 percent Capacity Level (71 instances)

Analysis of Costs

For both datasets, the FLR-LS formulation has the highest setup and lost sales percentage in the total cost. With the addition of backlog, the percentage of setup cost and lost sales decreases in the formulations that consider both backlog and lost sales. Models that consider backlog can use backlog as a strategy to cut costs. This strategy is most effective when only backlog is considered compared to when both backlog and lost sales are included simultaneously.

Analysis of Decisions

To evaluate the decision solution structure, we use the results obtained from the F-dataset. The F-dataset is preferred compared to the G-dataset in order to analyze the decision solution structure because the instances all have the same problem size. In Table 25, we summarize the average values for the number of setups, the number of units held in inventory, the number of units backlogged, and the number of units that became lost sales in the optimal or best found solution. The table provides for a better comparison of the solution structure between the different formulations.

Backlog	Lost Sales	Formulation	# of Setup	# of Holding	# of Backlog	# of Lost Sales	# of no B	# of no LS
No	Fixed	FLR-LS	43.60	3,929.33	N/A	58.51	N/A	31
Unlimited	No	FLR-B	40.17	3,573.80	668.86	N/A	0	N/A
	Fixed	FLR-B-FL	42.41	3,869.46	144.96	48.32	10	10
	Variable	FLR-B-VL	42.41	3,870.16	142.73	47.74	10	10
Restricted	No	FLR-BR	40.17	3,576.13	668.37	N/A	0	N/A
	Fixed	FLR-BR-FL	42.41	3,865.54	144.96	48.32	10	10
	Variable	FLR-BR-VL	42.41	3,870.16	142.73	47.74	10	10
Multiple	No	FLR-BM	40.36	3,762.14	475.91	N/A	0	N/A
	Fixed	FLR-BM-FL	42.14	3,910.82	137.05	45.68	10	10
	Variable	FLR-BM-VL	42.26	3,900.39	132.61	46.05	10	10

Table 25. Decisions Solution Structure Values for F-dataset

From the results in Table 25, we notice that FLR-BR-VL use slightly less backlog and lost sales than the FLR-BR-FL while having a lower average total cost in their optimal solution. The same is true when comparing FLR-B-VL and FLR-B-FL. Although variable lost sales must always have an equal or lower objective function value compared to fixed lost sales, the reason does not necessarily have to be due to a reduction in stock-outs. The change in the structure of the solution when going from a fixed lost sales formulation towards a variable lost sales formulation can go in different directions. The stock-outs for the variable lost sales model can either decrease or increase with the idea of using additional lost sales as a strategy to reduce total cost.

When we consider multi-customer types, the same possibilities remain true. However, as multiple customer formulations are inherently less flexible, the use of additional lost sales as a cost saving strategy when a stock-out occurs will increase. Table 25 supports this idea, as the number of lost sales are slightly higher while the number of backlog is lower for FLR-BM-VL compared to FLR-BM-FL.

The impact of multiple customers can be illustrated by comparing the solution structures of FLR-BM with FLR-BR. Although all customers are willing to wait for backlog, the added dimension of different customer types with different willingness's to wait has a significant effect on the structure of the solution. The added restrictions of multiple customer types discourage the use of backlog by approximately 30 percent. This is logical as backlog is more restricted.

Referring to Table 25, the last two columns indicate for how many instances did the optimal solution not include any stock-outs, in other words no lost sales (LS) or backlog (B). In the FLR-BR and FLR-B formulations, backlog is used in all instances because it can provide more flexibility and reduce total cost. In fact, backlog represents over 10 percent of total cost. In the FLR-LS formulation, lost sales is used less and is only approximately 3 to 5 percent of the total cost. Formulations that consider both backlog and lost sales have few instances that do not have any stock-outs, around 15 percent (10/70 instances). As for formulations that only consider backlog, all instances include at least one stock-out. The formulation that only considers lost sales has the largest percentage of instances without any stock-outs with 44.3 percent (31/70 instances).

Summary

From reviewing the structure of the solution for all formulations, we see an apparent trade-off between the cost of backlog and lost sales. The trade-off, however may be sensitive to certain parameters. Changing certain parameters can affect the total number of stock-outs that are included in the optimal solution. To further evaluate the trade-off between backlog and lost sales, we perform a sensitivity analysis on different parameters in the following chapter.

6. Sensitivity Analysis

In the previous section, we evaluated the optimal solutions and structure of the solutions across multiple models. The impact of the size of the models was also analyzed. In this section, we evaluate the impact that certain parameters have on the performance of the formulations. We conduct a sensitivity analysis on the following parameters: capacity, α and β_r, the lost sales costs and the value of r. Our goal for this section is to determine how and, to what extent, the performances and structures of the solutions are changed when certain parameters are altered. Although the G-dataset is included in the capacity sensitivity analysis, the large problem sizes made the additional sensitivity test's computing time very high. For this reason, we used the F-dataset for the majority of our sensitivity analysis.

6.1. Impact of Capacity

6.1.1. Performance Comparison

Capacity was discussed as an important factor for the cause of stock-outs. Up to this point, we have tested our models with an adjusted capacity level of 92.5 percent of the original values of the Trigeiro et al. (1989) datasets. For this sensitivity test, we perform tests using different capacity levels to evaluate the impact of capacity on all the formulations. The performance in terms of computing time and solutions of the models at different capacity levels for both datasets will be summarized. For each formulation, we list the results of the uncapacitated version along with the different capacity levels that were tested.

G-dataset

For the G-dataset, we test the models using a level of 95 percent of the original capacity and compare it to the results from the 92.5 percent capacity level. We anticipated to test the models using a 90 percent capacity level for the G-dataset. However, the large problem sizes would make the computing time for all instances excessively long. The number of instances that would reach the time limit without being solved to optimality would be very high. The results would therefore be too unreliable to make significant inferences. Furthermore, we have already observed two infeasible instances for formulation FLR-BM with a 92.5 percent capacity level. If we lower the capacity even further, we would obtain more infeasible solutions. For all above reasons, we deemed that the F-dataset would provide sufficient data to analyze the results for a 90 percent capacity level. To make comparisons between models clearer, we combine all problems size's results of the G-dataset together. The results may not be perfectly proportional between different sized problems. However, we feel that averaging all sized problems still provides a strong representations of the sensitivity of capacity. For a full summary of the results for the G-dataset at a 95 percent capacity level see Appendix A.

Backlog	Lost Sales	Formulation	Capacity level (%)	Avg. IP	Avg. Gap (%)	Avg. CPUTime (sec)	Avg. LP	Avg. LPTime (sec)	Avg. LP Gap (%)
		FLR-LS	Uncapacitated	72,181.45	0.000	0.04	72,181.45	0.01	0.000
No	Fixed	FLR-LS	95.0	75,520.18	0.068	381.66	74,183.72	0.03	2.844
		FLR-LS	92.5	77,167.17	0.179	683.57	75,466.65	0.05	3.262
		FLR-B	Uncapacitated	71,716.48	0.000	0.04	71,716.48	0.02	0.000
	No	FLR-B	95.0	73,511.69	0.032	372.42	72,728.84	0.07	1.722
		FLR-B	92.5	74,473.40	0.081	546.91	73,465.36	0.08	2.096
		FLR-B-FL	Uncapacitated	72,181.45	0.000	0.31	72,181.45	0.03	0.000
Unlimited	Fixed	FLR-B-FL	95.0	74,967.19	0.050	417.37	73,842.78	0.04	2.405
		FLR-B-FL	92.5	76,254.52	0.160	658.83	74,869.39	0.09	2.721
		FLR-B-VL	Uncapacitated	72,181.45	0.000	0.16	72,181.45	0.02	0.000
	Variable	FLR-B-VL	95.0	74,969.05	0.051	424.97	73,842.78	0.04	2.406
		FLR-B-VL	92.5	76,250.43	0.165	745.03	74,865.10	0.06	2.717
		FLR-BR	Uncapacitated	71,716.48	0.000	0.04	71,716.48	0.02	0.000
	No	FLR-BR	95.0	73,505.23	0.024	320.06	72,728.84	0.04	1.719
		FLR-BR	92.5	74,471.84	0.074	534.00	73,465.36	0.06	2.094
		FLR-BR-FL	Uncapacitated	72,181.45	0.000	0.07	72,181.45	0.01	0.000
Restricted	Fixed	FLR-BR-FL	95.0	74,967.19	0.050	401.85	73,842.78	0.04	2.405
		FLR-BR-FL	92.5	76,254.52	0.160	650.29	74,869.39	0.09	2.721
		FLR-BR-VL	Uncapacitated	72,181.45	0.000	0.21	72,181.45	0.03	0.000
	Variable	FLR-BR-VL	95.0	74,969.05	0.051	401.80	73,842.78	0.04	2.406
		FLR-BR-VL	92.5	76,250.43	0.165	624.93	74,865.10	0.06	2.717
		FLR-BM	Uncapacitated	71,758.07	0.000	0.06	71,758.07	0.02	0.000
	No	FLR-BM	95.0	73,879.31	0.029	365.71	72,932.58	0.04	2.099
		FLR-BM	92.5	75,126.73	0.557	775.86	73,695.94	0.14	2.941
		FLR-BM-FL	Uncapacitated	72,181.45	0.000	0.29	72,181.45	0.03	0.000
Multiple	Fixed	FLR-BM-FL	95.0	75,030.19	0.061	422.24	73,853.23	0.04	2.514
		FLR-BM-FL	92.5	76,476.28	0.184	694.92	74,928.98	0.06	3.039
		FLR-BM-VL	Uncapacitated	72,181.45	0.000	0.28	72,181.45	0.03	0.000
	Variable	FLR-BM-VL	95.0	75,025.67	0.058	439.67	73,853.23	0.04	2.510
		FLR-BM-VL	92.5	76,374.41	0.183	682.93	74,907.42	0.08	2.883

Table 26. Computational Results and Solutions for Different Capacity Levels for the G-dataset

In Table 26, the results show that a change in capacity levels has a significant impact on the performance of the formulations. This impact is made apparent by first examining the results of the uncapacitated formulations.

Uncapacitated Formulations

IP Computing Time

Without capacity restrictions, the results indicate that the formulations are able to solve each instance to optimality in on average under a second. Because of its efficiency, the optimality gap is zero percent for all formulations. Capacity is therefore a main factor in computing time.

IP Objective Function Value

For all models, the best found value of the objective function is obtained when there are no capacity limitations. The average IP value is also equal to the average LP value for all formulations.

The best found value of the objective function was the same for all uncapacitated models that considered both backlog and lost sales. The reason for this is due to the fact that none of the instances include stock-outs in there optimal solution. In addition, the uncapacitated FLR-LS formulation did not include any stock-outs either and therefore also had the same best found value of the objective function. When stock-outs are not included in the optimal solution, all our formulations we examine are the same. U-FLR-B, U-FLR-BR and U-FLR-BM are the only two formulations that produced a different optimal solution because the use of backlog is an effective cost saving strategy even when capacity is not considered, and backlog is therefore used in the optimal solutions.

LP Solution

For the uncapacitated problems, we observe that for the G-dataset, the LP gap is equal to zero for all instances. However, this is not a general result, as we will see later that in the F-dataset we have instances for which the LP relaxation of the uncapcitated problems has a strictly positive LP gap. When no lost sales or backlog is allowed, the uncapacitated multi-item lot-sizing problem decomposes into separate single-item uncapacitated lot-sizing problems for which the facility location reformulation is tight (Pochet and Wolsey, 2006).

As a note, the uncapacitated formulations did not require the use of any nodes when solving the datasets. Only when capacity was included did the program use nodes to solve the datasets.

Capacitated Formulations

IP Computing Time

There is a significant difference in computational time between the uncapacitated and capacitated models. Furthermore, the models take even longer to be solved up to optimality as the capacity level is decreased. The average increase in computing time going from 95.0 to 92.5 percent of the capacity level for the G-dataset is shown in Table 27. The computing time increased on average by 64.7 percent. In addition, the number of instances that were not able to reach the optimal solution within the time limit rose for all models, especially those which considered both backlog and lost sales. There is not enough evidence to determine if there are certain formulations that are more affected by a decrease in capacity than others.

Backlog	Lost sales	Formulation	Capacity Level (%)	Avg. IP	Increase in Avg. IP (%)	CPU Time (sec)	Increase in CPU Time (%)	# of Time Limits
No	Fixed	FLR-LS	95.0	75,520.18		381.66		15
		FLR-LS	92.5	77,167.17	2.18	683.57	79.10	20
Unlimited	No	FLR-B	95.0	73,511.69		372.42		10
		FLR-B	92.5	74,473.40	1.31	546.91	46.85	11
	Fixed	FLR-B-FL	95.0	74,967.19		417.37		14
		FLR-B-FL	92.5	76,254.52	1.72	658.83	57.85	20
	Variable	FLR-B-VL	95.0	74,969.05		424.97		14
		FLR-B-VL	92.5	76,250.43	1.71	745.03	75.31	24
Restricted	No	FLR-BR	95.0	73,505.23		320.06		10
		FLR-BR	92.5	74,471.84	1.32	534.00	66.84	11
	Fixed	FLR-BR-FL	95.0	74,967.19		401.85		12
		FLR-BR-FL	92.5	76,254.52	1.72	650.29	61.82	20
	Variable	FLR-BR-VL	95.0	74,969.05		401.80		11
		FLR-BR-VL	92.5	76,250.43	1.71	624.93	55.53	20
Multiple	No	FLR-BM	95.0	73,879.31		365.71		9
		FLR-BM	92.5	75,126.73	1.69	775.86	112.15	28
	Fixed	FLR-BM-FL	95.0	75,030.19		422.24		11
		FLR-BM-FL	92.5	76,476.28	1.93	694.92	64.58	22
	Variable	FLR-BM-VL	95.0	75,025.67		439.67		12
		FLR-BM-VL	92.5	76,374.41	1.80	682.93	55.33	22

Table 27. Impact on Avg. IP and CPU Time from Decreasing Capacity Levels for the G-dataset

IP Objective Function Value

By reviewing Table 26, we can make some general observations. With the addition of capacity, the results indicate an increase in the optimal value of the objective function across all models. The average total cost continues to increase as the capacity levels are lowered. On average for the G-dataset, the optimal solution of all formulations experienced an increase in cost of 3.59 percent with the introduction of capacity restrictions.

Table 27 summarizes the percent increase in the average best found value of the objective function when the capacity level is decreased from 95.0 to 92.5 percent. The FLR-LS model was the most affected with an increase of 2.18 percent, while FLR-B was the least with a 1.31 percent increase.

If we examine the formulations that consider both concepts simultaneously, we notice that the best found value of the objective function for formulations with multiple customers' types are more impacted than those with a single customer type. This is because in the formulation, the backlog constraints are more restrictive, and hence a lack of capacity will have bigger repercussions. There is also a slightly larger impact on the best found value of the objective function for models with fixed lost sales instead of variable lost sales. We attribute this difference to the added flexibility that the variable lost sales models offer.

LP Solution

The LP Gap, indicated in Table 26 is shown to have a small increase for all formulations when the capacity is decreased from 95 to 92.5 percent of the original capacity setting. The percentage decrease is fairly consistent across all formulations. The LP Time also increases consistently across all formulations when capacity is decreased, however the LP solution is still found very quickly.

In order to determine if the impact on the models are further worsened when the capacity levels decrease even more, we turn our focus on the results for the F-dataset.

F-dataset

For the F-dataset, we conduct the sensitivity analysis by testing the instances with a larger capacity level of 95 percent, and a smaller capacity level of 90 percent compared to the original capacity level used of 92.5 percent. The results are then combined and summarized in Table 28.

Backlog	Lost Sales	Formulation	Capacity level (%)	Avg. IP	Avg. Gap (%)	Avg. CPUTime (sec)	Avg. LP	Avg. LPTime (sec)	Avg. LP Gap (%)
No	Fixed	FLR-LS	Uncapacitated	40,639.50	0.000	0.03	40,639.50	0.01	0.000
		FLR-LS	95.0	43,044.79	0.000	38.27	41,987.59	0.02	2.457
		FLR-LS	92.5	43,974.15	0.032	168.41	42,479.13	0.02	3.393
		FLR-LS	90.0	45,211.67	0.025	224.11	43,325.73	0.02	4.116
Unlimited	No	FLR-B	Uncapacitated	39,879.83	0.000	0.03	39,879.83	0.01	0.000
		FLR-B	95.0	41,071.30	0.000	8.58	40,441.39	0.01	1.541
		FLR-B	92.5	41,460.68	0.000	30.24	40,654.04	0.02	1.944
		FLR-B	90.0	42,061.37	0.000	79.84	41,049.17	0.02	2.386
	Fixed	FLR-B-FL	Uncapacitated	40,625.60	0.000	0.08	40,622.67	0.03	0.005
		FLR-B-FL	95.0	42,417.96	0.000	33.24	41,549.20	0.02	2.082
		FLR-B-FL	92.5	42,975.89	0.000	82.29	41,832.22	0.01	2.701
		FLR-B-FL	90.0	43,890.93	0.007	142.72	42,516.08	0.02	3.140
	Variable	FLR-B-VL	Uncapacitated	40,625.60	0.000	0.10	40,622.67	0.04	0.005
		FLR-B-VL	95.0	42,417.96	0.000	25.03	41,549.20	0.02	2.082
		FLR-B-VL	92.5	42,974.98	0.000	114.15	41,832.22	0.03	2.699
		FLR-B-VL	90.0	43,888.01	0.005	141.01	42,515.59	0.02	3.136
Restricted	No	FLR-BR	Uncapacitated	39,879.83	0.000	0.03	39,879.83	0.01	0.000
		FLR-BR	95.0	41,071.30	0.000	7.83	40,441.39	0.01	1.541
		FLR-BR	92.5	41,460.68	0.000	29.27	40,654.04	0.02	1.944
		FLR-BR	90.0	42,061.37	0.000	79.30	41,049.17	0.02	2.386
	Fixed	FLR-BR-FL	Uncapacitated	40,625.60	0.000	0.08	40,622.67	0.03	0.005
		FLR-BR-FL	95.0	42,417.96	0.000	43.06	41,549.20	0.02	2.082
		FLR-BR-FL	92.5	42,975.89	0.000	80.10	41,832.22	0.02	2.701
		FLR-BR-FL	90.0	43,891.13	0.007	141.98	42,516.08	0.02	3.141
	Variable	FLR-BR-VL	Uncapacitated	40,625.60	0.000	0.07	40,622.67	0.03	0.005
		FLR-BR-VL	95.0	42,417.96	0.000	37.77	41,549.20	0.03	2.082
		FLR-BR-VL	92.5	42,974.98	0.000	78.79	41,832.22	0.02	2.699
		FLR-BR-VL	90.0	43,888.01	0.005	129.58	42,515.59	0.27	3.136
Multiple	No	FLR-BM	Uncapacitated	40,014.41	0.000	0.04	40,014.34	0.01	0.000
		FLR-BM	95.0	41,414.17	0.000	19.74	40,661.10	0.04	1.810
		FLR-BM	92.5	41,892.85	0.000	80.04	40,890.19	0.02	2.350
		FLR-BM	90.0	42,899.13	0.051	171.33	41,554.62	0.02	3.000
	Fixed	FLR-BM-FL	Uncapacitated	40,626.89	0.000	0.16	40,624.08	0.04	0.005
		FLR-BM-FL	95.0	42,439.91	0.000	22.75	41,555.69	0.01	2.114
		FLR-BM-FL	92.5	43,054.38	0.027	172.41	41,843.81	0.02	2.848
		FLR-BM-FL	90.0	44,055.50	0.005	212.20	42,545.78	0.02	3.390
	Variable	FLR-BM-VL	Uncapacitated	40,626.89	0.000	0.15	40,624.08	0.05	0.005
		FLR-BM-VL	95.0	42,437.14	0.000	43.39	41,555.69	0.03	2.109
		FLR-BM-VL	92.5	43,035.49	0.022	143.48	41,843.74	0.03	2.812
		FLR-BM-VL	90.0	43,996.97	0.017	243.76	42,543.99	0.03	3.294

Table 28. Impact on Avg.IP and CPU Time from Decreasing Capacity Levels for the F-dataset

First, we notice that the results for the F-dataset are consistent with those for the G-dataset. The Avg. IP, CPU Time, and Avg. LP Gap all increase as the capacity levels are decreased. The results from the F-dataset, however, allow us to identify trends in the results by further reducing the capacity levels to 90 percent. Before we review the result in detail, it is important to elaborate on the LP gap results for the uncapacitated models. Uncapacitated formulations that considered both backlog and lost sales have an average LP gap that is not equal to zero. By examining the result more closely, whenever an instance included a stock-out in the optimal solution, the LP gap was not equal to zero. This happened for all formulations that included both backlog and lost sales. The same result did not happen for formulations that just considered backlog. U-FLR-B and U-

FLR-BR had at least one stock-out in every optimal solution yet the LP gap remained at zero, while U-FLR-LS had no stock-outs.

IP Computing Time

In the G-dataset we saw a large increase in the computing time resulting from a drop in capacity. Using the F-dataset results, we can determine whether smaller size problems experience the same impact on computing time. From the result in Table 29, we can see the increase in CPU Time that was also found in the G-dataset. The average computing times are lower but that is due to the small problem sizes in the F-dataset. The impact that capacity has on computing time, is also shown in the F-dataset. If we look at the percentage increase more closely using Table 29, we can see some differences compared to the results from the G-dataset.

Backlog	Lost Sales	Formulation	Capacity Level (%)	Avg. IP	Increase in Avg. IP (%)	CPU Time (sec)	Increase in CPU Time (%)	# of Time Limits
		FLR-LS	95.0	43,044.79		38.27		0
No	Fixed	FLR-LS	92.5	43,974.15	2.159	168.41	340.05	3
		FLR-LS	90.0	45,211.67	2.814	224.11	33.07	2
		FLR-B	95.0	41,071.30		8.58		0
	No	FLR-B	92.5	41,460.68	0.948	30.24	252.62	0
		FLR-B	90.0	42,061.37	1.449	79.84	164.02	0
		FLR-B-FL	95.0	42,417.96		33.24		0
Unlimited	Fixed	FLR-B-FL	92.5	42,975.89	1.315	82.29	147.59	0
		FLR-B-FL	90.0	43,890.93	2.129	142.72	73.43	1
		FLR-B-VL	95.0	42,417.96		25.03		0
	Variable	FLR-B-VL	92.5	42,974.98	1.313	114.15	356.05	0
		FLR-B-VL	90.0	43,888.01	2.125	141.01	23.54	1
		FLR-BR	95.0	41,071.30		7.83		0
	No	FLR-BR	92.5	41,460.68	0.948	29.27	273.98	0
		FLR-BR	90.0	42,061.37	1.449	79.30	170.92	0
		FLR-BR-FL	95.0	42,417.96		43.06		0
Restricted	Fixed	FLR-BR-FL	92.5	42,975.89	1.315	80.10	86.02	0
		FLR-BR-FL	90.0	43,891.13	2.130	141.98	77.26	1
		FLR-BR-VL	95.0	42,417.96		37.77		0
	Variable	FLR-BR-VL	92.5	42,974.98	1.313	78.79	108.62	0
		FLR-BR-VL	90.0	43,888.01	2.125	129.58	64.47	1
		FLR-BM	95.0	41,414.17		19.74		0
	No	FLR-BM	92.5	41,892.85	1.156	80.04	305.47	0
		FLR-BM	90.0	42,899.13	2.402	171.33	114.06	2
		FLR-BM-FL	95.0	42,439.91		22.75		0
Multiple	Fixed	FLR-BM-FL	92.5	43,054.38	1.448	172.41	657.73	2
		FLR-BM-FL	90.0	44,055.50	2.325	212.20	23.08	1
		FLR-BM-VL	95.0	42,437.14		43.39		0
	Variable	FLR-BM-VL	92.5	43,035.49	1.410	143.48	230.67	2
		FLR-BM-VL	90.0	43,996.97	2.234	243.76	69.88	1

Table 29. Percentage Change in Optimal Solutions between Different Capacity Levels for the F-dataset

Across all formulations, the jump in computing time when capacity is decreased from 95 to 92.5 percent is approximately three times larger than in the G-dataset. The increase is significantly lower when capacity is further lowered from 92.5 to 90 percent. Figure 5 illustrates how the computing time increases as capacity is lowered.

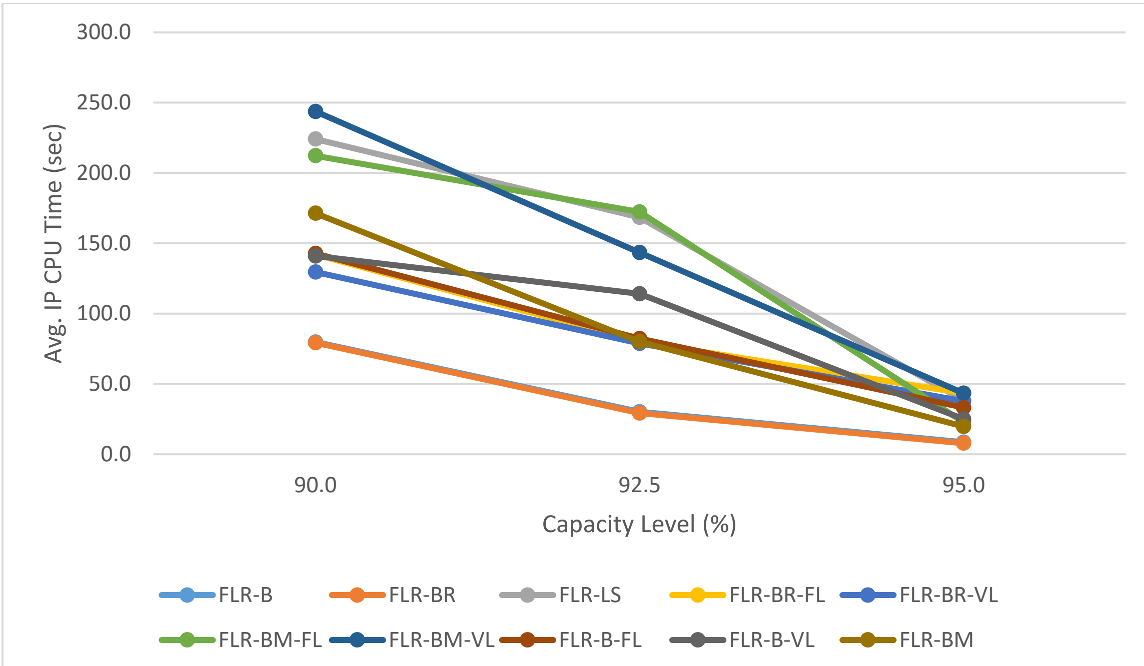

Figure 5. Comparison of Computing Time under Different Capacity Levels for F-dataset

The results indicate that there is a large decrease in CPU Time of on average 272.6 percent as the capacity level is changed from 92.5 to 95 percent and then a decrease of 532.2 percent with a change from 90 to 95 percent. There is however no evidence that the computing time decreases at an increasing rate and we do not have enough observations to test this.

Another difference in the F-dataset compared to the G-dataset is the number of instances that reached the time limit. The G-dataset had many instances that reached the time limit at 95 percent capacity level and the number rose for all models as the capacity level was changed to 92.5 percent. Although the number of instances reaching the time limit slightly increased for the F-dataset when changing the capacity from 92.5 to 95 percent, it is not to the same degree as the G-dataset. These results suggest that problems of larger size could be more time sensitive when capacity is lowered. Another possible explanation is that the F-dataset had a more "loose" capacity to begin with.

IP Objective Function Value

From the results in Table 29, we can see that the optimal value of the objective function increases as the capacity level decreases. In Figure 6, we see that the values rise at an increasing rate as the capacity levels decrease.

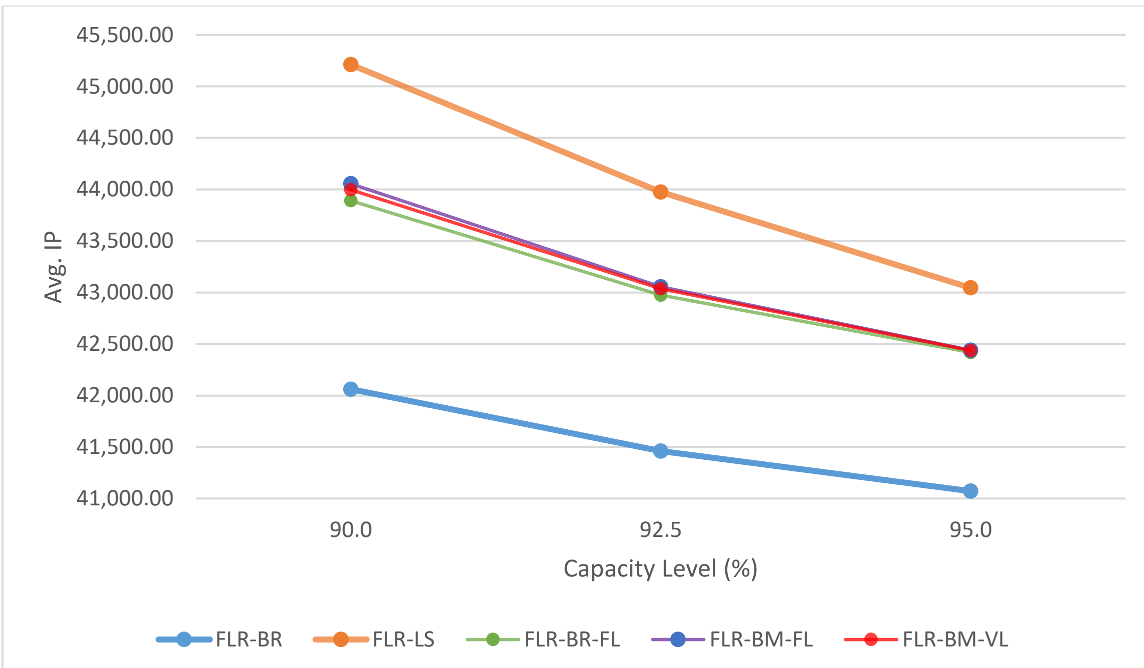

Figure 6. Comparison of Solution under Different Capacity Levels for F-dataset

For all formulations, the percentage increase was larger when we decreased the capacity levels by a second 2.5 percent (i.e. from 92.5 to 90 percent). Table 29 displays the percent increase in the Avg. IP for every 2.5 percent decrease in capacity. The formulation the most affected was again FLR-LS, which we attribute to the formulation's lack of flexibility. When faced with lower capacity levels, FLR-LS is forced into accepting more lost sales which have a very high cost. In contrast, FLR-BR, FLR-BR-FL, FLR-BR-VL, FLR-BM-FL and FLR-BM-VL have more flexibility with the option of backlog which has a lower cost then lost sales.

LP Solution

The LP Time across all formulations remained very low and fairly consistent as the capacity decreased. Meanwhile, the LP gap increased for all formulations as the capacity decreased.

Summary

To summarize, the capacity has a significant impact on the performance of the formulations. As capacity decreases, the computing time, the optimal value of the objective function, and the LP gap all significantly increase.

6.1.2. Comparison of the Structure of the Solution

The increase in total cost can be traced back to the change in the structure of the solution forced by the tightened capacity. Table 30 and Table 31 illustrate the information on the structure of the solution when capacity levels are decreased for the F-dataset and G-dataset respectively. The column "Setup Cost (%)" indicates the percentage of the total cost that is made up of the cost to setup production. The following three rows do the same for holding, backlog, and lost sales costs in that order.

Backlog	Lost Sales	Formulation	Capacity level (%)	Setup Cost (%)	Holding Cost (%)	Backlog Cost (%)	Lost Sales Cost (%)	# of no B	# of no LS
No	Fixed	FLR-LS	Uncapacitated	66.4	33.6	N/A	0.0	N/A	70
		FLR-LS	95.0	66.9	31.3	N/A	1.8	N/A	41
		FLR-LS	92.5	65.1	31.8	N/A	3.1	N/A	31
		FLR-LS	90.0	61.4	33.0	N/A	5.6	N/A	26
Unlimited	No	FLR-B	Uncapacitated	62.8	30.5	6.7	N/A	0	N/A
		FLR-B	95.0	62.5	29.3	8.2	N/A	0	N/A
		FLR-B	92.5	61.2	28.5	10.3	N/A	0	N/A
		FLR-B	90.0	58.6	28.5	12.9	N/A	0	N/A
	Fixed	FLR-B-FL	Uncapacitated	66.2	33.6	0.1	0.2	65	65
		FLR-B-FL	95.0	64.6	32.1	1.4	1.9	15	15
		FLR-B-FL	92.5	63.9	31.3	2.0	2.8	10	10
		FLR-B-FL	90.0	60.7	32.3	3.0	4.1	10	10
	Variable	FLR-B-VL	Uncapacitated	66.1	33.6	0.1	0.2	65	65
		FLR-B-VL	95.0	64.6	32.1	1.4	1.9	15	15
		FLR-B-VL	92.5	63.9	31.4	2.0	2.8	10	10
		FLR-B-VL	90.0	60.6	32.3	2.9	4.1	10	10
Restricted	No	FLR-BR	Uncapacitated	62.8	30.5	6.7	N/A	0	N/A
		FLR-BR	95.0	62.5	29.3	8.2	N/A	0	N/A
		FLR-BR	92.5	61.2	28.5	10.3	N/A	0	N/A
		FLR-BR	90.0	58.6	28.5	12.9	N/A	0	N/A
	Fixed	FLR-BR-FL	Uncapacitated	66.2	33.6	0.1	0.2	65	65
		FLR-BR-FL	95.0	64.6	32.1	1.4	1.9	15	15
		FLR-BR-FL	92.5	63.9	31.3	2.0	2.8	10	10
		FLR-BR-FL	90.0	60.7	32.3	3.0	4.1	10	10
	Variable	FLR-BR-VL	Uncapacitated	66.1	33.6	0.1	0.2	65	65
		FLR-BR-VL	95.0	64.6	32.1	1.4	1.9	15	15
		FLR-BR-VL	92.5	63.9	31.4	2.0	2.8	10	10
		FLR-BR-VL	90.0	60.6	32.3	2.9	4.1	10	10
Multiple	No	FLR-BM	Uncapacitated	63.6	31.1	5.3	N/A	0	N/A
		FLR-BM	95.0	63.1	30.5	6.4	N/A	0	N/A
		FLR-BM	92.5	61.8	31.3	6.9	N/A	0	N/A
		FLR-BM	90.0	59.6	31.8	8.6	N/A	0	N/A
	Fixed	FLR-BM-FL	Uncapacitated	66.2	33.6	0.1	0.1	66	66
		FLR-BM-FL	95.0	64.9	32.0	1.3	1.8	17	17
		FLR-BM-FL	92.5	63.4	32.1	1.9	2.6	10	10
		FLR-BM-FL	90.0	60.8	32.6	2.7	3.8	12	12
	Variable	FLR-BM-VL	Uncapacitated	66.2	33.6	0.1	0.1	66	66
		FLR-BM-VL	95.0	64.9	32.0	1.3	1.9	17	17
		FLR-BM-VL	92.5	63.6	31.9	1.8	2.7	10	10
		FLR-BM-VL	90.0	60.7	32.8	2.6	4.0	12	12

Table 30. Cost Solution Structure of the F-dataset with Different Capacity Levels

Backlog	Lost Sales	Formulation	Capacity level (%)	Setup Cost (%)	Holding Cost (%)	Backlog Cost (%)	Lost Sales Cost (%)	# of no B	# of no LS
		FLR-LS	Uncapacitated	68.0	32.0	N/A	0.0	N/A	75
No	Fixed	FLR-LS	95.0	65.8	31.5	N/A	2.7	N/A	36
		FLR-LS	92.5	62.6	32.3	N/A	5.1	N/A	31
		FLR-B	Uncapacitated	65.5	30.3	4.2	N/A	11	N/A
	No	FLR-B	95.0	62.2	30.0	7.8	N/A	0	N/A
		FLR-B	92.5	59.2	30.2	10.6	N/A	0	N/A
		FLR-B-FL	Uncapacitated	68.0	32.0	0.0	0.0	75	75
Unlimited	Fixed	FLR-B-FL	95.0	65.1	31.5	1.4	2.0	21	21
		FLR-B-FL	92.5	61.7	32.5	2.4	3.4	15	15
		FLR-B-VL	Uncapacitated	68.0	32.0	0.0	0.0	75	75
	Variable	FLR-B-VL	95.0	65.1	31.5	1.4	2.0	21	21
		FLR-B-VL	92.5	61.8	32.4	2.2	3.7	15	15
		FLR-BR	Uncapacitated	65.5	30.3	4.2	N/A	11	N/A
	No	FLR-BR	95.0	62.2	30.0	7.8	N/A	0	N/A
		FLR-BR	92.5	59.1	30.3	10.6	N/A	0	N/A
		FLR-BR-FL	Uncapacitated	68.0	32.0	0.0	0.0	75	75
Restricted	Fixed	FLR-BR-FL	95.0	65.1	31.5	1.4	2.0	20	20
		FLR-BR-FL	92.5	61.7	32.5	2.4	3.4	15	15
		FLR-BR-VL	Uncapacitated	68.0	32.0	0.0	0.0	75	75
	Variable	FLR-BR-VL	95.0	65.0	31.5	1.4	2.0	21	21
		FLR-BR-VL	92.5	61.8	32.4	2.2	3.7	15	15
		FLR-BM	Uncapacitated	66.2	30.3	3.4	N/A	0	N/A
	No	FLR-BM	95.0	62.5	31.3	6.1	N/A	0	N/A
		FLR-BM	92.5	60.2	31.3	8.3	0.1	1	N/A
		FLR-BM-FL	Uncapacitated	68.0	32.0	0.0	0.0	75	75
Multiple	Fixed	FLR-BM-FL	95.0	65.1	31.7	1.3	1.9	20	20
		FLR-BM-FL	92.5	61.6	31.0	2.5	3.5	17	17
		FLR-BM-VL	Uncapacitated	68.0	32.0	0.0	0.0	75	75
	Variable	FLR-BM-VL	95.0	65.1	31.7	1.3	1.9	20	20
		FLR-BM-VL	92.5	61.7	31.0	2.4	3.6	16	16

Table 31. Cost Solution Structure of the G-dataset with Different Capacity Levels

Table 30 and 31 show that the contribution of setup cost towards the total cost decreases when capacity is decreased because the backlog and/or lost sales costs become more important. The first thing to note is the structure of the solutions for the uncapacitated models. For the F-dataset we see some percentage backlog and lost sales cost in the optimal solutions for the uncapacitated models that consider both backlog and lost sales. We do not, however, see this in the results of the G-dataset. This could explain their differences in the LP gap as mentioned in the IP Objection Function Value subsection of Section 6.1.1.

Analysis of Costs

The results suggest that the proportion of setup cost does decrease as the capacity levels decrease, but not by a large amount. The proportion of backlog and lost sales cost rise significantly and this could explain the decrease in the setup cost's contribution to the optimal solution. The formulations with only backlog exhibit the smallest proportion of setup and holding cost at all capacity levels compared to the other model. The reason behind this is because for these formulations, the backlog cost constitutes a substantial proportion of the total costs. The combined percentage increase in the portion of backlog and lost sales from a decrease in capacity is seen to be similar for all the formulations. This could indicate that capacity has a similar impact on all formulations.

Analysis of Decisions

For greater insight, Table 32 summarizes the average decision variable values of all formulations for the different capacity levels for the F-dataset.

Backlog	Lost Sales	Formulation	Capacity level (%)	# of Setup	# of Holding	# of Backlog	# of Lost Sales	# of no B	# of no LS
No	Fixed	FLR-LS	Uncapacitated	42.2	4,012.8	N/A	0.0	N/A	70
		FLR-LS	95.0	44.0	3,896.5	N/A	34.9	N/A	41
		FLR-LS	92.5	43.6	3,929.3	N/A	58.5	N/A	31
		FLR-LS	90.0	42.0	4,059.6	N/A	102.4	N/A	26
Unlimited	No	FLR-B	Uncapacitated	40.3	3,682.8	509.4	N/A	0	N/A
		FLR-B	95.0	40.5	3,620.7	591.3	N/A	0	N/A
		FLR-B	92.5	40.2	3,573.8	668.9	N/A	0	N/A
		FLR-B	90.0	38.7	3,551.3	835.0	N/A	0	N/A
	Fixed	FLR-B-FL	Uncapacitated	42.2	4,002.8	11.0	3.7	65	65
		FLR-B-FL	95.0	42.6	3,918.3	95.1	31.7	15	15
		FLR-B-FL	92.5	42.4	3,869.5	145.0	48.3	10	10
		FLR-B-FL	90.0	40.9	3,927.3	214.2	71.4	10	10
	Variable	FLR-B-VL	Uncapacitated	42.2	4,004.1	11.0	3.7	65	65
		FLR-B-VL	95.0	42.6	3,918.6	95.1	31.7	15	15
		FLR-B-VL	92.5	42.4	3,870.2	142.7	47.7	10	10
		FLR-B-VL	90.0	40.9	3,932.5	211.3	72.8	10	10
Restricted	No	FLR-BR	Uncapacitated	40.3	3,682.2	510.0	N/A	0	N/A
		FLR-BR	95.0	40.5	3,628.4	584.2	N/A	0	N/A
		FLR-BR	92.5	40.2	3,576.1	668.4	N/A	0	N/A
		FLR-BR	90.0	38.7	3,552.6	835.4	N/A	0	N/A
	Fixed	FLR-BR-FL	Uncapacitated	42.2	4,002.8	11.0	3.7	65	65
		FLR-BR-FL	95.0	42.6	3,917.2	95.1	31.7	15	15
		FLR-BR-FL	92.5	42.4	3,865.5	145.0	48.3	10	10
		FLR-BR-FL	90.0	40.9	3,930.5	214.3	71.4	10	10
	Variable	FLR-BR-VL	Uncapacitated	42.2	4,004.1	11.0	3.7	65	65
		FLR-BR-VL	95.0	42.6	3,918.6	95.1	31.7	15	15
		FLR-BR-VL	92.5	42.4	3,870.2	142.7	47.7	10	10
		FLR-BR-VL	90.0	40.9	3,932.7	211.3	72.8	10	10
Multiple	No	FLR-BM	Uncapacitated	40.7	3,745.2	402.5	N/A	0	N/A
		FLR-BM	95.0	40.9	3,741.1	435.8	N/A	0	N/A
		FLR-BM	92.5	40.4	3,762.1	475.9	N/A	0	N/A
		FLR-BM	90.0	39.1	3,775.8	587.8	N/A	0	N/A
	Fixed	FLR-BM-FL	Uncapacitated	42.2	4,004.4	8.5	2.8	66	66
		FLR-BM-FL	95.0	42.8	3,915.7	89.1	29.7	17	17
		FLR-BM-FL	92.5	42.1	3,910.8	137.1	45.7	10	10
		FLR-BM-FL	90.0	41.0	3,940.4	200.6	66.9	12	12
	Variable	FLR-BM-VL	Uncapacitated	42.2	4,005.6	8.5	2.8	66	66
		FLR-BM-VL	95.0	42.8	3,913.4	88.0	29.8	17	17
		FLR-BM-VL	92.5	42.3	3,900.4	132.6	46.1	10	10
		FLR-BM-VL	90.0	40.9	3,962.4	189.1	69.6	12	12

Table 32. Decisions Solution Structure of the F-dataset with Different Capacity Levels

In the results from Table 32, the number of backlogged and lost demand increases as capacity ranges from uncapacitated to 90 percent capacity level. These results confirm the ideas discussed in the "*Analysis of Costs*" section, where the reduction of capacity leads to greater use of backlog and lost sales in the optimal solutions. As for the number of setups, the amount decreases when the capacity level is reduced from 95 to 90 percent. However, for all formulations, the number of setups slightly increase when going from the uncapacitated models to the models with 95 percent capacity levels. An explanation behind the difference in the number setups between the different capacity levels is not straightforward. There are two opposing tendencies in play. When capacity is reduced, the following possibilities can occur:

1. An increase in the number of production setups because of a reduced ability to satisfy demand for longer periods of time.
2. A decrease in the number of production setups because the capacity (which is already scarce) is consumed by setups through setup times.

The two tendencies play a role in determining the number of setups that are included in the optimal solutions.

Firstly it is important to note that in most cases, backlog is an effective strategy to reduce total cost. Therefore, for FLR-B and FLR-BR, the majority of the instances have stock-outs included in the optimal solution. This is the case even for the uncapacitated version of the models. Incurring backlogs instead of having to schedule an additional production setup can be cost efficient.

The “# of no B” column, shows the number instances out of 70 instances for the F-dataset and 75 for the G-dataset that do not include backlog in the optimal solution. The last column does the same for instances in which lost sales is not included. The number of instances that do not use backlog and lost sales in their optimal solution are shown to either decrease or stay the same as capacity is reduced. This is logical as lower capacity levels will force some backlog and/or lost sales because of feasibility. One exception is FLR-BM-FL and FLR-BM-VL in the F-dataset where the number of instances without a stock-out increases by two when going from 92.5 to 90 percent. On closer inspection, when capacity is lowered it may be optimal for some instances to have more frequent setups instead of paying for the backlog and lost sales penalty. Another reason could be because some instances were not solved to optimality which skewed the results. In most cases however, when capacity is tightened, lost sales and backlog will be used more frequently. Figure 7 illustrates the increasing trend in the total backlog and lost sales costs when capacity is decreased.

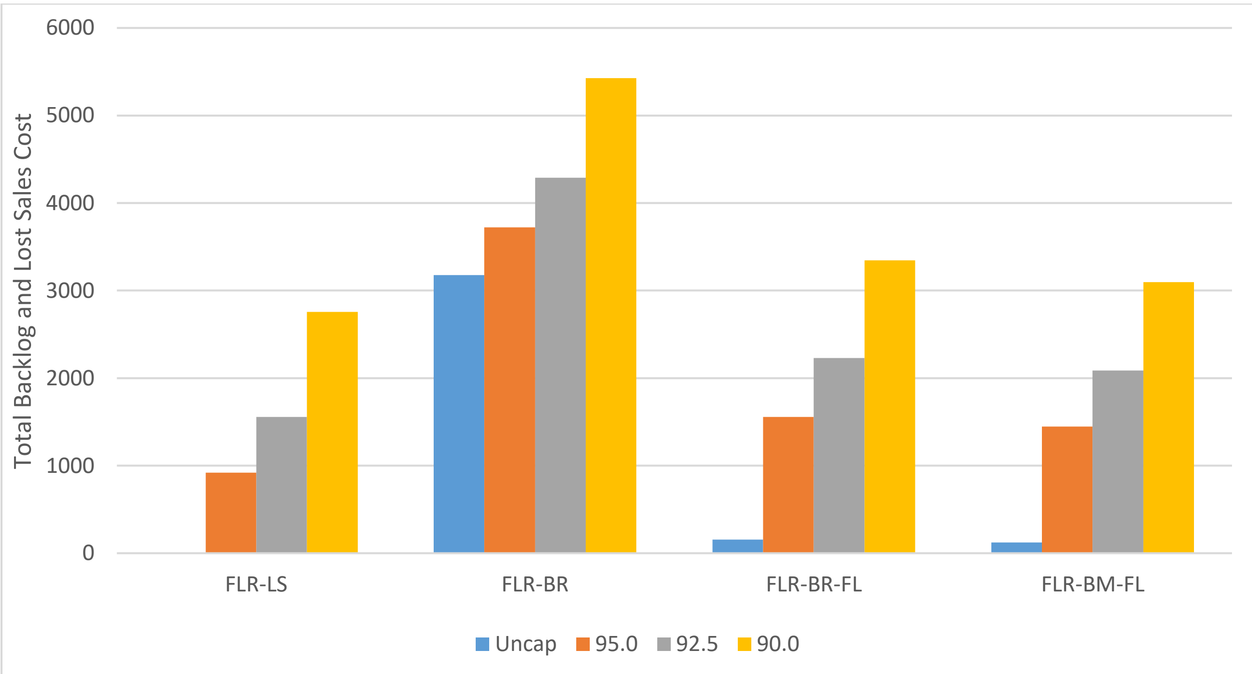

Figure 7. Comparison of Combined Total Backlog and Lost Sales Costs under Different Capacity Levels for the F-dataset

In the uncapacitated formulations, we can find the optimal combination of backlog and lost sales without any capacity restrictions. When capacity is reduced, stock-outs become more frequent, which leads to more backlog and lost sales. An alternative to stock-outs when capacity is reduced is scheduling additional production setups if feasible. Typically backlog and lost sales are the cheaper options and therefore this explains the rising trend that appears in Figure 7.

6.2. Impact of $\boldsymbol{\alpha}$

The parameter α indicates the percentage of customers willing to wait if a stock-out has happened. Up to this point, we have assumed that when a stock-out occurs, 75 percent of all customers are willing to wait for backlog (for the relevant formulations). For problems with only one customer type, the backlogged demand can be satisfied up to r periods later if backlog is restricted, or up to period m if backlog is unrestricted. To explore the impact of the α parameter on our formulations, we conduct an experiment with five different assumptions on the customer's willingness to wait when a stock-out occurs, as indicated in Table 33. For problems with multiple customers, we adjust each customer type proportionally, while maintaining the following condition: $\sum_{q\in R} B_q = \alpha$. Note that the case with α = 75% was the base case used in the previous experiments.

α(%)	β(%)
55	[18.33, 14.67, 11.00, 11.00]
65	[21.67, 17.33, 13.00, 13.00]
75	[25.00, 20.00, 15.00, 15.00]
85	[28.33, 22.67, 17.00, 17.00]
95	[31.67, 25.33, 19.00, 19.00]
100	[33.33, 26.67, 20.00, 20.00]

Table 33. Relationship between α and B_q

6.2.1. Performance Comparison

We test the five different assumptions for the customer's willingness to wait on all formulations that consider both backlog and lost sales simultaneously using the F-dataset. The results for the average best found solution and computational time are shown in Table 34.

Backlog	Lost Sales	Formulation	α	Avg. IP	Avg. Gap (%)	Avg. CPUTime	Avg. LP	Avg. LPTime	Avg. LP Gap (%)	# of Time Limits
Unlimited	Fixed	FLR-B-FL	55%	43412.36	0.031	179.41	42106.41	0.02	3.040	3
			65%	43228.20	0.013	144.42	41990.06	0.02	2.900	1
			75%	42975.89	0.000	82.29	41832.22	0.01	2.701	0
			85%	42605.86	0.000	65.58	41597.10	0.02	2.413	0
			95%	41970.01	0.000	47.46	41105.06	0.02	2.083	0
	Variable	FLR-B-VL	55%	43408.57	0.018	160.23	42106.41	0.02	3.033	1
			65%	43226.48	0.009	131.41	41990.06	0.03	2.897	1
			75%	42974.98	0.000	114.15	41832.22	0.03	2.699	0
			85%	42605.39	0.000	48.21	41597.10	0.02	2.412	0
			95%	41969.64	0.000	42.81	41105.06	0.02	2.083	0
Restricted	Fixed	FLR-BR-FL	55%	43412.36	0.040	215.05	42106.41	0.03	3.040	4
			65%	43228.20	0.016	129.97	41990.06	0.02	2.900	2
			75%	42975.89	0.000	80.10	41832.22	0.02	2.701	0
			85%	42605.86	0.000	56.19	41597.10	0.02	2.413	0
			95%	41970.01	0.000	42.16	41105.06	0.03	2.083	0
	Variable	FLR-BR-VL	55%	43408.57	0.017	125.78	42106.41	0.02	3.033	1
			65%	43226.48	0.010	133.88	41990.06	0.02	2.897	1
			75%	42974.98	0.000	78.79	41832.22	0.02	2.699	0
			85%	42605.39	0.000	46.12	41597.10	0.02	2.412	0
			95%	41969.64	0.000	27.95	41105.06	0.02	2.083	0
Multiple	Fixed	FLR-BM-FL	55%	43467.91	0.035	198.01	42112.64	0.02	3.132	2
			65%	43289.45	0.039	184.68	41996.74	0.02	3.010	4
			75%	43054.38	0.027	172.41	41843.81	0.02	2.848	2
			85%	42730.41	0.018	138.98	41625.93	0.03	2.623	2
			95%	42228.24	0.000	86.20	41218.53	0.02	2.395	0
	Variable	FLR-BM-VL	55%	43433.49	0.065	211.57	42112.63	0.03	3.077	4
			65%	43266.88	0.059	247.11	41996.73	0.05	2.970	4
			75%	43035.49	0.022	143.48	41843.74	0.03	2.812	2
			85%	42700.82	0.017	119.09	41625.74	0.02	2.566	1
			95%	42192.59	0.000	65.61	41217.97	0.02	2.326	0

Table 34. Optimal Solution and Computing Time for Different Values of α using the F-dataset

IP Computing Time

By examining the results in Table 34, we notice that some general trends remain consistent with the previous tests. Computing time generally increases as the value of α is decreased for all formulations. There are some exceptions for some formulations where the computing time is greater when the value of α is 65 percent compared to 55 percent. The results do not show enough evidence that the computing time of certain formulations are more affected than others when the value of α is reduced.

Figure 8 illustrates the impact of α on the computing time of the FLR-BM-FL and FLR-BM-VL formulations.

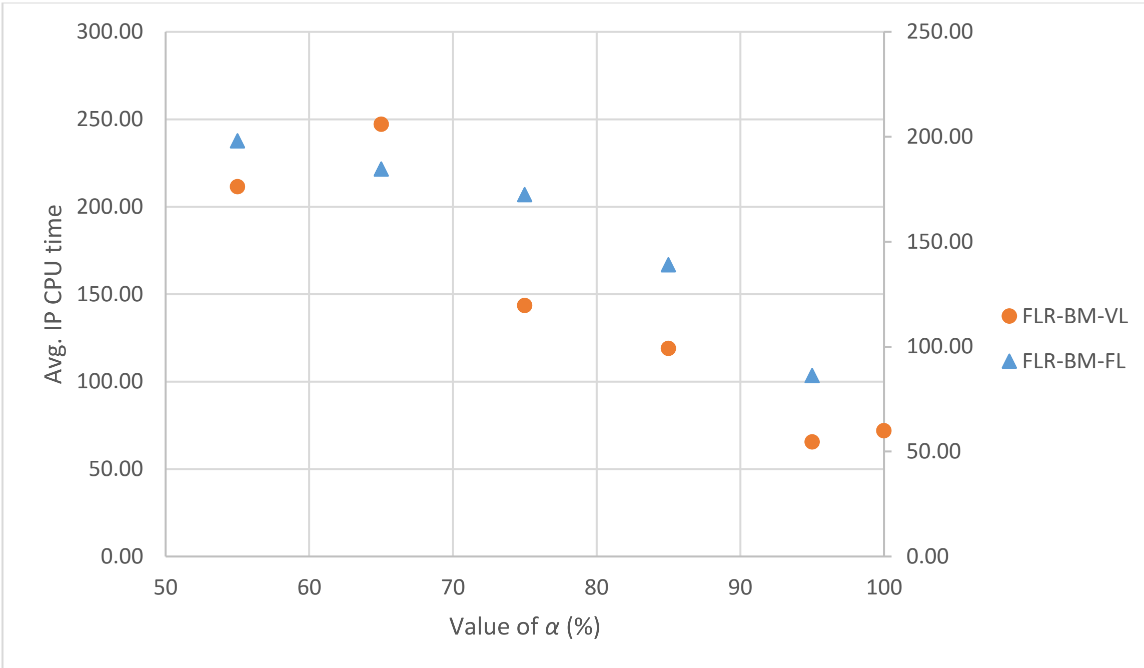

Figure 8. Comparison of Computing Time between FLR-BM-FL and FLR-BM-VL for Different Values of α using the F-dataset

Notice in Figure 8 the point at the right extremity of the chart: it represents the optimal solution for the formulations when α equals 100 percent. When α = 100%, all customers are willing to wait for backlog when a stock-out occurs, however different customers have different willingness' to wait. Only the FLR-BM-VL formulation is used to obtain the results for α = 100%, because the FLR-BM-FL formulation experienced feasibility issues when α = 100%.

From Figure 8, we observe that the computing times generally decrease when α is increased. Furthermore, FLR-BM-VL generally exhibits lower CPU times compared to FLR-BM-FL for the same level of α. An exception to both general trends is the change of α from 55 to 65 percent for FLR-BM-VL.

The computing time for formulations with a single customer type seems to exhibit a different trend than formulations with multiple customer types. Figure 9 compares the computing time of the FLR-BM-FL and FLR-BR-FL formulations.

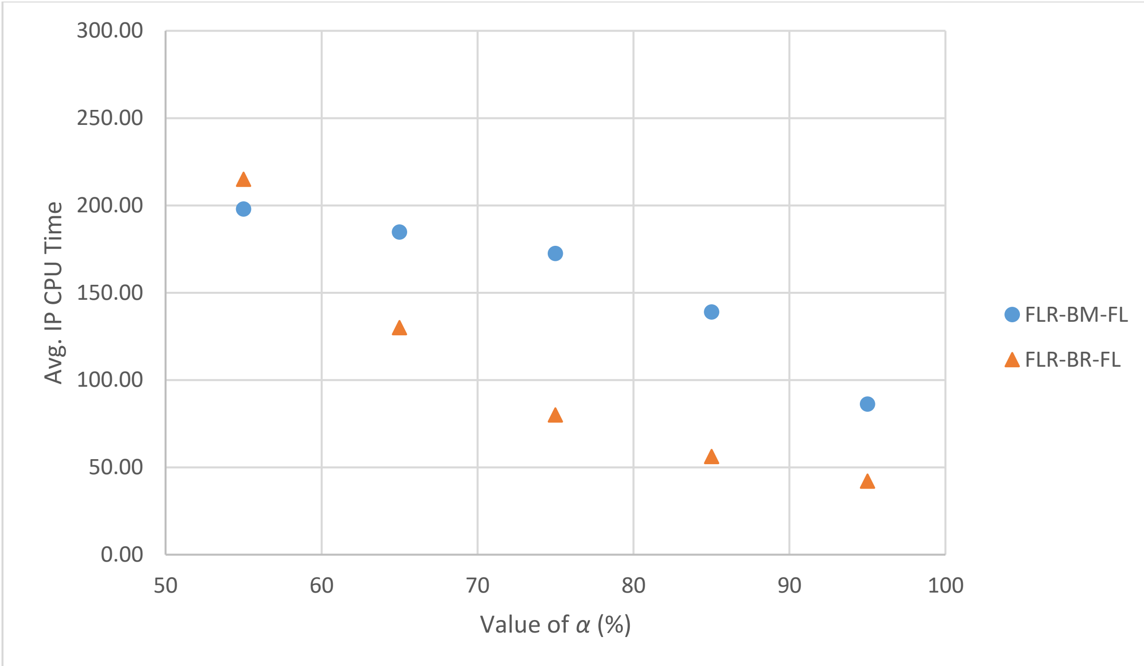

Figure 9. Comparison of Computing Time between FLR-BM-FL and FLR-BR-FL for Different Values of α using the F-dataset

The computing time decreases in a convex fashion for the FLR-BR-FL formulation compared to a concave fashion for the FLR-BM-FL formulation. The reason that the computational time is decreased quicker for FLR-BR-FL could be because of its ability to use backlog more liberally compared to FLR-BM-FL. This pattern between formulations with a single customer type versus multiple customer types is consistent across the formulations with different lost sales types and backlog assumptions. Based on our results there is strong indication that the computing time is impacted differently for models with multiple customer types compared to single customer types when the value of α is reduced. Results for α = 100% were not recorded because of feasibility issues that arise with the FLR-BM-FL and FLR-BR-FL formulations for when α = 100%.

IP Objective Function Value

By reviewing the results illustrated in Table 34, we can identify the apparent impact on the best found value of the objective function when the values of α and β are adjusted. In all cases as the value of α is reduced, the variable lost sales version of the formulation provides a better total cost than the fixed lost sales version. The difference is bigger for the multiple customer case compared to the restricted or unlimited backlog case. The backlog restriction does not have an impact on the best found value of the objective function, however this is true for when r = 4. In Section 6.3.1, we review if the best found value of the objective function is affected when the value of r is reduced. Lastly, the formulations with multiple customer types provide higher total costs than those that consider a single customer type, no matter the value of α and β.

We suggest that by allowing the possibility of more backlog, the formulation has more flexibility and can therefore provide a better solution in less time. In Figure 10, we graph the best found objective function value of the FLR-BM-FL and FLR-BM-VL formulations according to the value of α ranging from 55 to 100 percent.

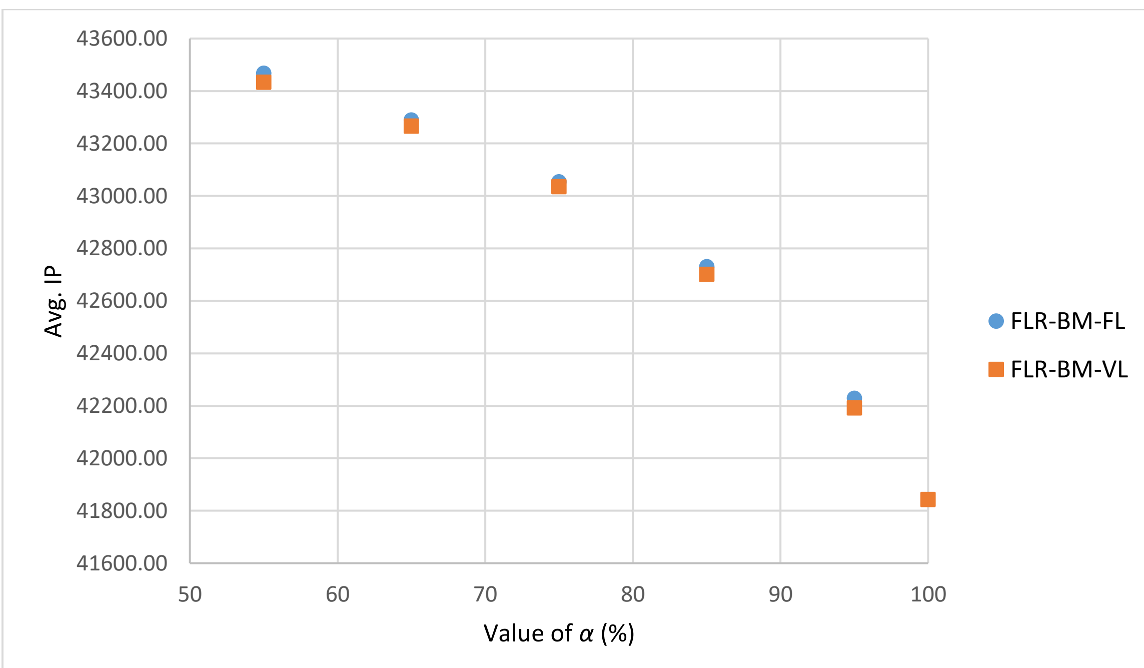

Figure 10. Comparison of Best Found Objective Function Values between FLR-BM-FL and FLR-BM-VL for Different Values of α using the F-dataset

Figure 10 shows that the total cost decreases progressively as more backlog is allowed in the formulation. This suggests that the impact of backlog is strengthen due to the added flexibility it provides to the formulation. Only the FLR-BM-VL formulation is used to obtain the results for α = 100%, because the FLR-BM-FL formulation experienced feasibility issues when α = 100%.

To determine if this trend holds true with formulations with single customer type, we compare the best found objective function values for the different values of α between the FLR-BM-FL and the FLR-BR-FL formulations.

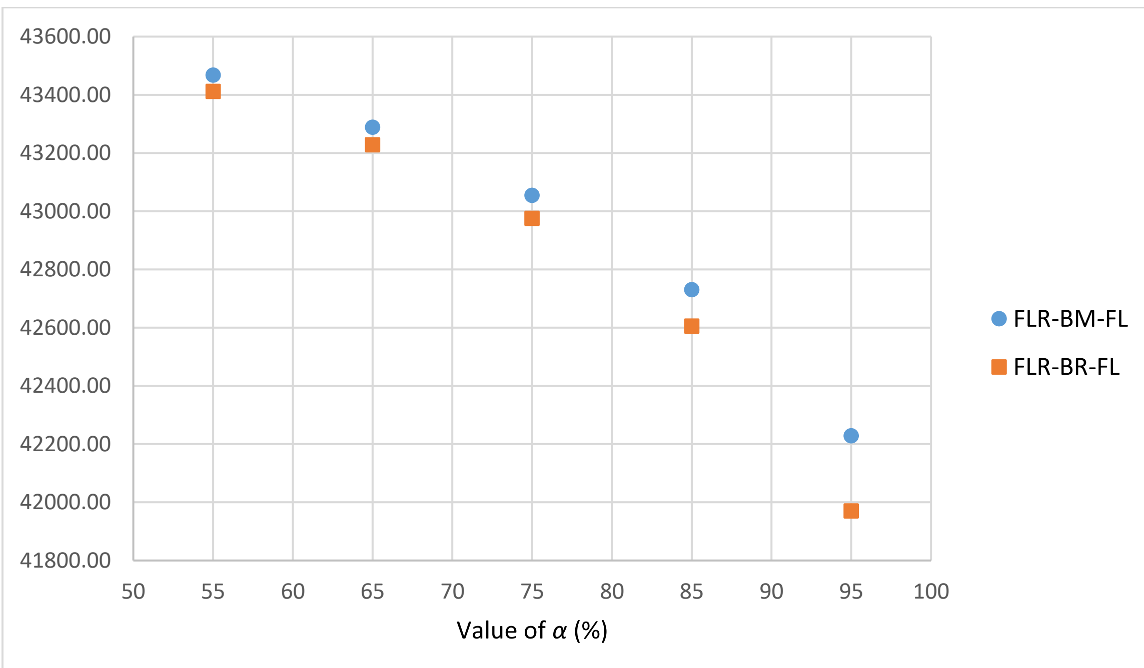

Figure 11. Comparison of Best Found Objective Function Values between FLR-BM-FL and FLR-BR-FL for Different Values of α using the F-dataset

Compared to the multiple customer type formulation, FLR-BR-FL exhibits a steeper decrease in the optimal value of the objective function when α is increased. The multiple customer type formulations have more restrictions in place, which could explain why it does not reduce the total cost as quickly. In the formulations with multiple customer types, there are customers that are only willing to wait one or two weeks for backlog, while the FLR-BR formulations assume that all customers are willing to wait at least four weeks. Therefore there is an inherent advantage for FLR-BR to lower the optimal solution through the use of backlog. Results for α = 100% were not recorded because of feasibility issues that arise with the FLR-BM-FL and FLR-BR-FL formulations for when α = 100%.

LP Solution

The LP gap follows a similar pattern as the best found objective function does. As the value of α increases, the LP gap decreases. The LP gap decreases more the closer it gets to α = 100%. As for the LP time, it stays fairly consistent throughout. The LP time does not seem to be affected by the type of formulation nor the value of α.

6.2.2. Comparison of the Structure of the Solution

Analysis of Costs

To further analyze the impact of α on the models, we look into the changes in the structure of the solutions. Table 35 summarizes the structure of the solution of all models that consider both backlog and lost sales for different values of α.

Backlog	Lost Sales	Formulation	α	Setup Cost (%)	Holding Cost (%)	Backlog Cost (%)	Lost Sales Cost (%)	# of no B	# of no LS
Unlimited	Fixed	FLR-B-FL	55%	64.26	31.94	0.91	2.90	18	18
			65%	64.06	31.83	1.31	2.81	16	16
			75%	63.87	31.34	2.01	2.78	10	10
			85%	63.00	31.30	3.31	2.39	8	8
			95%	61.76	30.36	6.47	1.41	1	1
	Variable	FLR-B-VL	55%	64.09	32.06	0.88	2.98	18	18
			65%	64.03	31.85	1.30	2.82	16	16
			75%	63.87	31.38	1.99	2.76	10	10
			85%	63.00	31.31	3.30	2.38	8	8
			95%	61.79	30.34	6.46	1.41	1	1
Restricted	Fixed	FLR-BR-FL	55%	64.23	31.97	0.91	2.90	18	18
			65%	64.06	31.83	1.31	2.81	16	16
			75%	63.87	31.34	2.01	2.78	10	10
			85%	63.00	31.30	3.31	2.39	8	8
			95%	61.76	30.36	6.47	1.41	1	1
	Variable	FLR-BR-VL	55%	64.09	32.06	0.88	2.98	18	18
			65%	64.03	31.85	1.30	2.82	16	16
			75%	63.87	31.38	1.99	2.76	10	10
			85%	63.00	31.31	3.30	2.38	8	8
			95%	61.79	30.34	6.46	1.41	1	1
Multiple	Fixed	FLR-BM-FL	55%	63.97	32.25	0.86	2.93	21	21
			65%	63.67	32.34	1.23	2.76	18	18
			75%	63.42	32.11	1.86	2.61	10	10
			85%	63.24	31.89	2.80	2.07	8	8
			95%	62.26	31.71	4.94	1.09	1	1
	Variable	FLR-BM-VL	55%	64.06	32.16	0.82	2.97	20	20
			65%	63.90	32.15	1.18	2.76	16	16
			75%	63.61	31.92	1.82	2.65	10	10
			85%	63.27	31.83	2.75	2.16	8	8
			95%	62.32	31.42	5.03	1.24	1	1

Table 35. Cost Solution Structure for Different Values of α using the F-dataset

We have suggested throughout the thesis that the reasoning for the better solutions is the added flexibility that backlogging provides. The results in Table 35 indicate that as the value of α increases up to 100 percent, the percentage of backlog cost increases. In Figure 12, we illustrate the impact of backlog on the optimal solutions for different values of α for both FLR-BM-FL and FLR-BR-FL.

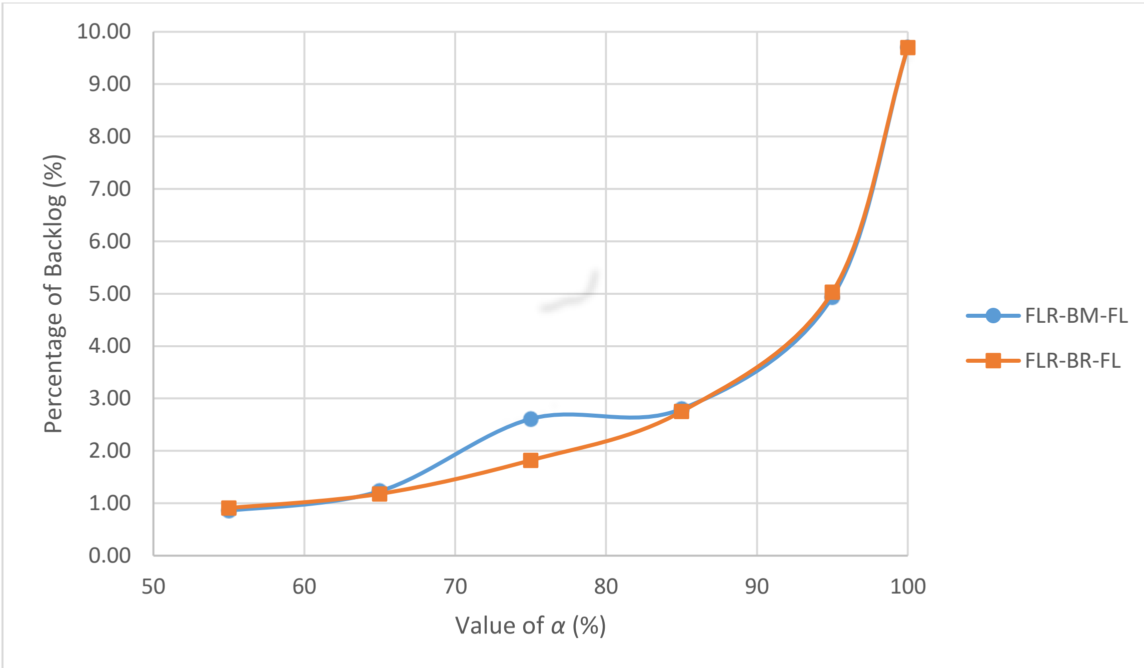

Figure 12. Percentage Impact of Backlog on the Optimal Solution for FLR-BM-FL and FLR-BR-FL for Different Values of α using the F-dataset

In Figure 12, we can clearly see a slope with increasing marginal impact as the value of α approaches 100 percent. This slope supports the notion that the impact of backlog grows increasingly as the value of α increases as well. In a scenario where both backlog and lost sales are present, backlog seems to thrive when customers are more patient therefore allowing the added flexibility to structure a production schedule that reduces overall cost.

Analysis of Decisions

For greater insight, Table 36 lists the average unit values of the decision variables. For the purpose of analyzing the disparity in the backlog and lost sales decision variables between the formulations with fixed and variable lost sales, we introduce a new "% Backlog of Stock-out" column. This column indicates the percentage of stock-out that is eventually satisfied by backlog. We can use this column to determine if there is a greater disparity in the usage of backlog between formulations with fixed and variable lost sales given different backlog assumption and different values of α.

Backlog	Lost Sales	Formulation	α	# of Setup	# of Holding	# of Backlog	# of Lost Sales	% Backlog of Stock-out	# of no B	# of no LS
Unlimited	Fixed	FLR-B-FL	55%	42.80	3,926.46	62.94	51.49	55.00	18	18
			65%	42.59	3,923.10	91.67	49.36	65.00	16	16
			75%	42.41	3,869.46	144.96	48.32	75.00	10	10
			85%	41.80	3,843.50	230.19	40.62	85.00	8	8
			95%	40.70	3,715.52	461.71	24.30	95.00	1	1
	Variable	FLR-B-VL	55%	42.70	3,937.86	61.15	52.82	53.65	18	18
			65%	42.57	3,919.94	90.71	49.74	64.58	16	16
			75%	42.41	3,870.16	142.73	47.74	74.94	10	10
			85%	41.80	3,849.79	229.40	40.54	84.98	8	8
			95%	40.71	3,717.24	460.94	24.29	94.99	1	1
Restricted	Fixed	FLR-BR-FL	55%	42.77	3,933.12	63.06	51.60	55.00	18	18
			65%	42.59	3,922.46	91.67	49.36	65.00	16	16
			75%	42.41	3,865.54	144.96	48.32	75.00	10	10
			85%	41.80	3,847.93	230.19	40.62	85.00	8	8
			95%	40.70	3,713.30	461.71	24.30	95.00	1	1
	Variable	FLR-BR-VL	55%	42.70	3,937.99	61.15	52.82	53.65	18	18
			65%	42.57	3,919.94	90.71	49.74	64.58	16	16
			75%	42.41	3,870.16	142.73	47.74	74.94	10	10
			85%	41.80	3,848.68	229.40	40.54	84.98	8	8
			95%	40.71	3,717.24	460.94	24.29	94.99	1	1
Multiple	Fixed	FLR-BM-FL	55%	42.70	3,938.36	64.21	52.54	55.00	21	21
			65%	42.39	3,938.28	91.97	49.52	65.00	18	18
			75%	42.14	3,910.82	137.05	45.68	75.00	10	10
			85%	41.86	3,875.17	201.53	35.56	85.00	8	8
			95%	40.87	3,827.00	346.79	18.25	95.00	1	1
	Variable	FLR-BM-VL	55%	42.73	3,939.63	59.40	52.79	52.95	20	20
			65%	42.54	3,933.01	84.58	48.81	63.41	16	16
			75%	42.26	3,900.39	132.61	46.05	74.22	10	10
			85%	41.87	3,877.14	197.99	37.05	84.24	8	8
			95%	40.87	3,809.60	357.90	21.04	94.45	1	1

Table 36. Decision Solution Structure for Different Values of α using the F-dataset

From Table 36, we observe that as the value of α increases, the number of setups decrease for all formulations. The results support the ideas discussed in the "*Analysis of Costs*" subsection, in which the use of backlog rises in an increasing fashion as the value of α increases from 55 to 95 percent. Lost sales as anticipated decreases when the value of α increases for all formulations, and the same occurs for the number of items held in inventory.

By examining the new "% Backlog of Stock-out" column, we notice that that discrepancy between the formulation with fixed lost sales versus variable lost sales decreases as the value of α increases. This indicates that additional lost sales are more likely to be incurred when the customer's willingness to wait is lower. The discrepancy between the formulations with fixed lost sales versus variable lost sales are larger for the formulations with multiple customer types compared to formulations with a single customer type. Therefore additional lost sales are more likely to be incurred when there are customers with different willingness to wait.

When the value of α increases, the number of instances without a single stock-out decreases. This trend is apparent for all formulation that include α in their formulation. As the value of α is increases, the maximum number of customers that must be lost when a stock-out occurs is reduced. Since the negative impact of the high lost cost is lessened, adding stock-outs in the optimal solution become more appealing because of the cost saving that backlogging can provide.

6.3. Impact of *r*

Up to this point, we have set our backlog restriction parameter (r) to 4 periods. This allows unsatisfied demand to be backlogged for a maximum of 4 periods. Throughout our results, formulations with backlog restrictions have produced the same optimal solution as formulations without backlog restrictions. In this section, we reduce the r parameter to determine if a lower backlog restriction will have an impact on the optimal solution.

6.3.1 Performance Comparison

The three models that incorporate backlog restrictions in their models are tested using the F-dataset at 92.5 percent capacity level with the value r ranging from 1 to 4. Table 37 summarizes the performance of the models.

Backlog	Lost Sales	Formulation	*r*	Avg. IP	Avg. Gap (%)	Avg. CPUTime (sec)	Avg. LP	Avg. LPTime (sec)	Avg. LP Gap (%)
Restricted	No	FLR-BR	4	41,460.68	0.000	29.27	40,654.04	0.02	1.944
		FLR-BR	3	41,460.68	0.000	26.91	40,654.04	0.02	1.944
		FLR-BR	2	41,460.68	0.000	22.67	40,654.04	0.01	1.944
		FLR-BR	1	41,490.87	0.000	27.04	40,673.46	0.03	1.966
	Fixed	FLR-BR-FL	4	42,975.89	0.000	80.10	41,832.22	0.02	2.701
		FLR-BR-FL	3	42,975.89	0.000	84.39	41,832.22	0.02	2.701
		FLR-BR-FL	2	42,975.89	0.000	89.92	41,832.22	0.01	2.701
		FLR-BR-FL	1	42,985.15	0.000	64.07	41,850.05	0.01	2.687
	Variable	FLR-BR-VL	4	42,974.98	0.000	78.79	41,832.22	0.02	2.699
		FLR-BR-VL	3	42,974.98	0.000	138.59	41,832.22	0.05	2.699
		FLR-BR-VL	2	42,974.98	0.000	133.94	41,832.22	0.03	2.699
		FLR-BR-VL	1	42,983.15	0.000	76.98	41,849.70	0.02	2.684

Table 37. Optimal Solution and Computing Time for Different Values of *r* using the F-dataset at a 92.5 percent Capacity Level

IP Computing Time

When the value of *r* is reduced, the computing time is affected, but not in a consistent way. FLR-BR has a very consistent computing time for all value of *r*. FLR-BR-FL and FLR-BR-VL both have their lowest computing time when the value of r is either 1 or 4, however there is not enough evidence to make any conclusions on the effect that the value of *r* has on computing time.

IP Objective Function Value

For all three formulations, the average optimal value of the objective function does not change when the value of *r* is reduced from 4 to 2. Only when the value of *r* is equal to 1, does the optimal value of the objective function change for the three formulations. The models that consider both backlog and lost sales experience a small increase in their optimal value when the value of *r* decreases from 2 to 1, while the FLR-BR formulation experiences a slightly greater increase.

For all models, there is only a change in the optimal value of the objective function when the value of *r* is reduced from 2 to 1. This indicates that in the original optimal solution when *r* = 4, backlog was held for a maximum of 2 periods. Therefore, the optimal solution would only differ significantly between an unlimited backlog model and a restricted backlog model if the value of *r* was less than 2. The impact of *r* can be higher if we would reduce the capacity further.

By reviewing the result we conclude that the value of *r* has a very minimal impact on the optimal value of the objective function. In most cases, the model aims to satisfy backlog immediately to minimize the backlog penalty cost. Therefore the impact of *r* is greatest the lower the value of *r* is.

LP Solution

The LP time remains consistent in all models for all values of *r*. The LP gap experiences a very small decline when the value of *r* decreases from 2 to 1.

6.3.2 Comparison of the Structure of the Solution

To explore in greater detail the impact of *r* on the optimal solution, we review the structure of the solutions illustrated in Table 38.

Backlog	Lost Sales	Formulation	*r*	Setup Cost (%)	Holding Cost (%)	Backlog Cost (%)	Lost Sales Cost (%)
Restricted	No	FLR-BR	4	61.19	28.47	10.34	N/A
		FLR-BR	3	61.19	28.47	10.34	N/A
		FLR-BR	2	61.16	28.44	10.40	N/A
		FLR-BR	1	61.43	28.40	10.17	N/A
	Fixed	FLR-BR-FL	4	63.87	31.34	2.01	2.78
		FLR-BR-FL	3	63.87	31.34	2.01	2.78
		FLR-BR-FL	2	63.83	31.38	2.01	2.78
		FLR-BR-FL	1	63.99	31.27	1.97	2.77
	Variable	FLR-BR-VL	4	63.87	31.38	1.99	2.76
		FLR-BR-VL	3	63.87	31.38	1.99	2.76
		FLR-BR-VL	2	63.87	31.38	1.99	2.76
		FLR-BR-VL	1	63.91	31.37	1.93	2.78

Table 38. Cost Solution Structure for Different Values of *r* using the F-dataset at a 92.5 percent Capacity Level

Analysis of Costs

The cost distribution does not experience any significant change when the value of *r* is reduced. The value of *r* does not seem to have an impact on the cost distribution of the optimal solution. In Table 38, we notice a small change in the cost percentages when changing *r* from 3 to

2. However there is no change in the best found objective function which therefore indicates the existence of alternative solutions.

Analysis of Decisions

For greater insight, Table 39 lists the unit values of the decision variables.

Backlog	Lost Sales	Formulation	*r*	# of Setup	# of Holding	# of Backlog	# of Lost Sales	# of no B	# of no LS
Restricted	No	FLR-BR	4	40.17	3575.07	668.66	N/A	0	N/A
		FLR-BR	3	40.17	3575.07	668.66	N/A	0	N/A
		FLR-BR	2	40.16	3567.93	672.33	N/A	0	N/A
		FLR-BR	1	40.29	3566.13	667.27	N/A	0	N/A
	Fixed	FLR-BR-FL	4	42.41	3865.10	144.96	48.32	10	10
		FLR-BR-FL	3	42.41	3865.10	144.96	48.32	10	10
		FLR-BR-FL	2	42.40	3869.17	144.96	48.32	10	10
		FLR-BR-FL	1	42.46	3862.79	144.19	48.06	10	10
	Variable	FLR-BR-VL	4	42.44	3866.03	142.64	47.71	10	10
		FLR-BR-VL	3	42.44	3866.03	142.64	47.71	10	10
		FLR-BR-VL	2	42.41	3865.73	142.73	47.74	10	10
		FLR-BR-VL	1	42.41	3865.51	140.58	48.35	10	10

Table 39. Decision Solution Structure Values for Different Values of *r* using the F-dataset at a 92.5 percent Capacity Level

By looking at the average unit values of the decision variables there are a few points to note. For the most part, the decisions remain the same as the value of *r* is reduced. The average units of backlog does not significantly drop which suggests that the majority of backlog is satisfied immediately. If we look at FLR-BR-VL, we see an increase of 0.61 units in the average number of lost sales used in the objective function, while the number of backlog decreases by 2.15 units when the value of *r* decreases from 2 to 1. This suggests that lost sales is used as strategy in some instances to reduce the total cost. The number of instances that do not have any stock-outs in their best found solution remains the same as the value of *r* changes.

Summary

The impact of *r* is very minimal as backlog is shown to be for the most part immediately satisfied. The impact of *r* is strongest when its value is low and we presume that a reduction in capacity would strengthen its impact.

6.4. Impact of Lost Sales Cost

Up to this point, we have set the lost sales cost the same for all items. In this sensitivity analysis, we aim to determine if there is a greater discrepancy in the best found objective function values between the formulation with fixed and variable lost sales if there is a difference in lost sales cost between items. In this section, using the F-dataset at a 92.5 capacity level, we introduce more variability between items with respect to the lost sales cost. Specifically, we multiplied the lost sales cost by 5 for the first three items. We aim to determine the impact that variability in lost sales costs has on the optimal solution on models with fixed lost sales compared to variable lost sales.

6.4.1 Performance Comparison

For this sensitivity analysis, we test the formulation using a different set of lost sales costs and compare it to our original results. Table 40 illustrates the performance of the models, consisting of the results of our original lost sales costs (Original) and our new lost sales costs (Adjusted).

Backlog	Lost Sales	Formulations	Lost Sales Cost	Avg. IP	Avg. Gap (%)	Avg. CPUTime (sec)	Avg. LP	Avg. LPTime (sec)	Avg. LP Gap (%)
No	Fixed	FLR-LS	Original	43,974.15	0.032	168.41	42,479.13	0.02	3.393
		FLR-LS	Adjusted	44,089.22	0.032	156.13	42,563.14	0.02	3.445
Unlimited	Fixed	FLR-B-FL	Original	42,975.89	0.000	82.29	41,832.22	0.01	2.701
		FLR-B-FL	Adjusted	43,286.16	0.000	114.25	42,055.65	0.02	2.847
	Variable	FLR-B-VL	Original	42,974.98	0.000	114.15	41,832.22	0.03	2.699
		FLR-B-VL	Adjusted	43,282.30	0.000	88.79	42,051.65	0.02	2.849
Restricted	Fixed	FLR-BR-FL	Original	42,975.89	0.000	80.10	41,832.22	0.02	2.701
		FLR-BR-FL	Adjusted	43,286.16	0.000	74.23	42,055.65	0.02	2.847
	Variable	FLR-BR-VL	Original	42,974.98	0.000	78.79	41,832.22	0.02	2.699
		FLR-BR-VL	Adjusted	43,282.30	0.000	88.55	42,051.65	0.02	2.849
Multiple	Fixed	FLR-BM-FL	Original	43,054.38	0.027	172.41	41,843.81	0.02	2.848
		FLR-BM-FL	Adjusted	43,400.71	0.013	180.20	42,061.99	0.02	3.057
	Variable	FLR-BM-VL	Original	43,035.49	0.022	143.48	41,843.74	0.03	2.812
		FLR-BM-VL	Adjusted	43,339.34	0.009	152.27	42,057.94	0.03	2.957

Table 40. Optimal Solution and Computing Time for Different Lost Sales Costs using the F-dataset at a 92.5 percent Capacity Level

IP Computing Time

The results in Table 40 indicate that an increase in the lost sales cost for certain items does not seem to impact the computing time of the formulations in a consistent way. The computing time remains fairly similar for all formulation. Some formulations have a longer computing time with the adjusted lost sales costs compared to the original lost sales cost, and others experience the opposite. The difference in computing between one set of lost sales cost versus the other is very small and therefore there is not enough evidence to make any conclusions.

IP Objective Function Value

Across all formulations the optimal value of the objective function experiences a significant increase when the lost sales costs are increased for certain items which is logical since some costs are increased. Formulations that consider both backlog and lost sales simultaneously exhibit a similar increase in the optimal value of the objective function, while FLR-LS is significantly less impacted.

When comparing the difference in the optimal value of the objective function for both the original and adjusted cases, we notice that the difference is larger in the adjusted case. The hypothesis for this observation would be that with more variability in the lost sales cost, the variable lost sales formulation has more flexibility to allow some extra lost sales for items with a low lost sales cost, and consequently freeing up capacity for production of items with a high lost sales cost. This remains true under all backlog assumptions. The difference between the variable and fixed lost sales model is greatest under the assumption of multiple customer types. This supports previous findings in this thesis where we believe the models with multiple customer types are the least flexible formulation and therefore are more likely to be impacted by a change in a parameter than models with a single customer type. The results in Table 40 indicate that formulations with restricted backlog provide the same optimal value of the objective function as formulations with unlimited backlog. This also support the idea that *r* has a minimal impact on the optimal value of the objective function when its value is high.

LP Solution

The LP time remains consistent for all formulations, between the models using the original lost sales cost versus the adjusted lost sales cost. All models using the adjusted lost sales cost have a slightly higher LP gap than models using the original lost sales cost. The difference is however small given the percentage change in the optimal value of the objective function.

6.4.2 Comparison of the Structure of the Solution

To explore in greater detail the impact of lost sales cost on the optimal solution, we review the structure of the solutions illustrated in Table 41.

Backlog	Lost Sales	Formulations	Lost Sales Cost	Setup Cost (%)	Holding Cost (%)	Backlog Cost (%)	Lost Sales Cost (%)
No	Fixed	FLR-LS	Original	65.08	31.82	N/A	3.10
		FLR-LS	Adjusted	65.43	31.58	N/A	2.99
Unlimited	Fixed	FLR-B-FL	Original	63.87	31.34	2.01	2.78
		FLR-B-FL	Adjusted	64.51	31.37	1.78	2.33
	Variable	FLR-B-VL	Original	63.87	31.38	1.99	2.76
		FLR-B-VL	Adjusted	64.58	31.42	1.68	2.33
Restricted	Fixed	FLR-BR-FL	Original	63.87	31.34	2.01	2.78
		FLR-BR-FL	Adjusted	64.51	31.37	1.78	2.33
	Variable	FLR-BR-VL	Original	63.87	31.38	1.99	2.76
		FLR-BR-VL	Adjusted	64.54	31.45	1.68	2.33
Multiple	Fixed	FLR-BM-FL	Original	63.42	32.11	1.86	2.61
		FLR-BM-FL	Adjusted	64.60	31.65	1.57	2.18
	Variable	FLR-BM-VL	Original	63.61	31.92	1.82	2.65
		FLR-BM-VL	Adjusted	64.45	31.70	1.49	2.36

Table 41. Cost Solution Structure for Different Lost Sales Costs using the F-dataset at a 92.5 percent Capacity Level

Analysis of Costs

For all cases, the proportion of backlog and lost sales cost decrease when we increase the lost sales cost for certain items. In turn, the percentage of setup cost increases as a consequence, while the percentage of holding cost changes at random. We believe that because holding cost is the lowest of all of them, it has the lowest priority and therefore its usage depends on the other decision variables.

The FLR-LS formulation has the smallest percentage cost decrease for backlog and lost sales while also having the smallest proportion of setup cost increase when there is an increase in the lost sales cost for certain items. The formulations that consider both backlog and lost sales exhibit a similar impact in its cost distribution when the lost sales costs are changed.

Analysis of Decisions

For greater insight, Table 42 lists the unit values of the decision variables.

Backlog	Lost Sales	Formulations	Lost Sales Cost	# of Setup	# of Holding	# of Backlog	# of Lost Sales	# of no B	# of no LS
No	Fixed	FLR-LS	Original	43.60	3,929.33	N/A	58.51	N/A	31
		FLR-LS	Adjusted	43.83	3,911.14	N/A	56.37	N/A	32
Unlimited	Fixed	FLR-B-FL	Original	42.41	3,869.46	144.96	48.32	10	10
		FLR-B-FL	Adjusted	42.76	3,885.87	119.82	39.94	14	14
	Variable	FLR-B-VL	Original	42.41	3,870.16	142.73	47.74	10	10
		FLR-B-VL	Adjusted	42.79	3,888.88	113.19	40.30	14	14
Restricted	Fixed	FLR-BR-FL	Original	42.41	3,865.54	144.96	48.32	10	10
		FLR-BR-FL	Adjusted	42.76	3,887.28	119.82	39.94	14	14
	Variable	FLR-BR-VL	Original	42.41	3,870.16	142.73	47.74	10	10
		FLR-BR-VL	Adjusted	42.77	3,888.18	113.19	40.30	14	14
Multiple	Fixed	FLR-BM-FL	Original	42.14	3,910.82	137.05	45.68	10	10
		FLR-BM-FL	Adjusted	42.91	3,890.34	112.13	37.38	16	16
	Variable	FLR-BM-VL	Original	42.26	3,900.39	132.61	46.05	10	10
		FLR-BM-VL	Adjusted	42.76	3,901.58	105.07	41.35	14	14

Table 42. Decision Solution Structure Values for Different Lost Sales Costs using the F-dataset at a 92.5 percent Capacity Level

From the results in Table 42, we see that the average units of backlog and lost sales significantly decrease when there is an increase in the lost sales cost for certain items. The number of setups also increases for all formulation. Although the increase in the number of setups is small, their high cost has a large impact on the total cost.

If we refer to Table 42, we notice an increase for all models in the number of instances that not include at least one stock-out in their optimal solution. All formulations that consider both backlog and lost sales simultaneously have the same increase in the number of instances except for FLR-BM-FL. This supports the idea that FLR-BM-FL is the least flexible formulation, and therefore is more likely to be impacted by a change in a parameter.

7. Conclusion

The objective of this thesis was to develop lot-sizing models that could simultaneously consider backlog and lost sales. In addition, we aim to generalize the traditional single customer backlog assumption by considering multiple customer types with a different willingness to wait. Both of these aspects are new in the lot-sizing literature. This thesis aims to contribute to the trend of adapting lot-sizing models to represent realistic situations in an effort to propose more practical solutions. We address the multi-item capacitated lot-sizing problem in its classical form, and review proposed formulations that consider either backlog or lost sales individually.

To accommodate the necessary capabilities required to consider both backlog and lost sales, such as the calculation of the stock-out levels, we use the facility location reformulation. When both concepts are considered simultaneously, there are two approaches that we consider for lost sales when backlog is also an option: *fixed proportion lost sales* and *variable proportion lost sales*. To properly explore the proposed problem, we develop extensions to our formulation using different assumptions on backlog: *unrestricted backlog*, *restricted backlog,* and *multiple customer types*.

As there is no available dataset that matches our proposed problem, we adapt a dataset intended for a CLSP by adding generated backlog and lost sales parameters. When analyzing our computational results we compare the best found solutions, computing times and structures of the solution amongst our different formulations. Our results showed that the formulations with fixed lost sales and variable lost sales provide similar optimal solutions, which we attribute to the high lost sales cost. The variable lost sales formulations however is shown to be a more flexible, which leads to slightly lower best found objective function values. The variable lost sales models will always provide a feasible solution, whereas the fixed lost sales models may face feasibility issues when capacity levels decrease.

By comparing the results of the formulations under different assumptions on backlog when lost sales is also an option, we determine that the results differ when considering multiple customer types as opposed to the traditional single customer type. The best found objective function value is either the same or higher for formulations with multiple customer types compared to formulations with a single customer type. By removing the traditional assumption of a uniform customer behaviour to stock-outs, we add realism to the model, but in return the model becomes less flexible and therefore results in higher total costs.

Different problem sizes were used to test the efficiency of our developed formulations. The results indicate that there is a significant relationship between the size of the problem and the average computing time per instance. The number of times the time limit was reached before the optimal solution could be found was more prevalent in problems of larger sizes. The optimal solution along with the computing times are especially sensitive to the number of items in a problem.

Given that the dataset we used was intended for a CLSP, we decided to conduct a sensitivity analysis with multiple capacity levels in an effort to find its impact on the performance of our formulations. Our results show that the capacity level is an important factor for our models. The computing time, structure of the solution, and the optimal solution are all affected by a change in the capacity level. Results suggest that the computing time and the best found objective function increase as the capacity levels decrease.

Our sensitivity analysis also explored the impact of the α parameter representing the percentage of customer willing to wait for backlog when a stock-out occurs. From our analysis, we find that the optimal or best found solution generally follows a progressively declining slope as α goes from 55 to 100 percent. We find that allowing for the possibility of backlog allows for more flexibility and in turn, better and quicker solutions. As more flexibility is added into the problem, the greater the marginal reduction in total cost will be.

The results of this research are limited to specific problems where there is a manufacturer that has multiple customers that each represent one unit of demand. The manufacturer has the option to satisfy the demand on time, satisfy it later through backlog, or not to satisfy the demand and incur lost sales instead. Another limitation of this thesis is that the sensitivity analysis only focuses on one parameter at a time. We suggest for future research to examine possible interactions between certain parameters. We believe there are combinations of parameters that can provide some interesting results, such as a simultaneous change in the capacity level and the backlog time restriction (r). Another interesting analysis would be to study a larger range of backlog and lost sales cost parameters.

To accommodate for our specific problem we made necessary assumptions. First, we assume that the unit selling price of an item would be greater or equal to the unit cost. This is a necessary assumption in order to use the lost sales variable in a minimization problem. It is an assumption that still encompasses the majority of realistic scenarios. Next, we assume that each unit of demand is represented by a separate customer. Lastly, we assume a scenario where the customer's willingness to wait for backlog is known only at a population level. Therefore the customer's willingness wait at an individual level is not known which prevents the ability to choose which customers to satisfy first in an effort to reduce total cost. Instead, we assume a first come first served approach so that the customers that are faced with a stock-out represent a mix of customers who are willing to wait and customers who will not buy the product. In practice however, this mix of customers that face the stock-out is not a fixed and known parameter. For future research, we therefore suggest to explore the uncertainty in the mix (i.e. the α parameter). In addition, it would interesting to allow multiple units of demand to be ordered by the same customer. This would add a new dynamic that could continue the trend of adapting lot-sizing models to represent realistic situations.

Finally, it would be interesting, especially in a business-to-business context, to consider that a manufacturer knows which customers are willing to wait (and for how long) and which customers are not willing to wait. In this case, the decisions that need to be taken will also include the allocation of the produced items to specific customers. The company can then exploit this customer specific knowledge, so that in case of a stock-out, the stock-outs are allocated (if possible) to customers that are willing to wait.

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Appendices

Appendix A: Computational Results and Solutions for the G-dataset at a 95 percent Capacity Level

Backlog	Lost sales	Formulation	Size (n, m)	CPU Time (sec)	IP	Gap (%)	LP	LP Time (sec)	LP Gap (%)	# of Time Limits
No	Fixed	FLR-LS	(6, 15)	105.34	39,787.48	0.031	38,210.20	0.01	3.883	1
			(12, 15)	284.39	70,907.80	0.000	70,027.29	0.02	1.228	0
			(24, 15)	769.80	146,238.20	0.010	145,827.70	0.04	0.275	2
			(6, 30)	545.89	68,588.80	0.166	67,139.87	0.03	2.152	1
			(12, 30)	1,800.00	146,352.00	0.400	145,159.21	0.07	0.796	5
			(24, 30)	1,050.35	274,255.00	0.104	273,720.90	0.14	0.204	2
Unlimited	No	FLR-B	(6, 15)	53.72	37,749.17	0.000	36,884.48	0.02	2.295	0
			(12, 15)	122.09	68,524.60	0.000	68,063.12	0.04	0.683	0
			(24, 15)	712.84	146,238.20	0.015	143,133.00	0.03	0.182	1
			(6, 30)	711.70	67,935.60	0.048	66,794.26	0.06	1.716	1
			(12, 30)	1,800.00	144,937.60	0.257	144,116.02	0.27	0.560	5
			(24, 30)	1,447.50	271,695.80	0.132	271,305.80	0.18	0.151	4
	Fixed	FLR-B-FL	(6, 15)	47.58	39,043.44	0.000	37,766.83	0.02	3.236	0
			(12, 15)	795.33	70,097.50	0.000	69,346.09	0.03	1.074	1
			(24, 15)	779.34	145,621.60	0.000	145,278.30	0.05	0.236	2
			(6, 30)	506.61	68,518.35	0.001	67,139.87	0.05	2.046	1
			(12, 30)	1,800.00	146,906.43	0.004	145,690.62	0.10	0.808	5
			(24, 30)	1,206.92	274,176.95	0.001	273,657.73	0.25	0.197	5
	Variable	FLR-B-VL	(6, 15)	41.14	39,043.44	0.000	37,766.83	0.01	3.236	0
			(12, 15)	723.30	70,097.50	0.000	69,346.09	0.03	1.074	0
			(24, 15)	785.49	145,621.60	0.028	145,278.30	0.06	0.236	2
			(6, 30)	505.39	68,518.35	0.152	67,139.87	0.04	2.046	1
			(12, 30)	1,800.00	146,951.00	0.436	145,690.62	0.10	0.834	5
			(24, 30)	1,205.08	274,172.35	0.104	273,657.73	0.21	0.196	3
Restricted	No	FLR-BR	(6, 15)	39.17	37,749.17	0.000	36,884.48	0.01	2.295	0
			(12, 15)	89.43	68,524.60	0.000	68,063.12	0.02	0.683	0
			(24, 15)	512.75	146,238.20	0.015	143,133.00	0.03	0.182	1
			(6, 30)	517.06	67,935.60	0.032	66,794.26	0.03	1.716	1
			(12, 30)	1,800.00	144,937.60	0.209	144,116.02	0.10	0.560	5
			(24, 30)	1,265.29	271,695.80	0.084	271,305.80	0.24	0.151	3
	Fixed	FLR-BR-FL	(6, 15)	51.77	39,043.44	0.000	37,766.83	0.02	3.236	0
			(12, 15)	829.05	70,097.50	0.000	69,346.09	0.06	1.074	1
			(24, 15)	783.54	145,621.60	0.027	145,278.30	0.05	0.236	2
			(6, 30)	547.12	68,518.35	0.154	67,139.87	0.04	2.046	1
			(12, 30)	1,800.00	146,920.10	0.421	145,690.62	0.09	0.816	5
			(24, 30)	1,270.22	274,176.95	0.108	273,657.73	0.23	0.197	3
	Variable	FLR-BR-VL	(6, 15)	66.67	39,043.44	0.000	37,766.83	0.02	3.236	0
			(12, 15)	787.18	70,097.50	0.000	69,346.09	0.03	1.074	0
			(24, 15)	788.88	145,621.60	0.028	145,278.30	0.05	0.236	2
			(6, 30)	507.97	68,518.35	0.152	67,139.87	0.03	2.046	1
			(12, 30)	1,800.00	146,951.00	0.436	145,690.62	0.10	0.834	5
			(24, 30)	1,208.08	274,172.35	0.104	273,657.73	0.20	0.196	3
Multiple	No	FLR-BM	(6, 15)	68.81	38,242.63	0.000	37,136.28	0.01	2.858	0
			(12, 15)	213.25	68,704.40	0.000	68,154.99	0.03	0.806	0
			(24, 15)	934.53	143,648.00	0.000	143,341.33	0.06	0.215	1
			(6, 30)	480.52	67,989.40	0.001	66,801.78	0.05	1.780	1
			(12, 30)	1,757.18	145,046.60	0.002	144,216.25	0.10	0.564	4
			(24, 30)	1,174.55	271,865.60	0.001	271,474.57	0.20	0.151	3
	Fixed	FLR-BM-FL	(6, 15)	39.13	39,125.59	0.000	37,782.36	0.02	3.393	0
			(12, 15)	672.83	70,097.50	0.000	69,351.57	0.03	1.067	0
			(24, 15)	808.87	145,626.00	0.030	145,278.30	0.06	0.240	2
			(6, 30)	596.39	68,547.35	0.219	67,139.87	0.04	2.086	1
			(12, 30)	1,800.00	147,038.26	0.518	145,690.62	0.12	0.887	5
			(24, 30)	1,269.26	274,164.15	0.105	273,657.73	0.24	0.192	3
	Variable	FLR-BM-VL	(6, 15)	101.88	39,125.59	0.000	37,782.36	0.01	3.393	0
			(12, 15)	836.83	70,097.50	0.011	69,351.57	0.05	1.067	1
			(24, 15)	829.22	145,626.00	0.031	145,278.30	0.06	0.240	2
			(6, 30)	581.48	68,531.30	0.192	67,139.87	0.03	2.064	1
			(12, 30)	1,800.00	146,990.23	0.486	145,690.62	0.13	0.858	5
			(24, 30)	1,258.53	274,164.15	0.104	273,657.73	0.21	0.192	3