HEC Montréal

The pricing and management of the credit valuation adjustment «CVA»

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Sciences de la gestion

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Mémoire présenté en vue de l'obtention du grade de maîtrise

en sciences (M.Sc.)

Août, 2011

Résumé

La tarification et la gestion de l'ajustment de l'évaluation de crédit «AEC»

L'objectif de ce mémoire est de mettre en évidence les défis, décisions et mesures de l'ajustement de l'évaluation de crédit pour les transactions de gré à gré. En premier lieu, la mesure et la gestion efficace de cette valeur à grande échelle pour des portefeuilles bancaires seront mis de l'avant. Par la suite, la valeur d'Aumann Shapley est présentée et utilisée comme une méthode cohérente et efficace par rapport aux autres méthodes d'allocation du risque de crédit. Finalement, les détails de cette méthodologie seront illustrés dans un contexte de portefeuilles de produit dérivés nantis et non-nantis.

Abstract

The pricing and management of the credit valuation adjustment «CVA»

The objective of this thesis is to highlight the various challenges, decisions, and methods of measuring the credit valuation adjustment for over-the-counter transactions. The large scale implementation issues that will allow for effective measurement and management of these values for large netting sets and bank portfolios will be treated. Aumann-Shapley price for CVA will be shown to be a coherent allocation method, and the most appropriate means of allocating CVA to transactions compared to other allocation methods whose drawbacks will be addressed. Aumann Shapley price implementation details will be shown in the context of collateralised and non-collateralised portfolio's.

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1 Introduction

The goal of this thesis is to provide a concise overview of the issues in credit management and pricing for practitioners through a selected review of the relevant literature of these issues. The contribution of this thesis is to present a pragmatic overview of the issues involved through a review of the selected literature given the practical experience of its author over the last eight years working in counterparty risk management at a mid tier Canadian bank. The core discussion area of this thesis is the application of the Aumman Shapley value to the allocation of credit charges to traders and trading desks for management and compensation purposes which is essential to the management of these risks.

Pykhtin and Rosen (2009) provide an initial allocation scheme using Euler allocations and was the impetus for this thesis. The paper addresses the problem of allocating the counterparty level credit charge to the individual trades composing the portfolio. The authors show that the problem can be reduced to calculating contributions of the trades to the counterparty level expected exposure conditional on the counterparties default. The authors show how to implement this into exposure simulations used at many financial institutions for both collateralised and non-collateralised counterparties. The authors also propose closed - form solutions when expected exposure is normally distributed, and the relaxation of independence between the trade values and counterparty credit quality.

Denault (2001) derives the equivalent of the Euler allocation principle (Aumann Shapley) by using game theoretic considerations. The author provides a set of axioms of the necessary properties for 'coherent' allocation of risks. The main result of the paper is that the Aumann Shapley value is both a coherent and practical approach to financial allocation. The author applies the axioms to a toy example of coherent risk measure based on the margin rules of the SEC, much in the same spirit as Artzner, Delbaen, Eber, and Heath (1999). In the section Alternative paths to the Aumann Shapley value the author indicates the key direction of this thesis's proposed risk measure allocation by indicating that the "Aumann-Shapley per unit allocation" or "Aumann-Shapely prices" is equivalent to the Euler principle. Thus under some differentiability conditions on the risk measure, the correct way of allocating risk capital is through Aumann Shapley prices.

Tasche (2004,2008) papers provide the link to game theoretic research of Denault (2001) that allowed us to identify the benefits of using Aumman Shapley values for allocation and the appropriate manner of deriving the values. These articles show that there is only one definition for risk capital allocations which is suitable for performance measurement, namely the derivative of the underlying risk measure with respect to the weight of the considered sub-portfolio or asset. The articles provide the theoretical background and practical aspects of implementing Euler allocation for risk measures such as standard deviation, Var, and expected shortfall. The articles also argue that inappropriate risk contribution calculations will likely lead to counter-intuitive results.

Larocque (2011) was the result of linking Pykhtin and Rosen (2009) and the Aumann Shapley derivation in Powers (2007) and Hougaard and Tind (2008) using the previous line of research Tasche (2004,2008) as the link to the game theoretic literature of Denault (2001) to derive an allocation scheme that does not require the artificial collateral trade constructions of Pykhtin and Rosen (2009), and is conceptually cleaner and easier to justify.

Picoult (2005), Gibson (2005) and Keenan (2009) all provide the frameworks for thinking about credit charges in this thesis and summarize the current practices within the industry regarding counterparty risk management and estimation, such as describing the use and impacts of margin agreements on counterparty credit risk. The authors explain the methods of measuring expected exposure, the credit charge, and economic capital for over-the-counter (OTC) derivative transactions from both a bilateral and unilateral perspective.

With the impetus of IAS 39: Financial Instruments: Recognition and Measurement rule from the International Accounting Standards Board (IASB) and especially FAS 157: Fair Value Measurements from the Financial Standards Board (FASB) from 2005 onward banks began taking into consideration counterparty non - performance risk in the valuation of mark to markets (MTM) that were presented as assets on the balance sheet; this adjustment became known as the credit valuation adjustment (CVA). During the crisis starting in 2007, not only was the counterparty risk recognized, but own counterparty risk began to be recognized as well. In order to benefit from widening spreads this became known as debit valuation adjustment (DVA) which is an offset to CVA.

In essence broker/dealers and banks began to actually incorporate true pricing for credit risk in their reported mark to market numbers. Consider for example a simple interest rate swap where a Canadian bank pays float and receives fixed for 10 years. On the day the swap is put on, there is a fair price plus a spread that covers the banks costs, and profit. On the close of the second day, the market has not moved significantly, but suddenly the counterparty to the swap, a manufacturer, has lost a significant multi - year production contract, and now the market recognizes that unless a new production contract is found, the counterparty is unlikely to survive the next two years. To the bank that put the swap on, the mark to market may not have moved much, but the value of future cash flows certainly look like they should be worth less when accounted for on the financial statements especially when recording profits and losses. If there were CDS prices for this counterparty, they would likely reflect the current credit deterioration and the expected future prospects by quoting a price for insuring a particular notional against default. These prices may provide the bank with some insight on how likely they are to receive future fixed payments from the counterparty on the swap.

We are now in a situation where a relatively plain vanilla interest rate swap depends upon the market swap rate and possibly on some information that may be embedded in a credit curve. Unfortunately, this is not all, as the bank may in the future be either paying or receiving net interest payments, and these future amounts are uncertain. Thus the market value of the credit risk on the swap also depends on the distribution of the future swap rate as determined by the market. Swaption markets may seem to be a good place to look to find information regarding this future expected behavior of swap rates. This is starting to get a little complicated to value, we started with a plain vanilla interest rate swap, now we are estimating multiple future swap rate distributions, while applying some sort of credit spread discounting to simply record a profit and loss that corresponds to the transaction on the financial statements.

To make matters worse, the counterparty has other transactions with the bank, such as FX forwards that also depend on the future evolution of swap rates for their valuation, as well as FX rates. These FX rates will have to have their future distributions estimated from the FX option markets as well. There likely will be an International Swaps and Derivatives Association (ISDA) agreement in place allowing for payment, and closeout netting. Thankfully, we'll stop there for now, and not include a CSA (collateral support annex) which would stipulate the possible exchange of margin depending on the mark to market of the transaction set as well as various other credit enhancement terms and conditions.

One begins to see that the calculation of CVA can quickly become extremely complex, involve a variety of legal documents/ terms, and be significantly more complex than valuing the instrument itself. Most of these problems can be solved or pragmatically remedied by resorting to a Monte – Carlo valuation of CVA, but this brings other problems and difficulties. So for now let's start with a simple representation of CVA and build up from there later on. CVA could be seen for each change in time, as the sum of the products of loss given default $(1 - R)$, exposure EPE, probability of default PD, all discounted. Here we will assume that the recovery rate is deterministic, and the probability of default PD deterministic and independent from the expected exposure of the transaction set. The recovery rate will be the same recovery rate that was assumed when the PDs were bootstrapped from the credit default swap prices. The expected exposure EPE will be the positive expected exposure value of the interest rate swap, and FX forwards whenever they are in the money for the bank, and will be zero otherwise. This expected exposure is built up from Monte – Carlo simulations of the correctly modeled underlying portfolio of transactions. The bilateral CVA (BCVA), or CVA that includes DVA (Debit/debt value adjustment) will be defined as follows: BCVA equals the difference of the CVA unilateral bank side, and DVA which is the unilateral CVA from the counterparty side.

The DVA is computed in the same way as the CVA, but from the counterparties point of view, and should be taken into consideration when the counterparty is in the money, and with the default estimation being made with respect to the bank's credit curve. The above representations are what most banks implement for their CVA calculation; they produce a positive and a negative expected exposure through time, and then apply the correct forward probability of default curves, assuming some recover rate, and discount curves.

Typically, there are different ways of possibly managing this CVA calculation, or in general, credit risk of counterparties. The first is the one classically performed by banks, which is credit rationing and mitigation with the possibility of special reserves based on CVA. This is where credit limits are set and monitored. For poorer credit counterparties, mitigation techniques such as MTM resets, collateralisation, term limits are implemented. This is the art of selecting appropriate counterparties to do a certain amount of business with, and protecting the bank by filtering out poorer counterparty and transaction decisions. This method has its advantages, but may also unnecessarily limit business, or due to concentrations create risks that are unseen or un-managed.

The second method is a risk warehousing approach, where the trades and dealers are charged appropriately, but the risk is warehoused at the bank. When making this decision the bank may decide not to actively hedge and pay the spread away, but to keep it and hope to manage it much like an insurance book, where the traders/ clients are charged appropriately, and these charges are treated as insurance fees to build a capital cushion in the event of possible losses, which will remain with the bank if they do not occur. This type of treatment is very similar to the typical loan banking book method, where expected loss reserves and bank capital provides the cushion against default.

The third method is the trading approach to credit management. The appropriate charge is made to clients/ traders, and the CVA exposure is actively managed in a trading format. Either the profit and loss reserve is actively hedged, or the total CVA is actively hedged. Either way the credit risk and exposure risk is hedged to lock in the amount that was charged to the client.

The fourth method is an extension of the trading approach, which we call the portfolio manager approach. Now the CVA desk has a its own profit and loss mandate, and acts by taking active positions on improving and deteriorating credits. This approach looks more like a credit portfolio manager, where trades are put on to change and optimise the risk return profile of the book of trades but instead of risk management begins looking more like a proprietary trading desk.

The choice of management method has great implications on how to measure the credit risk in the over the counter transactions of the bank. Essentially, if the bank is using traditional counterparty risk management, with credit mitigation, then some sort of maximal exposure should be used, in conjunction with loan equivalent amounts, where the CVA charge should be used in some cases as a reserve against profit and loss (P&L). Picoult (2005) defines the loan equivalent to be the fixed exposure profile, per counterparty, that would generate the same economic capital as the actual varying exposure. There may be many other measures to set limits on, such as notional and maturities.

If a warehousing method is used, one would assume that an actuarial based measure of credit would be more appropriate, since one would be interested in actual defaults and exposures happening in the book. Concentration analysis would be crucial. This would bring ones measurement in line with what was done for capital measurement, the expected losses, and unexpected losses would come from the same real loss distribution. This would imply also that the exposures should also come from the real distribution, where drifts would be estimated as real drifts, default probabilities would be actuarial ones etc... The drawback is that compensation for non-performance risk of counterparties placed in trades would still have to be estimated from the risk neutral distributions since they are pricing adjustments that are charged to counterparties, but would need to be sufficient to cover the actuarial cost.

If a trading approach is taken, the default risk would be measured in the risk neutral world, as well as the exposure calculations. This is because you would need the CVA to move as per market traded instruments and underlyings for the overlaying hedges to work. This type of pricing has direct impacts on the the type of simulation models used, since drifts and volatilities would be in the risk neutral world as with other pricing information. The simulation and pricing models should be calibrated to return observed market prices, swap transactions should return expected exposures in line with what the swaption market is indicating and this should be consistent with other observed market variables. However, since risk neutral pricing/ simulations are being used, your simulation paths are no longer back testable as per Basel and the Office of the Superintendent of Financial Institutions (OSFI) capital adequacy requirements, as forward prices are poor estimators of actual future price direction. This would imply the necessity of implementing one set of simulation models and calibrations for Basel capital measures, and a second set of calibrations and simulation models for CVA charges.

Since FAS 157 deals with estimating the fair market value of transactions, it would seem to indicate market prices should be used since that is what market participants would use in determining a fair price on the transaction. This is in conflict with the back-testing requirements of OSFI/BASEL. A large number of institutions would like to use their existing counterparty risk systems to calculate the CVA charge, but care must be taken since the system is typically set–up to satisfy OSFI/BASEL requirements, and hence models and calibrations do not typically reflect market dynamics. If a trading approach to management is taken, there will likely be a separate treatment, where the models are set–up to reflect market dynamics, and would then not be adequate to pass back testing.

If the trading or credit warehousing approaches are taken, the issues in the treatment of DVA will be discussed, as in general it is difficult to monetize ones own credit deterioration, and hence one has to be careful how the benefits of DVA are applied.

Once the decision has been made to manage the credit exposure, either through warehousing or by trading, it becomes important to be able to allocate the CVA to either the desk level or the trade level. This assigns specific responsibility for the CVA number, and motivates the trader to be aware of changes in credit, and of the power of different deal structures and credit mitigation techniques. The complication arises due to netting agreements, where multiple traders and/or desks can transact, and where the trades have possible offsetting effects. The allocation of CVA or trade capital then becomes non–trivial.

Given that the CVA risk measure is coherent, we will address what allocation properties we would like and then this should bring us to why Euler allocation (Aumann Shapley value) should be used. Typically, a lot of banks tend to use the with and without incremental trade method for charging and allocating CVA; however, it has been shown by Tasche (2004a) that this method underestimates the total CVA number, and hence is not an appropriate method to use unless modified in some way.

Once established as the most appropriate method of allocation, Aumann Shapley value allocation will be applied to a Monte Carlo estimated CVA. Some complications arise when the method is applied to counterparties that have collateral agreements, as this causes a discontinuity in the gradient of CVA with respect to the trade weights. Two methods of dealing with collateral will be addressed in the context of allocation, and the advantages / disadvantages of the two methods will be explored.

Once the important question of how to allocate the charge at the end of (day/month/quarter) has been addressed, then comes the decision on how to implement the various management methods. The two methods that will be explored are the warehousing method and the trading method. In the first case, one creates an insurance fund and ensures that the amount is adequate for expected defaults; maybe some large risk concentrations (so called tall tree risks) may be hedged out, but most of the credit spread is kept inhouse rather than paying away the hedging costs. In the second case, the variation in CVA is minimized through a hedging program. The traditional banking method will not be discussed in detail, as most banks have been running the method for some time.

The CVA hedging equation will be presented and the various sensitivities used for hedging will be discussed. Especially, we will discuss how these are derived, the limitations on the derivation, and how they will be implemented. The hedging program will be shown to be an optimization across a subset of hedging instruments, with the goal to minimize the volatility of CVA (and hedging costs, or frequency of re-balancing).

Once the hedging program has been established, a final question occurs on how much to charge for credit, which should include the identification of the total cost to be allocated back to the traders, the counterparty risk, capital charges, hedging costs, bid/ask, and risk premium for hedging mismatches.

Right / wrong way risk will be addressed in a cursory manner; while this is an extremely important aspect in the CVA charge, we will not go into the details of the estimation or management of this risk as this is a very complicated area and is outside the scope of this thesis.

The structure of this literature review is as follows:

Section 2 - of this thesis deals with the accounting standards that made it necessary to price credit in over-the-counter derivatives and put this fair value on the financial statements and also responsible for codifying the necessity of taking ones own credit quality into consideration when pricing.

Section 3 - provides the definitions that are necessary for the discussion, the definition of counterparty exposure, credit mitigation by modeling collateral, and at the end of section 3 the definition of CVA.

Section 4 - provides a generic and basic computational set-up for the estimation of credit exposure and the calculation of the CVA value; this is done so the reader has at least one possible estimation model in mind for the discussion.

Section 5 - goes on to describe the various approaches to the management of these exposures for a bank, with the end of Section 5 providing a quick summary of the possible credit decisions that are possible regardless which approach is taken.

Section 6 - provides a quick summary of the complications that arise when including ones own credit risk in the evaluation prices for accounting purposes, and possible management strategies.

Section 7 - describes one of the most basic approaches to allocation the with - without approach that is simple in theory to apply but extremely difficult in practice to deploy across large non centralised organisations.

Section 8 - provides and introduction to the term coherence in terms of risk measures, whose language will be brought over to the next section.

Section 9 - provides the criteria for 'coherent' allocations using the language of section 8, at the end of section 9 a coherent allocation method will be introduced formally, and is the method that will be applied later on in the thesis. Essentially, sections 8 and 9 are to highlight Denault (2001) as a key paper linking Euler allocation to the game theoretic Aumann Shapley value and the language of coherence for risk measures.

Section 10 - describes the Aumann Shapley value allocation method in a simple situation where the exposure measure has very nice properties, since it is not encumbered by a collateral agreement and is used for non - collateralised netted portfolios. This allocation method is applied to the full trade set.

Section 11 - describes the Aumann Shapley allocation in two situations where the exposure measure is either under a collateral agreement, or the allocation is done only for the new incremental trades occurring over a period of time rather than the entire trade set. This 'inhomogeneous' allocation methods are applied in section 11.1 to Allocation of Exposure in Collateralised Portfolios and in section 11.2 Allocation of End-of-Day Incremental Exposure.

Section 12 - describes the basic set - up if one were to begin dynamically hedging counterparty risk and the considerations one would have to take under various hedging strategies.

Section 13 - is the conclusion and provides some ideas for further investigation, and ongoing problems in counterparty risk management.

Appendix A - provides a detailed exposition of Larocque (2011) allocation of collateralised exposures under a close-out period larger than 0.

2 FAS 157 and IAS 39

Fair value measurements have been modified recently in order to take into consideration what assumptions market participants would use in determining fair value. Let us first recall the general definition of fair value from the Financial Accounting Standards Board (FASB): “A fair value measurement reflects current market participant assumptions about future inflows associated with an asset (future economic benefits) and the future outflows associated with a liability (future sacrifices of economic benefits)”.

IAS 39 covering fair value measurement of financial instruments became effective in January 2005, and covers similar territory as FAS 133 and 157. The modifications of fair value were made in FAS 157, whose guidance became effective after November 15th, 2007. For practical purposes we will concentrate on the implications of FAS 157.

FAS 157 does not include any new financial instruments to be measured at fair value, but does provide more accurate guidance on how to go about the measurement and what other pieces of information must be taken into account. The statement retains the notion of fair value as the exchange prices of financial contracts, but adds additional information, as this is the price of an orderly transaction between market participants to sell the asset or transfer the liability in the principal market, or the one that is most advantageous to the reporting entity. This transaction is assumed to continue and will be settled. The emphasis given by FAS 157 is an exit price to the transaction rather than an entry price.

The most important clarification in FAS 157 is that the assumptions for pricing must include assumptions about risks, and the appropriate pricing for such risks, that market participants would include when transacting in the market. These risks could include the effect of a restriction on the sale or use of an asset. The statement also provides clarification that a fair value measurement should incorporate non-performance risk (the risk that a contractual obligation will not be fulfilled). This statement clarifies that non-performance risk also includes the reporting entity’s credit risk, and should be included in fair value measurement. This is a key point that is a partial impetus for this thesis, and all of what will be discussed further on. Essentially, valuations must take into account the credit risk of the counterparty and the reporting entity when making a fair value measurement.

The guidance in FAS 157 applies to all instruments covered under FAS 133, "Accounting for Derivative Instruments" which requires that all derivative instruments be valued at fair value. Typically, derivative transactions are valued off of the swap curve, and/ or LIBOR rates, as well as many other market observed curves. These valuations are based on the settlement value of the trade, and do not include the counterparties credit risk, nor the reporting entities credit risk. Under FAS 157 this nonperformance risk must now be taken under consideration and will create a plethora of additional pricing curves, instead of a dozen or so pricing curves; every counterparty, of which a typical bank has tens of thousands, will require a credit pricing curve. Only a few hundred of these curves will be market observable, either from the credit default swap market or from implied liquid bond prices. If the credit risk is a considerable portion of the fair value, one could foresee the derivative being treated as a Level III asset due to the lack of observable market data on the credit worthiness of the counterparty, where Level III refers to the third level of the fair value hierarchy defined in FAS 157, and represents a fair value assessment based on unobservable market data such as most credit spreads applied by the bank for the calculation of CVA, and are based on subjective judgment. Market observable quoted prices in active markets are Level I, and valuations based on models that use and are supported by market observable data are considered Level II.

Since FAS 157 takes into consideration non-performance risk, is also linked to the legal implications of non-performance, such as the International Swaps and Derivatives Association (ISDA) conditions and close – out netting agreements. The valuation must also take into consideration the impact of credit enhancements, such as collateral or additional termination events, which may be included in the legal documentation. This implies that the fair value measurement adjustment should be made at the level of the netting set and counterparty, implying a portfolio-type credit adjustment to the fair value of the portfolio. FAS 157 appears to be active at the counterparty, netting pool portfolio level where credit enhancements may or may not be in place. In accounting terms this would mean that the unit of valuation is the counterparty portfolio which would generally at least be covered by an ISDA netting agreement. The derivative valuation credit adjustment is based on the portfolio of applicable transactions. FAS 133 deals with fair value for hedge effectiveness testing, and in the case of designated hedges the valuation is at the transaction level. One could argue that under FAS 133 fair value unit of account in some if not all cases would be considered on an individual transaction basis. This has lead to questions if FAS 157 is to apply to all FAS 133 fair value assessments, how to apply the credit adjustment to fair value when applying hedge effectiveness testing under the long haul method if applicable, and how to allocate from the counterparty portfolio to the individual derivatives for such purposes. I will leave the detailed discussion of these issues to accountants, but it is sufficient to point out that for accounting purposes there is a need to have a fair and equitable method of charging a portfolio level credit adjustment to the transaction level for the recording of accounting journal entries.

The net result is that financial academics realized quite early on that an adjustment in the pricing of over-the-counter (OTC) derivatives should take into consideration non-performance risk; an example of this would be the discussion by Sorensen and Bollier (1994) and numerous other researchers. However, in practice very few banks would take non-performance risk into consideration when marking their derivative books to market. For most banks November 15th 2007 would indicate the first time they would see non-performance risk hitting their financial statements. Due to this, everyone has become keenly interested in the practical application and management of the non-performance risk portion of their derivative valuations in financial statements, as it now is a number that is publicly available, and directly impacts financial performance and profit an loss numbers of trading desks.

It should be noted that it is not a foregone conclusion that the proper accounting treatment of the CVA adjustment should be based on market prices. There is a possibility of treating this component in a similar manner to the loan book of a bank. The CVA charge could be reported but held at cost, and managed through the capital management practices of the bank, where only material impairments to the credit worthiness of the counterparty would be written off. This would allow traders to charge BCVA to their clients but the bank would hold CVA capital and reserves, while not being subjected to the extreme swings due to changes in market rates. This would allow for much more prudent and stable financial planning, rather than counting on the inaccessible benefit of DVA and being subject to the overly large BCVA adjustment volatility.

This accounting change is what spurned numerous Credit Value Adjustment (CVA) desks to be formed to either administer the appropriate charges or manage the resulting fluctuations. Key to this management is the estimation of the CVA charges, and allocation of the CVA charge to various transactions and trading desks for management purposes. In the next section we will look at a more formal definition of counterparty risk and the credit adjustment itself.

3 Definition of Fair Value Adjustment "CVA"

3.1 Counterparty Exposure Measures

In this section we will define the risk measures that will be used for discussion later on. Most of these descriptions are fairly standard and can be seen similarly implemented in many vendor systems, though there is still room for more detailed improvement in a a lot of these measures.

Most systems at banks performing classical credit risk management, use Potential Future Exposure (PFE) as the risk measure; this is a percentile based measure, typically defined as a maximum exposure estimated to occur at dates in the future at a high confidence level. The confidence level is typically set to between 90% and 99%, most frequently 95% is selected. This confidence selection appears to be completely arbitrary in nature, and set small enough to avoid using an extremely rare and too large of an exposure, but large enough to cover typical market movements. Using a percentile eliminates the problem of using a max in a simulation based on distributions of infinite range, such as the normal distribution. The PFE is the X^{th} percentile of the positive exposure distribution, defined as $max\left(0, V\left(t\right)\right)$, where $V\left(t\right)$ would be defined as the mark-to-market value of a counterparties portfolio at time t. We will use their description of the market value of the counterparties portfolio in Pykhtin and Rosen (2009).

Denote the value of the i^{th}instrument of N instruments in the portfolio at time t from the banks perspective by $V_i\left(t\right)$. At each t, the value of the counterparty portfolio is:

$$V\left(t\right) = \sum_{i=1}^{N} V_i\left(t\right).$$

ISDA agreements allow for what is called close out netting, which allows for a single net payment for all transactions covered rather than multiple payments between counterparties. It should be noted that this netting is a conditional netting in the event of default, and care should be taken with that distinction, as most practitioners assume that this indicates netting for all purposes of exposure measurement, but is only applicable with the notion of exposure at default. When the portfolio of transactions are covered by a single netting agreement, the counterparty portfolio exposure $E(t)$ is defined as:

$$E\left(t\right) = max\left\{V\left(t\right), 0\right\}.$$

If there is no netting agreement, or if the legal department has defined the legal agreement as possibly unenforceable, then the counterparty exposure should rather be measured as follows:

$$E\left(t\right) = \sum_{i=1}^{N} max\left\{0, V_i\left(t\right)\right\},$$

where $E(t)$ represents the counterparty exposure both with and without netting sets, and with and without margin agreements.

3.2 Models of Collateral

Collateral is the right of recourse to an asset provided by the counterparty, that can be sold at a predictable value, and acts to offset the non-performance risk in the financial transaction. As per Gibson (2005) the terms for margin, or collateral agreements, are defined in annexes to the netting agreements. The credit support annex provides rules for computing the amount of collateral to be passed between counterparties on any given day. We will greatly simplify the document down to a few key terms as described in Gibson (2005) and ignore independent amounts for simplicity:

- Threshold: the exposure amount below which no margin (collateral) is held, denoted by H.

- Grace period or close-out period or margin period of risk: This is the number of days within which the realization that the counterparty is not posting called upon collateral, the legal department sends out the appropriate number of legal notices, the counterparty is then declared in default and the counterparty's position is liquidated or replaced. This indicates that the close-out period could possibly be different for counterparties where it would be difficult to realize a correct collateral call due to valuation disputes, as well as for positions that are extremely illiquid. This value can be determined from market practice and experience, and is not solely related to legally specified grace periods, but is based on a qualitative estimate of the amount of time it would take between perception of a problem to transaction liquidation. This value will be denoted by m.

- Re-margin period or valuation period: the interval in days at which collateral is monitored and called for. This value will be denoted by rm.

- Minimum transfer amount: the amount below which no collateral transfer is made. This is set so as not to overburden operations with small variations above and below the threshold. This value will be denoted by MTA.

Gibson (2005) defines the amount of collateral held by a bank from a non-defaulting counterparty as:

$$C(t) = max\{0, V(s) - H\}$$

which corresponds to what Pykhtin and Rosen (2009) call the instantaneous collateral model, where s is the re-margin date at or before t. Assuming that today$(t = 0)$ is a re-margin date, $s = t – t \, mod \, rm$. For the case of daily re-margining, $rm = 1$ and $s = t$.

This collateral model does not take into consideration the close-out period m. It is possible to specify a model for collateral that takes into consideration a deterministic close-out period as follows:

$$C(t) = max\{0, V(s - m) - H\} . \tag{3.1}$$

The counterparty collateralized exposure is given in both collateral model cases by:

$$E(t) = max\{V(t) - C(t), 0\}. \tag{3.2}$$

One must be careful to note what is represented under both collateral models specified. If the instantaneous collateral model is used, it assumes collateral is posted instantly, while typically in most large banks collateral would be received at $t+1$ or $t+2$ at the latest in normal operation. If a close-out period is taken into consideration, the exposure $E(t)$ then represents exposure at default at time t, where the collateral on hand is from the exposure m days before.

It is also important to note that the collateral held at time t, $C(t)$ is taken to be the cash equivalent amount. This is not necessarily the only way, as the full valuation of the bonds on hand can be taken into consideration as an additional trade for risk measurement against market variables. For future hypothetical collateral calls, assumptions would have to be made regarding the nature of the collateral pool posted or returned.

In the simple model of collateral above we have ignored very significant aspects of the collateral parameter decision set and their important impacts on exposure. To list just a few of these issues:

1. Collateral also moves with changes in interest rates and has some volatility. Even in the case of cash, there are interest rates to be applied making all collateral volatile in one manner or another. The question becomes how volatile is acceptable, and what overcollateralisation (haircuts) to apply for this intrinsic volatility. The collateral itself should also be relatively not affected by non-performance risks, related to the banks ability to realize the value of the collateral. The collateral posting currency option also has a value that should be monitored, as typically there is a choice under the CSA as to what currency of security or cash can be posted.

2. Since the collateral is volatile, it may also have correlation with the underlying transaction pool, as well as correlation with the counterparty's credit worthiness that should be taken into account to ensure that the collateral does not move in an undesirable manner when needed. An example of this would be a series of FX forwards and interest rate swap transactions with a large bank in a given country, while the bank is posting that country's government bonds. Depending on the

direction of the transactions, one can see a case where the collateral is worth less and the transactions worth more. (this is an example of wrong way risk involving collateral)

3. Collateral may become extremely concentrated in a particular currency or run of bonds and lead to a further concentration of risk, limiting the ability to liquidate the collateral at a predictable price when needed.

4. The collateral pools on hand should be such that it is easy to determine its value, possible to transfer custody, and have a precise determination of who actually owns it at different stages in the life cycle of counterparty relationship. The accounting concept of commingling has been known to come up in this regards, for example where cash ownership is indeterminate but the ownership of securities can be determined in bankruptcy proceedings.

It can also be argued that the collateral close-out period should be conditioned on the default process, which would be a very desirable direction to proceed in. For the sake of simplicity, we will assume that the close-out period is deterministic in our discussion, and where possible, discuss the impacts of this assumption. The collateral is also assumed to be in the reporting currency of the exposure, but frequently the exposure is in many currencies, and there are options available in the type of currency to be posted in the collateral. As well for systemically important counterparties it can be assumed that the exposure process can increase in volatility dramatically after default further increasing the exposure after a default, which we have not modeled. Again we note these possibilities, but for the simplicity of discussion we limit ourselves to the case where the collateral is in the same currency as the underlying exposure.

3.3 CVA Definition

We are now in a position to formalise the definition of CVA. First, define Expected Positive Exposure (EPE) at t:

$$EPE_B\left(t\right) = E\left[E\left(t\right)\right]$$

Where $E[\cdot]$ is the expectation operator, the subscript B indicates that this the the expected exposure viewed from the bank side, and the subscript C would indicate the expected exposure viewed from the counterparty's perspective.

In order to define the CVA value we will use a similar description as Pykhtin and Rosen (2009) for credit losses. In the possible event that the counterparty defaults at time τ before the bank defaults at time ϰ , the bank recovers a fractions R of the exposure $E(\tau)$. The banks discounted loss due to the counterparty's default would be measured as:

$$L = I_{\{\tau \leq T | \kappa \geq \tau\}} (1 - R) E(\tau) df(\tau)$$

where $I_{\{A\}}$is the indicator function that takes the value 1 when the logical variable A is true and the value 0 otherwise, $df(t)$ is the stochastic discount factor process at time t. The counterparty defaults at a random time τ with a known risk-neutral distribution $PDC(t) \equiv Pr[\tau \leq t]$.

By applying the expectation operator to the previous loss equation we obtain the following result:

$$CVA = (1 - R) \int_0^\infty EPE_B(t) dPDC(t)$$

Where:

$$EPE_B(t) = E[E(t) df(t) \mid \tau = t \wedge \kappa > t]$$

Where $EPE_B(t)$ is the risk neutral discounted expected exposure EPE_B at time t, conditional on the counterparty's default τ at time t, and the bank default κ not happening before time t. This definition would be the most complete version of unilateral CVA, but in practice is difficult to calculate as it assumes that the default processes are correlated with the exposure and discount factor. This could include "right-way" and "wrong-way" risk. BCBS (2005) defines general wrong (right)-way risk as arising when the probability of default of counterparties are positively (negatively) correlated with general market risk factors. Specific wrong (right)-way risk arises when the exposure to a particular counterpart is positively (negatively) correlated with the probability of default of the counterparty due to the nature of the transaction with the counterparty. In order to implement the calculation typically, we assume the variables are independent, which makes calculations dramatically easier. We also drop the conditional survival, and finally assume that the time between intervals $t_k - 1$ and t_k tends toward zero. In this case the integral is approximated by the following, assuming piecewise constancy:

$$CVA = (1 - R) \sum_{k=1}^{\infty} EPE_B(t_k) \left[PD(t_k) - PD(t_{k-1}) \right] df(t_k)$$

It should be noted that the assumption of independence between variables incorporates some inconsistent dynamics in the model; for example, interest rate swap exposures will be discounted by the same static discount curve which will be effected by shifts in the spot rate but not changes in risk factor volatility. This leads to difficulties in hedging the two sources of interest rate risk, one from the stochastic exposure and the second from the deterministic discount factors. A similar argument can be made for the probabilities of default, which may have an influence in the exposure calculation through such instruments as credit default swaps, credit indices and other products where there will be stochastic credit risk factors, but a deterministic probability of default curve will be applied in the calculation of CVA.This will lead to dichotomies between the CVA calculation and sensitivities and the exposure EPE estimation and sensitivities, that may have some repercussions in some applications.

If the above unilateral CVA is viewed from the bank side we will call it CVA credit value adjustment and if it is viewed from the counterparty's side we will call it DVA or debit value adjustment.

With these two values we will define the market clearing fair value adjustment that will take into consideration the possibility of the bank defaulting and the possibility of the counterparty defaulting, treated as independent events. The idea of incorporating jointly the probability of a counterparty defaulting and the cost (or impact) of the default for the solvent party was put forward in Sorensen and Bollier (1994) for the pricing of interest rate swaps. Sorensen and Bollier (1994) advocated the bilateral approach to include both counterparties credit conditions.

The fair value adjustment that takes into consideration both non-performance risks will be called Bilateral CVA (BCVA). From the equations given above, Bilateral CVA (BCVA) equals the difference between CVA (unilateral bank side) and DVA (unilateral CVA counterparty side).

This definition assumes that DVA is a positive value viewed from the counterparty's perspective. Again it must be noted that the recovery, expected exposure, and default probabilities in the above definition are all independent. The above formulation is what is observed as the standard bilateral fair value adjustment calculation. Remaining in the world where the individual components are still independent we can modify the equation slightly further to get a better estimate that includes bilateral defaults more accurately as follows:

$$\begin{aligned} BCVA &= (1 - R_B) \sum_{k=1}^{\infty} EPE_B(t_k)(1 - PDB(t_k)) \\ &\times [PDC(t_k) - PDC(t_{k-1})]\, df(t_k) \\ &- (1 - R_C) \sum_{k=1}^{\infty} EPE_C(t_k)(1 - PDC(t_k)) \\ &\times [PDB(t_k) - PDB(t_{k-1})]\, df(t_k). \end{aligned}$$

It is suspected that most banks do not implement this form calculation as it would reduce the beneficial impact of the DVA for fair value accounting purposes even though it would appear to be a rather simple calculation given the ability to calculate CVA and BVCA and represent a more precise estimate.

In the next section we will investigate a typical set-up for the estimation of PFE, EPE, and CVA. This description of the set-up will be typical to most banks who have not built a stand–alone CVA system but have modified their counterparty risk system to compute CVA.

4 Computational Set-Up for Exposure Estimation

Most banks have a typical computational set-up for the estimation PFE, a progress through a typical historical migration through methodologies. In the past, aside from pure notional risk measures, PFE could be calculated as a mark-to-market plus an add–on factor that was determined by a number of variables, such as instrument, maturity etc... Then banks migrated to Monte-Carlo based calculations and have for the most part remained at such a stage, there are other methods, but these will not be addressed in this paper. Here we will describe the typical Monte Carlo set-up as this is what will be seen in industry. It will be assumed that full Monte Carlo runs are performed at end of day, with the possibility of running ad–hoc limit and CVA trial checks during the day where only a single counterparty would be revalued. This would not be considered a real time set-up as we are describing the evolutionary step of systems between end of day portfolio valuation and a full real time system.

Most basic counterparty exposure set-ups have roughly five components that are always present. The set-up starts off with a database that houses an expansive set of data, typically much more than is required for a typical Value at Risk (VaR) calculation. This database contains trade data in sufficient detail to be able to re-price the deal, legal agreements and details, counterparty details such as legal structure, collateral details and finally risk limits. This database is typically fairly static, though it may be updated continuously during the day to take into consideration new counterparties. There would be storage for market data needed for trade valuation at end of day, with risk model details for the evolution of the stochastic processes, and the storage of the calibrated parameters for evolving processes.

To use the stochastic processes one would require a calculation framework that either generates Monte Carlo simulations on the fly or stores the simulation for future use, we will call this the Monte Carlo engine. The Monte Carlo engine computes an approximation of the evolution of correlated stochastic processes and makes available the results on a fixed time grid. This grid in practice tends to be very fine in the near term, typically daily, and gets progressively less dense as one goes out farther in the future to something like years at around the 30 year future point in time. This is done for computational tractability and is somewhat arbitrary. If market value parameters are needed between sampling points, an efficient and robust interpolation method is used.

Intimately entwined with the Monte Carlo engine is what we will call the valuation library. The valuation library is a set of valuation functions, which at first glance appear to be very similar to front office or VaR valuation libraries. The valuation functions used for counterparty risk management are a careful balance between accuracy and speed as they will be re-used billions of times. The valuation functions include approximations that allow for the aging and future valuation of possible resets, averages, and possible exercises etc... The valuation functions are crafted in conjunction with the Monte-Carlo engine so as to allow for reasonable trade-offs between accuracy and speed in the calculation. A lot of systems also provide for the valuation function to dictate valuation points in excess of the points dictated by the simulation time grid. These additional points represent deal events such as coupon payments, resets, etc... This deal information triggers a portfolio revaluation before and after such an event, in order to capture the impact of the event. Assume a typical calculation for a small counterparty with 100 deals, and assume it is a portfolio of swaps maturing in 30 years, given 50 fixed valuation grid points, and 10,000 simulation trials, there would be 50,000,000 revaluations. If one were to include the deal generated grid points assuming coinciding semi-annual coupon payments and excluding the fixed points, one could get 120 valuation points $\times$ 100 deals $\times$ 10,000 simulation trials 120,000,000 revaluations. This calculation is for a small counterparty that does not include a collateral agreement which would force revaluation at each rm point. Larger broker dealers could have thousands of trades that result in few thousand valuation points. So it is obvious that analytical valuation functions or approximations are absolutely necessary, and need to be fast given computational resources.

The next component is an exposure aggregator. This is a framework that allows for a very quick calculation of exposures based on the legal agreements (netting sets, collateral agreements) at the various sampling time points t. The first step is to add all the time t exposures given the applicability of netting sets. The next step would be to calculate the possible collateral given as received as per the applicable collateral agreements taking into account the close–out period m. The final step is to calculate the net exposure at time t given the netting and collateral effects. Based on this calculation the exposure measures are estimated, such as percentiles, and expected exposure measures, as well as derived measures such as CVA.

The final component is the reporting tool that allows one to observed risk limit violations and risk concentrations by numerous categories. This is the component that analysts treat as almost an after thought, but should be integral to the design of the system, as what and how you see something is linked inherently to how it is calculated and stored.

The following simulation set-up for the estimation of CVA comes from Pykhtin and Rosen (2009), where they consider the algorithm for calculating counterparty level CVA under the instantaneous collateral method, where $rm = 1$ and $m = 0$ and the exposures are independent of the counterparty's credit quality.

1. The initial simulation step one generates market scenarios j (interest rates, FX rates, equity prices etc...) for each future time point t_k
2. For each simulation time point t_k and scenario j:
   (a) For each trade i, calculate trade value $V_i^j(t_k)$
   (b) Calculate portfolio value $V^j(t_k) = \sum_{i=1}^{N} V_i^j(t_k)$
   (c) If there is a collateral agreement, calculate collateral $C^j(t_k) = max\{V^j(t_k) - H, 0\}$ available at time t_k, otherwise $C^j(t_k) = 0$.
   (d) Calculate counterparty level exposure
   $$E^j(t_k) = max\left\{V^j(t_k) - C^j(t_k), 0\right\}$$
3. After running a sufficiently large enough number M of market scenarios, compute the discounted expected exposure by averaging over all the market scenarios at each point: $EPE^*(t_k) = \frac{1}{M}\sum_{i=1}^{M} df^j(t_k) E^j(t_k)$
4. Compute CVA as:
   $$CVA = (1-R)\sum_{k=1}^{T} EPE^*(t_k)\left[PDC(t_k) - PDC(t_k - 1)\right]$$
   where as before R denotes the constant recovery rate and $PDC(t)$ is the unconditional cumulative probability of default up to time t.

Though an extremely simplified version of the simulation algorithm, this does incorporate the five operations discussed above. There are many modifications that can be made to this primary algorithm to increase speed, based on either memory or computational trade-offs for time efficiency, but for the most part traditional PFE simulators will look similar to the one described above. This will be the computational framework we will have in mind throughout this paper.

5 CVA Desk Mandate Approaches

In the following section four distinctly different CVA desk and credit risk management mandate approaches will be described. The core of this material has been adapted from Keenan (2009), where the nomenclature has been modified to allowed for a fourth mandate. The types of approaches are as follows: traditional bank manager, risk warehousing, trading, and finally credit portfolio manager approach. These four approaches will be explored in more detail in this section. The four approaches cover the spectrum of credit risk management from controlling and managing the risks to dynamically hedging these risks. In the final section we will enumerate the decision set that is available to the credit manager no matter which CVA management approach is embarked upon.

5.1 Traditional Bank Manager Approach

This approach relies on the existing risk framework of the bank credit analysis, operational constraints, legal analysis. New trades or sources of risk are added as credit lines are available. Potential future exposure, expected exposure, and loan equivalent amounts are used to monitor risk exposure to counterparties, and monitor concentrations.

Keenan (2009) lists several methods of credit pricing and reserving under the traditional approach. Under the traditional approach there is no explicit charging for risk, a special credit P&L reserve can be taken for selected deal counterparties. Credit risk is managed with peak exposure, potential future exposure methods. Reserve amounts are based on capital cost and possibly the credit charge on P&L. Potential future exposure and expected exposure for credit line reservation either by analytical methods, simulation-based methods or mark-to-market plus add–on approaches, but no explicit credit charge are otherwise applied, relies on traders to obtain appropriate spreads in deal pricing.

Keenan (2009) lists three issues with the traditional approach:

1. Without an explicit credit charge to traders or estimation of a credit charge there is a strong possibility that the bank could be consistently mis–pricing credit risk, and either end up not getting compensated enough, or losing trades due to excessive credit charging.

2. There is no overall view on the banks portfolio and this can lead to risk concentrations or deficiencies in the risk return profile. Typically, the credit officers look at the individual counterparty for approval and rarely take the allocation of credit in the context of the entire portfolio of credits. However, total exposure to industry limits and other such concentration global limits such as country exposure may exist.

3. Front office business lines do not have strong incentives to reduce counterparty risk through for example credit mitigants, restructuring opportunities, or legal documentation clean - ups. In this model credit decisions are seen as something to be overcome in order to book a new deal, and no further responsibility is given regarding the credit worthiness of the counterparty. In this case CVA adjusts accounting measures of profit and loss, but is not explicitly allocated back to specific traders or desks.

5.2 Risk Warehousing Approach

The risk warehousing approach is an attempt to essentially self-insure the credit risk by creating a suitably large capital base by charging for credit on the P&L of each trade. The counterparty credit risk is retained within the portfolio of the bank and is managed within a similar framework as the banking book. The risk can be aggregated with the banking book loan portfolio due to counterparty name overlaps and the possible efficiencies of optimising within the larger credit portfolio.

As with any insurance portfolio, management begins by ensuring that the portfolio is sufficiently diversified, with adequate credit selection criteria. Risk factor concentrations are monitored, as well as research and analysis on the counterparties and sectors. This book can be managed with traditional banking book credit management tools such as KMV, Credit Risk+, Credit Metrics for example. The management of the portfolio risk should be based on actuarial probabilities of default and under the real world risk measure. In this case credit charges on specific trades should be based on the real measure however what is supported by the market in terms of pricing on deals would likely be the risk neutral measure as this is what naturally lends itself to pricing. The risk would be managed as a warehoused risk under the through the cycle physical measure to identify risk concentrations and actuarial expected losses, but it appears that charges to the clients should be based on the risk neutral measure.

Since the benefits of own bank credit risk are not immediately realisable under bilateral CVA, it is likely that the credit charges based on the real measure with time will be insufficient to cover expected losses in the portfolio, and this issue should be monitored. Typically, corporate credits are price takers, and all attempts should be made to avoid giving the benefit of own counterparty risk to the counterparty. Since the goal is to insure the portfolio, expected losses and capital costs should be charged.

Larger credit concentrations could be hedged. Typically the risks to be hedged would be tall tree risks, that for one reason or another accumulated. Significantly strong negative credit views could be hedged in order to prevent further deterioration of the portfolio. Ideally, such hedges would be done taking the loan book into consideration as well. Essentially, if the credit does not have a place in the portfolio due to risk concentrations or default risk, the credit could be refused or priced under the assumption that it will be hedged.

Keenan (2009) lists the following components in determining the break–even pricing: Expected loss, Capital costs, Information costs (the cost of doing business), Premium if needed to cover possibility of hedging fallen angels, Premium if needed to cover possibility of needing to hedge tall tree risks.

In the risk warehousing approach, the standard tools used for the banking book can be used in making credit decision. New credits will need to exceed the Risk Adjusted Return on Capital (RAROC) hurdle rate, and fall within the typical credit allocation and credit line management process. It should be noted that correlations of risks among counterparties, especially non-performance risks should be measured and monitored, as these risks will directly impact the capital adequacy for the insurance book. A simple comparison can be made to tranched pricing technology in Collateralized Debt

Obligations CDO's where correlations are extremely important in the pricing of the tranche and risk management. The management of a risk warehousing approach would be very similar, as capital would have to be adequate to support some clustering of defaults under self insurance. One could also argue that charges for this correlation risk should be passed on through deal pricing, because even under the trading approach correlation risk would become hedgeable with a cost for hedging that is dependent on the value of correlation. This poses an interesting issue as the correlation risks are specific to the bank and any charges passed on would deviate from the market clearing price as discussed in Sorensen and Bollier (1994).

Front office traders are assigned the credit charges for their trades, relying on the banks ability to select good credits and allocate appropriate credit lines in creating an attractive credit portfolio. This avoids paying away hedging costs to other counterparties, and the benefits of credit selection, monitoring, and mitigation are accrued to the bank. There is a strong incentive from traders to charge appropriately for credit, and elect to put in place credit mitigants that would reduce the charge at trade inception. The main drawback of this method is having to maintain analysis and measurement based on two measures, the physical and risk neutral measures.

5.3 Trading Approach

The traditional trading approach is to treat credit as a risk that needs to be priced and hedged. Credit risk is treated as an additional trading desk that provides a service to the other desks, that provides credit transfer pricing services, and aggregates the credit risks which will then be managed through hedging. This mandate essentially defines how the risks will be measured and priced; the credit charge is based on the risk neutral measure and should represent the fair competitive charge to the client.

Keenan (2009) defines the possible components of a credit charge in the following manner, ignoring dependence between the variables:

$$
\begin{aligned}
Bilateral\,CVA = &(EPEcurve \times counterparty\,CDS\,curve \times discount\,factor) \\
&-(ENE\,curve \times ownrisk\,CDS\,curve \times discountfactor) \\
&+ rebalancing\,transaction\,costs \\
&+ cost\,of\,capital
\end{aligned}
$$

It seems unusual to include the possibility of your own default in pricing, more so given that the price includes the benefit in value of something that appears extremely difficult to realise, and for practical purposes effectively un-hedgeable. As previously discussed bilateral charges are required for market participants to agree upon a market clearing price, and its value is consistent with the fair value definition given by FAS 157 and IAS 39. Not including it in some cases may make the pricing uncompetitive.

As previously mentioned the management of credit risk through dynamic hedging has some implications on the measure being used. It essentially dictates the fact that the risk factor dynamics should be under the risk neutral measure, since most systems were created for the traditional approach, one would have to use two separate calibrations (models) one for capital purposes and credit reporting and another for BCVA pricing and accounting treatment. BCBS (2005) essentially stipulates this by their backtesting requirements, which are given as follows:

"Starting at a particular historical date, backtesting of an EPE model would use the internal model to forecast each portfolio's probability distribution of exposure at various time horizons. Using historical data on movements in market risk factors, backtesting then computes the actual exposures that would have occurred on each portfolio at each time horizon assuming no change in the portfolio's composition. These realised exposures would then be compared with the model's forecast distribution at various time horizons. The above must be repeated for several historical dates covering a wide range of market conditions (e.g. rising rates, falling rates, quiet markets, volatile markets). Significant differences between the realised exposures and the model's forecast distribution could indicate a problem with the model or the underlying data that the supervisor would require the bank to correct. Under such circumstances, supervisors may require additional capital."

The requirement to compare realised exposures to the models forecast distribution, essentially mandates the use of the real (actuarial) measure and not the risk neutral measure for capital purposes. This would indicate that if fair value adjustments under FAS 157 requires the use of market prices for credit, and will be dynamically hedged, then the risk neutral measure should be used, which will require two seperate models to measure these similar risks.

5.4 Credit Portfolio Approach

Keenan (2009) implies that the credit portfolio approach could be a logical extension of the trading approach. The CVA desk becomes an active credit portfolio manager and attempts to maximize returns on the counterparty portfolio and generate a positive alpha using the managers skill. The CVA desk should not make its profit on the charges made to the internal desks and traders but rather try to improve the portfolio quality through trade restructuring, credit mitigation, and collateral mechanisms. The manager will try to position the credit portfolio to profit either through natural selection criteria and credit improvement or by over hedging and benefiting from credit deterioration's. Possibly selling credit risk to investors where this optimises the portfolio characteristics in order to improve the risk return profile of the portfolio.

In the following section we will see that in over the counter derivatives banking the possible decision set is limited, no matter which CVA approach is embarked upon the fundamental decisions for the management of counterparty credit risk remain the same.

5.5 Credit Decision Set

In essence the decision set for managing credit, outside of being correctly remunerated to take on the credit risk, is quite limited, and can be listed almost in its entirety. The possible credit actions or decisions for a manager are as follows:

1. Decide to avoid the transaction or counterparty risk by not entering into the transaction. This decision is aided by classical credit analysis and decision making. It may also be made in the portfolio context and not solely related to the counterparties credit analysis.

2. Decide that the bank is large enough, or the transaction small enough, so that the bank is sufficiently capitalized to accept the non-performance risk. This is taking an outright credit position in the hope that the bank will be sufficiently be remunerated or at least gain ancillary business from the transaction.

3. Decide to engage in credit mitigation techniques to make the risk as small as possible through deal structuring such as resets, netting agreements, collateral support etc... This would also fall under traditional credit analysis and credit mitigation. It is important in all of the credit management approaches, though one should be wary of various interplay of early termination clauses and collateral concentrations, etc... and their possible impact on BCVA.

4. Decide to include the value of some asset as recourse against non-performance, so that the asset can be quickly sold at a predictable value and is not linked to the value of the transaction that is subject to collateral posting.

5. Decide to engage another legal entity to take on the credit risk through a financial guarantor, or insurance type arrangements as seen with CDS markets, or guarantors.

Within the CVA desk mandate approaches, decisions will have to be made regarding how to charge for existing trades on the book at the inception of the desk. A process would have to be in place to deal with trade unwinds, and negative CVA related to offsetting trades, who gets the benefit if any for DVA, how to treat credit mitigants such as additional termination events, possible changes in contractual terms, cancellation features such as break clauses, regulatory capital, and regulatory changes in capital or accounting treatment of the changes.

In all of the cases for counterparty credit management in the approaches described above, cost allocations for the credit charge in its entirety or in part must be made to the trade level, these allocations are either estimated pre-trade, or charged for post-trade. The necessity of allocating CVA to trade level is the impetus for this thesis, either allocating existing trades or allocating portfolios of new trades. No matter the cost allocation method the fundamental credit decision set remains the same.

In the next section, we will highlight the very real complications created by treating ones own credit risk in pricing as per FAS 157.

6 DVA Inconsistencies

Sorensen and Bollier (1994) highlighted the apparent need for a debt value adjustment or DVA in order to obtain a market clearing price of the credit charge. Accountants through FAS 157 and IAS 39 have appeared to embrace this adjustment wholeheartedly. Essentially, CVA losses can be offset by DVA gains, so counterparty credit deterioration will increase the CVA loss; under BCVA a deterioration in the banks or own counterparty risk will result in DVA gain. In the most extreme of cases with a bank whose credit worthiness has deteriorated greatly and whose counterparty's credit worthiness is impeccable, you could see the bank posting gains on the financial statement due to their credit deterioration. Keoun and Henry (2010) state that DVA was responsible for 18% of major banks pre-tax income, of which Morgan Stanley probably recorded one billion in such DVA adjustments, with Goldman Sachs, JP Morgan and Citigroup posting just under four hundred million in positive adjustments each. This situation would strike many readers as an extremely uncomfortable situation. Just because spreads widen does not mean that potential future liabilities (payments) will change.

From a basic accounting perspective this would seem to violate the notion of going concern and conservatism in reporting numbers, since there are very few ways for the bank to realise these gains in short of defaulting or coming extremely close to defaulting. Keenan (2009) raises the apparently valid question: is selling a derivative with a potential negative mark-to-market really a funding benefit to the bank?

Gregory (2009) attempts to list several possible ways in which a bank could realise the gain in BCVA that relates to their own non-performance risk:

- File for bankruptcy - this only serves to increase the recovery value of the firm and does not improve the credit quality of the firm. The author points out how humorous this argument is, once you imagine a firm that has a BCVA benefit so large that it can actually prevent their bankruptcy, but going bankrupt is the only way to realise those gains.

- Get close to bankruptcy - the trade can be unwound to realise the BCVA, which would cause the counterparty to realise CVA losses. But in order to realise this value the firm would have to be in all practicality bankrupt, and in that case we would see the first point.

- Return paid on collateral - in order to encourage the posting of collateral or the lowering of the agreed threshold, the counterparty could offer the bank to pay in excess of the overnight rate and this value would aid in monetizing BCVA. This may be easier to implement than paying the cost of buying a CDS on an uncollateralised exposure.
- Hedging - not really an option, since selling CDS protection on yourself is not really a practical solution.
- Funding arguments - if EPE_B represents a long term receivable, then EPE_C represents a long term payable which could be seen as a source of funding benefit.
- Buying your own debt back either through buying debt "cheaply" with cash, buy back debt synthetically. However, by buying back your own debt, it is possible for your own credit to improve and lead to BCVA losses on hedging. This type of gain should not be considered a sustainable business model.

Keenan (2009) lists some possible methods of hedging or realising the DVA in BCVA:

- Trade in and out of your own debt - this may cause problems with regulators, liquidity, compliance, and accounting.
- Proxy hedge - by using credit index tranches for financials. With this the basis risk is likely larger than the benefit, especially since large positions would be required. These trades would also have the potential to create massively crowded trades.
- Selling protection on your rivals or other highly correlated credits. Leads again to basis and event risk.

Picoult (2005) indicates that if the DVA gives rise to funding benefits then it could come in the form of cash from a margined counterparty which can be an immediate source of funding, or in the form of a receivable. It may also appear that treasury should be the owner of the apparent funding benefit since they are the ones who should be responsible for hedging their funding sources. The main point that I would like to highlight emphatically from Picoult (2005) is that a bank should not use BCVA unless it has a definite means in place for realising the value of the potential liability to its counterparty. Without such a means of realising the value, the bank would be putting an asset on its books that would be unrealisable and necessarily be eventually worth zero.

In the next section we begin our discussion of allocating CVA or EPE down to the trade level for management purposes. We begin with incremental allocation.

7 Incremental Allocation

The need to allocate the CVA or EPE to a the trade level is so far quite evident, and a simple method that many practitioners use to good effect is the incremental approach to allocation. This approach is mentioned in Tasche (2004a) as sometimes called the with without principle by some authors. The incremental allocation of an exposure measure in a netting set is essentially the impact of removing or adding 100% of a trades value:

$$\begin{aligned} \Delta EPE_B(t) &= EPE_B(t)\,[Portfolio\,of\,existing\,trades + New\,trade] \\ &- EPE_B(t)\,[Portfolio\,of\,existing\,trades] \end{aligned} \tag{7.1}$$

This method is convenient for the last trade in, as the sum of the incremental exposure and the existing exposure will equal the new exposure. The deficiency of this method is that the ordering of the trades is crucial to the allocation to the trades. System wise, a bank would have to have a real time trade capture system to keep the correct ordering of the trades. If the desire is to allocate using the incremental method to a portfolio of existing trades where the order is not kept, Tasche (2004a) shows that the risk contributions according to the incremental risk principle in the sense of 7.1 do not add up to the portfolio netting set measure if the measure is sub - additive, degree one homogeneous, and differentiable. We will give an example of the deficiencies of the incremental method when the ordering of the transactions are not known, or are completely arbitrary, or transacted at essentially the same time. The example we present is taken from Gregory (2009), where there are 4 trades, trade 1 is a 5 year payer interest rate swap (IRS), trade 2 is a 6 year payer IRS, trade 3 is a 6 year receiver IRS,

Table 1: Gregory(2009) Simulation Results at Time (t)

Gregory(2009) Simulation Results at Time (t)										
Trade	Sim 1	Sim 2	Sim 3	Sim 4	Sim 5	Sim 6	Sim 7	Sim 8	Sim 9	Sim 10
1	-655	-6810	-3062	-8056	-1958	3417	-2723	-4879	3788	-17622
2	2477	-7190	121	-6852	1519	8859	1206	-2586	7293	-18828
3	-2477	7190	-121	6852	-1519	-8859	-1206	2586	-7293	18828
4	-5556	-5148	-6152	-3303	1182	-5505	-5408	1732	2114	-2550

and trade 4 is a cross - currency swap. Trade 3 is essentially the reverse of trade 2. The simulation paths at time t are given in Table 1.

We present the results in the Table 2. The first case 'Incremental EPE' shows the ordered addition of each trade to the netting set portfolio where the total sum of all the trades allocated EPE would equal the total netting set EPE. This is the ideal case for applying the incremental allocation method, and is frequently used by banks in pricing deals when performing pre-deal analysis.

The second case 'Trade 4 traded before 3' assumes we reverse the ordering of trades 3 and 4. This means trade 4 arrives as the third trade, and trade 3 arrives as the fourth trade in the ordering of transactions. This case highlights the dramatic change in allocations that can happen when the ordering is not preserved for transactions. This could end up with a trader charging an amount to a counterparty that is significantly different from the amount that the trader is allocated internally at the bank.

The third case is the typical allocation that is performed when banks do not have the ordering of the transactions; the assumption is then that each transaction is considered the last transaction. It is this approach that Tasche (2004a) indicates as surely not adding up to the full exposure of the netting set. Typically, banks will attempt to pro-rate these values to the total EPE of the netting set. This type of pro-rata adjustment, though hiding some flaws, likely will not allow for an equitable allocation.

In the following sections we will investigate alternative methods of allocating the CVA so as to remedy the defects of the incremental allocation method. Hopefully, these remedies will be seen as superior by certain criteria that will be defined.

Table 2: Gregory(2009) Trade Examples Incremental Allocation Results

Gregory(2009) Trade Examples Incremental Allocation Results			
	Ordered Allocation	Trade 4 traded before 3	Every trade assumed as last trade
Trade	Incremental EPE	Incremental EPE	Incremental EPE
1	720.7	720.7	87.5
2	1797.4	1797.4	590.4
3	-1797.4	-446.9	-1480.8
4	-130.3	-1480.8	-130.3
Total:	590.4	590.4	-933.2

8 Coherent Measures

Denault (2001) provides an interesting framework by linking game theoretic concepts with the allocation of risk capital. It is through this framework that Larocque (2011) was able to link Euler allocation of Pykhtin and Rosen (2009) to the Aumann Shapley value and all of the corresponding research derived from Shapley (1953) and Aumann Shapley (1974).

Tasche (2004a) summarizes Denault (2001) by writing that he shows by arguments from game theory that in case of a degree-one homogeneous risk measure, its gradient is the only allocation principle that fulfills some coherence postulates. Tasche (2004a) states that the results only apply to coherent risk measures as per Artzner, Delbaen, Eber, and Heath (1999). So we begin by reviewing the concept of coherent risk measures for the reader, so that we can place them in the frame of thought when trying to define what is necessary to have what one could call a coherent risk allocation.

Denault (2001) defines X as a random variable representing a firms net worth at a specified point in the future, this can be seen as $V(t)$ in our case, and defines C as a risk measure that quantifies the level of risk, so that $C(X)$ is called the risk capital of the firm. He then summarizes the definition of a coherent risk measure by Artzner, Delbaen, Eber, and Heath (1999).

Definition 1. A risk measure $C : \mathcal{L}^{\infty} \to \mathbb{R}$ is coherent if it satisfies the following properties:

Subadditivity For all bounded random variables X and Y,

$$C(X+Y) \leq C(X) + C(Y)$$

in our case adding a netting set does not increase the risk exposure of the portfolio greater than the stand alone risks.

Monotonicity For all bounded random variables X, Y such that $X \leq Y$[21],

$$C(X) \geq C(Y)$$

in our case this relation is reversed, as if a trade Y is always worth more to the bank than trade X, then X cannot be riskier than Y. This is because exposure measurements for unilateral banks side are always positive.

Positive homogeneity For all $\lambda \geq 0$ and bounded random variable X,

$$C(\lambda X) = \lambda C(Y)$$

in our case this is the situation where there is no netting set in place and there is no portfolio diversification available. This is also known as the limit case of subadditivity.

Translation invariance For all $\alpha \in \mathbb{R}$ and bounded random variable X,

$$C(X + \alpha r_f) = C(X) - \alpha$$

where r_f is the price, at some point in the future, of a reference, riskless investment whose price is 1 today. In our case this would imply changing from a random variable to cash would be risk reducing. We can think of it as reducing the exposure.

These properties of a coherent risk measure, unfortunately do not specify an unique risk measure, but only give rise to classes of risk measures. Denault (2001), by using the concept of coherence to define risk measures, would like to use this same terminology and logic to define an allocation principle. In the next section we will show Denault's definition of a coherent allocation principle. We introduce this section by saying that

[1]As per Denault (2001), The Relation $X \leq Y$ between two random variables is taken to mean $X(\omega) \leq Y(\omega)$ for almost all $\omega \in \Omega$, in a probability space $(\Omega, \mathcal{F}, P)$.

we will only introduce the concept of coherent allocations through the work of Denault (2001), so that we can say that the allocation measures discussed later on using Aumann Shapley values are coherent in that sense.

9 Coherent Allocations

Denault (2001) defines an allocation principle as a solution to a risk capital allocation problem. The author argues that the set of axioms proposed are necessary properties of a “reasonable” allocation principle, which if satisfied the author will call a coherent allocation principle.

The following definitions are used in Denaults’ (2001) notation which will not be used outside of this section:

- X_i, $i \in \{1, 2, ..., n\}$, is a bounded random variable representing the net worth at time $T$of the i^{th}portfolio of a firm. We assume that the n^{th}portfolio is a riskless instrument with net worth at time T equal to $X_n = \alpha r_f$, where r_f is the time T price of a riskless instrument with price 1 today.
- X, the bounded random variable representing the firm’s net worth at some point in the future T, is defined as $X \triangleq \sum_{i=1}^{n} X_i$.
- $q$is the set of all portfolios of the firm.
- A is the set of risk capital allocation problems: pairs (q, C) composed of a set of n portfolios and a coherent risk measure C.
- $K = C(X)$ is the risk capital of the firm.

We can now define an allocation principle:

Definition 2. An allocation principle is a function $\Pi : A \to \mathbb{R}^n$ that maps each allocation problem (q, C) into a unique allocation:

$$\Pi : (q, C) \longmapsto \begin{bmatrix} \Pi_1(q, C) \\ \Pi_2(q, C) \\ \vdots \\ \Pi_n(q, C) \end{bmatrix} = \begin{bmatrix} K_1 \\ K_2 \\ \vdots \\ K_n \end{bmatrix}$$

such that $\sum_{i \in N} K_i = C(X)$.

The condition ensures that the risk capital is fully allocated.

Definition 3. An allocation principle Π is coherent if for every allocation problem (q, C), the allocation $\Pi(q, C)$ satisfies the properties

No Undercut

$$\forall M \subseteq q, \sum_{i \in M} K_i \leq C\left(\sum_{i \in M} X_i\right)$$

The no undercut property ensures that no portfolio can undercut the proposed allocation: an undercut would occurs when a portfolio's allocation is higher than the amount of risk capital it would face as an entity separate from the firm. In our case this would mean that a trade should not get allocated an amount larger than its stand alone value; any inclusion in a portfolio should equal or reduce the allocation.

Symmetry If by joining any subset $M \subseteq q \setminus \{i, j\}$, portfolios i and j both make the same contribution to risk capital, then $K_i = K_j$.

The symmetry property ensures that a portfolio's allocation depends only on its contribution to risk within the firm and nothing else.

Riskless allocation

$$K_n = C(\alpha r_f) = -\alpha$$

This relates to the n^{th} portfolio being a riskless instrument and implies that if a portfolio increases its cash position, it should see its allocated capital decrease by the same amount.

Denault (2001) goes on to describe game theory and allocation to atomic players in coalition games using the Shapley value, as per Shapley (1953) that meets the coherence axioms indicated in the above definition. The author shows that the Shapley value is the only value that satisfies a certain set of axioms. But due the the computational complexity this value will not be used. Instead, we will move along to coalition games with fractional players that have a scalable presence in the portfolio. In this case

Denault (2001) describes a set of conditions that are necessary for a coherent allocation in a coalition game with fractional players where members of the portfolio are infinitely divisible. We will summarize the demonstration and leave the reader to refer to Denault (2001) for the technical details. In essence the author defines 5 axioms that need to be met in order for an allocation with fractional players to be considered coherent.

1. **Aggregation invariance** which is similar to the symmetry property, that equivalent risks should receive equivalent allocations.
2. **Continuity** that the mapping of the function be continuous and differentiable, which is desirable to ensure that similar risk measures yield similar allocations.
3. **Non-negativity under non decreasing risk measures (Monotonicity)** is a requirement to enforce that more risk implies more allocation
4. **Dummy player allocation** this is equivalent to the riskless allocation definition.
5. **Fuzzy Core** allocations obtained from the fuzzy core allow no undercut from any player, coalition of players, nor coalition with fractional players.

Denault (2001) goes on to investigate the implications of these properties in defining an allocation principle. In essence, if one uses a coherent differentiable risk measure, and we deem the properties of coherent allocation important, then the Aumann Shapley value is the allocation principle to use.

9.1 The Aumann Shapley Value a Coherent Risk Allocator

From Hougaard and Tind (2008) we define the Aumann and Shapley (1974) cost / risk allocation method as follows:

Consider n different types of outputs and let $q \in \mathbb{R}_{+}^{n}$ be a (non - negative) output vector where q_i is the level of output i. The cost of producing any (output) vector q is given by a non-decreasing cost function $C : \mathbb{R}_{+}^{n} \rightarrow \mathbb{R}$ where $C(0) = 0$ (i.e. no fixed costs).

Denote by (q, C) a cost allocation problem and let Π be a cost allocation rule, which specifies a unique vector of costs related to each output $x = (x_1, ..., x_n) = \Pi(q, C)$ where

$$\sum_{i=1}^{n} x_i = C(q),$$

and x_i is the cost related to output i.

Consider the set of continuously differentiable cost functions C denoted by $\partial_i C(q)$ the first order derivative of C at q with respect to the i^{th} argument. Define the Aumann - Shapley value as Π^{AS} by allocated costs,

$x_i^{AS}(q, C) = \int_0^{q_i} \partial_i C\left(\frac{t}{q_i} q\right) dt = q_i \int_0^1 \partial_i C(tq)\, dt$ for all $i = 1, ..., n$.

Hougaard and Tind (2008) note that $\sum_{i \in N} x_i^{AS}(q, C) = C(q)$. In particular, $K_i^{AS} = \int_0^1 \partial_i C(tq)\, dt$ can be seen as the unit cost of output i also known as the Aumann Shapley value.

Note that Denault (2001) has defined what a coherent allocation measure is by borrowing the idea of coherence from Artzner, Delbaen, Eber, and Heath (1999) and defined the conditions for a coherent allocation, that will lead us to the Aumann Shapley value, which will be applied to the allocation of the counterparty risk exposure measure. The value will first be applied in the context of no collateral agreements in the homogeneous allocations section, and then the allocation method will be applied in two seperate ways in the inhomogeneous allocation section.

10 Homogeneous Allocations

We will follow with the derivation presented in Pykhtin and Rosen (2009) for marginal EPE allocations, for netted exposures with and without collateral agreements. We begin with the case where there is a netting agreement in place but no collateral agreement. Which Pykhtin and Rosen (2009) indicate that this un-collateralised exposure contribution is a risk function that is homogeneous (of degree) one.

Pykhtin and Rosen (2009) define homogeneity in a simple direct fashion as follows: A real function $f(\mathbf{x})$ of a vector $\mathbf{x} = (x_1, ..., x_n)$ is said to be homogeneous of degree β if for all $c > 0$, $f(c\mathbf{x}) = c^{\beta} f(\mathbf{x})$. The reader can find a very in depth discussion of this in Powers (2007) where the author lists three types of homogeneity, which are infrequently discussed in the literature. The three types of homogeneity discussed by Powers (2007) are homogeneity in distribution, in scale, and in mean.

10.1 Non Collateralised Portfolios - Continuous Marginal EPE Allocations for Netted Exposures Without Collateral Agreements

Define the weight α_i for trade i as a scale factor that represents the relative size of the trade in the portfolio, $V_i(\alpha_i, t) = \alpha_i V_i(t)$. These weights can assume any real value, with $\alpha_i = 1$ corresponding to the actual size of the trade and $\alpha_i = 0$ being removal of the trade from the set. The weight vector for adjusting the portfolio is described as $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_N)$, and the vector representing the original portfolio as defined as $\mathbf{1} = (1, ..., 1)$.

The existing netting set counterparty level exposure is a homogeneous function of degree one in trade weights:

$$E(c\boldsymbol{\alpha}, t) = cE(\boldsymbol{\alpha}, t). \tag{10.1}$$

Equation 10.1 is easily illustrated in the following example given by Pykhtin and Rosen (2009): if the bank uniformly doubles the size of its portfolio with the counterparty by doubling the notionals on the trade set, the bank's exposure would double. This concept is important and will be revisited in the demonstration of allocations in inhomogeneous circumstances where this is no longer the case.

The following demonstration holds for $EPE_B(t)$ and $EPE_C(t)$but for clarity we will use $EPE(t)$ for the expected exposure as the calculation can apply to either side. We will define the continuous marginal EPE contribution of trade i at time t as the infinitesimal increment of the EPE of the actual portfolio at time t resulting from an infinitesimal increase in trade $i's$ presence in the portfolio, scaled to the full trade amount.

$$EPE(t) = \lim_{(\delta\to 0)} \frac{EPE(t, \mathbf{1} + \delta \cdot \mathbf{u}_i) - EPE(t)}{\delta} = \left.\frac{\partial EPE(t, \boldsymbol{\alpha})}{\partial \alpha_i}\right|_{\boldsymbol{\alpha}=\mathbf{1}} \tag{10.2}$$

where $\mathbf{u}_i$describes a portfolio whose only component is one unit of trade i. Since the portfolio exposure is homogeneous in the trades' weights, the EPE contributions will sum up to the counterparty netting set EPE by Euler's theorem (Aumann Shapley value).

Pykhtin and Rosen (2009) derive an expression for the marginal EPE contributions as follows; by substituting $EPE_B(t) = E[max\{E(t), 0\}]$ into 10.2 and bringing the derivative inside the expectation we get:

$$EPE(t) = E_t\left[\left.\frac{\partial E(\boldsymbol{\alpha}, t)}{\partial \alpha_i}\right|_{\boldsymbol{\alpha}=\mathbf{1}}\right] \tag{10.3}$$

where the exposure of the adjusted portfolio (with weight vector $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_N)$) is given by

$$E(\boldsymbol{\alpha}, t) = max\left\{\sum_{i=1}^{N} \alpha_i V_i(t), 0\right\} \tag{10.4}$$

Calculating the first derivative of the exposure with respect to the weight α_i and setting all weights to one, we have:

$$\begin{aligned} \left.\frac{\partial E(t, \boldsymbol{\alpha})}{\partial \alpha_i}\right|_{\boldsymbol{\alpha}=\mathbf{1}} &= \left.\frac{\partial}{\partial \alpha_i} max\{V(\boldsymbol{\alpha}, t), 0\}\right|_{\boldsymbol{\alpha}=\mathbf{1}} \\ &= \left.\frac{\partial V(\boldsymbol{\alpha}, t)}{\partial \alpha_i} \mathbf{1}_{\{V(\boldsymbol{\alpha}, t)>0\}}\right|_{\boldsymbol{\alpha}=\mathbf{1}} \\ &= V_i(t)\, \mathbf{1}_{\{V(t)>0\}} \end{aligned} \tag{10.5}$$

Substituting equation 10.5 into equation 10.3, we obtain the EPE contribution of trade i:

$$EPE_i^*(t) = E_t\left[V_i(t)\, df(t)\, \mathbf{1}_{\{V(t)>0\}}\right] \tag{10.6}$$

The contribution of trade i is the expectation function which considers the discounted values of the trade on all scenarios where the total counterparty exposure is positive,

Table 3: Gregory(2009) Trade Examples Marginal Allocation

Gregory(2009) Trade Examples Marginal Allocation	
Trade	Marginal EPE
1	378.9
2	729.3
3	-729.3
4	211.3
Total:	590.4

or zero. The above EPE contributions sum up to the counterparty level discounted EPE:

$$\begin{aligned} EPE^{*}(t) &= \sum_{i=1}^{N} EPE_i^{*}(t) \\ &= E_t\left[V_i(t)\, df(t)\, \mathbf{1}_{\{V(t)>0\}}\right] \\ &= E_t\left[max\{V(t), 0\}\, df(t)\right] \end{aligned} \tag{10.7}$$

Using Gregory (2009)'s examples we can show the marginal allocation using the Aumann Shapley value to the trades using the previously defined simulation paths.

The first important point to notice in Table 3 is that the allocation to the first trade is not 720.7 which is what the stand-alone amount was, but rather 378.9. The reason for this is that the allocation was done for the entire portfolio once all the trades have arrived, so each new trade to the portfolio would change the allocation to past trades under the incremental approach. In the ordered trade incremental allocation this does not happen. The marginal allocation assigns allocations to the trades based on the the average of the marginal changes in the risk measure with respect to infinitesimally small changes in the participation level of each trade. The demonstration in Table 3 is an example of the simple homogeneous case without a margin or collateral agreement in place. The following section will deal with two types of inhomogeneity, and innovative solutions by Pykhtin and Rosen (2009) and Larocque (2011) to both cases.

11 Inhomogeneous Allocations

The first case of inhomogeneity to be treated is the case of collateral where an infinitesimal increase in trade weights may not have any impact on portfolio level exposure due to the margin threshold level. Much like the example given for homogeneous allocations, an inhomogeneous situation is when an addition of trades or a uniformly doubling in size of the notionals do not necessarily result in a doubling or even an increase in exposure. Pykhtin and Rosen (2009) provide two interesting solutions to this problem.

11.1 Allocation of Exposure in Collateralised portfolios

We will now look at the case where the counterparty has a netting agreement with a collateral agreement. We will still consider the instantaneous collateral model where $rm = 1$, and $m = 0$. As per Pykhtin and Rosen (2009), we substitute 7.1 and 3.1 the following equation is obtained for exposure under the instantaneous collateral model:

$$E\left(t\right) = V\left(t\right)\mathbf{1}_{\{0<V(t)<H\}} + H\mathbf{1}_{\{V(t)\geq H\}} \tag{11.1}$$

As can be seen from 11.1 the expected exposure is not a homogeneous function of the trades weight's; since the conditions for Euler's theorem are not satisfied, we are no longer able to increase trade weights infinitesimally above the threshold, since the exposure contribution is zero for all paths above the threshold.

Pykhtin and Rosen (2009) derive contributions for this non-homogeneous case by using the notion of an extended vector of weights, which are consistent with the marginal contributions described previously. Essentially, if the exposure in 11.1 is not homogeneous in the vector of weights $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_N)$, the function

$$E\left(\boldsymbol{\alpha'}, t\right) = V\left(\boldsymbol{\alpha}, t\right)\mathbf{1}_{\{0<V(\boldsymbol{\alpha},t)<\alpha_H H\}} + \alpha_H H\mathbf{1}_{\{V(\boldsymbol{\alpha},t)>\alpha_H H\}}$$

is a homogeneous function in the extended vector of weights $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_N, \alpha_H)$. The authors consider the scaling of each of the trades as well as the threshold that gets treated as another trade, and can think of the contribution of the threshold as well as the contribution of the trades. We then follow in a similar manner as in the non collateralised case; we take the first derivative of the exposure with respect to the trade weights

$$\left.\frac{\partial E\left(\boldsymbol{\alpha'},t\right)}{\partial \alpha_i}\right|_{\boldsymbol{\alpha'}=\mathbf{1}} = V_i\left(t\right)\mathbf{1}_{\{0<V(t)\leq H\}} \tag{11.2}$$

and the derivative with respect to the threshold weight is given by

$$\left.\frac{\partial E\left(\boldsymbol{\alpha'},t\right)}{\partial \alpha_H}\right|_{\boldsymbol{\alpha'}=\mathbf{1}} = H\mathbf{1}_{\{V(t)>H\}}. \tag{11.3}$$

These sum up to the netting set level exposure given by 11.1. By discounting and taking the conditional expectation of the right hand side of equations 11.3 and 11.2, the following expected exposure trade contributions are obtained:

$$EPE^{*}_{i,H}\left(t\right) = E_t\left[V_i\left(t\right)df\left(t\right)\mathbf{1}_{\{0<V(t)\leq H\}}\right] \tag{11.4}$$

and of the threshold

$$EPE^{*}_{H} = HE_t\left[df\left(t\right)\mathbf{1}_{\{V(t)>H\}}\right] \tag{11.5}$$

which satisfies

$$EPE^{*}\left(t\right) = \sum_{i=1}^{N} EPE^{*}_{i,H}\left(t\right) + EPE^{*}_{H} \tag{11.6}$$

The contribution of the threshold can be seen as the effect of an infinitesimal change in the threshold on the EPE, where the limiting case when the threshold is infinite is the base homogeneous formula. The final step of the Pykhtin and Rosen (2009) method is to allocate back the contribution adjustment of the collateral threshold given by 11.5 to the individual trades, so that 11.6 can be written as EPE contributions of only trades. The authors propose two methods of allocating EPE^{*}_{H} in meaningful proportions to each trade. The authors propose a weighting scheme that is given by the ratio of the individual instrument's expected discounted value when the threshold is crossed to the total counterparty discounted value when this occurs:

$$\overline{E}_t\left[df(t)\,\mathbf{1}_{\{V(t)>H\}}\right] = \sum_{i=1}^{N} E_t\left[df(t)\,\mathbf{1}_{\{V(t)>H\}}\right] \times \frac{E_t\left[df(t)\,\mathbf{1}_{\{V_i(t)>H\}}\right]}{E_t\left[df(t)\,\mathbf{1}_{\{V(t)>H\}}\right]} \tag{11.7}$$

and the individual trade contributions to EPE are given by

$$\begin{aligned} EPE_i^*(t) &= E_t\left[df(t)\,V_i(t)\,\mathbf{1}_{\{0<V(t)\leq H\}}\right] \\ &+ \frac{HE_t\left[df(t)\,\mathbf{1}_{\{V(t)>H\}}\right]}{E_t\left[df(t)\,V(t)\,\mathbf{1}_{\{V(t)>H\}}\right]} \\ &\times E_t\left[df(t)\,V_i(t)\,\mathbf{1}_{\{V(t)>H\}}\right] \end{aligned} \tag{11.8}$$

Both terms in 11.8 have basic interpretations according to Pykhtin and Rosen (2009), the first term is the contribution of all scenarios where the bank holds no collateral at time t, while the second term is the contribution of all scenarios where the bank holds non- zero collateral at time t. This allocation scheme is defined by the authors as type A allocation. The authors described a B allocation scheme, which is obtained by bringing the weighting scheme of the threshold contribution inside the expectation operations, so instead of 11.7 we now have:

$$\begin{aligned} \overline{E}_t\left[df(t)\,\mathbf{1}_{\{V(t)>H\}}\right] &= \sum_{i=1}^{N} E_t\left[df(t)\,\mathbf{1}_{\{V(t)>H\}}\frac{\sum_{i=1}^{N} V_i(t)}{V(t)}\right] \\ &= \sum_{i=1}^{N} E_t\left[df(t)\,\mathbf{1}_{\{V(t)>H\}}\frac{V_i(t)}{V(t)}\right], \end{aligned} \tag{11.9}$$

which will lead to the continuous marginal contributions given by

Table 4: Gregory(2009) Trade Examples, H=2000 Allocation Results

Gregory(2009) Trade Examples, H=2000 Allocation Results			
Trade	Incremental Ordered	Marginal A	Marginal B
1	400	128.35	128.35
2	182.2	247.05	247.05
3	-182.2	-247.05	-247.05
4	-200	71.65	71.65
Total:	200	200	200

$$EPE_i^*(t) = E_t\left[df(t)\,V_i(t)\,\mathbf{1}_{\{0<V(t)\leq H\}}\right] + HE_t\left[df(t)\,\mathbf{1}_{\{V(t)>H\}}\frac{V_i(t)}{V(t)}\right]. \tag{11.10}$$

The lagged collateral model where $m > 0$ will not be addressed as it is done so in Pykhtin and Rosen (2009).

Both methods give similar results for reasonable threshold levels H, but as the the threshold gets smaller the allocations may diverge in some instances, (in the case of Gregory (2009) examples this does not happen). Very low threshold levels can be seen with poor credit counterparties, or large counterparties where there is substantial enough volume to desire to trade collateral instead of taking on significant CVA charges. Table 4 illustrates the Pykhtin and Rosen (2009) solution A and B to a counterparty with a reasonable threshold.

Calculation of EPE and CVA contributions given computational set-up

Pykhtin and Rosen (2009) provide the following adjustments to our previously defined computational set-up for the estimation of EPE and CVA. These estimations can be incorporated by adding the following calculations to steps 2 - 4:

- Step 2: For each trade i, calculate the trade's exposure contribution for scenario j, $E_i^{*(j)}(t_k)$, which is equal to $V_i^{(j)}(t_k)$ if $0 < V^{(j)}(t_k) \leq H$, $\frac{HV_i^{(j)}(t_k)}{V^{(j)}(t_k)}$ if $V_i^{(j)}(t_k) > H$, and zero otherwise.

- Step 3: For each trade i, compute the discounted EPE contribution by averaging over all the market scenarios at each time point: $EPE_i^* (t_k) = \frac{1}{M} \sum_{j=1}^{M} df^{(j)} (t_k) E_i^{(j)} (t_k)$.

- Step 4: Compute CVA contributions as:

  $CVA_i = (1 - R) \sum_{k=1}^{T} EPE_i^* (t_k) [PDC (t_k) - PDC (t_k - 1)]$.

This completes our discussion of the Pykhtin and Rosen (2009) approach to CVA allocation.

11.2 Allocation of End-of-Day Incremental Exposure

The first method we have considered so far depends on the correct ordering of trades in their treatment, which as we have previously mentioned is difficult to perform in large financial institutions since no formal ordering may be in place or in most cases even possible given the division of front office and back office functions for trade input, as well as time zone and system limitations. The second method of marginal allocation allocates the exposure across all trades, and when a new batch of trades come in the allocation values changed for all trades, leading to cases where the trader may have quoted to a counterparty a certain CVA spread, only to be charged internally a different amount after the fact.

The method that will be discussed in this section allows for a consistent mixing of the two. Essentially a bank estimates the incremental effects of a group of trades over a given period, typical all trades made that day. This incremental CVA or exposure value then needs to be allocated in an equitable manner across all trades for that given period/day. The reasoning for this is that the bank should clearly be able to define what trades were made in that day or period but due to back office processing details cannot assure the exact sequence of trades, so the incremental impact of the CVA is allocated to all trades in the period using the Aumann Shapley value. We still have the problem of a trader charging his client one value only to receive another internal charge at a later date. This method allows for the difference in timing to be smaller and the allocation to be only for the incremental value, and should lead to smaller discrepancies, and a more equitable allocation of CVA when the precise order of trades is unknown.

In the paper of Pykhtin and Rosen (2009) the authors looked at the example of what to do in the specific problem of margin agreements creating an inhomogeneous case. Their solution was particular to margin agreements causing the inhomogeneity. Larocque (2011) used the Aumann Shapley value to determine a more generic allocation in the inhomogeneous case based upon Powers (2007) paper. The method here will be demonstrated for the case of allocating the incremental exposure of a set of trades, but without much adjustment to allocate an existing portfolio of trades with a collateral agreement. We will only document the more complicated case of allocating the incremental change in exposure. The problem is how to allocate at time t for the current trades, the incremental $EPE\left(t\right)$ of the n new trades $\alpha_1 V_1\left(t\right)+\ldots+\alpha_n V_n\left(t\right)$ at time t. The incremental $EPE\left(t\right)$ can be defined as:

$$E_t\left[min\left(H,\left(\sum_{i=1}^{n}\alpha_i V_i\left(t\right)+V\left(t\right)\right)^{+}\right)-min\left(H,E\left(t\right)\right)\right]$$

The problem solved in Larocque (2011) is how to allocate this value to the trades $V_1\left(t\right)$ to $V_n\left(t\right)$ in a way that is equitable as per Denaults' (2001) description.

The Aumann Shapley allocation method in the context of a cost function is assumed to be a continuously differentiable function $C\left(\alpha_1,\ldots,\alpha_n\right)$ with $C\left(0,\ldots,0\right)=0$. Fix q as a vector in $\mathbb{R}^n$and let $g\left(t\right)=C\left(tq\right)$. Basic calculus now gives:

$$C\left(q\right)=\sum_{i=1}^{n}\int_{0}^{1}\partial_i C\left(tq\right)q_i dt \qquad (11.11)$$

Powers (2007) writes that11.11 can be interpreted as the simple average (hence the integral with respect to the uniform density from 0 to 1) of the marginal changes in the risk measure as the participation level α of portfolio member i increases. For a given number α, we denote αV_i the i^{th} trade where the notional is multiplied by α as in Pykhtin and Rosen (2009). The incremental EPE can be written as $C\left(1,\ldots,1\right)$ where the function C is defined as:

$$C(\alpha_1,\ldots,\alpha_n) = E_t\left[min\left(H,\left(\sum_{i=1}^{n}\alpha_i V_i(t)+V(t)\right)^{+}\right) - min(H,E(t))\right] \tag{11.12}$$

The above C as defined from the incremental EPE is not continuously differentiable but it can be approximated arbitrarily close (in uniform norm) by a continuously differentiable function; such as for example, any sigmoid function such as tanh or the logistic function. However, for our application Larocque (2011) indicates that the fact that $\partial_i C$ is not defined at some points will not change the value of the resulting integral.

Larocque (2011) begins the derivation as follows:

$$\begin{aligned}\partial_i C(\alpha_1,\ldots,\alpha_n) &= \lim_{h\to 0} E_t\left[\frac{min\left(H,\left(\sum_{i=1}^{n}\alpha_i V_i(t)+V(t)+hV_i(t)\right)^{+}\right)}{h}\right. \\ &- \left.\frac{min\left(H,\left(\sum_{i=1}^{n}\alpha_i V_i(t)+V(t)+V_i(t)\right)^{+}\right)}{h}\right]\end{aligned} \tag{11.13}$$

$$= \lim_{h\to 0} E_t\begin{cases}0, & if\ \sum_{i=1}^{n}\alpha_i V_i(t)+V(t)>H\\ (min(H,H+hV_i(t))-H), & if\ \sum_{i=1}^{n}\alpha_i V_i(t)+V(t)=H\\ V_i(t) & if\ \sum_{i=1}^{n}\alpha_i V_i(t)+V(t)<H\\ \frac{(hV_i(t))^{+}}{h} & if\ \sum_{i=1}^{n}\alpha_i V_i(t)+V(t)=H\\ 0 & if\ \sum_{i=1}^{n}\alpha_i V_i(t)+V(t)<H\end{cases}$$

$$\begin{aligned}&= \lim_{h\to 0} E_t\left[\left(min(H,H+hV_i(t))-H\right)/h\right] P\left(\sum_{i=1}^{n}\alpha_i V_i(t)+V(t)=H\right)\\ &+ E_t\left[V_i(t)\,\mathbf{1}_{\left\{0<\sum_{i=1}^{n}\alpha_i V_i(t)+V(t)<H\right\}}\right]\\ &+ E_t\left[\frac{(hV_i(t))^{+}}{h}\right] P\left(\sum_{i=1}^{n}\alpha_i V_i(t)+V(t)=0\right)\end{aligned}$$

The limit of the first and last expectations do not exist in general. For $> H$ we have

$$\lim_{h\to 0} E_t\left[\frac{min\left(H, H+hV_i\left(t\right)-H\right)}{h}\right]$$
$$= \lim_{h\to 0} E_t \begin{cases} V_i\left(t\right), & if\ hV_i\left(t\right)\leq 0 \\ 0, & if\ hV_i\left(t\right)>0 \end{cases}$$

and this limit breaks down as follows

$$\lim_{h\to 0^-} E_t\left[\frac{min\left(H, H+hV_i\left(t\right)-H\right)}{h}\right] = E_t\left[V_i\left(t\right)^+\right]$$

and

$$\lim_{h\to 0^+} E_t\left[\frac{min\left(H, H+hV_i\left(t\right)-H\right)}{h}\right] = E_t\left[V_i\left(t\right)^-\right]$$

while

$$\lim_{h\to 0^+} E_t\left[\frac{\left(hV_i\left(t\right)\right)^+}{h}\right] = E_t\left[V_i\left(t\right)^+\right]$$

and

$$\lim_{h\to 0^-} E_t\left[\frac{\left(hV_i\left(t\right)\right)^+}{h}\right] = E_t\left[V_i\left(t\right)^-\right] = 0.$$

If the distribution of $\sum_{i=1}^{n}\alpha_i V_i\left(t\right)+V\left(t\right)$ is continuous, the probabilities of it being equal to H or 0 are zero, and the limits are irrelevant. But in practice $\sum_{i=1}^{n}\alpha_i V_i\left(t\right)+V\left(t\right)$ within a discretized numerical simulation could offset $V\left(t\right)$ in such a way that these probabilities are not 0. It could be possible to specify $\partial_i C$ at these points, but since it is only the integral of $\partial_i C$ that is needed, Larocque (2011) chose to ignore this technicality as the numerical results remain the same. The above demonstration finally gives the partial differential of the cost function as follows:

$\partial_i C\left(\alpha_1,\ldots,\alpha_n\right) = E_t\left[V_i\left(t\right)\mathbf{1}_{\left\{0<\sum_{i=1}^{n}\alpha_i V_i(t)+V(t)<H\right\}}\right]$

and hence

$$Allocation_i = \int\limits_0^1 \partial_i C\left(u\left(1,\ldots,1\right)\right)\mathbf{1}du \tag{11.14}$$
$$= \int_0^1 E_t\left[V_i\left(t\right)\mathbf{1}_{\left\{0<\sum_{i=1}^n \alpha_i V_i(t)+V(t)<H\right\}}\right]du$$

It is important to note as that, it is also the case in Pykhtin and Rosen (2009), the $V_i\left(t\right)$ and the indicator function are not independent, which is the reason it is not possible to allocate only using information at the netting set exposure level. This means that the computation requires the $V_i\left(t\right)$ and $V\left(t\right)$ to be available, and hence is memory intensive. In 11.15 the simulations themselves are needed to compute the allocation. So in a Monte Carlo set of of M simulations, $s=1$ to M, we get

$$Allocation_i = \frac{1}{M}\sum_{s=1}^{M} V_i\left(s\right)\int_0^1 \mathbf{1}_{\left\{0<\sum_{i=1}^n \alpha_i V_i(t)+V(t)<H\right\}}du \tag{11.15}$$

where $V_i\left(s\right)$is the value of trade i, in simulation s, at time t. The last integral can be calculated as:

$$Integral_i\left(s\right) = \begin{cases} min\left(1,\left(\frac{H-V(s)}{\sum_{j=1}^n V_j(s)}\right)^+\right) & if\sum_{j=1}^n V_j\left(s\right)>0 \\ -min\left(1,\left(\frac{-V(s)}{\sum_{j=1}^n V_j(s)}\right)^+\right) & \\ \frac{sign(H-V(s))-sign(-V(s))}{2} & if\sum_{j=1}^n V_j\left(s\right)=0 \\ min\left(1,\left(\frac{-V(s)}{\sum_{j=1}^n V_j(s)}\right)^+\right) & if\sum_{j=1}^n V_j\left(s\right)<0 \\ -min\left(1,\left(\frac{H-V(s)}{\sum_{j=1}^n V_j(s)}\right)^+\right) & \end{cases} \tag{11.16}$$

where we define:

$$sign\left(x\right) = \begin{cases} 1, & if\ x>0 \\ 0, & if\ x=0 \\ -1, & if\ x<0. \end{cases}$$

Table 5: Gregory(2009) Simulation Results at Time (t) (Incremental trade allocation), $H = 2000$

Gregory(2009) Simulation Results at Time (t) (Incremental trade allocation), $H = 2000$										
Trade	s 1	s 2	s 3	s 4	s 5	s 6	s 7	s 8	s 9	s 10
$V(t)$	1822	-14001	-2941	-14909	-439	12277	-1517	-7465	11082	-36451
3	-2477	7190	-121	6852	-1519	-8859	-1206	2586	-7293	18828
4	-5556	-5148	-6152	-3303	1182	-5505	-5408	1732	2114	-2550
Total:	-6212	-11959	-9216	-11361	-776	-2087	-8132	-3147	5904	-20173
Δ:	-8034	2042	-6275	3548	-337	-14364	-6615	4318	-5178	16278
Δ, H:	-8034	2042	-6275	3548	-337	-4087	-6615	4318	0	16278

Table 6: Gregory(2009) Trade Examples: Marginal (∂), Incremental (Δ), Allocation Results

Gregory(2009) Trade Examples: Marginal (∂), Incremental (Δ), Allocation Results				
	Ordered Alloc.	Trade 3,4 simult.	Ordered Alloc.	Trade 3,4 simult.
Trade	Δ EPE	∂ Δ EPE	Δ EPE, $H = 2000$	∂ Δ EPE, $H = 2000$
$V(t)$	2518.1	2518.1	582.2	582.2
3	-1797.4	-1542.66	-182.2	-179.52
4	-130.3	-385.04	-200	-202.68
Total:	590.4	590.4	200	200

We will now give an application of the allocation described by Larocque (2011). This example will use the same simulation paths, where the first two trades in the Gregory (2009) example will be taken to be the existing portfolio $V(t)$, and trades 3 and 4 will be seen to be the incremental trades, in table 5.

Table 6 presents the results for the ordered incremental and the marginal allocation of the incremental effect of adding trades 3 and 4 simultaneously. Results are presented in the case with and without collateral and we can see that the incremental allocation of both trades simultaneously is able to reconcile with the ordered allocation method totals.

We have now established the Pykhtin and Rosen (2009) and Larocque (2011) methods of allocating the EPE or CVA to individual trades. This allocation allows for internal charging of the CVA and for the accountability of the exposure. Once an institution can begin to see the sources of exposure in detail, it is in a position to begin managing the exposure in an active manner, and begin taking a view on the CVA management mandates. In the next section we will discuss one of the possible methods of managing CVA. Appendix A provides Larocque's (2011) derivation of the Aumann Shapley allocation for the m day close out period case. It is presented, solely for the interested reader who may be implementing the method under such conditions, as the derivation is involved with extensive conditions.

12 Dynamically Hedging Counterparty Credit Exposure Risk

This section deals primarily with the trading approach to pricing and managing credit exposure, as it provides the most natural grounds for discussing the hedging of credit risk. The hedging of tall tree risks (risks that are few in number, isolated and large in nature) in the traditional and risk warehousing approach is a limited case as the hedging is more static in nature, whereas the credit portfolio approach lends itself to taking skewed bets on credit and market variables with the goal of producing alpha returns. The trading approach allows for a clear exposition in a simplified manner, in the sense that the estimation of exposure is done in the risk neutral measure and is related to the market prices on instruments. This gives the measure for the estimation of simulation parameters, and a direct link to the hedging instruments.

Essentially, dynamic hedging is an attempt to lock in the CVA that is being charged to the client and to the trader. It is a trading-based solution as to what to do with CVA and DVA, and their associated implications with pricing. The hedging costs should also form the basis of the charge to the traders, as well as associated risk premiums.

Gregory (2009) provides the following simple generic hedging formula for CVA at a single time period t to which we have added the higher order term:

$$
\begin{aligned}
\Delta CVA &= EPE\left[\frac{\partial CRD}{\partial h}\triangle h + \frac{\partial CRD}{\partial \delta}\triangle \delta + \frac{\partial CRD}{\partial r}\triangle r\right] \\
&+ CRD\left[\sum_{i=1}^{n}\frac{\partial EPE}{\partial x_i}\triangle x_i\right] \\
&+ \left[\sum_{i=1}^{n}\frac{\partial^2 EPE}{\partial CRD \partial x_i}\Delta CRD \Delta x_i\right] \\
&+ \ldots
\end{aligned}
$$

where CRD refers to the values of a premium leg of the CDS on the counterparty, and where the following derivatives refer to:

- $\frac{\partial CRD}{\partial h}$ = sensitivity of the CRD to the hazard rate,
- $\frac{\partial CRD}{\partial \delta}$ = sensitivity of the CRD to the recovery rate,
- $\frac{\partial CRD}{\partial r}$ = sensitivity of the CRD to interest rates,
- $\frac{\partial EPE}{\partial x_i}$ = sensitivity of the EPE to the ith risk factor,
- $\frac{\partial^2 EPE}{\partial CRD \partial x_i}$= cross sensitivities between the exposure, CRD and the ith risk factor.

In the above equation we have ignored the time dependence of the hedge parameters for clarity. This hedging equation seems pretty straightforward, but actually represents by definition the most complicated derivative possible as it comprises the super set of all other exotic derivatives. The equation contains many variables and many risks, as well as significant cross dependencies, with many term structures, where the market may not actually provide instruments that are available for adequate hedging. For example the banks own non-performance risks. Most importantly Gregory (2009) points out that the lack of arbitrage in mis-pricing, if the counterparty or bank mis-prices the counterparty risk with each other an arbitrageur is unlikely to be able to profit from such a mis-pricing. Which means that pricing CVA is not strictly a risk neutral problem. Not to mention that some of the market rate hedges for CVA essentially unwind the hedges put on by other desks at the institution.

So lets begin by a point made by Gregory (2009) that is very important. The dynamic hedging of a simple fixed rate bond with CDS's is non trivial in practice, for the following reasons:

- Bonds trading away from par: CDS protection is based on a fixed notional, protection value has to be rebalanced and is not static.
- Duration mis-match between CDS and bond due to annuity risk.
- The notionals will vary significantly to hedge credit risk changes on the bond with CDS credit changes.
- If collateral is involved on the CDS contract there are funding issues.
- There may be an economic loss on the bond but no credit event triggered, of vice versa.
- Interest rate risks, legal risks, and liquidity.

There are just a few issues to contend with on the hedging of a simple fixed rate bond with a financial instrument that was meant to hedge credit risk on bonds. The bond can be seen a the most simple case of a CVA exposure that requires dynamic hedging.

We will add further difficulty by adding the volatility and term variables. Sorensen and Bollier (1994) showed that the value of the CVA for a swap position could be seen as the counterparty having the option to default at any point in the future and therefore unwind the trade. So the future value of the swap exposure to the counterparty could be seen a default probability weighted position in swaptions to reverse the position of the swap. Sorensen and Bollier (1994) write that in effect, a swap party is short an option to receive (pay) fixed and long an option to pay (receive) fixed, while the counterparty simultaneously owns the opposite pair of options, and due to the term structure of interest rates these position exposures are not necessarily symmetrical. The authors go on to write that the values of these options will have an impact on fixed-coupon swaps rates negotiated. Their prices will depend on a number of factors - the swap parties' default probabilities, interest rate volatility and the term structure of interest rates. This set-up will provide the mini-example where we will center our discussion of hedging on.

The CVA formula $CVA = (1-R)\sum_{k=1}^{\infty} EPE_B(t_k)\left[PD(t_k) - PD(t_{k-1})\right] df(t_k)$ of the single swap position, in the Sorensen and Bollier (1994) example becomes:

$$CVA_{swap} = (1-R)\sum_{k=1}^{\infty} V_{swaption}(t_k, t^*, T)\left[PD(t_k) - PD(t_{k-1})\right] df(t_k)$$

.

where $V_{swaption}(t_k, t^*, T)$ is the value today of the option to reverse the position of the swap with maturity date T, where the swaption has the maturity date t^*. The bank ends up being short swaption volatility, that will have to be purchased in order to negate the changes related to volatility. Gregory (2009) remarks that the exposure of the swap will be defined by the interaction between the swaption payoff and the underlying swap duration. If one assumes the Black framework for hedging of swaptions a few possible methods may be used, and are independent of any assumptions on the term structure dynamics. These methods are:

- Swap Method - the price of a swaption can be decomposed as the sum of the underlying swap and a portfolio of bonds
- Zero-coupon bond method - Can be hedged using portfolios consisting of zero coupon bonds
- Forward swap method - Can be hedged using two forward starting swaps
- Swaption Method - Sell or Buy the reverse swap position
- Delta hedging - hedging with just swaps

In our case since the match is perfect one would likely use swaptions, or similar exchange traded instruments such as interest rate futures options, as shorting zero coupon bonds is difficult for many reasons, and swaps would incur further CVA charges. Essentially, when simulating the exposure for an interest rate swap one would like to minimize the mismatch between the simulation model exposure and the value of underlying hedging instrument. This would mean that one's simulation model should be able to replicate the swaption pricing to have consistent hedging. Trade-offs would have to be made in order to have the simulation model successfully model swap, cap/floor, swaption, and other interest rate derivatives parsimoniously with traded derivative prices. By using the reverse swaption, there is also the added benefit of reducing interest rate sensitivity in addition to locking in the volatility, and thus would appear to be an efficient hedge instrument.

Picoult (2005) gives the description of how to hedge this simplified portfolio consisting of a single interest rate swap in which the counterparty pays fixed and receives float. The author also makes a simplifying assumption that the value of the swap is a function of a single interest rate r, there is a single volatility σ, used to simulate the EPE. Picoult (2005) along the lines of Sorensen and Bollier (1994) indicates that such a simplified portfolio can be hedged using the following instruments:

- Buying a CDS on a bond issued by the counterparty with a duration that matches the counterparties exposure profile, and a par amount equal to the EPE, which will have to be changed as the EPE changes. Consideration should be given to the value of cheapest to deliver option, and possible delivery squeezes that could impact hedging performance. As well the jump to default risk will not likely be hedged adequately, and will likely be relegated to unexpected losses to which capital must be held against.

- The sensitivity of CVA to the changes in r, where in our example the EPE increases if r increases and hence CVA will increase. To hedge this, one would enter into another swap in the same direction (pay fixed, receive float) with a different counterparty but with a notional amount proportional with the spread. Picoult (2005) does not go in the details of this being a margined counterparty or using futures for this risk and the corresponding funding requirements of such hedging.

- The sensitivity of CVA to the changes in σ, the implied interest rate volatility. This can be hedged with r by buying a call option on a swap where we pay fixed and receive floating, with notional proportional to the spread. One should note this hedge will change the sensitivity of the CVA to the changes in r.

- The cross gamma changes in CVA and changes in spread. Picoult (2005) indicates that if the correlation of changes in CVA and changes in credit spreads were positively correlated we would have wrong-way risk, if negatively correlated we would have right-way risk. Typically in an economic downturn increases in credit spreads are correlated with falling interest rates. Indicating some right-way risk in our example, if implied volatility is uncorrelated with the level of interest rates.

If we were hedging BCVA instead of just CVA Picoult (2005) provides the following:

- The market components of BCVA can be hedged in similar manner to CVA. Thus to hedge the interest rate and implied volatility components of the BCVA, one could sell a put option on the swap in which one would (pay fixed, receive float), with the notional proportional to the banks spread.

- To hedge the credit spread one would attempt to sell protection on one's own firm, in order to realise this value. To date we have not found a valid method of doing this regularly.

There is also the possibility of purchasing or selling contingent credit default swaps (CCDS) which for complete protection can be bought directly on the portfolio of OTC derivatives from a margined counterparty, leaving only residual hedge counterparty risks. Finally, attempts could be made to securitize the BCVA risks entirely, by issuing a credit linked note backed by the portfolio of OTC derivatives. Both of these methods provide some manner of obtaining limited price discovery for a portfolio of OTC derivative BCVAs, but also introduce significant information asymmetries and moral hazards that will have to be dealt with.

This should illustrate to the reader that the dynamic hedging of CVA and BCVA is not as straightforward as a simple application of the hedging equation, as small changes in one variable due to to cross dependencies can incur significant re-balancing actions and costs. The risks that immediately need to be dealt with that are not included in the hedging equation explicitly, but may need to be priced, are as follows:

- Liquidity risks - which includes transaction costs bid / ask spreads, and squeezes. Large scale hedging of CVA will result in single direction hedging for large financial institutions, typically in the sovereign CDS market, Corp CDS market, CDS index markets, FX, and interest rates. The effects of this hedging could be substantial especially since large amounts of it is unidirectional. Liquidity issues and crowded trades become important issues.

- Credit correlation risks within the portfolio - it is extremely difficult to estimate the joint non-performance risks and market risk prices. Correlation can be substantial and most practitioners treat each credit as independent for pricing, but in reality hedges are usually put on with indices, and the total CVA is dependent on clustering of defaults which is likely given the standard corporate use of OTC markets to hedge FX, interest rate, and some production risks. Given the different composition of various bank books, and different correlation risks, BCVA including these risks is no longer a market clearing price.

- Model risks - on how well the models are able to reflect pricing realities. The impossibility of pricing some instruments in a coherent manner jointly, think of Caps, and Swaption models and their assumptions which also have to be consistent with swap curve evolution dynamics, which also have to be consistent with FX forward pricing and evolution dynamics. In essence, finance has not reached a level of coherence in modeling for all these risk factors and pricing functions to play coherently with each other in a correlated and consistent fashion.

- Qualitative risks - such as legal risks, netting and documentation risks and their eligibility and enforceability compared to what is represented in the systems.

In order to price CVA (BCVA) correctly the hedging costs and risk premiums must be reflected in the price charged. This section has highlighted the difficulties in performing that hedging cost estimation. Identification of the total cost to be allocated back to the traders, includes: the counterparty risk, capital charges, and hedging costs bid/ask, and risk premiums for hedging mismatches. Overall, general sensitivities to market factors can be hedged in aggregate across all counterparties.

13 Conclusion

Essentially, this thesis should be seen as a selected literature review guideline for more research and thought in the area of credit and counterparty risk. The discussion of implementation requirements sheds light immediately on the unknowns and limitations of modern finance. All assumptions should be questioned, as we have seen a large number of problems that have been inadequately, or incompletely dealt with by just using common financial analysis and techniques, without truly questioning the deeper underlying issues. Presently, if one reads www.defaultrisk.com one can see that research is just beginning in earnest in the area. Most researchers suspected that non-performance risks should be included in pricing, but large portions of the how to's such as pricing, accounting treatment, capital treatment, P&L treatment for various cases have not been investigated in detail.

Some areas for future research spring immediately to mind as a result of the issues addressed in this paper, of which just a few are provided below:

- Stochastic close-out period m related to the default process.
- Stochastic collateral modeling that incorporates the currency option and other issues. How to address the relationship between collateral, exposure and credit worthiness in a consistent manner for risk measurement and allocating collateral useage costs.
- How to deal with hypothetical collateral pool composition modeling in the collateral call simulation for exposure measurement.
- How to correlate the variables in the CVA equation in a pragmatic manner.
- How to address general and specific wrong and right way risk in the estimation of CVA and pragmatic calibration of these models.
- Address the problem and inconsistencies of DVA from its accounting and P&L to capital treatment.
- Explore other possible allocation or management methods that solve the problem of differences between what is charged to the client to what is allocated.
- More sophisticated methods rather than brute force Monte - Carlo for the estimation of CVA.

- How to estimate correct hedge sensitivities in a parsimonious manner given technical constraints.

Gregory (2009) summarizes precisely why this thesis is relevant by collecting various considerations in the literature for the management of CVA. A CVA (BCVA) desk must manage and operate under the most unique set of conditions. They manage the most complicated trading book that is comprised of the super set of risks generated by exotic derivatives, and plain vanilla trades. The desk must take on the internal trade risk of the bank and provide fair equitable pricing and CVA allocations, to which they are unable to outrightly say no to or have the luxury of being extremely conservative in pricing as a default reaction. The desk must understand accounting, capital, legal and the pricing impact of such credit mitigants and termination events, and give a price to these elements that is fair yet sufficient. As well, numerous risks are not hedgeable in the market, are only partially hedgeable, incur negative gamma hedges if hedged, and are extremely sensitive and incur significant re-balancing costs to market changes due to cross dependencies.

The CVA (BCVA) desk operates in a non arbitrageable pricing world where most information is only partially available on risks to be hedged, and runs the real possibility of entering markets that may exist solely for hedging these risks likely creating crowded trades for risk mitigation. This thesis has highlighted different conceptual ideas on the management of CVA and has provided a quick overview of Larocque's (2011) contribution to the discussion of allocating CVA to the trade level by providing an alternative implementation of Pykhtin and Rosen (2009)'s allocation by using the Aumann Shapely value in inhomogeneous circumstances.

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A Larocque (2011) Aumann Shapley m day close out period case

This is to consider a collateral model where the collateral is delivered m days after the collateral call is made. At time t, the exposure is now driven by the portfolio value at time t, the threshold H and the portfolio value at time $t-m$. We have V_1 to V_n the new trades, and V the current portfolio. So the exposure at time t is:

$max\left(0, V_1\left(t\right) + \ldots + V_n\left(t\right) + V\left(t\right) - max\left(0, V_1\left(t-m\right) + \ldots + V_n\left(t-m\right) + V\left(t-m\right) - H\right)\right)$

Note that this formulation is exactly equivalent to equation (33) in Pykhtin and Rosen's (2009) paper. The inner max is simply to compute the collateral called at time $t-m$, which is the difference between the total exposure and the threshold H, if positive. Now the EPE at time t is simply:

$$
\begin{aligned}
EPE\left(t\right) &= E_t\left[max\left(0, V_1\left(t\right) + \ldots + V_n\left(t\right) + V\left(t\right)\right.\right. \\
&- \left.\left. max\left(0, V_1\left(t-m\right) + \ldots + V_n\left(t-m\right) + V\left(t-m\right) - H\right)\right)\right]
\end{aligned}
$$

In order to simplify the equations, lets define:

$\tilde{V}\left(t\right) = V_1\left(t\right) + \ldots + V_n\left(t\right)$

For a given number α, we denote αV_i the i^{th} trade where the notional is multiplied by α (this is considered the infinitely divisible case in some papers). Alternatively, one can consider αV_i to be the trade defined by taking the values of trade V_i across simulations, and multiplying them by α. The incremental EPE can be written as $C\left(1, \ldots, 1\right)$ where the function C is defined as:

$$
\begin{aligned}
C_t\left(\alpha_1, \ldots, \alpha_n\right) &= E_t\left[max\left(0, \alpha_1 V_1\left(t\right) + \ldots + \alpha_n V_n\left(t\right) + V\left(t\right)\right.\right. \\
&- \left.\left. max\left(0, \alpha_1 V_1\left(t-m\right) + \ldots + \alpha_n V_n\left(t-m\right) + V\left(t-m\right) - H\right)\right)\right] \\
&- E_t\left[max\left(0, V\left(t\right) - max\left(0, V\left(t-m\right) - H\right)\right)\right]
\end{aligned}
\tag{A.1}
$$

As before, Larocque (2011) demonstrated the Aumann Shapley allocation method in the context of a cost function as follows. A cost function is assumed to be a continuously differentiable function $C(\alpha_1, \ldots, \alpha_n)$ with $C(0, \ldots, 0) = 0$. Now fix q as a vector in $\mathbb{R}^n$and let $g(t) = C(tq)$ .for t any real number between 0 and 1. Basic calculus taking the derivative of g with respect to t now gives:

$$C(q) = \sum_{i=1}^{n} \int_{0}^{1} \partial_i C(tq)\, q_i dt \tag{A.2}$$

The above C as defined from the incremental EPE is not continuously differentiable but it can be approximated arbitrarily close (in uniform norm) by a continuously differentiable function. Such as for example, by any sigmoid function such as tanh or the logistic function. However, for our application Larocque (2011) indicates that the fact that $\partial_i C$ is not defined at some points will not change the value of the resulting integral. To compute the partial derivatives, let's first fix a point $(\alpha_1, \ldots, \alpha_n)$ and to ease the notation, define

$A(t) = \alpha_1 V_1(t) + \ldots + \alpha_n V_n(t) + V(t)$.

We now have:

$$\begin{aligned}
\partial_i C_t(\alpha_1, \ldots, \alpha_n) &= \lim_{h \to 0} \left(\frac{C_t(\alpha_1, \ldots, \alpha_i + h, \ldots, \alpha_n) - C_t(\alpha_1, \ldots, \alpha_n)}{h} \right) \\
&= \lim_{h \to 0} \left[\frac{E\left[max\left(0, A(t) - max\left(0, A(t-m) + hV_i(t-m) - H\right)\right)\right.}{h} \right. \\
&\quad \left. \frac{\left. -max\left(0, A(t) - max\left(0, A(t-m) - H\right)\right)\right]}{h} \right]
\end{aligned}$$

$$= \lim_{h \to 0} E \begin{cases} \left[\frac{(max(0, A(t) + hV_i(t) - A(t-m) - hV_i(t-m) + H) -}{h} \right. \\ \left. \frac{max(0, A(t) - A(t-m) + H))}{h} \right] & C1 \\ \left[\frac{(max(0, A(t) + hV_i(t) - max(0, hV_i(t-m))) -}{h} \right. \\ \left. \frac{max(0, A(t)))}{h} \right] & C2 \\ \frac{(max(0, A(t) + hV_i(t)) - max(0, A(t)))}{h} & C3 \\ V_i(t) - V_i(t-m) & C4 \\ \frac{max(0, h(V_i(t) - V_i(t-m)))}{h} & C5 \\ 0 & C6 \\ V_i(t) - \frac{max(0, hV_i(t-m))}{h} & C7 \\ \frac{max(0, hV_i(t) - max(0, hV_i(t-m)))}{h} & C8 \\ 0 & C9 \\ 0 & C10 \\ \frac{max(0, hV_i(t))}{h} & C11 \\ V_i(t) & C12 \end{cases}$$

Where the conditions are defined as follows:

$$
\begin{cases}
C1 = & if\, A(t-m) - H > 0 \\
C2 = & if\, A(t-m) - H = 0 \\
C3 = & if\, A(t-m) - H < 0 \\
C4 = & if\, A(t) > A(t-m) - H > 0 \\
C5 = & if\, A(t) = A(t-m) - H > 0 \\
C6 = & if\, A(t-m) - H > 0 \wedge A(t) < A(t-m) - H \\
C7 = & if\, A(t-m) - H = 0 \wedge A(t) > 0 \\
C8 = & if\, A(t-m) - H = 0 \wedge A(t) = 0 \\
C9 = & if\, A(t-m) - H = 0 \wedge A(t) < 0 \\
C10 = & if\, A(t-m) - H < 0 \wedge A(t) < 0 \\
C11 = & if\, A(t-m) - H < 0 \wedge A(t) = 0 \\
C12 = & if\, A(t-m) - H < 0 \wedge A(t) > 0.
\end{cases}
$$

As in the text, out of the nine limits, some don't exist, namely 2,4,5, and 8. According to Larocque (2011) this is not an issue since we will be integrating $\partial_i C_t$. Only the cases 1 and 9 are non-zero so the last equation can be written as:

$$
\partial_i C_t (x_1, \ldots, x_n) = E\left[V_i(t)\, \mathbf{1}_{A(t-m)<H \wedge A(t)>0} + \left(V_i(t) - V_i(t-m)\right) \mathbf{1}_{A(t)>A(t-m)-H>0}\right] \tag{A.3}
$$

.

Larocque (2011) remarks that $V_i(t)$ and $V_i(t-m)$ and their indicator functions are not independent, which is the reason why its not possible to allocate using only information at solely the exposure level. In formulaA.3 the individual simulations themselves are needed to compute the allocation. The allocation of EPE at time t and for trade i is:

$$
\begin{aligned}
Allocation_i(t) &= \int_0^1 \partial_i C_t (u(1, \ldots, 1))\, \mathbf{1} du \\
&= \int_0^1 \partial_i C_t (u, \ldots, u)\, du
\end{aligned}
$$

Where we remind the reader that $\tilde{V}$was the defined as the sum of all the V_i's. So in a Monte - Carlo set - up of M simulations, $s = 1$ to M , we get:

$$\begin{aligned}
Allocation_i(t) &= \frac{1}{M}\sum_{s=1}^{M}\int_0^1 V_i(t,s)\,\mathbf{1}_{u\tilde{V}(t-m,s)+E(t-m,s)<H\wedge u\tilde{V}(t,s)+E(t,s)>0}du \\
&+ \frac{1}{M}\sum_{s=1}^{M}\int_0^1 (V_i(t,s)-V_i(t-m,s))\,\mathbf{1}_{u\tilde{V}(t,s)+E(t,s)>u\tilde{V}(t-m,s)+E(t-m,s)-H>0}du
\end{aligned}$$

where $V_i(t,s)$ is the value of trade i , in simulation s, at time t.

The integrals can be calculated using an intermediate result. For four numbers a, b, a', b', let us define:

$F(a,b,a',b') = \int_0^1 \mathbf{1}_{ua+b>0\wedge ua'+b'>0}du.$

We can write the integrals using F :

$$\begin{aligned}
Allocation_i(t) &= \frac{1}{M}\sum_{s=1}^{M} V_i(t,s)\,F\left(-\tilde{V}(t-m,s), H-E(t-m,s), \tilde{V}(t,s), E(t,s)\right) \\
&+ \frac{1}{M}\sum_{s=1}^{M}(V_i(t,s)-V_i(t-m,s)) \\
&\times\ F\left(\tilde{V}(t,s)-\tilde{V}(t-m,s), E(t,s)-E(t-m,s)\right. \\
&+\ \left. H, V(t-m,s), \tilde{E}(t-m,s)-H\right)
\end{aligned}$$

Now the last step is to compute F.

In order to compute F, we start by computing:

$$\begin{aligned}
\int_0^1 \mathbf{1}_{u<x\wedge u<y\wedge u>\alpha\wedge u>\beta}du &= \int_{-\infty}^{\infty}\mathbf{1}_{u<min(1,x,y)\wedge u>max(0,\alpha,\beta)}du \\
&= max(0, min(1,x,y)-max(0,\alpha,\beta))
\end{aligned}$$

The difficulty in computing F is that the variable u is multiplied by a or a' and this can be negative, which reverses the inequality, or can even be 0. Larocque (2011) points out that the trick is to realise that the inequality $ua + b > 0$ can be broken up into 3 cases: $ua + b > 0$ is equivalent to $\begin{cases} u > -\frac{b}{a} & if\, a > 0 \\ b > 0 & if\, a = 0 \\ u < -\frac{b}{a} & if\, a < 0 \end{cases}$. Essentially, depending on the sign of a, the condition $ua + b > 0$ becomes an upper or lower bound. The middle condition is always satisfied if $b > 0$, and never if $b <= 0$.

So to compute F, initialise $\alpha = \beta = 0, x = y = 1$ and multiplier $= 1$ (only one multiplier). Then,

if $a > 0$, let $\alpha = \frac{-b}{a}$

if $a < 0$, let $x = \frac{-b}{a}$

if $a = 0$: if $b <= 0$, let multiplier $= 0$, otherwise do nothing.

Next, do the same thing with a', b' (resulting in possible changes to β, y or the same multiplier). Then, $F(a, b, a', b') = multiplier \times max(0, min(1, x, y) - max(0, \alpha, \beta))$ .Which ends the calculation of F.